A HOMOTOPY$^+$ SOLUTION TO THE A-B SLICE PROBLEM

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Dedicated to the memory of Tim Cochran

Abstract. The A-B slice problem, a reformulation of the 4-dimensional topological surgery conjecture for free groups, is shown to admit a link-homotopy$^+$ solution. The proof relies on geometric applications of the group-theoretic 2-Engel relation. Implications for the surgery conjecture are discussed.

1. Introduction

Four-dimensional surgery is known to work in the topological category for a class of *good* fundamental groups. This result was originally established in the simply-connected case in [3], and it is currently known to hold for groups of subexponential growth and a somewhat larger class generated by these [4, 10, 15]. The A-B slice problem [5, 6] is a reformulation of the surgery conjecture for free groups, which is the most difficult case.

The A-B slice problem concerns decompositions of the 4-ball. The handle structure of a decomposition, interpreted as a Kirby diagram, gives rise to a *stabilization* of a given link L, see section 2 for a precise definition. In these terms, to show that L is A-B slice one needs to find a stabilization and band-sums between the components so that the resulting link is slice. The Generalized Borromean Rings (GBRs) are a collection of links any coinitial subset of which is universal for surgery. In a recent work [7] we showed that GBRs have a coinitial subset admitting a link-homotopy solution to the A-B slice problem. In other words, given such a GBR there exists a stabilization and band sums so that the resulting link is homotopically trivial (h-trivial) in the sense of Milnor [18]. Here we sharpen this result:

Theorem 1. *The A-B slice problem for a coinitial collection of generalized Borromean rings, forming universal surgery problems, admits a (link-homotopy)$^+$ solution.*

The Generalized Borromean Rings are the collection of links obtained from the Hopf link by (any non-trivial amount of) iterated ramified Bing doubling. There is a natural partial order on GBRs where more ramification and more Bing doubling means "less than", see figure 1.1.

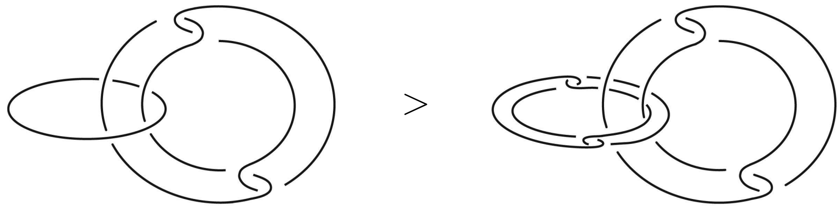

FIGURE 1.1. Partial order on Generalized Borromean Rings

An n-component link K is called *h-trivial*$^+$ if each one of the n links obtained by adding to K a parallel copy of a single component is homotopically trivial. The extension from h-trivial to h-trivial$^+$ is of interest in part due to the theorem [11] that untwisted Whitehead doubles of h-trivial$^+$ links are topologically slice. This means that the strongest possible version of the A-B slice problem for such links has a solution. That is, given an h-trivial$^+$ link, there exists a stabilization and band sums giving a slice link. It is interesting to relate this to Theorem 1 which, starting with a GBR, finds a stabilization yielding an h-trivial$^+$ link. A natural question is whether these two stabilizations can be combined to give a genuine A-B slice solution for GBRs.

As discussed in [7], if a link homotopy solution can be sufficiently improved, this could lead to an affirmative resolution to the surgery conjecture for all groups. Our result may be seen as a step in this direction. In terms of Milnor's $\bar{\mu}$-invariants, [7] constructs a stabilization giving a link with trivial $\bar{\mu}$-invariants with non-repeating indices. Theorem 1 improves this to a link with trivial $\bar{\mu}$-invariants with at most two repeating indices. An interesting question is whether there exists a stabilization giving a link with *all* vanishing $\bar{\mu}$-invariants.

The proof of theorem 1 may be extended to give a link-homotopy^{+k} solution, where $+k$ means that any link obtained by adding a total of k parallel copies of various components is homotopically trivial. However this gain comes at a price: the amount of Bing doubling in GBRs for which our methods give a link-homotopy^{+k} solution grows with k. (The simplest representative link has $2^{2k}+1$ components.) It follows from grope height raising [9] that such a collection of links, for a fixed k, is still universal for surgery, see [7, Proposition 4.1].

The novel ingredient in the construction of a link-homotopy solution in [7] is a geometric use of the group-theoretic 2-Engel relation $[[y,x],x]$, in conjunction with handle slides. An important algebraic feature underlying this construction is the fact that any 2-Engel group is nilpotent of a fixed class.

One natural way to approach the h-trivial$^+$ condition is to use the 3-Engel relation. However, as discussed in [7], n-Engel relations for $n > 2$ are generally not well understood. Instead, in this paper h-triviality$^+$ is achieved through a systematic application of the 2-Engel relation. More precisely, we study links modulo a more subtle relation which may be termed "$(2,2)$-Engel". Roughly, a group element satisfies this relation if it is trivial modulo the 2-Engel relation in two different ways. h-triviality$^+$ is seen to be a geometric consequence of this relation.

The background material on the A-B slice problem and the 2-Engel relation is summarized in sections 2, 3. Theorem 1 is proved in section 6.

2. A-B slice links and the relative slice problem

This section gives a brief summary of the relevant background on the A-B slice problem, the reader is referred to [8], for a more detailed exposition. We also state the notion of a link-homotopy$^+$ solution to the A-B slice problem, used in theorem 1.

A *decomposition* of D^4 is a pair of compact codimension zero smooth submanifolds with boundary $A, B \subset D^4$, satisfying conditions (1)-(3) below. Denote

$$\partial^+ A = \partial A \cap \partial D^4, \;\; \partial^+ B = \partial B \cap \partial D^4, \;\; \partial A = \partial^+ A \cup \partial^- A, \;\; \partial B = \partial^+ B \cup \partial^- B.$$

(1) $A \cup B = D^4$,
(2) $A \cap B = \partial^- A = \partial^- B$,
(3) $S^3 = \partial^+ A \cup \partial^+ B$ is the standard genus 1 Heegaard decomposition of S^3.

The "attaching curves" α, β of A, B (the cores of the solid tori $\partial^+ A$, respectively $\partial^+ B$) form the Hopf link in $S^3 = \partial D^4$.

Given a k-component link $L = (l_1, \dots, l_k) \subset S^3$, let $L' = (l'_1, \dots, l'_k)$ be its untwisted parallel copy.

Definition 2.1. A link L is *A-B slice* if there exist decompositions (A_i, B_i), of D^4 and self-homeomorphisms ϕ_i, ψ_i of D^4, $i = 1, \dots, k$ such that all sets $\phi_1 A_1, \dots, \phi_k A_k$, $\psi_1 B_1, \dots, \psi_k B_k$ are disjoint and satisfy: $\phi_i(\partial^+ A_i)$ is a tubular neighborhood of l_i and $\psi_i(\partial^+ B_i)$ is a tubular neighborhood of l'_i, for each i.

The collection of $2k$ manifolds $\{A_i, B_i\}$ are disjointly embedded into D^4 by the restrictions $\phi_i|_{A_i}$, $\psi_i|_{B_i}$. Since these maps are restrictions of self-homeomorphisms of D^4, the embeddings are *standard*, in the sense that the complement $D^4 \smallsetminus \phi_i(A_i)$ is homeomorphic to B_i, and $D^4 \smallsetminus \psi_i(B_i)$ is homeomorphic to A_i. (This condition that the embeddings are standard is important: it was shown in [14] that any link with trivial linking numbers is *weakly* A-B slice, when this condition is omitted.) A version of this requirement, in the link-homotopy setting, is stated as condition 2.4.

The 4-dimensional topological surgery conjecture for free groups was reformulated in [5, 6] in terms of the A-B slice problem for the Generalized Borromean rings (GBRs), the collection of links formed from the Borromean rings by iterated ramified Bing doubling. An example is shown in figure 6.1.

The notion of a link-homotopy$^+$ A-B slice link relies on a choice of handle decompositions of the submanifolds $\{A_i, B_i\}$. We will analyze them in the context of the *relative-slice problem*, introduced in [8].

Given a decomposition $D^4 = A \cup B$, without loss of generality it may be assumed [8] that each side A, B has a handle decomposition (relative to the collar $S^1 \times D^2 \times I$) with only 1- and 2-handles. Denote $A = (\partial^+ A) \times I \,\cup\, H_1 \,\cup\, H_2$. As usual, the 1-handles will be considered as standard 2-handles H_1^* removed from the collar, $A = (\partial^+ A \times I \setminus H_1^*) \cup H_2$. The decompositions constructed in this paper (see section 6) have the property that the 2-handles H_2 of each side do not go through the handles H_1^* of the same side. (See [7] for a discussion of this terminology.)

Suppose an n-component link L is A-B slice, with decompositions $D^4 = A_i \cup B_i$, $i = 1, \ldots, n$. Denote by D^4_0 a smaller 4-ball obtained by removing from D^4 the collars on the attaching regions $\phi_i(\partial^+ A_i), \psi_i(\partial^+ B_i)$ of all submanifolds $\{\phi_i(A_i),\ \psi_i(B_i)\}$. Let $\mathcal{H}_2$ denote the 2-handles of all these submanifolds, and $\mathcal{H}_1^*$ the 2-handles removed from the collars, corresponding to the 1-handles. Consider $\mathcal{H}_1^*$ as zero-framed 2-handles attached to D^4_0. (See [7, Section 3.1] for more details.)

Consider the following two links J, K in $S^3 = \partial D^4_0$. J denotes the attaching curves of the 2-handles $\mathcal{H}_2$, and K the attaching curves of the 2-handles $\mathcal{H}_1^*$. (Here $\mathcal{H}_2$ are 2-handles contained in D^4_0, and $\mathcal{H}_1^*$ are attached to D^4_0 with zero framings along K.) We call the pair (J, K) a *stabilization* of the original link L. The structure of the stabilization links which is a consequence of the duality between the 1- and 2-handles of the two sides of each decomposition is shown in figure 2.1.

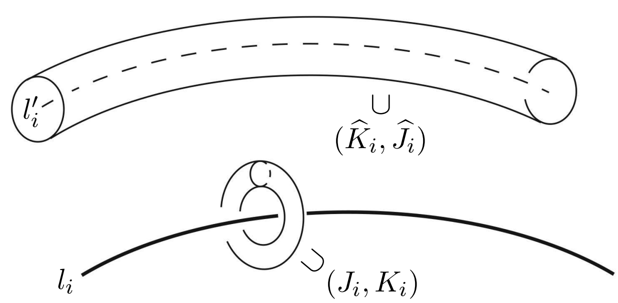

FIGURE 2.1. Stabilization corresponding to an A-B slice link $L = \{l_i\}$: link pairs $(J_i, K_i) \subset$ solid torus neighborhood of a meridian to l_i, $(\widehat{K}_i, \widehat{J}_i) \subset$ solid torus neighborhood of a parallel copy l'_i, and a diffeomorphism between the solid tori exchanging their meridian and longitude takes K_i to $\widehat{K}_i$ and J_i to $\widehat{J}_i$.

Definition 2.2. A link pair (J, K) in $S^3 = \partial D^4_0$ is called *relatively slice* if the components of J bound disjoint, smoothly embedded disks in the handlebody

$$H_K = D^4_0 \cup \text{zero-framed 2-handles attached along } K.$$

If a link L is A-B slice, the associated link pair (J, K) is relatively slice. We now turn to the definition of a link-homotopy$^+$ solution to the A-B slice problem, referred to in the statement of theorem 1.

Definition 2.3. A k-component link L is *link-homotopy$^+$ A-B slice* if there exist decompositions $D^4 = A_i \cup B_i$, $i = 1, \ldots, k$ and handle decompositions of the submanifolds A_i, B_i so that the associated relative-slice problem (J, K) has a link-homotopy$^+$ solution. In other words, in the context of definition 2.2 for each component l of J the link $J \cup l'$ bounds disjoint maps of disks Δ in the handlebody H_K. Here l' denotes an untwisted parallel copy of l. Moreover, the disks Δ are subject to condition 2.4 below.

Recall that in the formulation of the A-B slice problem the disjoint embeddings of the manifolds $\{A_i, B_i\}$ are required to be standard, see the paragraph following definition 2.1. We formulate a version of this condition in the link-homotopy setting (see [7, Section 3.1] for a more detailed discussion):

Condition 2.4. Let S be any submanifold in the collection $\{\phi_i(A_i),\ \psi_i(B_i)\}$. Then the maps of disks Δ for the components of $J \cup l'$ corresponding to S do not go through the 2-handles attached to D^4_0 along the components of K corresponding to the same submanifold S.

3. The Milnor group and the 2-Engel relation

We start by giving a brief overview of the Milnor group [18]. Let π be a group normally generated by a fixed finite collection of elements $g_1, \ldots, g_k$. The *Milnor group* of π, defined with respect to a given normal generating set $\{g_i\}$, is defined by

$$M\pi = \pi \,/\, \langle\langle\, [g_i, g_i^y] \;\; i = 1, \ldots, k, \;\; y \in G \rangle\rangle. \tag{3.1}$$

The Milnor group ML of a link L in S^3 is set to be the Milnor group $M\pi$ where $\pi = \pi_1(S^3 \smallsetminus L)$, defined with respect to meridians to the link components.

Denote by $F_{g_1,\ldots,g_k}$ the free group generated by the $\{g_i\}$, $i = 1, \ldots, k$. The Magnus expansion

$$M\colon\ F_{g_1,\ldots,g_k} \longrightarrow \mathbb{Z}[\![x_1, \ldots, x_k]\!] \tag{3.2}$$

into the ring of formal power series in non-commuting variables $\{x_i\}$ is given by

$$M(g_i) = 1 + x_i, \;\; M(g_i^{-1}) = 1 - x_i + x_i^2 - x_i^3 \pm \ldots$$

The Magnus expansion induces an injective homomorphism

$$MF_{g_1,\ldots,g_k} \longrightarrow R_{x_1,\ldots,x_k}, \tag{3.3}$$

into the quotient $R_{x_1,\ldots,x_k}$ of $\mathbb{Z}[\![x_1, \ldots, x_k]\!]$ by the ideal generated by all monomials $x_{i_1} \cdots x_{i_k}$ with some index occuring at least twice. Milnor's $\bar{\mu}$-invariants of a link are defined in terms of coefficients of the Magnus expansion [19].

Two links are *link-homotopic* if they are connected by a 1-parameter family of link maps where different components stay disjoint for all values of the parameter. If L, L' are link-homotopic then their Milnor groups ML, ML' are isomorphic. Moreover, a k-component link L is homotopically trivial (h-trivial) if and only if ML is isomorphic to the free Milnor group $MF_{m_1,\ldots,m_k}$. Equivalently, a link is h-trivial if and only if all its $\bar{\mu}$-invariants with non-repeating indices are trivial.

This paper concerns a stronger version of this equivalence relation. An n-component link L is called *h-trivial*$^+$ if each one of the n links obtained by adding to L a parallel copy of a single component is homotopically trivial. A link is h-trivial$^+$ if and only if all its $\bar{\mu}$-invariants with at most two repeating indices are trivial.

3.1. **2-Engel groups.** Given a group π, consider its lower central series defined by $\pi^1 = \pi, \pi^n = [\pi^{n-1}, \pi]$. It is convenient to introduce a concise commutator notation

$$[g_1, g_2, \ldots, g_n] := [[\ldots[g_1, g_2], \ldots, g_{n-1}], g_n].$$

This paper concerns geometric applications of the 2-Engel relation $[[y, x], x] = 1$, or equivalently $[x, x^y] = 1$. A 2-*Engel group* π is a group satisfying this relation for all $x, y \in \pi$. Note the difference with the definition of the Milnor group (3.1) where this relation is imposed only on x in a fixed set of normal generators.

The free Milnor group on n generators MF_n is nilpotent of class n [18]. In contrast, the nilpotency class of 2-Engel groups is independent of the number of generators. This result, building on earlier work of Burnside [1], is due to Hopkins [12] (also see [16]):

Lemma 3.1. *Any 2-Engel group is nilpotent of class* ≤ 3.

A proof in the context of the Milnor group is given in [7]. The following corollary of the proof [7, Corollary 2.3] will be useful for applications in the next section.

Corollary 3.2. *Suppose π is a group normally generated by $g_1, \ldots, g_n$. Let $g \in \pi^k$ be an element of the k-th term of the lower central series, $4 \leq k \leq n$. Then g may be represented in the Milnor group $M\pi$ as a product of (conjugates of) k-fold commutators of the form $[h_1, \ldots, h_k]$ where two of the elements h_i are equal to each other and to a product of two generators, $h_j = h_m = g_{i_1} g_{i_2}$ for some $j \neq m$, and each other element h_i is one of the generators $g_1, \ldots, g_n$.*

Figure 3.1 shows examples of links which are a geometric realization (for $k = 4$) of the types of commutators that appear in the statement of corollary 3.2. A central feature of these homotopically *essential* links is that a 0-framed handle slide (in the notation of the figure, of the z-curve over the y-curve) gives a split link consisting of an unknot and a homotopically *trivial* link pictured in figure 6.2.

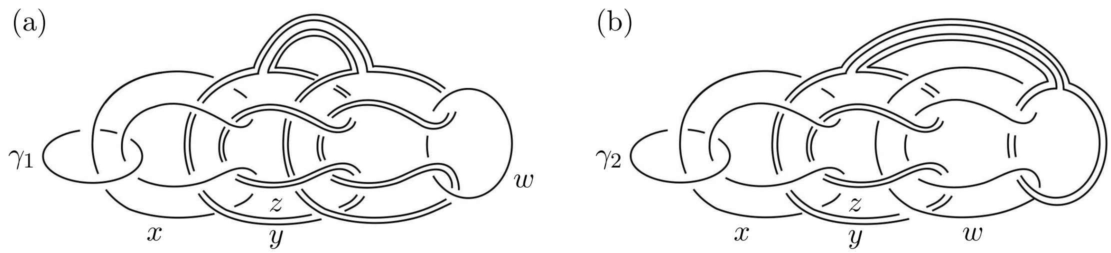

FIGURE 3.1. (a): $\gamma_1 = [x, yz, yz, w]$, (b): $\gamma_2 = [x, yz, w, yz]$.

4. A motivating example

Before giving a formal proof of theorem 1 we illustrate the idea underlying h-triviality$^+$ in the set-up in figure 4.1. Start with the Borromean rings and let T_1, T_2 denote solid torus neighborhoods of two of the components. Let $L_i \subset T_i$ be two links embedded in these solid tori. Denote by Λ_i a meridian of T_i: a curve in ∂T_i bounding a disk in T_i, figure 4.1.

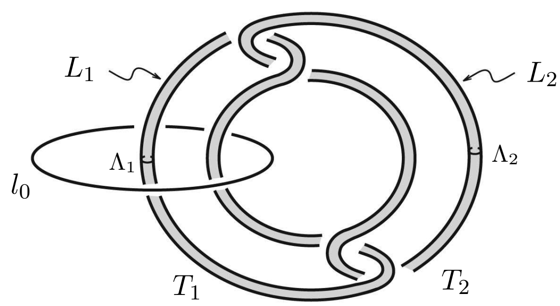

FIGURE 4.1. The link $L = l_0 \cup L_1 \cup L_2$ in lemma 4.1.

Lemma 4.1. *Consider the link $L = l_0 \cup L_1 \cup L_2$, figure 4.1. Suppose that for each $i = 1, 2$,*

(1) $L_i \cup \Lambda_i$ is h-trivial, and
(2) L_i is h-trivial$^+$ in S^3, where $L_i \subset T_i \subset S^3$ and $T_i \subset S^3$ is the standard inclusion.

Then L is h-trivial$^+$.

Figure 4.2 shows an example of a link L_i in the solid torus satisfying the assumption in lemma 4.1: the Whitehead double of the core of the solid torus, and a parallel copy. (Note that in this example $L_i \cup \Lambda_i$ is h-trivial, but L_i is not h-trivial in the solid torus.) Other examples are given by the links in figure 6.2, where the three components on the right form the link in the solid torus = complement in S^3 of the left-most component.

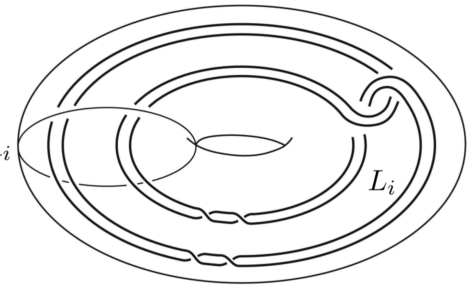

FIGURE 4.2. An example of a link L_i in the solid torus satisfying the assumptions in lemma 4.1.

Proof lemma 4.1. There are two separate cases to consider: when a parallel copy of a component of $L_1 \cup L_2$ is added, and when a parallel copy of l_0 is added. First consider $L' := L \cup$ parallel copy l'_1 of a component of L_1. We start with a geometric argument to show that L' is h-trivial. The steps below are labeled for referencing in follow-up sections.

(1) The link L may be built starting with $L_2 \cup \Lambda_2$ as follows: Bing double Λ_2, denote one of the resulting components by l_0 and insert L_1 in a tubular neighborhood of the other component. Pictured this way, $l_0 \cup L_1$ is contained in a tubular neighborhood of Λ_2. Consider a link null-homotopy of $L_2 \cup \Lambda_2$ and extend it to $L_2 \cup (l_0 \cup L_1)$. Self-intersections of Λ_2 during the link-homotopy are implemented by self-intersections of l_0. This gives a link-homotopy of L where the components of L_1 have no self-intersections, so the same argument goes through when a parallel copy l'_1 is added to L_1. The assumption (2) of the lemma completes the proof that L' is h-trivial. h-triviality of $L \cup$ parallel copy of a component of L_2 is established analogously.

(2) Now we give another, algebraic, proof that L' is h-trivial. This argument will be applicable in the more general setting of theorem 1. Abusing the notation, let l_0, Λ_i refer to based loops. Then

$$l_0 = [\Lambda_1, \Lambda_2], \tag{4.1}$$

where

$$\Lambda_1 = \Lambda_2 = 1 \in M\pi_1(S^3 \smallsetminus (L_1 \cup L_2)). \tag{4.2}$$

It follows that every monomial (other than 1) in the Magnus expansion of l_0 has two sets of repeated variables: one pair corresponding to a component of L_1 and another pair corresponding to a component of L_2. This implies that the link remains h-trivial when a parallel copy is added to one component of either L_1 or L_2.

(3) Now consider $L \cup l'_0$, where l'_0 is a parallel copy of l_0. Note that there exist maps of disks Δ into D^4 bounded by $L_1 \cup L_2$ and a capped punctured torus T^c bounded by l_0 in D^4 such that all disks and T^c are pairwise disjoint. The body of T^c is an embedded genus 1 surface bounded by l_0 in $S^3 \smallsetminus (T_1 \cup T_2)$, with a symplectic

basis of curves isotopic to Λ_1, Λ_2. Extend $L_1 \cup \Lambda_1 \cup L_2 \cup \Lambda_2$ by a product in a collar $S^3 \times I \subset D^4$, where $S^3 \times 0$ is identified with ∂D^4. Then $L_1 \cup \Lambda_1$ and $L_2 \cup \Lambda_2$ form a split link in $S^3 \times 1$ which is h-trivial by the assumption (1) of the lemma. The null-homotopies for Λ_1, Λ_2 give the caps for T^c. Contraction/push-off [9] applied to T^c and its parallel copy give disjoint maps of disks for all components of $L \cup l_0'$. □

Remark. Contraction/push-off in part (3) of the proof, if desired, could be iterated to show that $L \cup$ (any given number of parallel copies of l_0) is a homotopically trivial link.

5. A lemma in commutator calculus.

5.1. Lemma 4.1 above illustrates the idea that the improvement from a homotopy solution in [7] to a homotopy$^+$ solution will follow from allowing two "parallel channels" or "two participants in a commutator" by which an element can die. More precisely, the key features of the link $l_0 \cup L_1 \cup L_2$ in figure 4.1 are the expression (4.1) for l_0, subject to (4.2), and the fact that $L_1 \cup L_2$ is h-trivial$^+$. The link that will come up in the proof of theorem 1 is more general than the basic example in lemma 4.1. This section develops the relevant algebraic framework which generalizes Corollary 3.2 using the main features described above.

We will refer to the commutators of the form $[h_1, \dots, h_k]$ in the statement of Corollary 3.2 as *almost basic commutators*. (This term is meant to avoid confusion with commutators $[g_{i_1}, \dots, g_{i_k}]$ which are usually called *basic*.) Almost basic commutators are geometrically realized by standard links (figure 3.1) which will be used to construct decompositions of D^4.

Note that the representation of g in Corollary 3.2 as a product of conjugates of almost basic commutators holds *in the Milnor group*, in general it is not valid in the group π. For the purpose of proving Theorem 1 it is insufficient to work modulo the Milnor relation, a more subtle equivalence relation is needed.

Let π be a group normally generated by a fixed set of elements $\{g_1, \dots, g_n\}$. Motivated by (4.1), (4.2), consider the relation

$$\big[[g_i, g_i^{y_1}]^{z_1}, [g_j, g_j^{y_2}]^{z_2}\big], \tag{5.1}$$

where $1 \le i, j \le n$, and y_k, z_k are arbitrary elements of π. The notation $f \equiv g$ for two elements $f, g \in \pi$ will indicate that $f \cdot g^{-1}$ is in the normal subgroup generated by the relations (5.1). Technically we will not consider quotients of groups by these relations, rather (5.1) will be used in section 6 to construct specific h-trivial$^+$ links.

The following lemma establishes a version of Corollary 3.2 in the setting of the relations (5.1). A useful fact about the lower central series to keep in mind is that $[\pi^p, \pi^q] \subset \pi^{p+q}$.

Lemma 5.1. *Let π be a group normally generated by $\{g_1, \dots, g_n\}$. Fix $k \geq 4$ and consider a commutator $[\alpha, \beta]$ where α, β are both elements of the kth term of the lower central series π^k. Then there exists $W \in \pi^{2k}$ such that*

(1) $[\alpha, \beta] \equiv W$, and

(2) W equals in π a product of conjugates of elements of the form $[C, \beta]$, $[\alpha, C']$, $[C, C']$ where C, C' are almost basic commutators $[h_1, \dots, h_k]$ in Corollary 3.2.

Proof of lemma 5.1. Let α', $\beta' \in \pi$ denote products of conjugates of almost basic commutators representing α, β, given by Corollary 3.2,

$$\alpha \cdot (\alpha')^{-1} = 1 \in M\pi, \;\; \beta \cdot (\beta')^{-1} = 1 \in M\pi.$$

Note that in general this does not imply $[\alpha, \beta] \equiv [\alpha', \beta']$. Recall the basic commutator identities, cf. [17, Theorem 5.1]:

$$[x, yz] = [x, z] \cdot [x, y]^z, \;\; [xz, y] = [x, y]^z \cdot [z, y]. \tag{5.2}$$

Use (5.2) to observe that

$$1 \equiv [\alpha \cdot (\alpha')^{-1}, (\beta')^{-1} \cdot \beta]. \tag{5.3}$$

Again using (5.2),

$$[\alpha \cdot (\alpha')^{-1}, (\beta')^{-1} \cdot \beta] = [\alpha, \beta]^{\gamma_1} \cdot [(\alpha')^{-1}, \beta] \cdot [\alpha, (\beta')^{-1}]^{\gamma_2} \cdot [(\alpha')^{-1}, (\beta')^{-1}]^{\gamma_3} \tag{5.4}$$

for some $\gamma_i \in \pi$, determined by (5.2). Set $W \in \pi$ to be (a conjugate of) the inverse of the product of the three right factors in (5.4):

$$W := \Big(\big([(\alpha')^{-1}, \beta] \cdot [\alpha, (\beta')^{-1}]^{\gamma_2} \cdot [(\alpha')^{-1}, (\beta')^{-1}]^{\gamma_3}]\big)^{-1}\Big)^{\gamma^{-1}}. \tag{5.5}$$

It follows from (5.3), (5.4) that

$$[\alpha, \beta] \equiv W. \tag{5.6}$$

The second statement of the lemma follows from an application of the identities (5.2) to the definition (5.5) of W. $\square$

5.2. The proof of theorem 1 in section 6 will require an application of lemma 5.1 in a slightly more general setup. In the context of the lemma, suppose $g \in \pi$ is of the form

$$g = \big[[\alpha_1, \beta_1], [\alpha_2, \beta_2]\big], \tag{5.7}$$

where each element α_i, β_i is in π^k, $k \geq 4$. Then lemma 5.1 gives $W_1, W_2 \in \pi^{2k}$ such that $[\alpha_i, \beta_i] \equiv W_i$, $i = 1, 2$. Observe that given $x, y, z \in \pi$, $x \equiv y$ implies $[x, z] \equiv [y, z]$. It follows that

$$g = \big[[\alpha_1, \beta_1], [\alpha_2, \beta_2]\big] \equiv \big[W_1, [\alpha_2, \beta_2]\big] \equiv \big[W_1, W_2\big]. \tag{5.8}$$

Moreover, both W_1, W_2 satisfy the conclusion (2) in the lemma.

6. Proof of theorem 1

As discussed in the introduction (also see [7, Proposition 4.1]) any coinitial subset of the Generalized Borromean Rings forms a collection of links universal for surgery. A homotopy A-B slice solution in [7] applies to links obtained from the Hopf link by keeping one of its components l_0 intact and Bing doubling the other components at least twice, see figure 6.1 for an example of such a link. A homotopy$^+$ solution constructed in sections 6.1 - 6.3 applies to a collection of higher Bing-doubled links (still universal for surgery). We start by briefly summarizing the construction of [7].

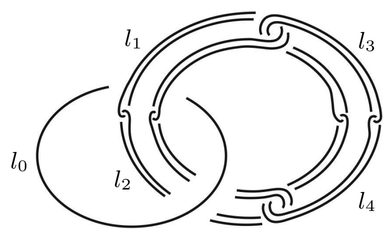

Figure 6.1. A link in the collection of GBRs.

The decompositions $D^4 = A_i \cup B_i$ for all link components other than the fixed component l_0 are set to be the trivial decomposition, A_i = 2-handle and B_i = collar on the attaching curve β_i. The entire complexity of the construction is in the decomposition $D^4 = A_0 \cup B_0$ for the component l_0. A_0 will be defined to be the collar $l_0 \times D^2 \times I$ with a single 2-handle attached to the core of the solid torus $l_0 \times D^2 \times \{1\}$, and many 1-handles governed by the algebraic outcome of lemma 5.1, as explained below. In terms of figure 2.1 (where the index i is understood to equal 0) the 2-handle is attached to the curve labeled l_i, J_i is empty since there are no other 2-handles, and K_i is the Kirby diagram representation of the 1-handles. Correspondingly, the other side B_0 of the decomposition has no 1-handles ($\widehat{J}_i$ in the figure is empty) and the attaching curves of its 2-handles form the link $\widehat{K}_i$.

The link-homotopy solution in [7] uses a geometric implementation of Corollary 3.2, where each almost basic commutator of the form $[h_1, \dots, h_k]$ is realized by a standard link illustrated in figure 3.1. More precisely, building blocks in the construction of the link K_0, describing the 1-handles of A_0, are shown in figure 6.2. These are *h-trivial* counterparts of the links in figure 3.1 where one of the two parallel curves labeled y, z is removed.

A key point, using the terminology of definition 2.2, is that K_0 is the attaching link for 0-framed 2-handles attached to D^4_0, and parallel copies of each component bound disjoint copies of the core of the attached 2-handle. The links in figure 3.1 then may be recovered from links in figure 6.2 by adding the relevant parallel copy.

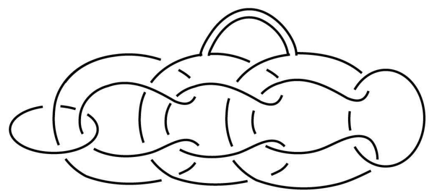

FIGURE 6.2. Links in figure 3.1, with one of the parallel components y, z removed.

The homotopy solution from [7, Section 4] is schematically shown in Figure 6.3 (a). This construction will be refined to achieve h-triviality$^+$ in three steps in sections 6.1 - 6.3.

6.1. Consider the link (l_0, L) in figure 6.3 (b), obtained by Bing doubling of the components $l_1, \ldots, l_4$ of the link in figure 6.1. The argument applies to GBRs that are more Bing-doubled and ramified; to be concrete we focus here on this simplest representative link. This section constructs a stabilization of l_0, a link K_0 in a solid torus linking l_0, and a band-sum $L^\sharp$ of L with ($K_0 \cup$ parallel copies) with the property:

$(l_0 \cup L^\sharp)$ *is h-trivial*$^+$ *with respect to adding a parallel copy of a component of* $L^\sharp$.

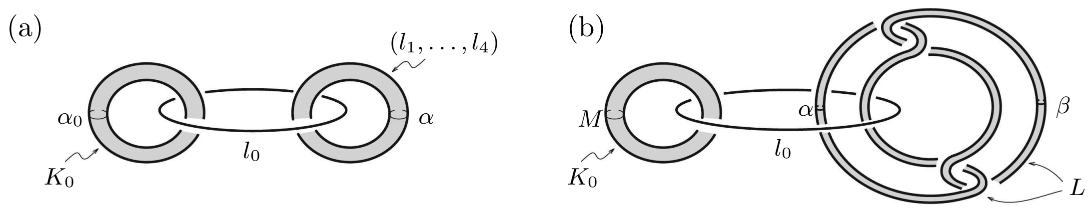

FIGURE 6.3.
(a): A schematic representation of the link-homotopy A-B slicing in [7], where $(l_0, \ldots, l_4)$ is the GBR is figure 6.1. (The link $\widehat{K}_0$, and also bands connecting $l_2, \ldots, l_4$ with K_0 are not shown.)
(b) The modified stabilization K_0, constructed in section 6.1, for the GBR (l_0, L). Here L is obtained by Bing doubling twice the cores of the two solid tori, and α, β (considered as based loops) represent 4-fold commutators in the meridians of L.

The dual part $\widehat{K}_0$ of the stabilization (figure 6.4) is analyzed in section 6.2. The addition of a parallel copy of l_0 is addressed in section 6.3.

To apply Lemma 5.1, consider $\pi := \pi_1(S^3 \smallsetminus L)$. L is the unlink, π is the free group, and l_0 is an element of the form $[\alpha, \beta]$ as in the statement of the lemma. In the concrete example discussed above, both α, β are 4-fold commutators (more generally the argument applies to higher Bing doubles where α, β are k-fold commutators, $k \geq 4$.) According to the statement (2) of the lemma, W is a product of conjugates of the elements of the form

$$[C, \beta],\ [\alpha, C'],\ [C, C'], \tag{6.1}$$

where C, C' are almost basic commutators as in Corollary 3.2. To construct K_0, consider several parallel copies of the meridian to l_0. For each element of the form (6.1) in the expression for W, take a Bing double of a meridian and thicken the two resulting components to solid tori. Next we define links K', K'', geometrically representing the given element (6.1) and insert them into these two solid tori. For each element of the form $[C, \beta]$ consider a link of the type shown in figure 6.2, corresponding to the almost basic commutator C. More precisely, K' consists of three components on the right in a link in figure 6.2, considered in the solid torus complement of the leftmost component. K'' is the iterated Bing double of the core of the solid torus corresponding to the 4-fold commutator β. The analogous Bing double of links (K', K'') is created for each factor of the form (6.1) in the expression for W, completing the construction of K_0.

The construction in the preceding paragraph is a generalization of that in [7, Section 4], in particular see figure 4.3 in that reference. The stabilization in [7] was defined in terms of almost basic commutators C, while here we have Bing doubles of links corresponding to commutators (6.1).

For each constituent link of K_0 of the type in figure 6.2 add a parallel copy to recreate a link as in figure 3.1. In the relative slice setting these parallel copies bound disjoint disks in the zero-framed 2-handles attached to D^4_0 along K_0. To find a homotopy$^+$ solution to the relative-slice problem, L will be band-summed with the components of K_0 and their parallel copies. In the homotopy solution in [7] the choice of bands was immaterial. This is due to the fact that all commutators in the construction are of maximal length, so conjugation does not affect calculations in the Milnor group. The only relevant constraint for a homotopy solution is $l_0 = \alpha \cdot (\alpha')^{-1} = 1 \in M\pi_1(S^3 \smallsetminus L)$, in the notation of figure 6.3 (a). (The equality $l_0 = \alpha \cdot (\alpha')^{-1}$ is established using additivity of $\bar{\mu}$-invariants [2, 13], or by directly reading off the element represented by l_0 in the Milnor group [7, Proof of theorem 1].) The homotopy$^+$ problem is more sensitive to the choice of bands.

Suppose the bands in the definition of $L^\sharp$ could be chosen so that the meridian M of the solid torus in figure 6.3 (b) *precisely* matched the element W in lemma 5.1. Then in π one would have $l_0 = [\alpha, \beta] \cdot W^{-1} \equiv 1$, see (5.6). In this case the h-triviality$^+$ with respect to parallel copies of $L^\sharp$ is proved exactly the same way as in part (2) of the proof of lemma 4.1.

In fact, to establish h-triviality$^+$ with respect to parallel copies of $L^\sharp$ it suffices to choose bands so that M suitably approximates W. Recall that W is a product of conjugates of the elements (6.1). The Milnor group $M\pi = M\pi_1(S^3 \smallsetminus L)$ is nilpotent of class equal to the number of components of L. For any choice of bands, each commutator (6.1) in the expression for W is of maximal length in the power series of non-repeating monomials. Considering the Magnus expansion (3.2) of the free group π, note that only the homology class of conjugating elements is relevant. Indeed, suppose an element W' is created by some conjugating elements that agree homologically with those defining W in Lemma 5.1. Then each term (other than 1) in the Magnus expansion of $W \cdot (W')^{-1}$ contains either three copies of a variable, or two pairs of different repeated variables. In either case adding a parallel copy preserves the condition of being h-trivial. Finally, choose arcs connecting L with the relevant components of K_0 and its parallel copies, homologically matching the conjugating elements. Perform the band sums along these arcs; such operations do not interfere with each other since only homological information is relevant. This establishes h-triviality$^+$ with respect to components of $L^\sharp$.

6.2. **The link $\widehat{K}_0$.** This section modifies the stabilization K_0 constructed above, using the extension of Lemma 5.1 in section 5.2. The goal is to establish h-triviality$^+$ of the link $\widehat{K}_0$ in the solid torus, figure 6.4.

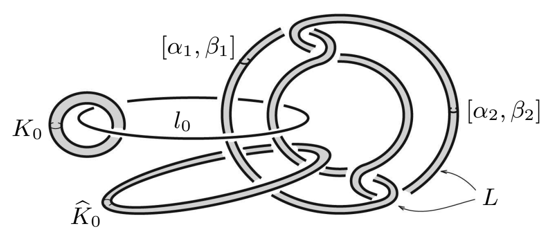

FIGURE 6.4.

Bing double the link L in figure 6.3 one more time, so that l_0 represents the commutator $l_0 = \big[[\alpha_1, \beta_1], [\alpha_2, \beta_2]\big]$, where each α_i, β_i is a 4-fold commutator as in (5.7). Now define the stabilization link in the solid torus linking l_0 to be the Bing double of two copies of K_0 from section 6.1. The key point now is that the link K_0 was defined in the previous section using links in figure 6.2 which are h-trivial in the solid torus, so K_0 is also is h-trivial in the solid torus. It follows that the Bing double of this link is h-trivial$^+$ in the solid torus. The link $\widehat{K}_0$ is a copy of K_0 in a solid torus parallel to l_0, figures 2.1, 6.4. Therefore $\widehat{K}_0$ is h-trivial$^+$ in this solid torus, and it does not affect the analysis of the the link $l \cup L^\sharp$. Here $L^\sharp$ is created using bands analogously to section 6.1, implementing the algebraic equation (5.8).

6.3. **h-triviality$^+$ with respect to l$_0$.** Finally, consider $l_0 \cup l_0' \cup L$, where $l_0 \cup L$ is the link in section 6.2 (figure 6.4), and l_0' is a parallel copy of l_0. The goal is to show that $l_0 \cup l_0' \cup L^\sharp$ is h-trivial. To achieve this, the stabilization will require an additional slight modification. (This argument is independent of sections 6.1, 6.2; it can be carried out in the setting of [7].)

First consider the Borromean rings with a parallel copy l_0' of one of its components l_0, figure 6.5. Denoting by m_1, m_2 the meridians suitably connected to a basepoint and similarly regarding l_0 as a based loop, note that the expression $l_0 = [m_1, m_2]$ holds regardless of whether l_0 is considered as an element of the Milnor group $M\pi_1(S^3 \smallsetminus (l_1 \cup l_2))$ or as an element of $M\pi_1(S^3 \smallsetminus (l_0' \cup l_1 \cup l_2))$. This expression can be read off from the torus bounded by l_0 in the complement of l_1, l_2, figure 6.5.

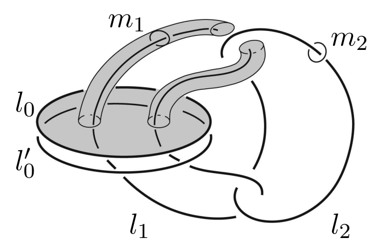

FIGURE 6.5.

Similarly, a calculation explained in the following paragraph shows that the expressions read off by the components γ_i of the elementary Engel links in figure 3.1 are unchanged when a parallel copy of γ_i is present, provided that the bands do not involve the component labeled x (the figure only illustrates the links of this type). Considering 5-fold rather than 4-fold commutators in Corollary 3.2, the links analogous to the elementary links in figure 3.1 satisfy this property on bands. To be more precise, consider a 5-fold commutator $[g_i, g]$ where g is a 4-fold commutator and g_i is a generator. Then according to Corollary 3.2, g is a product of conjugates of almost basic commutators C. Using commutator identities (5.2), $[g_i, g]$ is then seen to be a product of conjugates of elements of the form $[g_i, C]$. In the corresponding link the element g_i is represented by a curve which is not part of a band sum.

We will now give details of the calculation mentioned in the previous paragraph. This calculation may be given in terms of the Milnor group, or it can be read off from a grope bounded by l_0 in the complement of the link. The links discussed above may be represented as a "composition" (in the sense of [8, Theorem 2.3]) where the components labeled γ_i and x of a link in figure 3.1 are identified with the components l_0, l_1 of the Borromean rings in figure 6.5, and the rest of the link (denote it by K) is inserted in a solid torus neighborhood T of l_2. The following argument applies to any link K in T. As in figure 6.5,

$$l_0 = [m_1, m_2]. \tag{6.2}$$

Consider the meridian m_2 as an element in the Milnor group of the complement of K in the solid torus T, $M\pi_1(T \smallsetminus K)$. The generators of this Milnor group are meridians to K and a longitude of the solid torus. Considered as part of figure 6.5, this longitude represents the commutator $[m_1, m_0 m_0']$. Substituting this into (6.2), observe that the only way that the meridian m_0' appears in the expression for l_0 is as part of the commutator

$$[m_1, \ldots \cdot [m_1, m_0 m_0'] \cdot \ldots].$$

Applying the commutator identities (5.2) and the Milnor relation (3.1), it follows that omitting $[m_1, m_0 m_0']$ from this expression does not change the element in the Milnor group, establishing the desired claim.

Now instead of iterated Bing doubles in sections 6.1, 6.2 corresponding to 4-fold commutators consider links corresponding to 5-fold commutators. (For example in figure 6.1 Bing double any one of the components $l_2, \ldots, l_4$.) Given such GBR $l_0 \cup L$ and stabilization K_0, l_0 then may be assumed to be in the subgroup $M\pi_1(S^3 \smallsetminus (L \cup K_0))$ of $M\pi_1(S^3 \smallsetminus (l_0' \cup L \cup K_0))$. As in the proof in section 6.1, only the homological information about bands connecting L and K_0 and forming $L^\sharp$ is relevant. Choose arcs connecting L with K_0 so that the corresponding conjugating elements are in the subgroup $M\pi_1(S^3 \smallsetminus (L \cup K_0))$ of $M\pi_1(S^3 \smallsetminus (l_0' \cup L \cup K_0))$. Now all calculations in the preceding sections, establishing that $l_0 = 1 \in M\pi_1(S^3 \smallsetminus L^\sharp)$, do not involve the meridian to l_0', so $l_0 = 1 \in M\pi_1(S^3 \smallsetminus (l_0' \cup L^\sharp))$. This shows that $l_0 \cup l_0' \cup L$ is h-trivial. □

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