Topology Vol. 15, pp. 343–347. Pergamon Press, 1976. Printed in Great Britain.

HOMEOMORPHISMS OF SUFFICIENTLY LARGE P^2-IRREDUCIBLE 3-MANIFOLDS

ALLEN HATCHER

(*Received* 11 *November* 1975)

LET V BE a compact connected PL 3-manifold which is irreducible, sufficiently large, and contains no embedded projective plane having a trivial normal bundle. Denote by $PL(V \text{ rel } \partial)$, $G(V \text{ rel } \partial)$ the simplicial spaces of PL homeomorphisms, respectively, homotopy equivalences of V which restrict to the identity on ∂V. Waldhausen[3] showed that the inclusion $PL(V \text{ rel } \partial) \to G(V \text{ rel } \partial)$ induces an isomorphism on π_0. Laudenbach[1] extended this to π_1. Pushing their techniques further, we prove in this paper:

THEOREM. *The inclusion* $PL(V \text{ rel } \partial) \to G(V \text{ rel } \partial)$ *is a homotopy equivalence.*

Since V is a $K(\pi, 1)$, it is an easy application of obstruction theory to determine the homotopy type of $G(V \text{ rel } \partial)$ when $\partial V = \emptyset$. One finds that $\pi_0 G(V)$ is isomorphic to the group of outer automorphisms of $\pi_1 V$, $\pi_1 G(V)$ is isomorphic to the center of $\pi_1 V$, and all higher homotopy groups of $G(V)$ vanish. Furthermore, Waldhausen[4] showed that when the center of $\pi_1 V$ is nontrivial and V is orientable, V is a Seifert manifold[2]. In this case either V is the 3-torus, or the center of $\pi_1 V$ is $\mathbf{Z}$, generated by an orbit of an S^1 action on V. Thus the identity component of $PL(V)$ contains a Lie subgroup ($\{1\}$, S^1, or $S^1 \times S^1 \times S^1$) as a deformation retract, when V is closed and orientable. Similar statements probably hold also in the non-orientable case. When $\partial V \neq \emptyset$, the components of $G(V \text{ rel } \partial)$ are contractible, though $G(V \text{ rel } \partial)$ itself need not be, e.g. for $V = S^1 \times S^1 \times I$.

It is a well-known consequence of the triangulation theorems of Bing and Moise that the simplicial spaces of PL and topological homeomorphisms of any 3-manifold have the same homotopy type, so our theorem holds also in TOP. To pass to the differentiable category one would need to know that $\text{Diff}(D^3 \text{ rel } \partial)$ is contractible (the "Smale conjecture").

The significance of the theorem is that it gives the first examples of manifolds M^n, $n > 2$, for which the full homotopy type of $PL(M^n \text{ rel } \partial)$ or $TOP(M^n \text{ rel } \partial)$ is known, excluding of course the trivial case $M = D^n$.

In some cases when $V \subset \mathbf{R}^3$ the theorem has also been obtained by T. Akiba.

§1. A SPECIAL CASE

For notational simplicity we will omit "rel ∂" from now on. Thus $PL(V)$ means $PL(V \text{ rel } \partial)$, etc.

Let V be a 3-disc with a finite number of 1-handles attached. In this section we prove that $PL(V)$, like $G(V)$, is contractible. This special case of the theorem contains the essential ideas of the general case while avoiding most of the technical complexities. Also, it will be used in the proof of the general case.

Let $M \subset V$ be the co-core of one of the 1-handles of V. Thus M is a 2-disc. Denote by $E(M, V)$ the simplicial space of PL embeddings $M \to V$ which are proper ($M \cap \partial V = \partial M$) and agree with the given $M \subset V$ on ∂M. Restriction to $M \subset V$ gives a fibration:

$$PL(V') \to PL(V) \to E(M, V),$$

where V' is V cut open along M. If we can show that $E(M, V)$ is contractible, then $PL(V)$ deforms into $PL(V')$ which is contractible by induction on the number of handles. (The induction starts with $PL(D^3) \simeq *$ by the Alexander trick.)

Let $M \times I \subset V$, $I = [0, 1]$, be a collar on the given $M = M \times 1$ and set $N = M \times 0 \subset V$.

PROPOSITION 1. *The subspace* $E(M, V - N)$ *of* $E(M, V)$ *consisting of embeddings disjoint from* N *is a deformation retract of* $E(M, V)$.

Assuming this, it remains to show that $E(M, V - N)$ is contractible. Consider the fibration

$PL(M\times I)\to E \xrightarrow{r} E(M, V-N)$, where E is the space of embeddings $M\times I\to V$ agreeing with the given $M\times I\subset V$ on $M\times 0\cup \partial M\times I$, and r is restriction to $M\times 1$. Note that r is surjective since V is irreducible. It is not hard to see that E is contractible. Since $PL(M\times I)\simeq *$ by the Alexander trick, the result follows.

Proof of Proposition 1. Let $M_t\to V$, $t\in D^k$, represent an element of $\pi_k(E(M, V), E(M, V-N))$. Let $(N\times[-1,1], \partial N\times[-1,1])$ be a bicollar neighborhood of $(N,\partial N)$ in $(V,\partial V)$, which we may assume to be disjoint from ∂M_t.

LEMMA 1. *There exist finite covers $\{B_i\}$ and $\{B'_i\}$ of D^k by k-balls, with $B'_i\subset \text{int}\, B_i$, such that M_t is transverse to a slice $N_i = N\times s_i\subset N\times[-1,1]$ for $t\in B_i$. Also, we may assume that $N_i\neq N_j$ if $i\neq j$.*

Proof. Triangulate so that the composition

$$\bigcup_t M_t\cap(N\times[-1,1]\times D^k)\to N\times[-1,1]\times D^k\to[-1,1]\times D^k$$

is simplicial. The triangulation T of $[-1,1]\times D^k$ intersects each slice $[-1,1]\times t$ in a triangulation T_t. For fixed t and for s in the interior of a one-simplex of T_t, $M_t\cap N\times s$ is independent of s, up to isotopy, and hence M_t is transverse to $N\times s$. Thus in $[-1,1]\times D^k$ we need only avoid a subpolyhedron X which is zero-dimensional in each t-slice, namely, the simplices of T which project to D^k non-degenerately. So choose finite covers $\{B_i\}$, $\{B'_i\}$ and distinct $s_i\in[-1,1]$ such that $s_i\times B_i\cap X=\emptyset$ for each i. □

To prove the proposition we will construct a family h_{tu}, $0\le u\le 1$, of isotopies of M_t which for $t\in B'_i$ eliminates all the circles of intersection of M_t with N_i. This is sufficient to make M_t disjoint from N over D^k: "average" the N_i's together via a partition of unity subordinate to $\{B'_i\}$ to get a slice $N\times s(t)$ disjoint from the isotoped M_t, then ambient isotope $N\times s(t)$ to N. (If M_t is disjoint from N_i and N_j, it is disjoint from the region between N_i and N_j since M_t is connected and $\partial M_t\cap N\times[-1,1]=\emptyset$.)

Let $\mathscr{C}_t$ be the collection of circles in $M_t\cap N_i$ for all i such that $t\in B_i$. Each circle $C\in\mathscr{C}_t$, with $C\subset N_i$ say, belongs to a family $C_t\subset N_i$, $t\in B_i$, which varies by a k-parameter isotopy. C_t bounds unique discs $D_M^2(C_t)\subset M_t$ and $D_N^2(C_t)\subset N_i$. The inclusion relations among the D_M^2's define a partial ordering $<_M$ on $\mathscr{C}_t$: $C_t <_M C'_t$ if if $D_M^2(C_t)\subset D_M^2(C'_t)$. Similarly we have $<_N$ on $\mathscr{C}_t$.

The basic construction is the following. Fix a $t\in D^k$. If $C_t\in\mathscr{C}_t$ is minimal in $<_N$, then $D_M^2(C_t)\cup D_N^2(C_t)$ is an embedded 2-sphere in V, which bounds a (unique) 3-disc $D^3(C_t)$ in V since V is irreducible. Choose a homeomorphism of $(D^3(C_t); D_M^2(C_t), D_N^2(C_t))$ with a standard lens-shaped model $(D^3; D_+^2, D_-^2)$, where $\partial D^3 = D_+^2\cup D_-^2$, $D_+^2\cap D_-^2 = S^1$. There is then an evident isotopy of $D_M^2(C_t)$ across $D^3(C_t)$ to $D_N^2(C_t)$ which, if continued slightly to the other side of $D_N^2(C_t)$, gives an isotopy of M_t eliminating C_t.

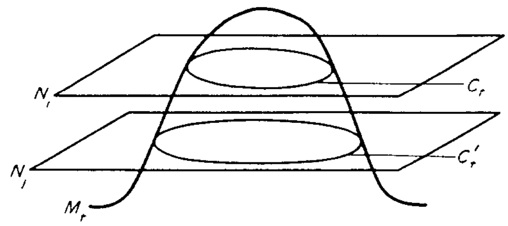
We must somehow piece together all these little isotopies for the various $C_t\in\mathscr{C}_t$ in such a way that the resulting isotopy h_{tu} is continuous in t. To begin, we construct a (PL) family of functions $\varphi_t:\mathscr{C}_t\to(0,2)$ which are to tell in what order the circles of $\mathscr{C}_t$ are to be eliminated. φ_t is to satisfy:

(1) $\varphi_t(C_t)<\varphi_t(C'_t)$ whenever $C_t <_N C'_t$,
(2) $\varphi_t(C_t)<1$ if $C_t\subset N_i$ and $t\in B'_i$,
(3) $\varphi_t(C_t)>1$ if $C_t\subset N_i$ and $t\in\partial B_i$.

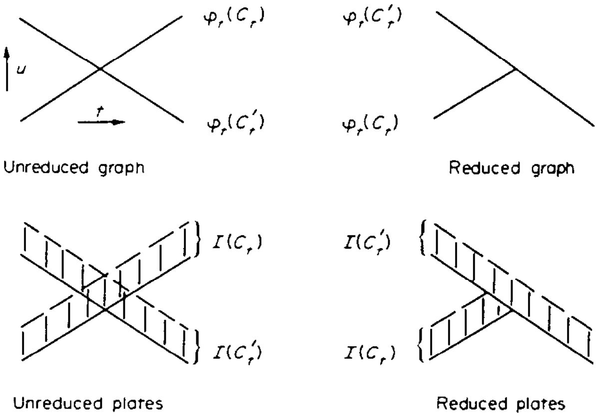
Such a family φ_t clearly exists.

Let $C_t\in\mathscr{C}_t$ with $C_t\subset N_i$. In the graph $\Gamma\subset D^k\times[0,2]$ of $\bigcup_t \varphi_t$, the points $\varphi_t(C_t)$ form a sheet lying over B_i. Thicken this sheet to a "plate" intersecting $t\times[0,2]$ in an interval $I(C_t)=[\varphi_t(C_t), \varphi_t(C_t)+\epsilon]$. We can assume ϵ is chosen small enough so that $I(C_t)\cap I(C'_t)\neq\emptyset$ only near the intersection of the sheets of Γ corresponding to C_t and C'_t, so that neither $C_t <_N C'_t$ nor $C'_t <_N C_t$.

$I(C_t)$ is intended to be the u-support of the little isotopy $h_{tu}(C_t)$ of M_t which eliminates C_t as above, assuming we have already eliminated all other circles of $\mathscr{C}_t$ with smaller φ_t-value, thereby making C_t minimal in $<_N$. However it might happen that for some $C'_t \in \mathscr{C}_t$ with $\varphi_t(C'_t) < \varphi_t(C_t)$, we also have $C_t <_M C'_t$.



In this case the isotopy which eliminates C'_t would automatically eliminate C_t. So we reduce the graph Γ by deleting all $\varphi_t(C_t)$ such that $\varphi_t(C_t) > \varphi_t(C'_t)$ for some C'_t with $C_t <_M C'_t$. Also, we reduce the intervals $I(C_t)$ by deleting points $u \in I(C_t)$ such that $u > \varphi_t(C'_t)$ for some C'_t with $C_t <_M C'_t$.



As a result, if two reduced intervals $I(C_t)$ and $I(C'_t)$ overlap, then C_t and C'_t are unrelated both in $<_M$ and in $<_N$. Hence $D^3(C_t) \cap D^3(C'_t) = \emptyset$ and the isotopies $h_{tu}(C_t)$ and $h_{tu}(C'_t)$ are completely independent of each other, having disjoint supports in V.

Now for fixed $t \in D^k$, the $h_{tu}(C_t)$'s can be strung together in the obvious way: Proceeding upward in the reduced Γ from $u = 0$ to $u = 1$, on the reduced interval $I(C_t)$, which is an initial segment of $[\varphi_t(C_t), \varphi_t(C_t) + \epsilon]$, take the restriction of $h_{tu}(C_t)$ to this initial segment. And on the overlaps of these reduced $I(C_t)$'s, do (the initial segments of) the corresponding $h_{tu}(C_t)$'s concurrently.*

To get h_{tu} simultaneously for all $t \in D^k$, first triangulate Γ and D^k so that the projection $\Gamma \to D^k$ is simplicial. (The reduced graph is then a subcomplex of Γ.) Assume inductively that h_{tu} has already been defined over the boundary of an l-simplex Δ^l in D^k. In particular, the $h_{tu}(C_t)$'s have been defined over $\partial\Delta^l$ as $PL(l-1)$-parameter families. To extend $h_{tu}(C_t)$ to an l-parameter family over the interior of Δ^l it is only necessary to extend the homeomorphism of $(D^3(C_t); D_M^2(C_t), D_N^2(C_t))$ with the standard model $(D^3; D_+^2, D_-^2)$ from $t \in \partial\Delta^l$ to $t \in \Delta^l$. This is done as follows. A family of homeomorphisms $f_t: D_M^2 \to D_+^2$ for $t \in B_i$ is chosen at the start by isotopy extension. Over $\partial\Delta^l$ we have by induction an extension of f_t to $F_t: D^3(C_t) \to D^3$, which we wish to extend over Δ^l. By isotopy extension f_t can be extended to F'_t: $D^3(C_t) \to D^3$ for $t \in \Delta^l$. The obstruction to extending F_t from $\partial\Delta^l$ to Δ^l is the homotopy class of $\partial\Delta^l \to PL(D^3 \text{ rel } D_+^2)$, $t \mapsto F'_t \circ F_t^{-1}$. But $PL(D^3 \text{ rel } D_+^2)$ is contractible by the Alexander trick. Thus each isotopy $h_{tu}(C_t)$ for $t \in \partial\Delta^l$ extends over Δ^l, and the same prescription for building h_{tu} from the $h_{tu}(C'_t)$'s for fixed t now works for all $t \in \Delta^l$ simultaneously. So h_{tu} for $t \in \partial\Delta^l$ extends to h_{tu} for $t \in \Delta^l$.

One final remark: If $C_t \in \mathscr{C}_t$, with $C_t \subset N_i$, then as t crosses ∂B_i, C_t is suddenly dropped from $\mathscr{C}_t$. This could cause a discontinuity in the family h_{tu}. However at ∂B_i, $\varphi_t(C_t) > 1$. So since we restrict u to $[0, 1]$ this difficulty does not actually occur. (The requirement that $\varphi_t(C_t) < 1$ for

*Note that if $\varphi_t(C_t)$ is not deleted from Γ, then C_t is not moved by h_{tu} for $u < \varphi_t(C_t)$.

$t \in B'_i$ guarantees that h_{tu} for $u \in [0, 1]$ will eliminate C_t over B'_i, as desired, provided that ϵ is small enough so that $I(C_t) = [\varphi(C_t), \varphi_t(C_t) + \epsilon] \subset [0, 1]$ for $t \in B'_i$.) □

§2. THE GENERAL CASE

Let V be P^2-irreducible and sufficiently large. We distinguish two cases: (1) $\partial V \neq \emptyset$, (2) $\partial V = \emptyset$. In (1), $\pi_k G(V) = 0$ for $k \geq 1$ and in (2), $\pi_k G(V) = 0$ for $k \geq 2$. We will prove the corresponding statements with G replaced by PL. This suffices to prove the theorem since Waldhausen and Laudenbach have already shown that $PL(V) \to G(V)$ induces an isomorphism on π_0 and π_1. *From now on we assume* $k \geq 1$ *in case* (1) *and* $k \geq 2$ *in case* (2).

As in §1, consider the fibrations

$$PL(V') \to PL(V) \to E(M, V),$$
$$PL(M \times I) \to E \to E(M, V - N),$$

where now M is an incompressible surface in V (and N is a parallel copy of M). If $\pi_k E(M, V) = 0$, then $\pi_k PL(V) \approx \pi_k PL(V')$. By hypothesis, V possesses a hierarchy, so that V can be reduced to a disjoint union of discs by a finite number of such cutting operations $V \to V'$. Hence we would have $\pi_k PL(V) \approx \pi_k PL(D^3) = 0$.

PROPOSITION 2. $\pi_k(E(M, V), E(M, V - N)) = 0$ *for* $k \geq 1$ *if* $\partial M \neq \emptyset$ *and for* $k \geq 2$ *if* $\partial M = \emptyset$.

Assuming this, we have $\pi_k E(M, V) \approx \pi_k E(M, V - N) \approx \pi_{k-1} PL(M \times I)$. In case (1), $\partial M \neq \emptyset$, and so $\pi_{k-1} PL(M \times I) = 0$ by §1. In case (2), where $\partial M = \emptyset$ for the first cut, $\pi_{k-1} PL(M \times I) = 0$ ($k \geq 2$) by case (1).

It remains only to prove Proposition 2, which is a simple matter of combining Laudenbach's methods with the machinery of §1. Let $M_t \to V$, $t \in D^k$, representing an element of $\pi_k(E(M, V), E(M, V - N))$, be transverse to N_i over $B_i \subset D^k$ as before. The proof of Proposition 1 applies in this setting to eliminate over $B'_i \subset B_i$ all the circles in $M_t \cap N_i$ which are contractible in V (hence also in M_t and N_i since $\pi_1 M_t$ and $\pi_1 N_i$ inject into $\pi_1 V$). So suppose only non-contractible circles remain in $M_t \cap N_i$.

Let $p: \tilde{V} \to V$ be the covering with $p_* \pi_1 \tilde{V} = \pi_1 M_t \subset \pi_1 V$. Let $\tilde{M}_t \to \tilde{V}$ be a lift of $M_t \to V$ (so p restricts to a homeomorphism of $\tilde{M}_t$ onto M_t. Denote by $\{N_{i,\alpha}\}$ the components of $p^{-1}(N_i)$. Each $N_{i,\alpha}$ separates V into two components. Let $Y_{i,\alpha}$ be the closure of the component not containing ∂M_t if $\partial M_t \neq \emptyset$, or if $\partial M_t = \emptyset$, the closure of the component not containing $\tilde{M}_t$ for $t \in \partial D^k$. ($Y_{i,\alpha}$ is well-defined in the latter case since ∂D^k is connected if $k \geq 2$. This is the only place where $k \geq 2$ in case (2) is required. It is not hard to extend this case to $k = 1$ by including some extra data about paths to a basepoint. (See [1], II.7.3.)

Now fix a $t \in B_i$. Let Y_{i,α_0} be minimal, with respect to inclusion, among the (finite number of) $Y_{i,\alpha}$'s intersecting $\tilde{M}_t$, and let C_t be a component of $\tilde{M}_t \cap Y_{i,\alpha_0}$. Assuming only that there exists a homotopy of M_t (rel ∂) into $V - N$, Laudenbach shows ([1], Corollary II.4.2 and Lemmas II.5.4(3) and (5)) that C_t is one end of a (unique) h-cobordism $W(C_t) \subset Y_{i,\alpha_0}$, the other end C'_t lying in N_{i,α_0}, and moreover that $p | W(C_t)$ is a homeomorphism. Since V is irreducible, $p(W(C_t))$ is a product h-cobordism. Choosing a product structure determines an isotopy of M_t carrying $p(C_t)$ across $p(W(C_t))$, eliminating $C_t \subset \tilde{M}_t \cap Y_{i,\alpha_0}$. Now repeat the process for the remaining components of $\tilde{M}_t \cap Y_{i,\alpha_0}$ until $\tilde{M}_t \cap Y_{i,\alpha_0} = \emptyset$. Then continue with the other $Y_{i,\alpha}$'s until $\tilde{M}_t$ is disjoint from $Y_{i,\alpha}$ for all α, and hence M_t is disjoint from N_i.

Adding parameters presents no new difficulties. Let Y_{i,α_0} be minimal among the $Y_{i,\alpha}$'s intersecting $\tilde{M}_t$ for some $t \in D^k$. The inductive step is to produce a k-parameter family of isotopies h_{tu} of M_t, $t \in D^k$, which eliminates $\tilde{M}_t \cap Y_{i,\alpha_0}$ over B'_i. Let $\mathscr{C}_t$ now denote the components C_t of $\bigcup_i(\tilde{M}_t \cap Y_{i,\alpha_0})$, the union over all i such that $t \in B_i$. Then $\mathscr{C}_t$ again has two partial orderings, $<_M$ and $<_N$, given by inclusion among the C_t's in $\tilde{M}_t$, or among the corresponding h-cobordant C'_t's in the $N_{i,\alpha}$'s. The only modification needed in the arguments of §1 is the following replacement for the contractibility of $PL(D^3 \text{ rel } D_+^2)$.

LEMMA 2. *Let* C *be a compact connected surface other than* S^2 *or* P^2. *Then* $PL(C \times I \text{ rel } C \times 0)$ *is contractible.*

Proof. Let F be the homotopy fiber of $PL(C) \to G(C)$ and consider the map of fibrations

$$\begin{array}{ccccc} PL(C\times I) \to & PL(C\times I \text{ rel } C\times 0 \cup \partial C\times I) & \to & PL(C\times 1) & \\ \downarrow & \downarrow & & \| & \\ \Omega G(C) \longrightarrow & F & \longrightarrow & PL(C) & \longrightarrow G(C) \end{array}$$

The vertical map to F comes from interpreting F as the space of pairs (f, γ), where $f \in PL(C)$ and γ is a path in $G(C)$ from f to 1. An element of $PL(C\times I \text{ rel } C\times 0 \cup \partial C\times I)$ determines such a pair by projection of $C\times I$ to C. The map $PL(C\times I)\to\Omega G(C)\simeq G(C\times I)$ is a homotopy equivalence by §1 if $\partial C\neq\emptyset$ and by case (1) if $\partial C=\emptyset$. Hence $PL(C\times I \text{ rel } C\times 0)$, which can be identified with $PL(C\times I \text{ rel } C\times 0\cup\partial C\times I)$, is homotopy equivalent to F. By surface theory $PL(C)\to G(C)$ is a homotopy equivalence if C is not S^2 or P^2, so F is contractible. □

REFERENCES

1. F. LAUDENBACH: Topologie de la dimension trois: homotopie et isotopie, *Astérisque* 12 (1974), Soc. math. de France.
2. P. ORLIK: Seifert manifolds, *Springer Lecture Notes* 291 (1972).
3. F. WALDHAUSEN: On irredicuble 3-manifolds which are sufficiently large, *Ann. Math.* **18** (1968), 56–88.
4. F. WALDHAUSEN: Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten, *Topology* **6** (1967), 505–517.

Princeton University
Institute for Advanced Study