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# ON THE DIFFEOMORPHISM GROUP OF A REDUCIBLE 3-MANIFOLD

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The loop space of the configuration space of César de Sá-Rourke-Hendriks-Laudenbach is identified as a factor (up to homotopy) of a certain space of imbeddings of a punctured 3-cell $B$ into the compact 3-manifold $M$. The restriction map from the diffeomorphism group of $M$ to this space of imbeddings is shown to be a product fibration. As an application, it is proved that the imbedding $\mathrm{Diff}(M \text{ rel } B \bmod \partial M) \to \mathrm{Diff}(M \bmod \partial M)$ induces injective homomorphisms of homotopy groups.

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## 1. Introduction

Let $M$ be a connected compact orientable 3-manifold with a sphere boundary component $S_0$. Then $M$ can be constructed as a connected sum of a disk $P_0$ whose boundary is $S_0$, with say $n$ irreducible 3-manifolds $P_1, P_2, \ldots, P_n$ ($P_i \not\approx S^3$) and say $g$ copies of $S^1 \times S^2$. Let $D_i$ be the connected sum disk in $P_i$. There is a compact codimension-zero submanifold $B$ of $M$, diffeomorphic to the closure of the complement of $n+2g$ disjointly imbedded disks in $P_0$, so that

(1) $S_0 = \partial B \cap \partial M$, and

(2) $M - B$ is the disjoint union of $P_i - D_i$, $i = 1, 2, \ldots, n$ and $g$ diffeomorphs of $(0,1) \times S^2$.

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Consider the space $\text{Imb}(B, M \bmod S_0)$ of imbeddings $B \to M$ which restrict to the inclusion on $S_0$. Let $\text{Imb}_e(B, M \bmod S_0)$ be the subspace of imbeddings that are extendible to a diffeomorphism of $M \bmod \partial M$ (i.e. restricting to the inclusion on $\partial M$).

As direct consequence of [5], we will prove the following:

**Theorem 1.** *The restriction map*

$$\text{Diff}(M \bmod \partial M) \xrightarrow{\rho} \text{Imb}_e(B, M \bmod S_0)$$

*which is a principal fibre bundle with fibre*

$$\prod_{i=1}^{n} \text{Diff}(P_i \bmod \partial P_i \cup D_i) \times \prod_{j=1}^{g} \text{Diff}([0,1] \times S^2 \bmod \{0,1\} \times S^2),$$

*is a product fibration.*

**Remark 1.** The analogous principal fibration

$$\text{Diff}(\hat{M} \bmod \partial \hat{M}) \to \text{Imb}_e(\hat{B}, \hat{M})$$

where

$$\hat{M} = M \bigcup_{S_0} D^3 \quad \text{and} \quad \hat{B} = B \bigcup_{S_0} D^3$$

is not necessarily a product fibration. For example, if $P_1$ and $P_2$ are aspherical then in $\hat{M} = P_1 \# S^3 \# P_2$ the rotations along the connected sum spheres of $P_1$ and $P_2$ with support in $P_1$ and $P_2$ respectively are isotopic, but there is no isotopy in the fibre of $\rho$.

**Remark 2.** A fibration like the one in Remark 1 has been directly studied by Jahren and by Hatcher in case $\hat{M} = S^1 \times S^2$ or $\hat{M} = P_1 \# P_2$ (see [6, 3]). In particular, Hatcher shows that in these cases the space of images of $\hat{B}$ by diffeomorphisms of $\hat{M}$ is contractible.

As an application consider the group $\text{Diff}(M \text{ rel } B \bmod \partial M)$ of diffeomorphisms of $M$ that map $B$ onto $B$ and that restrict to the inclusion on $\partial M$. We will show the following:

**Theorem 2.** *Suppose none of the irreducible summands $P_i$ has as universal cover a homotopy 3-sphere non-diffeomorphic to $S^3$. Then the inclusion map*

$$\text{Diff}(M \text{ rel } B \bmod \partial M) \to \text{Diff}(M \bmod \partial M)$$

*induces injective homomorphisms of homotopy groups.*

In fact, the hypothesis is not needed to prove the injectivity at the $\pi_0$-level. On the other hand, we will show that injectivity fails at the $\pi_i$-level for $i > 0$ if some $P_i$ is allowed to be a fake 3-sphere (see Section 4).

## 2. The configuration space

We will explain here how Theorem 1 fits into the context of configuration spaces as introduced by [1] and elaborated by [5]. In [5] is defined a semi-simplicial complex $C_1$, the configuration space, depending on the manifold $M$. The following difficult result is proved:

*There are H-space morphisms*:

$$\alpha:\ (F_g)^n \to \mathrm{Diff}(M \bmod \partial M),$$

$$\beta:\ \Omega C_1 \to \mathrm{Diff}(M \bmod \partial M),$$

$$\gamma = \text{inclusion:}\ \prod_{i=1}^{n} \mathrm{Diff}(P_i \bmod \partial P_i \cup D_i) \times \Omega O(3)^g \subset \mathrm{Diff}(M \bmod \partial M)$$

*so that the map defined by*

$$h(x, y, z) = \alpha(x) \circ \beta(y) \circ \gamma(z)$$

*is a homotopy equivalence.*

Here, $F_g$ denotes the free group on $g$ generators. It may be considered as a free factor of $\pi_1(M)$ corresponding to $\pi_1(\#_g S^1 \times S^2)$. For $(x_i)_{i=1}^n \in (F_g)^n$, $\alpha(x)$ corresponds to the composition of the slidings of the summands $P_i$ along a loop representing $x_i$. $\Omega O(3)$ corresponds to $\mathrm{Diff}([0, 1] \times S^2 \bmod \{0, 1\} \times S^2)$ by the Smale Conjecture (see [4]).

For applications of the theorem of [5] one may need knowledge of the space $C_1$. Although we feel that the space $C_1$ does not have a very complicated homotopy type, its definition is unhappily very delicate. Our main theorem will describe $\Omega C_1$ in more familiar terms.

**Theorem 3.** *The composition*

$$(F_g)^n \times \Omega C_1 \xrightarrow{\alpha\cdot\beta} \mathrm{Diff}(M \bmod \partial M) \xrightarrow{\rho} \mathrm{Imb}_e(B, M \bmod S_0)$$

*is a homotopy equivalence.*

Theorems 1 and 3 follow immediately from the result of [5] stated above by using the following lemma (where we take $S = (F_g)^n \times \Omega C_1$ and $G = \prod_{i=1}^n \mathrm{Diff}(P_i \bmod \partial P_i \cup D_i) \times \Omega O(3)^g$ and $\rho$ as in Theorem 1).

**Lemma 1.** *Let $G$ be a group and $\rho: E \to B$ a principal $G$-bundle, with $G$ acting on the right on $E$. Let $S$ be a space and $h: S \times G \to E$ a $G$-equivariant map which is a weak homotopy equivalence. Then*

(1) *$\mu: S \to B$ defined by $\mu(s) = \rho h(s, 1)$ is a weak homotopy equivalence.*

(2) *If $S$ and $B$ are dominated by CW-complexes, then $\mu$ is a homotopy equivalence and $\rho$ admits a section.*

**Proof.** Part (1) is immediate. If $S$ and $B$ are dominated by CW-complexes, then $\mu$ admits a homotopy inverse $\nu$. Then, $b \mapsto h(\nu(b), 1)$ is a homotopy section to $\rho$. By the homotopy lifting property of $\rho$ it is homotopic to a section. □

## 3. Application

We will prove Theorem 2. Recall that a 3-manifold is said to satisfy the *Poincaré Conjecture* if every compact contractible codimension-zero submanifold is a 3-ball (see [7]). The next two lemmas show that under our hypothesis, every compact subset of the universal cover $\tilde{M}$ of $M$ can be imbedded in $R^3$.

**Lemma 2.** *Suppose $M$ is a compact 3-manifold none of whose irreducible summands has universal cover a fake 3-sphere. Then $\tilde{M}$ satisfies the Poincaré Conjecture.*

**Proof.** Passing to the orientable double cover if necessary, we may assume that $M$ is orientable. Let $P$ be an irreducible summand of $M$. By Theorem 3 of [9], the universal cover of $P$ is irreducible. Since it is not a fake 3-sphere it must satisfy the Poincaré Conjecture. Since $\tilde{M}$ can be constructed by removing open 3-balls from the universal covers of the irreducible summands (including $S^3$) and then identifying copies of the resulting manifolds along some of their 2-sphere boundary components, results from Appendix I of [7] now show that $\tilde{M}$ satisfies the Poincaré Conjecture.

**Lemma 3.** *Suppose $N$ is a noncompact simply connected 3-manifold which satisfies the Poincaré Conjecture. Let $K$ be a compact subset of $N$. Then $K$ is contained in a submanifold which is a connected sum of punctured handlebodies.*

**Proof.** $K$ is contained in a compact codimension-zero submanifold $V$. We will use the terminology and results of Section 2 of [8]. By Lemma C and the fact that $N$ is simply connected, there is a sequence of simple moves (of type 2) which changes $V$ into a union $H_0$ of compact simply connected submanifolds. Since $N$ satisfies the Poincaré Conjecture, each component of $H_0$ is a punctured 3-cell. By Theorem 1 of [8], we can attach 1-handles to $H_0$ and then move the result $H$ by isotopy to contain $V$. We also may assume $H$ to be connected. Since $N$ is simply connected, $H$ must be a connected sum of punctured handlebodies.

We now proceed with the proof of Theorem 2. Let $\mathrm{Diff}_e(B \bmod S_0)$ denote the union of the components of $\mathrm{Diff}(B \bmod S_0)$ of diffeomorphisms which extend to a diffeomorphism of $M \bmod \partial M$. Note that $\rho^{-1}(\mathrm{Diff}_e(B \bmod S_0)) = \mathrm{Diff}(M \operatorname{rel} B \bmod \partial M)$ and denote by $\rho'$ the restriction of $\rho$ to this space. Since

$\rho$ and $\rho'$ are product fibrations with the same fiber, the theorem follows readily from the next lemma:

**Lemma 4.** *Suppose every compact subset of $\tilde{M}$ can be imbedded in $R^3$. Then the inclusion*

$$\mathrm{Diff}_e(B \bmod S_0) \to \mathrm{Imb}_e(B, M \bmod S_0)$$

*induces injective homomorphisms of homotopy groups.*

**Remark 3.** By the Smale Conjecture, $\mathrm{Diff}(B \bmod S_0)$ has the homotopy type of $\Omega B_{0,n+2g}$ where $\Omega B_{0,n+2g}$ is the classical configuration space [2] of subsets of $n+2g$ elements of $R^3$ ($\cong \mathrm{Int}(P_0)$).

**Proof of Lemma 4.** Consider the universal covering $p: \tilde{M} \to M$. Let $\tilde{S}_0$ be a component of $p^{-1}(S_0)$, and let $\tilde{B}$ be the lift of $B$ such that $\tilde{S}_0 \subset \partial\tilde{B}$. Let $\tilde{S}_i$ $(1 \leq i \leq n+2g)$ be the other boundary components of $\tilde{B}$. There is a well defined map

$$\lambda: \mathrm{Imb}(B, M \bmod S_0) \to \mathrm{Imb}(\tilde{B}, \tilde{M} \bmod \tilde{S}_0)$$

defined by the unique path lifting property of $p$ (as $B$ is simply connected).

Let $\langle\phi_t\rangle \in \pi_q(\mathrm{Diff}(B \bmod S_0))$, with $q \geq 0$, and suppose that $\langle\phi_t\rangle$ is trivial in $\pi_q(\mathrm{Imb}(B, M \bmod S_0))$. Then the lifted element $\lambda_{\#}(\langle\phi_t\rangle)$ is trivial in $\mathrm{Imb}(\tilde{B}, \tilde{M} \bmod \tilde{S}_0)$. The image of a trivializing homotopy lies in a compact subset $K \subset \tilde{M}$, containing $\tilde{B}$, and from Lemmas 2 and 3 we may choose an imbedding of $K$ into a 3-ball $D$ so that $\tilde{S}_0$ is carried onto $\partial D$. Regarding $\mathrm{Imb}(\tilde{B}, K \bmod \tilde{S}_0)$ as a subspace of $\mathrm{Imb}(\tilde{B}, \tilde{M} \bmod \tilde{S}_0)$, we find that $\lambda_{\#}(\langle\phi_t\rangle)$ is trivial in $\pi_q(\mathrm{Imb}(\tilde{B}, K \bmod \tilde{S}_0))$.

Since each $P_i$ is either non simply connected or has nonempty boundary, none of the $\tilde{S}_i$ bounds a ball in $\tilde{M}$. Therefore for each $i$, $1 \leq i \leq n+2g$, there exists a point $x_i \in D$ such that

(1) $x_i \notin K$, and

(2) $x_i$ lies in the ball in $D$ bounded by $\tilde{S}_i$.

Now consider the composition

$$\mathrm{Diff}(B \bmod S_0) \to \lambda^{-1}(\mathrm{Imb}(\tilde{B}, K \bmod \tilde{S}_0)) \to \mathrm{Imb}(\tilde{B}, K \bmod \tilde{S}_0)$$
$$\to \mathrm{Imb}(\tilde{B}, D - \{x_1, x_2, \ldots, x_{n+2g}\} \bmod \tilde{S}_0).$$

Using the Smale Conjecture [4] it is easy to show that this composition is a homotopy equivalence. This shows that $\langle\phi_t\rangle$ is trivial.

## 4. Remarks

We will make two remarks. Without the hypothesis in Theorem 2 we still have the following remark.

**Remark 4.** *The following map induced by inclusion is injective*:

$$\pi_0 \operatorname{Diff}(M \text{ rel } B \bmod \partial M) \to \pi_0 \operatorname{Diff}(M \bmod \partial M).$$

**Proof sketch.** Let $f \in \operatorname{Diff}(M \text{ rel } B \bmod \partial M)$ with $f$ isotopic to Id mod $\partial M$. We have to see that $f|_B$ is isotopic to $\operatorname{Id}|_B$ in $\operatorname{Diff}_e(B \bmod S_0)$. As $f$ induces the identity homomorphism of the fundamental group of $(M, S_0)$, $f$ preserves each boundary component of each handle component (i.e. diffeomorph of $[0, 1] \times S^2$) of $M - \operatorname{Int}(B)$.

Let $S$ be the permutation group of indices $\{1, 2, \ldots, n\}$ generated by the transpositions $(i, j)$ for which $P_i$ and $P_j$ are closed and orientation preservingly diffeomorphic. Then it follows from [5] that there is a surjective homomorphism $\pi_0 \Omega C_1 \to S$. (In fact, $C_1$ admits an $S$-covering, which corresponds to a concept of configuration distinguishing diffeomorphic factors.) Now $f|_B \in \operatorname{Imb}_e(B, M \bmod S_0)$ comes from a particularly simple loop $\lambda$ in $\Omega C_1$, namely one with 'support' in $P_0$ (under $\rho\ \alpha \cdot \beta$). It can be seen that the permutation determined by $\lambda$ is exactly the permutation of the components of $\partial B$ determined by $f|_B$. As $f|_B$ is isotopic to $\operatorname{Id}|_B$ in $\operatorname{Imb}(B, M \bmod S_0)$, this permutation will be the identity and therefore $f|_B$ is isotopic to Id in $\operatorname{Diff}_e(B \bmod S_0)$.

Let $M = P_1 \# P_0 \# P_2$, so that $g = 0$ and there are only two irreducible summands (except $P_0$). We will prove the following:

**Remark 5.** *If $P_1$ or $P_2$ is a fake 3-sphere, then the inclusion*

$$\operatorname{Diff}_e(B \bmod S_0) \to \operatorname{Imb}_e(B, M \bmod S_0)$$

*induces the trivial homomorphism of homotopy groups in dimensions greater than* 0.

**Proof.** Suppose for the moment that it is not the case that $P_1$ and $P_2$ are closed and orientation preservingly diffeomorphic. Then it follows from Hatcher [3] (see Remark 2) that $\operatorname{Imb}_e(B, M \bmod S_0)$ is homotopy equivalent to $\Omega C'$ with $C' = P_1 \# P_2$. Furthermore in this case we have

$$\operatorname{Diff}_e(B \bmod S_0) \simeq \operatorname{Diff}(P_0 \bmod \{x_1, x_2\} \cup S_0) \simeq \Omega S^2.$$

And $\operatorname{Diff}_e(B \bmod S_0) \to \operatorname{Imb}_e(B, M \bmod S_0)$ corresponds to the map $\Omega S^2 \to \Omega(P_1 \# P_2)$ induced by a homeomorphism from $S^2$ onto the connected sum sphere of $P_1 \# P_2$. If $P_1$ or $P_2$ is homotopy equivalent to $S^3$, then this map is nullhomotopic!

Now we consider the case when $P_1$ and $P_2$ are closed and orientation preservingly diffeomorphic. Then $C'$ will be a quotient of $P_1 \# P_2$ by a fixed point free involution $T$ that permutes $P_1$ and $P_2$. Moreover, $\operatorname{Diff}_e(B \bmod S_0) \simeq \Omega RP^2$ and the map $\operatorname{Diff}_e(B \bmod S_0) \to \operatorname{Imb}_e(B, M \bmod S_0)$ corresponds to the map $\Omega RP^2 \to \Omega(P_1 \# P_2/T)$ induced by a homeomorphism from $RP^2$ onto the quotient of the connected sum sphere. Again, if $P_1$ and $P_2$ are homotopy spheres this mapping will give trivial homomorphisms of homotopy groups of dimension greater that 0.

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