ON HOMOTOPY TORI II

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In the preprint **[2]** of Kirby and Siebenmann, it is shown that $\pi_i(\mathrm{Top}/PL)$ vanishes for $i \neq 3$ and has order at most 2 in that case. It is further shown that isotopy classes of PL structures on a topological manifold of dimension $\geqslant 5$ correspond bijectively to reductions from Top to PL of the structure group of its stable tangent bundle. The same authors, using the same techniques, have since shown that $\pi_3(\mathrm{Top}/PL) = \mathbb{Z}_2$. We first use this to make some elementary observations on the homotopy type of certain spaces related to Top, and then apply this to study *topological* manifolds homotopy equivalent to the torus T^n ($n \geqslant 5$): *Our main result states that they are all homeomorphic to* T^n.

The argument showing that $\pi_3(\mathrm{Top}/PL) \cong \mathbb{Z}_2$ shows also that $\pi_4(G/\mathrm{Top})$ is infinite cyclic, and contains $\pi_4(G/PL)$ as a subgroup of index 2. We take this as our starting point.

It is well known that $\pi_3(PL) \cong \mathbb{Z}$ and $\pi_4(G/PL) \to \pi_3(PL)$ is an inclusion with index 24. The exact sequence

$$\pi_4(G/PL) \to \pi_3(PL) \oplus \pi_4(G/\mathrm{Top}) \to \pi_3(\mathrm{Top}) \to \pi_3(G/PL) = 0 \qquad (1)$$

now shows $\pi_3(\mathrm{Top}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$; it follows that

$$\pi_3(PL) \to \pi_3(\mathrm{Top}) \to \pi_3(\mathrm{Top}/PL) \qquad (2)$$

is a split short exact sequence; moreover, there is a unique splitting.

Next, we consider mod 2 cohomology. Now by the result of Boardman and Vogt **[1]**, all the above have classifying spaces. Since Top/PL is of type $(\mathbb{Z}_2, 3)$ there is, up to homotopy, a fibration

$$BPL \to B\,\mathrm{Top} \to K(\mathbb{Z}_2, 4). \qquad (3)$$

Now, in low dimensions, the cohomology of BPL coincides with that of BO, so is generated by Stiefel–Whitney classes. But these survive to BG (*c.f.* **[3]**), so certainly to $B\,\mathrm{Top}$; thus the differentials in the Serre spectral sequence of the above fibration vanish on them. Hence the fundamental class of $K(\mathbb{Z}_2, 4)$ survives to a class in $B\,\mathrm{Top}$, yielding a unique class $\tau \in H^4(\mathrm{B\ Top};\ \mathbb{Z}_2)$.

Evidently, if $f : M^m \to B\,\mathrm{Top}$ classifies the tangent (micro) bundle of the topological manifold M, the unique obstruction in $H^4(M;\ \mathbb{Z}_2)$ to lifting through BPL is precisely $f^*\tau$. By the results of **[2]**, if $m \geqslant 5$ (or 6 if M has boundary) the vanishing of this class is necessary and sufficient for triangulability of M. (This result has a relative version, yielding a corresponding statement for the Hauptvermutung.)

We next discuss how this obstruction can be computed. It will be simpler if we restrict ourselves to the case $w_1(M) = w_2(M) = 0$. Write $S\,\mathrm{Top}$ for the subgroup

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of Top of orientation-preserving maps and Spin-Top for its (universal) double covering group. Then B(Spin-Top) is 3-connected, so has a fundamental class in $H^4(B(\text{Spin-Top});\ \pi_3(\text{Spin-Top}))$. The coefficient group here equals

$$\pi_3(\text{Top}) = \mathbb{Z} \oplus \mathbb{Z}_2,$$

so the class has two components, of which the second is clearly the lift to B(Spin-Top) of τ, and the first—which we write q_1—is such that $2q_1$ is induced from

$$p_1 \in H^4(BS\,\text{Top};\ \mathbb{Z}) \cong H^4(BSO;\ \mathbb{Z}),$$

as we see by comparing with $H^4(B\,\text{Spin};\ \mathbb{Z})$. Now the map $\pi_3(\text{Top}) \to \pi_3(G) = \mathbb{Z}_{24}$ induced by inclusion is composed of a surjection $p : \mathbb{Z} \to \mathbb{Z}_{24}$ and the injection $i : \mathbb{Z}_2 \to \mathbb{Z}_{24}$. Thus if ξ is a topological bundle with fibre $(\mathbb{R}^m, 0)$ such that $w_1(\xi) = w_2(\xi) = 0$, $p_* q_1(\xi) + i_* \tau(\xi)$ depends only on the fibre homotopy type of ξ (with zero section deleted). In particular, if ξ is the tangent bundle of a closed manifold M^m, this depends only on the homotopy type of M^m.

Now finally let M^m be a closed *topological* manifold homotopy equivalent to T^n, $n \geqslant 5$. Let $\tilde{M}$ be the covering space (of degree 2^n) corresponding to

$$\text{Ker}\left(\pi_1(M) \to H_1(M;\ \mathbb{Z}_2)\right),$$

$\pi : \tilde{M} \to M$ the projection. Then the tangent bundle of $\tilde{M}$ is induced from that of M, so $\tau(\tilde{M}) = \pi^* \tau(M) = 0$ (since $\pi^* = 0$). Hence $\tilde{M}$ is triangulable as a closed PL-manifold. By [4, 5], since $\tilde{M}$ is a PL homotopy torus, it is parallelisable, so $0 = p_1(\tilde{M}) = \pi^* p_1(M)$. But π^* is injective on $H^4(M;\ \mathbb{Z})$, which is torsion-free, so $p_1(M) = 0$ and $q_1(M) = 0$. Now $p_* q_1 + i_* \tau$, which is homotopy invariant by the above, vanishes for T^n and hence for M. Since $q_1(M) = 0$, $i_* \tau(M) = 0$. But i_* is injective, so $\tau(M) = 0$. Hence M is triangulable as a PL-manifold. Now by [2], the triangulations of T^n are classified by $H^3(T^n;\ \mathbb{Z}_2)$. But by [4, 5], the same group classifies all PL-manifolds homotopy equivalent to T^n. Moreover, a PL-manifold homeomorphic to T^n has the same invariant under both classifications; indeed, the calculation of the groups $\pi_i(\text{Top}/PL)$ depends precisely on this interpretation of these results of [4, 5]. Thus M with its PL-structure is PL-homeomorphic to T^n with some PL-structure; in particular, M is homeomorphic to T^n.

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