DIFFEOMORPHISM GROUPS OF WALDHAUSEN MANIFOLDS

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In this paper the homotopy type of the group of diffeomorphism of sufficiently large irreducible three-dimensional manifolds is described and the space of incompressible surfaces in such manifolds is studied.

1. Notation and Definitions

We use differential-topological C^∞-terminology. Let V be a compact manifold, F a submanifold of it, N and A closed sets in V and F, respectively. We denote by $\mathrm{Diff}(V,N)$ the group of all diffeomorphisms of the manifold V, fixed on ∂V and N, by $Em_A(F,V)$ the set of imbeddings of the manifold F in V, which coincide on ∂F and A with the inclusion, and by $H(V)$ the set of homotopy equivalences of the manifold V, fixed on ∂V; $\mathrm{Diff}(V,N)$ and $Em_A(F,V)$ are given the C^∞-topology, and $H(V)$ the compact-open topology. We set $\mathrm{Diff}(V)=\mathrm{Diff}(V,\varnothing)$.

Let V be a compact three-dimensional manifold. A surface (a compact two-dimensional submanifold) F in V is called proper if $\partial F=\partial V\cap F$ and F is transverse to ∂V. It is called incompressible if: (*i*) F is connected, proper, and lies two-sidedly in V; (*ii*) F is not the boundary of a contractible submanifold and either F is not a disk or the inclusion $(F,\partial F)\to(V,\partial V)$ does not admit a homotopy into a map with image lying in V; (*iii*) the inclusion homomorphism $\pi_1(F)\to\pi_1(V)$ is a monomorphism.

The manifold V is called irreducible if any two-dimensional sphere imbedded in V bounds a ball in V. It is called a Waldhausen manifold if V is irreducible, contains an incompressible surface, and $\mathbb{R}P^2$ cannot be imbedded two-sidedly in V.

2. History of the Question

If N is a submanifold of the manifold V, then $\mathrm{Diff}(V,N)$ has the structure of a Frechet manifold; cf. Leslie [9]. In particular, in this case $\mathrm{Diff}(V,N)$ has the homotopy type of a countable cellular space (countable " CW-complex") and is determined by its homotopy type up to homeomorphism; cf. Burghelea and Kuiper [3]. The homotopy type of $\mathrm{Diff}(V)$ has been studied in a series of papers. The first were the results of Milnor on $\pi_0\mathrm{Diff}(S^n)$ (cf. [7, 10]). On the basis of them it was proved by Novikov [11], Antonelli, Burghelea, and Kahn [1], and other authors that a large number of homotopy groups $\pi_i\mathrm{Diff}(V)$ for $V=D^n$ and some other V were nontrivial.

One has complete information on the homotopy type of $\mathrm{Diff}(V)$ only in a few cases. Essentially they reduce to the following. It is well known (and easily proved) that

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$\mathrm{Diff}(S^1) \approx S^1$. Smale [12] proved that $\mathrm{Diff}(S^2) \approx O(3)$, and for the remaining closed surfaces, the homotopy type of the group of diffeomorphisms was calculated by Eells and Earle [5]. As to the three-dimensional case, Akiba [2] announced a proof of Smale's conjecture according to which $\mathrm{Diff}(D^3)$ is contractible (the author is not familiar with it, so Theorems 2 and 3 below are formulated with caution).

In addition, in the three-dimensional case there are partial results. Laudenbach obtained significant information about $\pi_0 \mathrm{Diff}(S^1 \times S^2 \# \cdots \# S^1 \times S^2)$. In the case when V is a Waldhausen manifold, Waldhausen and Laudenbach reduced the calculation of $\pi_0 \mathrm{Diff}(V)$ to a homotopy problem, and Laudenbach proved that $\pi_1 \mathrm{Diff}(V, \{v\}) = 0$, where $v \in V$. There is an account of these results in Laudenbach [8].

3. Basic Result

Let V be a Waldhausen manifold.

THEOREM 1. (Separation Theorem). Let F be an incompressible surface in V and F' be the result of a small motion of F along a normal field. If $v \in F$, then the inclusion

$$Em_{\{v\}}(F, V \setminus F') \subset Em_{\{v\}}(F, V)$$

induces an isomorphism of homotopy groups.

Now let us assume that Smale's conjecture is true.

THEOREM 2 (Basic Theorem). If $\partial V \neq \emptyset$, then the components of the group $\mathrm{Diff}(V)$ are contractible. If $\partial V \neq \emptyset$ and $v \in V$, then the components of the group $\mathrm{Diff}(V, \{v\})$ are contractible.

THEOREM 3. The inclusion $\mathrm{Diff}(V) \subset H(V)$ is a homotopy equivalence.

Theorem 3 is an easy consequence of Theorem 2. Theorem 2 is derived without great difficulty from Theorem 1 and the existence of Haken hierarchies on Waldhausen manifolds [6]. The main difficulty is the separation theorem which is of independent interest. Section 4 is devoted to a sketch of its proof. It can be interpreted as the assertion that any finite-parametric family of imbeddings $F \to V$ can be pushed off F'. A special case of it was proved by Laudenbach [8], who constructed pushings for 0 and I -parameter families.

4. Basic Steps in the Proof of the Separation Theorem

The proof consists of two parts. The first constructs a more or less canonical pushing isotopy for one imbedding. Then from several such isotopies, the pushing of a whole family is glued together.

4a. Covering. We denote by $p : (V^\sim, w) \to (V, v)$ the covering associated with the image of the group $\pi_1(F, v)$ in $\pi_1(V, v)$. We set $F'' = p^{-1}(F)$ and we denote by Π the set of components of the manifold F''. Each surface C from Π divides $V^\sim$ into two parts X_C and Y_C; by X_C we denote that parts which contains w. The relation $Y_C \subset Y_D$ determines an order relation $D < C$ on Π.

Laudenbach [8] suggested using this construction to prove the separation theorem.

4b. Imbedding of General Form. We call a point y of F singular for the imbedding $f: F \to V$, if f is not transverse to F' at y. A singular point y of an imbedding f will be called of finite multiplicity, if the germ at $f(y)$ of the quadruple $(V, F', f(F), f(y))$ is diffeomorphic with the germ at 0 of some quadruple of the form $(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2 \times 0, \Gamma, 0)$ with Γ being the graph of a function $\mathbb{R}^2 \to \mathbb{R}$ such that 0 is a critical point of finite multiplicity of it. One says that the imbedding f is of general form if all its singular points are of finite multiplicity.

The complement in $Em_{\{v\}}(F,V)$ to the set of imbeddings of general form has codimension ∞ and hence to prove the separation theorem it suffices to prove that finite-parametric families of imbeddings of general form can be pushed off F'.

4c. Basic Lemma. Let $f: F \to V$ be an imbedding of general form, isotopically connected at v to the inclusion $F \subset V$, $f^{\sim}: F \to V$ be a lifting of it such that $f^{\sim}(v) = w$, A be a closed set in $V^{\sim}$ and let C be a component of the manifold F''. If $A \cap f^{\sim}(F) = \emptyset$, $C \cap f^{\sim}(F) \neq \emptyset$ and $D \cap f^{\sim}(F) = \emptyset$ for $C < D$, then there exist an imbedding $g: F \to V^{\sim}$ transverse to F'', a component G of the set $g(F) \setminus F''$, and a compact manifold with corners W in $V^{\sim}$ such that: (i) $g(v) = w$ is an imbedding; (ii) $\bar{G}$ intersects exactly one component of the manifold F'' and $\partial W = G \cup (\partial W \cap F'')$; (iii) $\bar{G} \cap (A \cup f^{\sim}(F)) = \emptyset$, $\partial W \cap F'' \cap f^{\sim}(F) \neq \emptyset$; (iv) either $\partial \bar{G} \subset C$ and $W \subset Y_c$ or there exists a disk E such that $G \subset E \subset g(F)$; (v) if the component S of the manifold $\partial \bar{G}$ bounds a disk F'' in D_s, then either $D_s \cap W = S$, or there exists a disk E_s such that $\partial E_s = S$ and $G \subset E_s \subset g(F)$.

[A disk E_s with $\partial E_s = S$ and $G \subset E_s \subset g(F)$ exists for not more than one component S of the manifold $\partial \bar{G}$).]

The proof is based on the study of the approximation of the imbedding f by imbeddings transverse to F'.

4d. Pushing Isotopies. In the situation of the basic lemma, the restriction $p|_W$ is an imbedding. We choose an imbedding $\varphi: F' \times [0,1] \to V$ such that $\varphi^{-1}(p(\bar{G})) = p(\partial \bar{G}) \times [0,1]$.

For each components S of the manifold $p(\partial \bar{G})$ bounding a disk F' in D_s such that $D_s \cap p(W) = S$, we add to the manifold $p(W)$ the cylinder $\varphi(D_s \times [0, \varepsilon_s])$, where $0 < \varepsilon_s < 1$, and we smooth the corners of the manifold obtained which do not lie on F'. As a result, we get a manifold Z, which is a trivial cobordism between $\partial Z \cap F'$ and $\overline{\partial Z \setminus F'}$. With its help, one can construct an isotopy of the manifold V, carrying $\overline{\partial Z \setminus F'}$ into $\partial Z \cap F'$. It follows from (iii) that if the number ε_s is sufficiently small, then this isotopy in some sense (which we will not make precise here) simplifies the intersection of the image of the imbedding f with F'. Hence, performing several such isotopies in succession, we push the imbedding off F'.

4e. Pushing Families. We restrict ourselves to the case of a one-parameter family (cf. Laudenbach [8]); the general case is analogous but rather awkward. We construct a pushing isotopy for each imbedding of the family. The isotopy pushing a certain imbedding also pushes all close imbeddings. Hence one can choose several isotopies λ^i and a partition $\{T_i\}$ of the domain of definition of the family into closed segments such that λ^i pushes f_t for $t \in T_i$, where $\{f_t : t \in T\}$ is the family considered. If $t \in T_i \cap T_j$ and $i \neq j$, then there are two isotopies pushing f_t. They can be considered as a map $\lambda : I \times 0 \cup 0 \times I \to \mathrm{Diff}(V)$. In order to construct the pushing needed, it suffices to extend this map to I^2 so that $\lambda(x) \circ f_t \in \mathrm{Em}_{\{v\}}(F, V \setminus F')$ for $x \in I \times 1 \cup 1 \times I$. Such an extension consists of a map $\sigma : I^2 \to \mathrm{Diff}(V)$ of the form $\sigma(t_1, t_2) = \lambda_1(t_1) \circ \lambda_2(t_2)$, where λ_1 and λ_2 are either constructed with the help of the basic lemma as in 4d or leave F' fixed.

[In the case of a multiparametric family one requires maps of the form $\sigma(t_1, \ldots, t_n) = \lambda_1(t_1) \circ \ldots \circ \lambda_n(t_n)$.]

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