

ON NORMAL MICROBUNDLES

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§1. INTRODUCTION

JOHN MILNOR invented the microbundle [6, 7], and proved the following basic theorems:

(A). (EXISTENCE OF INVERSE MICROBUNDLES). *For any microbundle ξ over a polyhedron, there exists a microbundle η such that $\xi \oplus \eta$ is trivial.*

(B). (STABLE EXISTENCE OF NORMAL MICROBUNDLES). *Let M be a submanifold of V . Then $M \times 0$ has a normal microbundle in $V \times \mathbf{R}^q$ for some q .*

Lashof and Rothenberg [5] proved:

(C) (STABLE ISOTOPY OF NORMAL MICROBUNDLES). *If ξ_0 and ξ_1 are normal microbundles on M in V , then for some k , $\xi_0 \oplus \epsilon^k$ and $\xi_1 \oplus \epsilon^k$ are isotopic normal microbundles on $M \times 0$ in $V \times \mathbf{R}^k$.*

The purpose of this article is to give new proofs of these facts. The ingenious constructions of Milnor and Lashof–Rothenberg are replaced by the standard approach of first proving the results for coordinate neighborhoods and then inducting on the number of coordinate neighborhoods needed to cover M . As byproducts of the proofs, new upper bounds are found for the dimension of η in (A), for q in (B), and for k in (C). The earlier bounds were exponential in $\dim \xi$ or $\dim M$, while the new ones are quadratic, and, in many special cases, linear. (In the piecewise linear case, however, the results of Haefliger–Wall [2] are much stronger. They prove, for example, that if $\dim V \geq 2 \dim M$, then M must have a normal microbundle.)

All definitions, theorems and proofs are set in both the piecewise linear and topological categories.

§2. OUTLINE OF PROOFS

The logical order of the proofs of Milnor and Lashof–Rothenberg is, roughly, (A) \Rightarrow (B) \Rightarrow (C). In the present paper this order is reversed. More precisely, a relative form of (C) is proved for the special case where M is an open set in Euclidean space. Next an isotopy extension theorem is proved, for isotopies of normal microbundles; its use requires addition of a trivial line bundle. Since every manifold is covered by a finite number of open sets that are homeomorphic to open sets in Euclidean space, (C) and (B) follow. It is well known that

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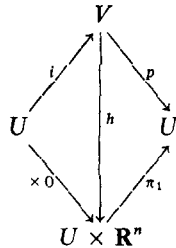
(B) \Rightarrow (A), since any microbundle ξ over a polyhedron B is stably isomorphic to $\tau_M|_B$, where M is a manifold containing B , and τ_M is its tangent microbundle. By (B), M has a normal microbundle in some Euclidean space; therefore τ_M , and hence ξ , has an inverse.

§3. TERMINOLOGY

A *microbundle* ξ of dimension n is a diagram

$$\xi : B \xrightarrow{i} E \xrightarrow{p} B$$

having the property that for each $x \in B$ there exist neighborhoods $U \subset B$ of x and $V \subset E$ of $i(U)$, and a homeomorphism $h : V \rightarrow U \times \mathbf{R}^n$ such that $p(V) = U$, and the diagram



commutes, where π_1 denotes projection on the first factor. Thus $p \circ i = 1_B$, the identity map of B . In the *PL* category, naturally, the maps i, p and h are piecewise linear. The *total space* of ξ is E , the *base space* is B , the *projection* is p , and i is the *zero section*. We may write $E = E\xi$ or $p = p_\xi$, etc. Usually B is identified with $i(B)$. If i is understood we may write $\xi = (p, E, B)$.

The *trivial* microbundle over B of dimension n is

$$\varepsilon^n(B) : B \xrightarrow{\times 0} B \times \mathbf{R}^n \xrightarrow{\pi_1} B.$$

If M is a manifold without boundary, the *tangent microbundle* of M is

$$\tau_1(M) = \tau_M : M \xrightarrow{d} M \times M \xrightarrow{\pi_1} M,$$

where d universally denotes the diagonal map $d(x) = (x, x)$. In addition, the *second tangent microbundle* of M is

$$\tau_2(M) : M \xrightarrow{d} M \times M \xrightarrow{\pi_2} M.$$

Let V be a manifold containing M . A *normal microbundle* on M in V is a microbundle

$$M \xrightarrow{i} E \longrightarrow M$$

where E is a neighborhood of M in V and $i : M \rightarrow E$ is the inclusion.

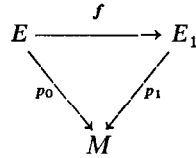
Let $P \subset V$ be a subset. An *isotopy* of P in V is a homotopy $f_t : P \rightarrow V$ such that $f_1 = 1_P$, and the map

$$F : P \times I \rightarrow V \times I, F(x, t) = (f_t(x), t)$$

is an embedding.

Let ξ_0 and ξ_1 be normal microbundles on M in V ; let $K \subset M$ be a subset. A *rel K isotopy* from ξ_0 to ξ_1 is an isotopy $f_t : E \rightarrow V$ such that, putting $E\xi_i = E_i$ for $i = 0, 1$,

- (i) $E \subset V$ is a neighborhood of M in E_0 ;
- (ii) f_t is fixed on M and on $E_0 \cap p_0^{-1}K$;
- (iii) f_1 is a microbundle isomorphism $\xi_0 \sim \xi_1$. That is, $f_1(E) \subset E_1$, and the diagram



commutes.

If such an isotopy exists, then ξ_0 is *rel K isotopic* to ξ_1 , written $\xi_0 \cong \xi_1 \text{ rel } K$. If $\xi_0 \cong \xi_1 \text{ rel } a$ neighborhood of K , we write $\xi_0 \cong \xi_1 \text{ rel } [K]$. In this case $p_0 = p_1$ in a neighborhood of K in $E_0 \cap E_1$, or in other words, ξ_0 and ξ_1 have the same *K germ*. In most instances the germ of a microbundle is the object of interest, rather than the whole microbundle. The notion of "germ" is made precise as follows. Given the submanifold $M \subset V$ and a subset $K \subset M$, consider pairs (U, ξ) where $U \subset M$ is an open neighborhood of K , and ξ is a normal microbundle on U in V . Define (U, ξ) and (U', ξ') to *have the same K germ* if there exists a neighborhood W of K in $E\xi \cap E\xi'$ such that $p_\xi|_W = p_{\xi'}|_W$. The equivalence class of (U, ξ) under this equivalence relation is the *K germ* of ξ , denoted by $[\xi]_K$. We say that $[\xi]_K$ *extends over M* if there exists a normal microbundle η on M in V such that $[\eta]_K = [\xi]_K$.

Given microbundles $\xi : B \xrightarrow{i} E \xrightarrow{p} B$ and $\eta : E \xrightarrow{j} D \xrightarrow{q} B$, the composition $\xi \circ \eta$ is $\xi \circ \eta : B \xrightarrow{j \circ i} D \xrightarrow{p \circ q} B$.

Given ξ and a map $f : B' \rightarrow B$, the *induced microbundle* $f^*\xi : B' \xrightarrow{i'} E' \xrightarrow{p'} B'$ is defined in the usual way:

$$\begin{aligned}
 E' &= \{(x, y) \in B' \times E \mid f(x) = p(y)\}, \\
 p'(x, y) &= x, \\
 i'(x) &= (x, i f(x)).
 \end{aligned}$$

The projection $\pi_2 : B' \times E \rightarrow E$ induces the *natural map* $E' \rightarrow E$.

The *Whitney sum* $\xi \oplus \eta$, where η is a microbundle over B , is defined to be the composition $\xi \circ (p^*\eta)$. We identify $\xi \oplus \eta$ with the microbundle $B \xrightarrow{(i, 0)} E \times \mathbf{R}^n \xrightarrow{p \circ \pi_1} B$ in the natural way.

For any space Y in the category, define the microbundle

$$Y \times \xi : Y \times B \xrightarrow{1 \times i} Y \times E \xrightarrow{1 \times p} Y \times B.$$

This is essentially the same as $\pi_2^*\xi$, induced by the projection $\pi_2 : Y \times B \rightarrow B$.

The composition of germs of microbundles is defined in the obvious way as the germ of the composition of microbundles.

§4. STATEMENT OF THE MAIN THEOREMS

Throughout the rest of the paper it is assumed that

$$K \subset U \subset M \subset V$$

where M is an m -manifold and V is a manifold, both without boundaries; U is an open subset of M and K is a closed subset of M .

For any k , we shall customarily identify a set A with $A \times 0 \subset A \times \mathbf{R}^k$.

THEOREM 1. *Let $q = (m + 1)^2$. Then:*

- (a) M has a normal microbundle in $V \times \mathbf{R}^{q-1}$
- (b) If ξ is a normal microbundle on U in V , the K germ of $\xi \oplus \varepsilon^q$ extends over M in $V \times \mathbf{R}^q$.
- (c) If ξ_0 and ξ_1 are normal microbundles on M in V having the same K germ, then

$$\xi_0 \oplus \varepsilon^{q-1} \cong \xi_1 \oplus \varepsilon^{q-1} \text{ rel } [K].$$

These results can be improved for smoothable manifolds:

THEOREM 2. *Let $q = 4m - 1$. Then (a) of Theorem 1 is true if M is smoothable, and (b) and (c) are true if $M - K$ is smoothable.*

Remark. Theorem 2 can be proved with "smoothable" replaced by "homeomorphic to a PL manifold", and with $q = 4m$.

Still better estimates hold for spheres:

THEOREM 3. *Let $M = S^m$. Then:*

- (a) S^m has a normal microbundle in $V \times \mathbf{R}^{m+2}$, and in $V \times \mathbf{R}^m$ if $m = 1, 3$ or 7 .
- (b) If $K \neq \emptyset$ and ξ is a normal microbundle on U in V , the K germ of $\xi \oplus \varepsilon^{m+1}$ extends over S^m in $V \times \mathbf{R}^{m+1}$.
- (c) If ξ_0 and ξ_1 are normal microbundles on S^m in V having the same K germ, then

$$\xi_0 \oplus \varepsilon^{m+2} \cong \xi_1 \oplus \varepsilon^{m+2} \text{ rel } [K].$$

Moreover, if $M = 1, 3$ or 7 , then

$$\xi_0 \oplus \varepsilon^m \cong \xi_1 \oplus \varepsilon^m \text{ rel } K.$$

THEOREM 4. *Let ξ be a microbundle of dimension n over a polyhedron B of dimension d . There exists a microbundle η over B of fibre dimension $(2 + n)(2n + d) + 2n$ such that $\xi \oplus \eta$ is trivial.*

§5. SYMMETRIC MANIFOLDS

An open subset of Euclidean space has two important properties: it is *parallelizable* and *symmetric*. A manifold M is parallelizable if τ_M is trivial. It is symmetric if $\tau_1(M) \cong \tau_2(M)$, considering $\tau_1(M)$ and $\tau_2(M)$ as normal microbundles on the diagonal $d(M) \subset M \times M$. It is easy to see that an open submanifold of a symmetric manifold is symmetric; more

generally, a manifold is symmetric if it immerses in a symmetric manifold of the same dimension. It is clear that the cartesian product of symmetric manifolds is symmetric.

A proof or disproof of the conjecture that all manifolds are symmetric would be interesting.

LEMMA 5. (a) *Every open subset of \mathbf{R}^m is symmetric.*

(b) *Every smoothable topological manifold is symmetric.*

Proof. (a) By the preceding discussion it suffices to prove that $\mathbf{R} = \mathbf{R}^1$ is symmetric. In the topological case this is obvious, since one can rotate counterclockwise each vertical line in \mathbf{R}^2 about its intersection with the diagonal until it is horizontal. The resulting isotopy from $\tau_1(\mathbf{R})$ to $\tau_2(\mathbf{R})$ is not piecewise linear, however. To take care of the PL case, define a trivialization of $\tau_1(\mathbf{R})$ by

$$g_1 : \mathbf{R} \times (-1, 1) \rightarrow \mathbf{R} \times \mathbf{R}; \quad G_1(x, y) = (x, x - y).$$

Define a PL isotopy by

$$g_t(x, y) = \begin{cases} (x + y, x) & \text{if } 0 \leq |y| \leq t \\ (x + t, x + t - y) & \text{if } 0 \leq t \leq y \\ (x - t, x - t - y) & \text{if } 0 \leq t \leq -y, \end{cases}$$

for $0 \leq t \leq 1$. Then g_0 trivializes $\tau_2(\mathbf{R})$, and g_t is fixed on $\mathbf{R} \times 0$. Therefore $g_t g_0^{-1}$ is a PL isotopy from $\tau_1(\mathbf{R})$ to $\tau_2(\mathbf{R})$.

Part (b) follows from the uniqueness of tubular neighborhoods of smooth submanifolds; see [4].

Remark. It is true that a compatibly smoothable PL manifold is symmetric as a PL manifold; the proof requires the triangulation of vector bundles [3, 5].

Next we consider normal microbundles on $M = d(M)$ in $M \times V$, where $d : M \rightarrow M \times V$ is the diagonal embedding. There is always

$$\tau_V | M : M \xrightarrow{d} M \times V \xrightarrow{\pi_1} M.$$

On the other hand, suppose that M has a normal microbundle ξ in V . We may assume that $E\xi = V$. Then there is also the normal microbundle

$$\xi^* : M \xrightarrow{d} M \times V \xrightarrow{p \circ \pi_2} M, \quad p = p_\xi.$$

To see that ξ^* is actually a microbundle, observe that it is the composition

$$\xi^* = \tau_2(M) \circ (M \times \xi)$$

of

$$M \times \xi : M \times M \subset M \times V \xrightarrow{1 \times p} M \times M$$

and

$$\tau_2(M) : M \xrightarrow{d} M \times M \xrightarrow{\pi_2} M.$$

To compare ξ^* with $\tau_V|_M$, express $\tau_V|_M$ as the composition

$$\tau_V|_M = \tau_1(M) \circ (M \times \xi).$$

LEMMA 6. *If M is symmetric then $\tau_V|_M$ and ξ^* are isotopic normal microbundles on $d(M) \subset M \times V$.*

Proof. By definition of "symmetric", $\tau_1(M) \cong \tau_2(M)$. Therefore the compositions of $\tau_1(M)$ and $\tau_2(M)$ with $M \times \xi$ are isotopic.

COROLLARY 7. *If ξ and η are normal microbundles on M in V , then $\xi^* \cong \eta^*$.*

Proof. Each is isotopic to $\tau_V|_M$.

Corollary 7 can be proved in a relative form:

LEMMA 8. *Assume M is symmetric. Let ξ_0 and ξ_1 be normal microbundles on M in V having the same K germ. Then*

$$\xi_0^* \cong \xi_1^* \text{ rel } [K].$$

Proof. We may assume that $\xi_0|_U = \xi_1|_U$, and that $E\xi_i = V$. Let the projection of ξ_i be $p_i: V \rightarrow M$. We shall find a neighborhood E of $d(M)$ in $M \times V$, and isotopies

$$f_{i,i}: E \rightarrow M \times V, \quad i = 0, 1$$

from ξ_i^* to $\tau_V|_M$, such that

(i) $f_{0,i} = \text{identity}$,

(ii) there is an open set $U_1 \subset U$ with $K \subset U_1$ and $\bar{U}_1 \subset U$, such that $f_{i,0}$ and $f_{i,1}$ restrict to the same isotopy from $(\xi_i|_{U_1})^*$ to $\tau_V|_{U_1}$.

Here $(\xi_i|_{U_1})^* = \tau_2(U_1) \circ (U_1 \times (\xi_i|_{U_1}))$. Recall the assumption that $\xi_0|_U = \xi_1|_U$.

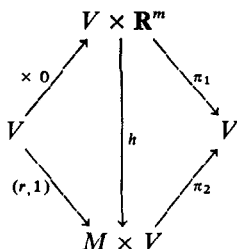
Assuming for the moment that such isotopies have been chosen, it is easy to see that $h_i = f_{i,1}^{-1} \circ f_{i,0}$ is a rel $[K]$ isotopy from ξ_0^* to ξ_1^* .

To define $f_{i,0}$ and $f_{i,1}$, let $A \subset M \times M$ be a neighborhood of the diagonal, and let $g_i: A \rightarrow M \times M$ be an isotopy from $\tau_2(M)$ to $\tau_1(M)$. The "covering homotopy theorem" provides a neighborhood E of $d(M)$ in $(1_M \times p_i)^{-1}A \subset M \times V$, and homotopies $f_{i,i}: E \rightarrow M \times V$ of microbundle maps $M \times \xi_i \rightarrow M \times \xi_i$ such that the diagrams

$$\begin{array}{ccc} M \times V \supset E & \xrightarrow{f_{i,i}} & M \times V \\ \downarrow & & \downarrow \\ M \times M \supset A & \xrightarrow{g_i} & M \times M \end{array}$$

commute, the vertical maps all being $1_M \times p_i$. Since g_i is an isotopy, so is $f_{i,i}$. Moreover, we can first choose $f_{i,0}$ and then choose $f_{i,1}$ to agree with it over a neighborhood of K in A . It is easy to see that $f_{i,i}$ is an isotopy from ξ_i^* to $\tau_V|_M$ as required.

LEMMA 9. Assume that M is parallelizable. Let $r : V \rightarrow M$ be a retraction. There is an embedding $h : V \times \mathbf{R}^m \rightarrow M \times V$ that makes commutative the diagram



Proof. As Milnor observed in [7], the total space of $r^*\tau_2(M)$ can be identified with $M \times V$ in a natural way making $r^*\tau_2(M)$ isomorphic to the microbundle

$$V \xrightarrow{(r, 1)} M \times V \xrightarrow{\pi_2} V,$$

which is the bottom edge of the diagram in the Lemma. Since M is parallelizable, $r^*\tau_2(M) \sim \varepsilon^m(V)$; the diagram of $\varepsilon^m(V)$ is the top edge. Therefore h exists as required.

THEOREM 10. Let M be parallelizable.

- (a) M has a normal microbundle in $V \times \mathbf{R}^m$.
- (b) Assume that M is also symmetric. If ξ_0 and ξ_1 are normal microbundles on M in V having the same K germ, then

$$\xi_0 \oplus \varepsilon^m \cong \xi_1 \oplus \varepsilon^m \text{ rel}[K].$$

Proof. (a). Follows from Lemma 9, because the embedding $h : V \times \mathbf{R}^m \rightarrow M \times V$ takes $M \times 0$ onto $d(M)$, which has the normal microbundle $\tau_V | M$.

(b). It is easy to see that if h is as in Lemma 9, then, for any normal microbundle ξ on M in V , h is an isomorphism $\xi \oplus \varepsilon^m \rightarrow \xi^*$. Hence the fact that $\xi_0^* \cong \xi_1^* \text{ rel}[K]$ (Lemma 8) implies that $\xi_0 \oplus \varepsilon^m \cong \xi_1 \oplus \varepsilon^m \text{ rel}[K]$.

Further progress requires an extension theorem for isotopies. The one given below is unsatisfactory in that it requires the addition of a trivial line bundle. This could perhaps be avoided by closer examination of the specific isotopies used in Lemmas 6 and 8.

First a definition. Two isotopies, defined in neighborhoods of K in a space X , have the same K germ if they agree in some neighborhood of K in X .

THEOREM 11. (ISOTOPY EXTENSION THEOREM). Let E be a neighborhood of K in V , and let $f_t : E \rightarrow V$ be an isotopy fixed on $M \cap E$. There exists a neighborhood E' of M in $V \times \mathbf{R}$ and an isotopy $H_t : E' \rightarrow V \times \mathbf{R}$ which is fixed on M and has the same K germ as $f_t \times 1$.

Proof. We first point out that V and M need not be manifolds; it is only necessary that V be a normal space.

Choose open neighborhoods E_0, E_1, E_2 of K in E such that $E_0 \subset E$, $E_1 \subset E_0$, and $f_t(E_2) \subset E_1$ for all $t \in I$. Let $\varphi : V \rightarrow \mathbf{R}$ be such that $\varphi(V - E_0) = 3$ and $\varphi(E_1) = 0$. Define

$T: V \times \mathbf{R} \rightarrow V \times \mathbf{R}$ by $T(x, s) = (x, s - \varphi(x))$. Obviously T is a homeomorphism. Choose $\tau: \mathbf{R} \times I \rightarrow I$ so that $\tau(s, t) = t$ for $s \geq -\frac{1}{2}$, and $\tau(s, t) = 0$ for $s \leq -1$.

Define $F_t: E \times \mathbf{R} \rightarrow V \times \mathbf{R}$ by $F_t(x, s) = (f_{\tau(s,t)}(x), s)$. Clearly $F_t|E \times (-\frac{1}{2}, \infty)$ is an isotopy.

Define $G_t: E \times \mathbf{R} \rightarrow V \times \mathbf{R}$ by $G_t = T^{-1}F_tT$. Clearly G_t is an isotopy on $E \times (-\frac{1}{2}, \infty)$. To obtain a formula for G_t , first define $\rho = \rho(x, s, t) = \tau(s - \varphi(x), t)$. Then

$$G_t(x, s) = (f_\rho(x), s - \varphi(x) + \varphi f_\rho(x)).$$

If $x \in E_2$ then $\varphi(x) = 0$, and, since $f_t(x) \in E_1$, also $f_t(x) = 0$. If $t \geq \frac{1}{2}$ then $\rho = t$. Therefore

$$G_t|E_2 \times (-\frac{1}{2}, \infty) = f_t \times 1.$$

In other words, G_t has the same K germ as $f_t \times 1$.

If $x \in M$ then $f_\rho(x) = x$, making G_t fixed on $M \times \mathbf{R}$.

If $x \in E - E_0$ then $\varphi(x) = 3$; if also $s < 1$, then $s - \varphi(x) < -1$ making $\rho = 0$. Hence

$$G_t|(E - E_0) \times (-\infty, 1) = \text{identity}.$$

Therefore $G_t|E \times (-\frac{1}{2}, 1)$ extends to an isotopy H_t of a neighborhood E' of M in $V(-\frac{1}{2}, 1)$ which is the identity on $(V - E_0) \times (-\infty, 1)$. The theorem is proved.

Applying Theorem 11 to the case where f_t is an isotopy of $\xi|U$, ξ being a normal microbundle on M in V , we arrive at the following corollary.

COROLLARY 12. *Let $\xi: M \subset V \xrightarrow{p} M$ be a normal microbundle, and $E \subset p^{-1}U$ an open neighborhood of K . If $f_t: E \rightarrow p^{-1}U$ is an isotopy of $\xi|U$ to a normal microbundle η on U in V , there is an isotopy h_t of $\xi \oplus \varepsilon^1$ to a normal microbundle ζ on M in $V \times \mathbf{R}$ such that*

- (a) h_t has the same K germ as $f_t \times 1: E \times \mathbf{R} \rightarrow V \times \mathbf{R}$;
- (b) ζ has the same K germ as $\eta \oplus \varepsilon^1$.

In particular,

- (c) the K germ of $\xi \oplus \varepsilon^1$ extends over M in $V \times \mathbf{R}$.

With the Isotopy Extension Theorem at our disposal, we can supplement Theorem 10 with the following result.

THEOREM 13. *Assume that $M - K$ is parallelizable and symmetric. If ξ is a normal microbundle on U in V , the K germ of $\xi \oplus \varepsilon^{m+1}$ extends over M in $V \times \mathbf{R}^{m+1}$.*

Proof. Let K' be a closed neighborhood of K in M which lies in U . By replacing M by $M - K$, U by $U - K$, and K by $K' - K$, we may assume that M is parallelizable and symmetric.

Recall that $\tau_V|M$ is a normal microbundle on $d(M)$ in $M \times V$. Moreover, by Lemma 6, $\tau_V|U$ and ξ^* are isotopic normal microbundles on $d(U)$ in $M \times V$. By definition, $\tau_V|U$ extends to a normal microbundle on $d(M)$ in $M \times V$. Lemma 9 shows that $\xi \oplus \varepsilon^m$ is isotopic to a normal microbundle on U in $V \times \mathbf{R}^m$ whose K germ extends over M . By Corollary 12c, the K germ of $\xi \oplus \varepsilon^{m+1}$ extends over M , proving the theorem.

§6. PROOF OF THE MAIN THEOREMS

The *breadth* $b(M)$ of a manifold M is the smallest number b such that M is covered by b open, symmetric, parallelizable submanifolds.

LEMMA 14. $b(M) \leq m + 1$.

Proof. Every m -manifold is covered by $m + 1$ open sets each of which is contained in some coordinate neighborhood; see [8]. By Lemma 5, such submanifolds are symmetric, and they are obviously parallelizable.

THEOREM 15. Put $b(M) = b$. Then M has a normal microbundle in $V \times \mathbf{R}^{b(m+1)-1}$.

Proof. By induction on b . The case $b = 1$ is covered by Theorem 10a. The inductive step is handled by Theorem 13. The details are left to the reader.

THEOREM 16. Put $b(M - K) = b$. Then:

(a) If ξ is a normal microbundle on U in V , the K germ of $\xi \oplus \varepsilon^{b(m+1)}$ extends over M in $V \times \mathbf{R}^{b(m+1)}$.

(b) If ξ_0 and ξ_1 are normal microbundles on M in V having the same K germ, then

$$\xi_0 \oplus \varepsilon^{b(m+1)-1} \cong \xi_1 \oplus \varepsilon^{b(m+1)-1} \text{ rel}[K].$$

Proof. By induction on b . The proof of (a) is similar to that of Theorem 15 and is left to the reader. The proof of (b) uses Theorem 10b and the Isotopy Extension Theorem (Corollary 12).

Observe that Theorem 1 follows from Theorems 15 and 16 and Lemma 14.

To prove Theorem 3, concerning the case $M = S^m$, first consider the cases $m = 1, 3$ or 7 —that is, assume that S^m is parallelizable. Then apply Theorem 10 to prove the relevant parts of (a) and (c). To prove (b), observe that if $K \neq \emptyset$, then $S^m - K$ is symmetric and parallelizable, and hence Theorem 16a applies. Finally, to prove the general case of (a) and (c), observe that $S^m \times \mathbf{R}$ is a symmetric, parallelizable submanifold of $V \times \mathbf{R}$. Hence by Theorem 10a, $S^m \times \mathbf{R}$ has a normal microbundle in $(V \times \mathbf{R}) \times \mathbf{R}^{m+1}$. Since obviously S^m has a normal microbundle in $S^m \times \mathbf{R}$, Theorem 3a follows. Part (c) is proved similarly.

Theorem 2 depends on the following generalization of the idea of replacing S^m by the symmetric parallelizable manifold $S^m \times \mathbf{R}$.

THEOREM 17. Suppose there is a symmetric parallelizable manifold N of dimension n , and a microbundle $\nu: M \rightarrow N \xrightarrow{r} M$. Then M has a normal microbundle in $V \times \mathbf{R}^{2n}$. If μ is a microbundle over M with $\nu \oplus \mu \sim \varepsilon^k$, then M has a normal microbundle in $V \times \mathbf{R}^{n+k}$.

Proof. We may assume a retraction $p: V \rightarrow M$. Consider the induced microbundle

$$p^*\nu: V \xrightarrow{i} W \xrightarrow{s} V$$

Since p is a retraction there is a natural embedding $N \subset W$; the natural map $t: W \rightarrow N$ covering $p: V \rightarrow M$ is then a retraction, and the following diagram commutes:

$$\begin{array}{ccccc}
 N & \subset & W & \xrightarrow{t} & N \\
 \downarrow r & & \downarrow s & & \downarrow r \\
 M & \longrightarrow & V & \longrightarrow & M
 \end{array}$$

Since N is symmetric and parallelizable, N has a normal microbundle in $W \times \mathbf{R}^n$, by Theorem 10a. If M is identified with the zero section of v , then M has a normal microbundle in $W \times \mathbf{R}^n = E(p^*v) \times \mathbf{R}^n$. If $v \oplus \mu \sim \varepsilon^k$, then M has a normal microbundle in $E(p^*(v \oplus \mu)) \times \mathbf{R}^n = V \times \mathbf{R}^{k+n}$. Since N is parallelizable, $\tau_M \oplus v \sim \varepsilon^n$, so that one may always take $k \leq n$.

Next, a relative form of the last theorem.

THEOREM 18. *Let ξ be a normal microbundle on U in V . Let N be a symmetric parallelizable n -manifold; let $r: N \rightarrow M - K$ be the projection of a microbundle v . Then the K germ of $\xi \oplus \varepsilon^{2n+1}$ extends over M in $V \times \mathbf{R}^{2n+1}$. If μ is a microbundle over $M - K$ such that $v \oplus \mu = \varepsilon^k$, then the K germ of $\xi \oplus \varepsilon^{n+k+1}$ extends over M in $V \times \mathbf{R}^{n+k+1}$.*

Proof. As in the proof of Theorem 13, we may replace M by $M - K$, etc. Thus we assume that the projection of v is $r: N \rightarrow M$, and that μ is defined over M also. The rest of the proof is similar to that of Theorem 17. We may assume that $p = r$ on $r^{-1}U$. Observe that the natural map $t: W \rightarrow N$ is a microbundle projection over a neighborhood of K in $r^{-1}U$. Call this microbundle η . By Theorem 13, N has a normal microbundle ζ in $W \times \mathbf{R}^{n+1}$ having the same K germ as $\eta \oplus \varepsilon^{n+1}$. It is easy to see that $\eta = (p|p^{-1}U)^*\xi$. In $W \times \mathbf{R}^{n+1}$, M has the composite normal microbundle $v \circ \zeta$. Over a neighborhood of K , $v \circ \zeta = v \oplus \xi \oplus \varepsilon^{n+1}$. Thus M has a normal microbundle in

$$E(p^*(v \oplus \mu)) \times \mathbf{R}^{n+1} = V \times \mathbf{R}^{k+n+1}$$

having the same K germ as $\xi \oplus \varepsilon^{k+n+1}$. Since N is parallelizable we may always take $k \leq n$.

Parts (a) and (b) of Theorem 2 now follow because a smooth m -manifold immerses in \mathbf{R}^{2m-1} ; the total space of the normal vector bundle of the immersion is a symmetric parallelizable manifold N of dimension $n = 2m - 1$. Part (c) is a consequence of the next theorem.

THEOREM 19. *Let ξ_0 and ξ_1 be normal microbundles on M in V having the same K germ. Let N be a symmetric parallelizable n -manifold, and $r: N \rightarrow M - K$ the projection of a microbundle v . Then $\xi_0 \oplus \varepsilon^{2n} \cong \xi_1 \oplus \varepsilon^{2n} \text{ rel}[K]$. If $v \oplus \mu = \varepsilon^k$, then $\xi_0 \oplus \varepsilon^{n+k} \cong \xi_1 \oplus \varepsilon^{n+k} \text{ rel}[K]$.*

Proof. Again we may assume that the projection of v is $r: N \rightarrow M$. We may also assume that ξ_0 and ξ_1 have projections $p_i: V \rightarrow M$, for $i = 0, 1$, that are homotopic $\text{rel}[K]$ (that is, rel a neighborhood of K).

Let W be the total space of p_0^*v . Thus $W = E(\xi_0 \oplus v)$. Let $q_0: W \rightarrow N$ be the natural

microbundle map $p_0^*v \rightarrow v$ that covers p_0 . Let $s : W \rightarrow V$ be the projection of p_0^*v . The following diagram is commutative:

$$\begin{array}{ccc} W & \xrightarrow{s} & V \\ q_0 \downarrow & & \downarrow p_0 \\ N & \xrightarrow{r} & M \end{array}$$

Since $p_0 \cong p_1 \text{ rel}[K]$, there is a map $q_1 : W \rightarrow N$ which is a microbundle map $p_0^*v \rightarrow v$ covering $p_1 : V \rightarrow M$, and which agrees with q_0 in a neighborhood of K in W .

It is easy to see that q_0 and q_1 are projections of normal microbundles η_0 and η_1 on N in W having the same K germ. Since N is symmetric and parallelizable, $\eta_0 \oplus \varepsilon^n \cong \eta_1 \oplus \varepsilon^n \text{ rel}[K]$, by Theorem 10b. Therefore the composite normal microbundles

$$t_i : W \times \mathbf{R}^n \rightarrow W \xrightarrow{s} V \xrightarrow{p_i} M, \quad i = 0, 1,$$

are isotopic $\text{rel}[K]$. Now $W = E(\xi_0 \oplus v)$. Therefore the composite normal microbundles

$$E(\xi_0 \oplus v \oplus \mu) \times \mathbf{R}^n \xrightarrow{\lambda \times 1} E(\xi_0 \oplus v) \times \mathbf{R}^n = W \times \mathbf{R}^n \xrightarrow{t_i} M$$

are isotopic $\text{rel}[K]$, where $\lambda : E(\eta_0 \oplus v \oplus \mu) \rightarrow E(\xi_0 \oplus v)$ is the natural projection. The theorem is now proved by an identification

$$E(\xi_0 \oplus v \oplus \mu) \times \mathbf{R}^n \approx E(\eta_0 \oplus \varepsilon^k) \times \mathbf{R}^n = V \times \mathbf{R}^{n+k}$$

which comes from an isomorphism $v \oplus \mu \sim \varepsilon^k$. The two resulting microbundles $V \times \mathbf{R}^{n+k} \rightarrow M$ are $\xi_i \oplus \varepsilon^{n+k}$.

To prove Theorem 4, start with a parallelizable symmetric $2n$ -manifold M containing B as a subcomplex and retracting onto B . For example, let M be the interior of a "regular neighborhood" of a PL immersion $B \rightarrow \mathbf{R}^{2n}$. The microbundle ξ extends to a microbundle over M , also denoted by ξ . Put $E = E\xi$. Observe that $\tau_E|_B \sim \xi \oplus \varepsilon^{2n}$, since M is parallelizable. Therefore if η is a normal microbundle for E in some \mathbf{R}^q , then

$$\xi \oplus \varepsilon^{2n} \oplus (\eta|_B) \sim \varepsilon^q.$$

It is easy to see that the breadth of the manifold E is $\leq n + 1$, because B , and hence M , can be covered by $n + 1$ opens sets over each of which ξ is trivial. The dimension of E is $2n + d$. Therefore, by Theorem 10a, if $E \subset \mathbf{R}^s$, there is a normal microbundle η on E in $\mathbf{R}^{s+(n+1)(2n+d)-1}$. The fibre dimension of η will be $s + n(2n + d) - 1$. A well known embedding theorem embeds E in \mathbf{R}^s with $s = 2 \dim E + 1 = 2(2n + d) + 1$. This proves Theorem 4.

Remark. With a little more work one can prove a relative form of Theorem 4. This is left to the reader as an exercise.

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