# HOMOTOPY OF SPACES OF DIFFEOMORPHISMS OF SOME THREE-DIMENSIONAL MANIFOLDS

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The goal of the paper is to calculate the homotopy type of the space of diffeomorphisms for most orientable three-dimensional manifolds with finite fundamental group containing the Klein bottle. The fundamental group of such a manifold Q has the form $\langle a, b \mid aba b^{-1} = 1, a^m b^{2n} = 1\rangle$. As m and n one can have any relatively prime natural numbers; these numbers m, n determine the manifold Q up to diffeomorphism. Let K be a Klein bottle lying in Q and let P be a closed tubular neighborhood in Q of this Klein bottle K. We denote by $\mathfrak{Diff}_0(Q)$ the connected component of the space of diffeomorphisms $Q \to Q$ containing $\mathrm{id}\, Q$, and by $E_0(K, Q)$ the connected component of the space of imbeddings $K \to Q$ containing the inclusion $K \longrightarrow Q$; analogously we define $E_0(K, P)$. The main results of the paper are the following two theorems. THEOREM 1. If m, n $\neq$ 1, then the space $\mathfrak{Diff}_0(Q)$ is homotopy equivalent with a circle. THEOREM 2. If m, n $\neq$ 1, then the inclusion $E_0(K, P) \hookrightarrow E_0(K, Q)$ is a homotopy equivalence. With the help of familiar results on spaces of diffeomorphisms of irreducible manifolds which are sufficiently large, Theorem 1 reduces without difficulty to Theorem 2. The main difficulty is the proof of Theorem 2. This proof develops a technique of Hatcher and the author which deals with spaces of PL-homeomorphisms and diffeomorphisms of irreducible manifolds which are sufficiently large. In the paper we use a different structure definition of the class of manifolds considered. It is easy to verify that these definitions are equivalent.

## 1. Introduction

(1.1) The study of the homotopy type of spaces of diffeomorphisms of compact smooth manifolds attracts more and more attention. A survey of recent results in this area can be found in Proc. Symp. in Pure Math., Vol. 32, Part 1; cf. [3, 7, 9, 25]. At the same time in only a small number of cases has there been success up to now in calculating completely the homotopy type of spaces of diffeomorphisms. Namely, such a calculation has been made for manifolds of dimension 1 and 2 and for certain three-dimensional manifolds.

The case of one-dimensional manifolds is simple and is well known. The homotopy type of the space of diffeomorphisms of surfaces was calculated by Eells and Earle [6], and their paper was preceded by Smale's paper [24], in which the cases of the two-dimensional sphere and the two-dimensional disk were considered. As to three-dimensional manifolds, here the homotopy type of the space of diffeomorphisms is known in the following cases. Firstly, according to the famous "Smale conjecture," recently proved by Hatcher [10, 11], the space of diffeomorphisms of the sphere $S^3$ is homotopy equivalent with O(4), and the space of diffeomorphisms of the disk $D^3$ is homotopy equivalent with O(3) (the cases of $S^3$ and $D^3$ reduce to one another easily). Further, the homotopy type of the space of diffeomorphisms has been calculated for the case of Waldhausen manifolds, i.e., irreducible compact three-dimensional manifolds containing a noncontractible surface and not admitting a two-sided imbedding of the projective plane. This calculation was made independently by Hatcher [8] and the author [13, 14]; it is based on the Smale conjecture. (Hatcher [8] is devoted to the corresponding piecewise-linear problem; in order to get differential results from it, it suffices to use the PL/Diff comparison theorem of Burghelea–Lashof–Morley [4] and the Smale conjecture). Finally, Hatcher [12] calculated the homotopy type of the space of diffeomorphisms of the manifold $S^1 \times S^2$ and certain other reducible manifolds. (In [5] a series of results were announced

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dealing with reducible manifolds, but according to [12], Laudenbach discovered an essential lacuna in the proof of the results of [5].) There is a more detailed discussion of these results in Hatcher's report to the congress in Helsinki [10].

The goal of the present paper is to calculate the homotopy type of the space of diffeomorphisms for a certain infinite family of irreducible three-dimensional manifolds with finite fundamental group (we note that all Waldhausen manifolds have infinite fundamental group). These manifolds are defined in (1.2), and the main results of the paper are formulated in (1.3).

We note that all these results, including the results of the present paper, have analogs dealing with spaces of homeomorphisms and with spaces of piecewise-linear homeomorphisms. In fact, as is well known, for manifolds of dimension $\leqslant 3$, all three spaces: of diffeomorphisms, homeomorphisms, and piecewise-linear homeomorphisms, are homotopy equivalent. In dimension 3 this fact depends on the Smale conjecture. Here in most cases the results on spaces of homeomorphisms and piecewise-linear homeomorphisms can be obtained directly, without turning to the differential results.

(1.2) The manifolds with which we shall be concerned in the present paper are defined as follows. Let K be a Klein bottle, $\pi: T \to K$ be the orienting covering, so that T is a torus, and let $P = T \times I \cup K/(x,0) \sim \pi(x)$, where I is the segment [0, 1], be the mapping cylinder of $\pi$. It is clear that P is a three-dimensional manifold with boundary T, containing K as a deformation retract. The manifolds of interest to us are obtained by gluing on the solid torus $\mathcal{D}^2 \times \mathcal{S}^1$ to P by all diffeomorphisms $\partial(\mathcal{D}^2 \times \mathcal{S}^1) \to T$.

It is easy to list these manifolds. In fact, the result of gluing $\mathcal{D}^2 \times \mathcal{S}^1$ to P is determined up to a diffeomorphism by the isotopy class of the image of the meridian $\partial\mathcal{D}^2 \times x$, where x is (any) point of the circle $S^1$, and in addition its homotopy class (since homotopic curves on the torus are isotopic). Let $a$ and b be standard generators of the group $\pi_1(K)$, so that $\pi_1(K) = \langle a, b \mid abab^{-1} = 1\rangle$. Then $a$ and $b^2$ generate the image of the group $\pi_1(T)$ in $\pi_1(K)$, which we shall identify with the group $\pi_1(T)$ itself. Thus, the homotopy class of the image of the meridian has the form $a^m b^{2n}$, and thus the two numbers m and n are defined. Since the class $a^m b^{2n}$ contains an imbedded circle, the numbers m and n are relatively prime. Actually, the result of gluing depends up to a diffeomorphism only on $|m|$, $|n|$ (cf. the following paragraph). The result of gluing corresponding to positive relatively prime m and n will be denoted by Q(m, n).

The fact that the result of gluing depends only on $|m|$, $|n|$ follows from the existence of a diffeomorphism $P \to P$ acting on $\pi_1(T)$ by the formula $a \to a^{-1}$, $b^2 \to b^2$ and a diffeomorphism acting on $\pi_1(T)$ by the formula $a \to a$, $b^2 \to b^{-2}$. For example, in order to get the first diffeomorphism, it suffices to take a diffeomorphism $K \to K$ acting on $\pi_1(K)$ by the formula $a \to a^{-1}$, $b \to b$, take a covering diffeomorphism $T \to T$ of it, and finally take the diffeomorphism $P \to P$ induced by the latter. The diffeomorphisms $K \to K$ which are needed are induced by symmetries of the standard model of the Klein bottle as a rectangle with sides identified.

(1.3) Now we formulate the main results of the paper, Theorems 1 and 2. As usual, we denote by $\mathcal{D}iff(\mathcal{M})$ the space of diffeomorphisms of the manifold $\mathcal{M}$, and by $\mathcal{D}iff_0(\mathcal{M})$ the component of the space $\mathcal{D}iff(\mathcal{M})$ containing $id_{\mathcal{M}}$. For a submanifold N of the manifold M, we denote by $E(N, \mathcal{M})$ the space of imbeddings $N \to M$, and by $E_0(N, \mathcal{M})$ the component of the space E(N, M) containing the inclusion $N \hookrightarrow M$.

THEOREM 1. If m, n $\neq$ 1, then the space $\mathcal{D}iff_0(Q(m,n))$ is homotopy equivalent with a circle.

THEOREM 2. If m, n $\neq$ 1, then the inclusion $E_0(K, P) \hookrightarrow E_0(K, Q(m, n))$ is a homotopy equivalence.

Theorem 2 can be considered as a "separation theorem": it is easy to verify that the image of the imbedding $K \to Q(m, n)$ lies in int P if and only if this image does not intersect $\partial P$. With the help of the results on Waldhausen manifolds mentioned in (1.1), Theorem 1 can be reduced without difficulty to Theorem 2. This is one of the main sources of interest in Theorem 2.

The technique of the proof of these theorems can be supplemented so as to include the case of the manifolds Q(1, n) with n $\neq$ 1. Since the manifolds Q(1, n) are lens spaces [namely, Q(1, n) is diffeomorphic with L(4n, 2n − 1)], this case is of particular interest. A separate paper of the author will be devoted to it. The main result here is that $\mathfrak{Diff}_0$ (Q(1, n)) for n ≠ 1 is homotopy equivalent with the two-dimensional torus. Probably the methods of this paper can be extended to the remaining manifolds Q(m, n) also. Of the others the most interesting is the manifold Q(2, 1), which is diffeomorphic with the quaternion space, i.e., the quotient-space of the sphere $S^3$ by the action of the group of eight unit quaternions.

The proof of Theorems 1 and 2 recounted in the present paper is based on the Smale conjecture, whose proof, obtained by Hatcher [10, 11], is very difficult. The use of the Smale conjecture in the proof of Theorem 1 is apparently necessary. However, Theorem 2 can also be proved without using the Smale conjecture. Namely, such a proof was originally obtained by the author. The proof given in the present paper is based on the combination of the ideas of this original proof and the ideas of Hatcher [8]. It is essentially simpler in technical terms than the original proof. The use of the Smale conjecture allows us to concentrate on the difficulties connected with the fact that K lies in a one-sided way in Q(m, n). At the same time, essentially all the ideas needed to prove Theorem 2 without being based on the Smale conjecture are contained in the present paper and [14].

Theorem 1 can be supplemented by the calculation of the group $\pi_0(\mathfrak{Diff}(Q(m,n)))$. It is equal to $(\mathbb{Z}/2\mathbb{Z})^2$ for m ≠ 1, 2 and n ≠ 1 and is equal to $S_3 \oplus \mathbb{Z}/2\mathbb{Z}$, where $S_3$ is the symmetric group, for m = 2 and n ≠ 1. For other m and n this group is also known; cf. the following subsection.

(1.4) Theorems 1 and 2 were announced by the author in [15, 16] along with the calculation of the groups $\pi_0(\mathfrak{Diff}(Q(m,n)))$ for m ≠ 1, 2 and n ≠ 1. The manifolds Q(m, n) were also introduced there. The manifolds Q(m, n) were also introduced independently by Kim [17], Rubinstein [22] and Asano [2]. Kim [17] and Rubinstein [22] applied the manifolds Q(m, n) to the study of the actions of some finite groups on $S^3$. Rubinstein [22], Asano [2], and Cappel and Shaneson (unpublished) calculated the groups $\pi_0(\mathfrak{Diff}(Q(m,n)))$ for all m and n. These papers also contain a series of elementary facts about the manifolds Q(m, n): the calculation of the groups $\pi_1(Q(m, n))$ and $H_1(Q(m, n))$ — it shows that the manifolds Q(m, n) are pairwise homotopically inequivalent; the identification of the manifolds Q(1, n) with the lens spaces and of the manifold Q(2, 1) with the quaternionic space; the fact that the sphere is the universal covering of all the manifolds Q(m, n), from which it follows that these manifolds are irreducible. We shall not repeat the proofs of these results here, or the calculation of the groups $\pi_0(\mathfrak{Diff}(Q(m,n)))$, referring the reader to the papers cited.

The splitting of the quaternion space into two parts, diffeomorphic with P and $\mathfrak{D}^2 \times S^1$, from which it is evident that it is diffeomorphic with Q(2, 1), is already contained in Price [21]. Thinking about this paper led the author to the definition of the manifolds Q(m, n) and was the initial point in the study of their spaces of diffeomorphisms.

(1.5) The rest of the paper is devoted to the proof of Theorems 1 and 2. They are completed in Secs. 10 and 9, respectively. One of the main steps in the proof is the Basic lemma, proved in Sec. 8. Sections 3 and 4 are devoted to specific questions connected with the manifolds Q(m, n). Sections 5, 6, and 7 contain a series of general constructions needed in the proof of Theorems 1 and 2. Finally, Sec. 2 has an auxiliary character. It contains, in particular, the conventions and terminology used throughout the entire paper.

## 2. Preliminary Material

(2.1) We use differential-topological terminology. Manifolds, diffeomorphisms, etc. are assumed to be of class $C^\infty$. The spaces of diffeomorphisms and embeddings are provided with the $C^\infty$-topology as usual.

(2.2) As usual, we denote by I the segment [0, 1], by $\mathfrak{D}^n$ and $S^{n-1}$, respectively, the unit ball and sphere in $\mathbb{R}^n$. A manifold which is diffeomorphic with I, $S^1$, $\mathfrak{D}^2$, or $S^2$, respectively, will be called a *segment*, *circle*, *disk*, or *sphere*. By a *disk with holes* we mean a manifold diffeomorphic with a disk with the interiors of some pairwise disjoint disks lying in its interior discarded. By a *ring* we mean a manifold diffeomorphic with $S^1 \times I$.

A manifold diffeomorphic with $\mathfrak{D}^2 \times S^1$ is called a *solid torus*. A circle Y, lying in the boundary ∂X of a solid torus X, is said to be a *parallel* of the solid torus X if the inclusion $Y \hookrightarrow X$ is a homotopy equivalence, and a *meridian* of the solid torus $X$ if it is null-homotopic in $X$ and not null-homotopic in $\partial X$. The intersection index (in $\partial X$) of a meridian with a parallel is always equal to $\pm 1$.

By a *surface* we mean a compact two-dimensional manifold. A surface $F$ imbedded in a three-dimensional manifold $V$ is called a *regular surface* in $V$ if $\partial F = \partial V \cap F$ and $F$ is transverse to $\partial V$ in $V$.

By $\mathcal{D}^2_+$ we shall denote the half-disk $\mathcal{D}^2 \cap \mathbb{R}^2_+$, where $\mathbb{R}^2_+$ is the half-space $\{(x,y) \in \mathbb{R}^2 : y \geq 0\}$. By $S^1_+$ we shall denote the semicircle $S^1 \cap \mathbb{R}^2_+$, and by $S^1_-$ the segment $\mathcal{D}^2_+ \cap \partial\mathbb{R}^2_+$, so that $\partial\mathcal{D}^2_+ = S^1_+ \cup S^1_-$. It is clear that $\mathcal{D}^2_+$ is a smooth manifold with corners.

(2.3) In what follows we assume fixed the relatively prime natural numbers $m$ and $n$ such that $m, n \neq 1$ and we shall denote the manifold $Q(m, n)$ simply by $Q$.

The notation and conventions of (1.2) are used for the duration of the paper. In particular, the group $\pi_1(T)$ can be identified with its image in $\pi_1(K)$; $a$ and $b^2$ can be considered as its standard generators. The boundary $\partial(\mathcal{D}^2 \times S^1)$ of the solid torus $\mathcal{D}^2 \times S^1$ will be identified with the boundary $T$ of the manifold $P$ by means of the gluing diffeomorphism [corresponding to $Q = Q(m, n)$]. We note that a meridian of the solid torus $\mathcal{D}^2 \times S^1$ has in $T$ the homotopy class $a^m b^{2n}$.

<ins>(2.4) The Covering $q: Q^\sim \to Q$.</ins> We recall that $P = T \times I \cup K/(x,0) \sim \pi(x)$, where $\pi: T \to K$ is the orienting covering. Let $\rho: T \times I \to P$ be the canonical map and $\tau: T \to T$ be the involution permuting the sheets of the covering $\pi: T \to K$. We set $P^\sim = T \times [-1, 1]$ and we define the map $p: P^\sim \to P$ by

$$p(x,t) = \begin{cases} \rho(x,t) & \text{for } t \geq 0 \\ \rho(\tau(x), -t) & \text{for } t \leq 0. \end{cases}$$

It is clear that $p$ is a covering. Here $p^{-1}(K) = T \times 0$ and the map $T \times 0 \to K$, induced by $p$, is obviously the orienting covering. Moreover, $p$ induces a homeomorphism between each of the components of the boundary $\partial P^\sim$ and the boundary $\partial P$. Hence one can glue to $P^\sim$ two copies of the solid torus $\mathcal{D}^2 \times S^1$ so that the manifold $Q^\sim$ obtained is a two-sheeted covering of $Q$. Let $q: Q^\sim \to Q$ be the covering obtained. It is clear that $Q^\sim$ is a lens space. The torus $p^{-1}(K)$ divides $Q^\sim$ into two solid tori, which we denote by $R$ and $R'$. Further, we set $r = q|R$ and $r' = q|R'$. The map $r: R \to Q$ obviously induces a homeomorphism $R \setminus \partial R \to Q \setminus K$ and a two-sheeted covering $\partial R \to K$; this covering is orienting.

## 3. Klein Bottles in Q

<ins>(3.1) LEMMA.</ins> A circle lying in a Klein bottle either bounds a disk in $K$ or is isotopic with one of the four circles $\alpha$, $\beta$, $\alpha\beta$, $\beta^2$ indicated in Fig. 1: in Fig. 1a there is shown the identification of sides of a rectangle giving $K$; in Fig. 1b the circles are indicated.

There is a proof in [21]; cf. [21, Lemma 2.1].

We note that the circle $\alpha$ (respectively, $\beta$, $\alpha\beta$, $\beta^2$) is isotopic with the image of some loop of the homotopy class $a$ (respectively, $b$, $ab$, $b^2$). This explains our notation. We note in addition that the circles isotopic with $\beta$ or $\alpha\beta$ lie in $K$ in a one-sided fashion, and the others in a two-sided fashion.

<ins>(3.2) LEMMA.</ins> A connected, orientable surface $A$, lying in $K$, is a disk with holes. Here either all components of the boundary $\partial A$ are contractible in $K$, or exactly two components are not contractible in $K$.

For the proof we apply Lemma (3.1) to the components of the boundary $\partial A$ and we consider the possibilities which arise. We get that $A$ is isotopic in $K$ with either a tubular neighborhood of the curve $\alpha$ with some holes, or a tubular neighborhood of the curve $\beta^2$ with holes.

(3.3) In the rest of this section $L$ denotes a Klein bottle in $Q$, isotopic with $K$.

<ins>(3.4) LEMMA.</ins> If a circle on $L$ is null-homotopic in $Q$, then it bounds a disk in $L$.

<ins>Proof.</ins> It suffices to consider the case $L = K$.

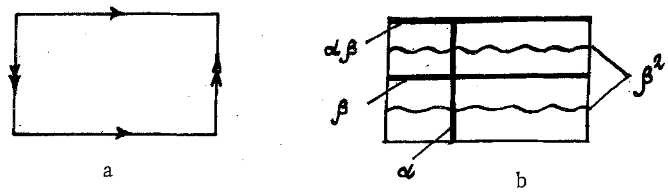



Fig. 1

Applying van Kampen's theorem to the subdivision of the manifold Q into P and $\mathfrak{D}^2 \times S^1$, we get the following representation of the group $\pi_1(Q)$ by generators and relations: $\pi_1(Q) = \langle a, b \mid aba b^{-1} = 1,\ a^m b^{2n} = 1\rangle$. From this one gets a representation of the group $H_1(Q)$ by generators and relations: $H_1(Q) = \langle a, b \mid ab = ba,\ a^2 = 1,\ a^m b^{2n} = 1\rangle$. These $a$ and b are the images of the generators $a$ and b of the group $\pi_1(K)$. Since m, n ≠ 1, one has that $a$, b, $a$b, and $b^2$ are not equal to one in $H_1(Q)$. From this it follows that the circles α, β, αβ, and $\beta^2$ are not null-homotopic (or even null-homologous) in Q. To complete the proof it remains to apply Lemma (3.1).

(3.5) LEMMA. If L is transverse to K, then any component of the surface $r^{-1}(L)$ is a disk with holes.

[We recall that $r: R \to Q$ is defined in (2.4).]

Proof. Since L is isotopic with K, one has $q^{-1}(L)$ is a torus. Since $r^{-1}(L) = q^{-1}(L) \cap R$, it follows that the surface $r^{-1}(L)$ is orientable. On the other hand, a slight reduction of the surface $r^{-1}(L)$ can be imbedded in the Klein bottle L. By (3.2) it follows from this that $r^{-1}(L)$ is a disk with holes.

(3.6) LEMMA. If L is transverse to K, then no component of the boundary $\partial r^{-1}(L)$ is either a parallel or a meridian of the solid torus R.

Proof. We can assume that $L \cap P$ is a tubular neighborhood in L of the intersection $L \cap K$. In this case the pairs $(R, r^{-1}(L))$ and $(\mathfrak{D}^2 \times S^1, L \cap \mathfrak{D}^2 \times S^1)$ are obviously diffeomorphic. Hence it suffices for us to show that no component of the intersection $L \cap T$ [which coincides with $\partial(L \cap \mathfrak{D}^2 \times S^1)$ ] is either a parallel or a meridian of the solid torus $\mathfrak{D}^2 \times S^1$.

It is clear that each component of the intersection $L \cap T$ is isotopic in T with a curve covering (under the covering $\pi: T \to K$) some component of the intersection $L \cap K$. Using Lemma (3.1), we see that each component of the intersection $L \cap T$ lies in one of the following homotopy classes: $1, a^{\pm 1}, (b^2)^{\pm 1}$. Since m, n ≠ 1, none of these classes coincides with the homotopy class $a^m b^{2n}$ of a meridian of the solid torus $\mathfrak{D}^2 \times S^1$. Hence none of the components of the intersection $L \cap T$ is a meridian of the solid torus $\mathfrak{D}^2 \times S^1$. Further, since m, n ≠ 1, the intersection index in T of any of the classes $1, a^{\pm 1}, (b^2)^{\pm 1}$ with the class $a^m b^{2n}$ of a meridian is equal to ±1. Hence, none of the components of the intersection $L \cap T$ is a parallel of the solid torus $\mathfrak{D}^2 \times S^1$.

(3.7) LEMMA. Let L be transverse to K and let A be a component of the surface $r^{-1}(L)$. Then either all components of the boundary ∂A are contractible in ∂R, or exactly two components are not contractible (in ∂R).

Proof. It is clear that $\partial A \subset r^{-1}(L \cap K)$ and that each component S of the boundary ∂A covers a component r(S) of the intersection $L \cap K$. Here S is contractible in ∂R if and only if r(S) is contractible in K. In fact, in case r(S) either bounds a disk in K or is one of the standard curves α, β, αβ, $\beta^2$, this assertion is obvious (we recall that $\partial R \to K$ is an orienting covering); the general case reduces to this by (3.1).

Further, by (3.4), r(S) is contractible in K if and only if r(S) is contractible in L.

Now let B be a slight reduction of the surface A; more precisely, let B be obtained by imbedding in A the interior of some collar of the boundary. Let S' be the component of the boundary ∂B, corresponding to the component S of the boundary ∂A. It is clear that either r(S) and r(S') bound a ring in L and thus are isotopic in L, or r(S') is the boundary of a

Möbius strip with axis $r(S)$ lying in L. From this it is clear that $r(S)$ is contractible in L if and only if $r(S')$ is contractible in L.

Combining the equivalences obtained, we see that S is contractible in $\partial R$ if and only if $r(S')$ is contractible in L. By (3.5), the surface A, and hence the surface $r(B)$ also, is a disk with holes. It remains to use Lemma (3.2).

(3.8) LEMMA. If L is transverse to K, then at least one of the components of the boundary $\partial r^{-1}(L)$ is not contractible in $\partial R$.

Proof. Obviously it suffices to show that if L is transverse to K, then at least one of the components of the intersection $L \cap K$ is not contractible in K. Let us assume the contrary. Then by Lemma (3.4) each component of the intersection $L \cap K$ is contractible not only in K, but also in L. Standard arguments (cf., e.g., [19, Chap. 2, Sec. 5.2]) show that then L is isotopic with a surface L' such that $L' \cap K = \emptyset$. Thus we get an imbedding of a Klein bottle in $Q \setminus K$. Since $Q \setminus K$ is diffeomorphic with $\operatorname{int} \mathcal{D}^2 \times S^1$, this is impossible. The contradiction obtained completes the proof.

## 4. Admissible Surfaces in a Solid Torus

(4.1) Let X be a solid torus and A be a surface in X. We shall say that A is an *admissible surface in* X if the following conditions hold:

(4.1.1) A is a regular surface in X;

(4.1.2) A is a disk with holes;

(4.1.3) none of the components of the boundary $\partial A$ is either a parallel or a meridian of the solid torus X;

(4.1.4) either all components of the boundary $\partial A$ are contractible in $\partial X$, or exactly two components are not contractible in $\partial X$.

We shall call an admissible surface A in X *inessential* if all components of the boundary $\partial A$ are contractible in $\partial X$, and *essential* in the opposite case.

The following lemma shows how admissible surfaces arise from Klein bottles in Q.

(4.2) LEMMA. Let L be a Klein bottle in Q, isotopic with K. If L is transverse to K, then any component of the surface $r^{-1}(L)$ is an admissible surface in R.

This is a consequence of Lemmas (3.5), (3.6), and (3.7).

(4.3) Let A be an essential admissible surface in X. In addition, let no component of the boundary $\partial A$ be contractible in $\partial X$. Then obviously A is a ring and A is not contractible in X [i.e., the inclusion homomorphism $\pi_1(A) \to \pi_1(X)$ is a monomorphism]. As is well known, a regularly imbedded noncontractible ring in a solid torus is a parallel of the boundary, more precisely, of a part of the boundary. In other words, A divides X into two parts, at least one of which, say W, has standard form: the triple $(W; A, \partial X \cap W)$ is diffeomorphic with the triple $(\mathcal{D}^2_+ \times S^1; S^1_+ \times S^1, S^1_- \times S^1)$. Since the components of the boundary $\partial A$ are not parallels of the solid torus X, only one of these parts has standard form. We shall denote this part by $W_X(A)$ or simply $W(A)$.

(4.4) LEMMA. Let A be an essential admissible surface in X. Let $h_1, h_2 : \mathcal{D}^2_+ \to X$ be two imbeddings such that:

$$h_i(\mathcal{D}^2_+ \cap A) = h_i(S^1_+),\quad h_i(\mathcal{D}^2_+) \cap \partial X = h_i(S^1_-),\ \text{in particular, } h_i(\mathcal{D}^2_+) \cap \partial A = h_i(\partial S^1_-);$$

$h_i(\mathcal{D}^2_+)$ is transverse to A and to $\partial X$;

the two points which constitute $h_i(\partial S^1_-)$ lie on two different components of the boundary $\partial A$ which are not contractible in $\partial X$.

Then $h_1(\mathcal{D}^2_+)$ and $h_2(\mathcal{D}^2_+)$ lie on one side of A.

Proof. We set $\mathcal{D}_i = h_i(\mathcal{D}^2_+)$, $i = 1, 2$. We glue disks to all components of the boundary $\partial A$ which are contractible in $\partial X$; we denote the surface obtained by $A^+$. Obviously $A^+$ is a ring. The inclusion $A \hookrightarrow X$ obviously extends to a map $A^+ \to X$; let i be some extension.

Each of the segments $\mathcal{D}_1 \cap A$, $\mathcal{D}_2 \cap A$ joins two components of the boundary $\partial A^+$, and hence these segments are homotopic in $A^+$ in the class of segments joining two components of the boundary. Let $h: I \times I \to A^+$ be the corresponding homotopy, so that h induces homeomorphisms $I \times 0 \to \mathcal{D}_1 \cap A$, $I \times 1 \to \mathcal{D}_2' \cap A$ and $h(0 \times I), h(1 \times I) \subset \partial A^+$. We glue $\mathcal{D}_1$ and $\mathcal{D}_2$ to $I \times I$ by means of these homeomorphisms $I \times 0 \to \mathcal{D}_1 \cap A$, $I \times 1 \to \mathcal{D}_2 \cap A$; let $\mathcal{D}$ be the space obtained. Obviously $\mathcal{D}$ is homeomorphic with a disk. The inclusions $\mathcal{D}_1 \hookrightarrow X$, $\mathcal{D}_2 \hookrightarrow X$ together with the composition $i \circ h: I \times I \to X$ determine a map $\mathcal{D} \to X$, which we denote by j. If $\mathcal{D}_1$ and $\mathcal{D}_2$ lie on different sides of A, then the intersection index in $\partial X$ of the loop $j \mid \partial \mathcal{D}$ with any component of the boundary $\partial A^+$ is equal to $\pm 1$ (cf. Fig. 2). Since the loop $j \mid \partial \mathcal{D}$ is null-homotopic in X (j defines the homotopy), it follows that in this case the components of the boundary $\partial A^+$ are parallels of the solid torus X. Since these components are simultaneously components of the boundary $\partial A$, this contradicts the admissibility of A.

(4.5) LEMMA. Let A and B be essential admissible surfaces in X. Suppose none of the components of the boundary $\partial A$ and none of the components of the boundary $\partial B$ is contractible in $\partial X$. If $A \cap B = \emptyset$, then either $W(A) \cap W(B) = \emptyset$, or $W(A) \subset W(B)$, or $W(B) \subset W(A)$.

Proof. We note first that obviously the components of the boundary $\partial A$ are isotopic with components of the boundary $\partial B$. If $W(A) \cap W(B) \neq \emptyset$, then either $A \cap W(B) \neq \emptyset$ or $B \cap W(A) \neq \emptyset$. In the first case obviously $A \subset W(B)$ and $\partial A \subset W(B) \cap \partial X$. Here $\partial A$ bounds some ring C in $W(B) \cap \partial X$. Moreover, A divides W(B) into two solid tori. Let Y be the one which contains C. It is clear that $C = Y \cap \partial X$ and that the triple (Y; A, C) is homeomorphic with the triple $(\mathcal{D}^2_+ \times S^1;\ S^1_+ \times S^1,\ S^1_- \times S^1)$. Hence Y = W(A) and $W(A) \subset W(B)$. Analogously, if $B \cap W(A) \neq \emptyset$, then $W(B) \subset W(A)$.

## 5. General Position

(5.1) Let M and N be closed surfaces in the closed three-dimensional manifold Z. We shall say that the point $x \in M \cap N$ is a *singular point of intersection of M with N* if M is not transverse to N at x. In this case the germ at x of the triple (Z; N, M) is obviously diffeomorphic with the germ at 0 of some triple of the form $(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2 \times 0, \Gamma)$ such that $\Gamma$ is the graph of some function $\mathbb{R}^2 \to \mathbb{R}$ having 0 as a critical point.

If this function has 0 as a critical point of finite multiplicity (this circumstance does not depend on the choice of triple $(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2 \times 0, \Gamma)$), then we shall say that x is a *singular point of finite multiplicity*. (For the definition of critical points of finite multiplicity, cf., e.g., [1].) We shall say that M *is in general position with* N if all singular points of the intersection of M with N are of finite multiplicity.

Let F be some closed surface. We shall say that the *imbedding* $F \to Z$ *is in general position with* N if its image is in general position with N.

(5.2) LEMMA. Any finite-parametric family of imbeddings $F \to Z$ admits an approximation by a family of imbeddings in general position with N.

This follows from the Thom transversality theorem and the fact that the set of all $\infty$-jets of functions at critical points of infinite multiplicity has codimension $\infty$ in the space of all jets of functions (cf. [1]).

(5.3) For a function $f: \mathcal{D}^2 \to [-1,1]$ we shall denote by $\Gamma_f$ its graph; $\Gamma_f \subset \mathcal{D}^2 \times [-1,1]$.

Let $x$ be a singular point of finite multiplicity of the intersection of M with N. By definition, for some neighborhood U of the point $x$ in Z the quadruple $(x, N \cap U, M \cap U, U)$ is diffeomorphic with some quadruple of the form $(0, \mathcal{D}^2 \times 0, \Gamma_f, \mathcal{D}^2 \times [-1,1])$, where $f: \mathcal{D}^2 \to (-1,1)$ is a function having 0 as a critical point of finite multiplicity. If here 0 is the unique critical point of the function f and $f^{-1}(0)$ is a union of several segments joining 0 with $\partial \mathcal{D}^2$,

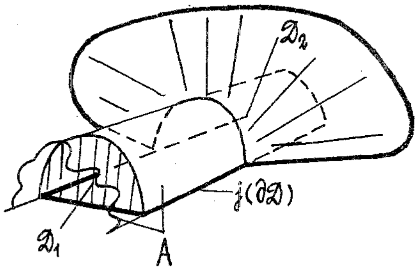


Fig. 2

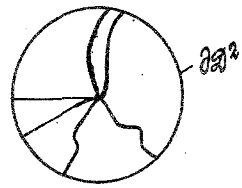


Fig. 3

transverse to $\partial\mathcal{D}^2$, and intersecting only at 0 (cf. Fig. 3), then we shall say that U is a *regular neighborhood* of the point $\varkappa$.

We shall show that $\varkappa$ always has a regular neighborhood. Firstly, since a critical point of finite multiplicity is isolated in the set of all critical points (cf. [1]), we can assume that 0 is the unique critical point of the function f. Further, since for a function $g:\mathbb{R}^2 \to \mathbb{R}$ with critical point of finite multiplicity at 0 there exists a diffeomorphism $h:(\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ such that the germ of the function $g \circ h$ at 0 is polynomial (cf. [1]), one can assume in addition that the function f is polynomial. Then by Theorem 2.10 of [20] for a disk $\mathcal{D}$ of sufficiently small radius with center at 0 the pair $(\mathcal{D}, \mathcal{D} \cap f^{-1}(0))$ is homeomorphic with the pair $(\mathcal{D}, \mathrm{con}(\partial\mathcal{D} \cap f^{-1}(0)))$, where con is a rectilinear cone with vertex at 0. Moreover, as is evident from the proof of this theorem, the homeomorphism $(\mathcal{D}, \mathcal{D} \cap f^{-1}(0))$ $(\mathcal{D}, \mathrm{con}(\partial\mathcal{D} \cap f^{-1}(0)))$ can be chosen so that it is a diffeomorphism away from 0. From this it is evident that the image of the cylinder $\mathcal{D} \times [-1,1]$ under the diffeomorphism $\mathcal{D}^2 \times [-1,1] \to U$ is a regular neighborhood of the point $\varkappa$.

We note for what follows that if the function $f:\mathcal{D}^2 \to (-1,1)$ satisfies the conditions from the definition of regular neighborhood, then 0 is a regular value of the function $f|\partial\mathcal{D}^2$; moreover, $f^{-1}(0) \cap \partial\mathcal{D}^2$ consists of an even number of points. The second assertion obviously follows from the first. The first, in its own right, follows from the fact that f has no critical points on $\partial\mathcal{D}^2$ and $f^{-1}(0)$ is transverse to $\partial\mathcal{D}^2$.

(5.4) LEMMA. Let M be in general position with N; $M, N \subset Z$. Let $\varphi:\mathcal{D}^2 \times [-1,1] \to Z$ be an imbedding such that $\mathrm{Im}\,\varphi \cap N = \varphi(\mathcal{D}^2 \times 0)$. Then for all sufficiently small u the disk $\varphi(\mathcal{D}^2 \times u)$ is transverse to M.

Proof. Let $y \in \mathcal{D}^2$. If $\varphi(y,0)$ is not a singular point of the intersection of M with N, then for some neighborhood $U_y$ of the point y, obviously $\varphi(U_y \times u)$ is transverse to M for all sufficiently small u. Now let $\varphi(y,0)$ be a singular point. If $U_y$ is a sufficiently small neighborhood of the point y and $\varepsilon$ is sufficiently small, then $\varphi^{-1}(M) \cap U_y \times [-\varepsilon,\varepsilon]$ is the graph of some function $U_y \to (-\varepsilon, \varepsilon)$. This function has y as a critical point of finite multiplicity. Since critical points of finite multiplicity are isolated in the set of all critical points, we can assume that y is the unique critical point of this function. Then $\varphi(U_y \times u)$ is transverse to M for $|u| < \varepsilon$. In order to complete the proof, it suffices to note that from the family $\{U_y\}_{y \in \mathcal{D}^2}$ one can choose a finite covering of the disk $\mathcal{D}^2$.

## 6. Reconstruction along the Boundary

(6.1) Let M be a regular surface in the three-dimensional manifold Z. Let $h:\mathcal{D}^2_+ \times I \to Z$ be an imbedding such that $h(S^1_+ \times I) = M \cap h(\mathcal{D}^2_+ \times \dot{I})$ and $h(S^1_- \times I) = \partial Z \cap h(\mathcal{D}^2_+ \times I)$. Let M' be the result of smoothing corners in the surface $(M \setminus h(S^1_+ \times I)) \cup h(\mathcal{D}^2_+ \times \partial I)$. In this situation we shall say that M' is obtained from M by *reconstruction along the boundary in Z by means of the imbedding h*. See Fig. 4.

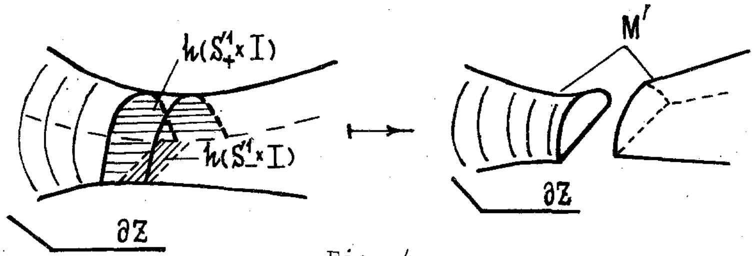



Fig. 4

It is clear that M' is a regular surface in Z. The boundary $\partial M'$ is obtained from $\partial M$ by Morse reconstruction of index 1: $h(S^1_- \times I)$ is the corresponding handle.

(6.2) Let M' be obtained from M by reconstruction (along the boundary) by means of h, and M'' be obtained from M' by reconstruction by means of h'. If the images of the imbeddings h and h' do not intersect, then obviously these reconstructions can be commuted: M'' can be obtained from M by first reconstructing by means of h' and then reconstructing by means of h. Actually, these reconstructions can be done simultaneously.

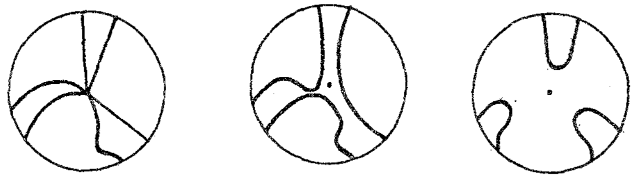
More generally, if several imbeddings such as in (6.1) are given, and their images are pairwise disjoint, then reconstruction by means of these imbeddings can be done simultaneously.

## 7. Special Approximations of Surfaces

(7.1) Let N be a closed surface in the closed three-dimensional manifold Z. Each surface M in Z can be approximated by a surface transverse to N. In case M is in general position with N [in the sense of (5.1)], there is a particularly convenient class of such approximations, the class of special approximations. This section is devoted to defining it.

Let M be a closed surface in Z, in general position with N. Let $\varkappa$ be a singular point of intersection of M with N. We choose a regular neighborhood U of the point $\varkappa$ and a corresponding function f, such that the quadruples $A=(\varkappa, N\cap\mathcal{U}, M\cap\mathcal{U}, \mathcal{U})$ and $B=(0, \mathscr{D}^2\times 0, \Gamma_f, \mathscr{D}^2\times[-1,1])$ are diffeomorphic and f satisfies the conditions from the definition of regular neighborhood [cf. (5.3)]. In particular, the preimage $f^{-1}(0)$ is the union of some segments joining 0 to $\partial\mathscr{D}^2$, and since $f^{-1}(0)\cap\partial\mathscr{D}^2$ consists of an even number of points [cf. (5.3)], the number of these segments is even. Cf. Fig. 5a. Let $g:\mathscr{D}^2\to(-1,1)$ be some function coinciding with f near $\partial\mathscr{D}^2$, having 0 as a regular value, and such that the preimage $g^{-1}(0)$ consists of several pairwise disjoint segments, each of which joins two neighboring points of the intersection $f^{-1}(0)\cap\partial\mathscr{D}^2$, so that up to an isotopy which is the identity on $\partial\mathscr{D}^2$, one has two possibilities for $g^{-1}(0)$, which are pictured in Fig. 5b and c. Since 0 is a regular value of the function $f|\partial\mathscr{D}^2$ [cf. (5.3)], such functions g obviously exist. It is clear that for such a function g the graph $\Gamma g$ is transverse to the disk $\mathscr{D}^2\times 0$ in $\mathscr{D}^2\times[-1,1]$. In the surface M we replace the piece $M\cap\mathcal{U}$ by the surface corresponding to the graph $\Gamma g$ under the diffeomorphism of quadruples $A\to B$. We perform the analogous operation for the remaining singular points of intersection of M with N. The surface obtained is obviously transverse to N; by the choice of the functions g, we can arrange that it will be arbitrarily close to M. We shall call surfaces obtained from M in the way indicated *special approximations of the surface* M *(with respect to* N*)*.

(7.2) We note that for the quadruple $(0, \mathscr{D}^2\times 0, \Gamma_g, \mathscr{D}^2\times[-1,1])$, where g is a function participating in the construction of a special approximation of the surface M near some singular point, there are, up to an isotopy which is the identity on $\partial(\mathscr{D}^2\times[-1,1])$, two possibilities for the preimage $g^{-1}(0)$. Hence, close to a given singular point a special approximation M' of the surface M can be constructed in only two essentially different ways. If U is a regular neighborhood of a singular point, taking part in the construction of a special approximation, and $U_0$ is one of the two halves into which N divides U, then these two ways are obviously distinguished by whether $M'\cap\mathcal{U}_0$ is connected or disconnected.



a b c

Fig. 5

(7.3) Now let the surface N divide Z into two parts $Z_0$ and $Z_1$, so that $Z = Z_0 \cup Z_1$ and $N = \partial Z_0 = \partial Z_1$. Let M' be a special approximation of the surface M such that for any singular point $\varkappa$ the intersection $M' \cap \mathcal{U}_\varkappa \cap Z_0$ is disconnected, where $U_\varkappa$ is the regular neighborhood of the point $\varkappa$ taking part in the construction of M'. Let M" be some other special approximation of the surface M. It is clear that $M' \cap Z_0$ and $M'' \cap Z_0$ are regular surfaces in $Z_0$.

(7.4) LEMMA. Under the conditions of (7.3), one can get from the surface $M'' \cap Z_0$ by simultaneous realization of several reconstructions along the boundary in $Z_0$ [by means of imbeddings with pairwise disjoint images; cf. (6.2)] a surface isotopic with the surface $M' \cap Z_0$ in the class of regular surfaces in $Z_0$.

Proof. We can assume that in the construction of the approximations M' and M" the same regular neighborhoods and the same diffeomorphisms of quadruples $(\varkappa, N \cap \mathcal{U}, M \cap \mathcal{U}, \mathcal{U}) \to (0, \mathcal{D}^2 \times 0, \Gamma_f, \mathcal{D}^2 \times [-1,1])$ from (7.1) are used. In fact, the choice of other regular neighborhoods and diffeomorphisms in the construction of the approximation M' leads to replacement of the surface $M' \cap Z_0$ by a surface which is isotopic (in the class of regular surfaces in $Z_0$). (The same is true for M".)

Thanks to this, it suffices to consider our situation in a regular neighborhood of a singular point and even the corresponding situation in $\mathcal{D}^2 \times [-1,1]$. The latter is pictured in Fig. 6. By F we denote the homeomorphism of quadruples $(\varkappa, N \cap \mathcal{U}, M \cap \mathcal{U}, \mathcal{U}) \to (0, \mathcal{D}^2 \times 0, \Gamma_f, \mathcal{D}^2 \times [-1,1])$ of (7.1). Here it is assumed that $F(\mathcal{U} \cap Z_0) = \mathcal{D}^2 \times [0,1]$ (this obviously does not restrict the generality) and only those parts of the surfaces are pictured which lie in $\mathcal{D}^2 \times [0, 1]$ (it is precisely these in which we are interested). It is clear from the figure that the reconstructions and isotopies needed exist.

## 8. Basic Lemma

(8.1) Let $u \in (0,1]$. We denote by $P_u$ the image of the product $T \times [0, u]$ under the canonical map $T \times I \to P$ [cf. (1.2)], so that $P_u$ is diffeomorphic with P and $P_1 = P$. We set $R_u = Cl(Q \setminus P_u)$; it is clear that $R_u$ is a solid torus.

(8.2) Basic Lemma. Let L be a Klein bottle in Q, isotopic with K and in general position with K. If u is sufficiently small, then each component of the surface $L \cap R_u$ is an admissible surface in the solid torus $R_u$.

The rest of this section is devoted to the proof of this lemma.

(8.3) We recall that $q: Q^\sim \to Q$ is the standard two-sheeted covering from (2.4). We set $L^\sim = q^{-1}(L)$ and $R_u' = r^{-1}(R_u)$ [where $r = q|R$; cf. (2.4)]. It is clear that the pair $(R_u, L \cap R_u)$ is diffeomorphic with the pair $(R_u', L^\sim \cap R_u')$. Hence it suffices for us to prove that for sufficiently small u each component of the surface $L^\sim \cap R_u'$ is an admissible surface in $R_u'$.

Further, let $L_1^\sim$ be a special approximation of the surface $L^\sim$ with respect to $q^{-1}(K)$, such that for any singular point the intersection $L_1^\sim \cap \mathcal{U}_\varkappa \cap R$ is not connected, where $U_\varkappa$ is the regular neighborhood of the point $\varkappa$ which takes part in the construction of $L_1^\sim$.

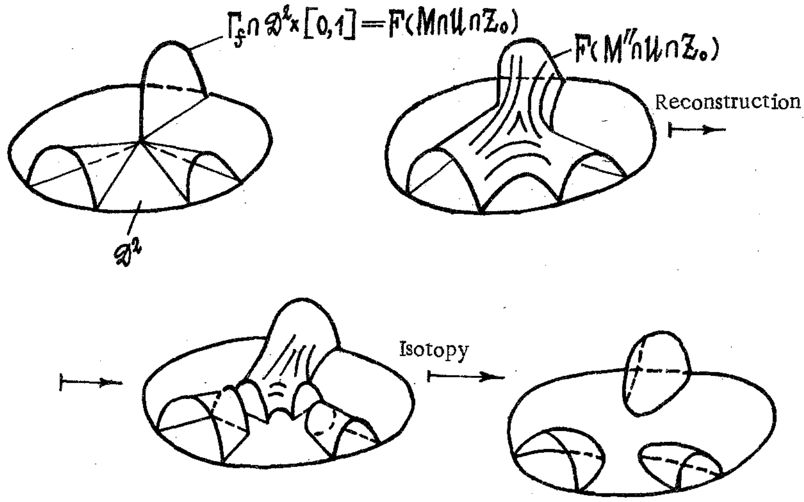



Fig. 6

Obviously for sufficiently small u the pairs $(R, \tilde{L}_1 \cap R)$ and $(R'_u, \tilde{L} \cap R'_u)$ are diffeomorphic. Thus it suffices for us to show that each component of the intersection $\tilde{L}_1 \cap R$ is an admissible surface in R.

(8.4) Let $L_0$ be some special approximation of the surface L with respect to K and let $\tilde{L}_0 = q^{-1}(L_0)$. It is clear that $\tilde{L}_0$ is a special approximation of the surface $L^{\sim}$ [with respect to $q^{-1}(K)$]. We set $H_0 = \tilde{L}_0 \cap R$ and $H = \tilde{L}_1 \cap R$. The surfaces $H_0$ and H are regular surfaces in R. By Lemma (7.4), from $H_0$ one can get, by simultaneous realization of several reconstructions along the boundary in R, a surface which is isotopic with H in the class of regular surfaces in R. Let this be reconstructions by means of the imbeddings $h_1, \ldots, h_m$; the images of these imbeddings are pairwise disjoint. Let $H_1$ be the result of reconstruction of the surface $H_0$ by means of $h_1$, $H_2$ be the result of reconstruction of the surface $H_1$ by means of $h_2, \ldots, H_m$ be the result of reconstruction of the surface $H_{m-1}$ by means of $h_m$; $H_m$ is isotopic with H. Changing the indexing of the imbeddings $h_i$, if necessary, we can assume that for some $n \leqslant m$ one has the following:

(8.4.1) for $i = 1, \ldots, n$ the image $\operatorname{Im} h_i$ intersects no more than one component of the boundary $\partial H_i$ which is not contractible in $\partial R$;

(8.4.2) for $i = n + 1, \ldots, m$ the image $\operatorname{Im} h_i$ intersects two components of the boundary $\partial H_n$ which are not contractible in $\partial R$ [we stress that (8.4.2) is concerned with $\partial H_n$, and not $\partial H_i$ as in (8.4.1)].

(8.5) We shall show that for $i = 0, \ldots, n$, each component of the surface $H_i$ is an admissible surface (in R). We shall prove this by induction on i simultaneously with the following assertion:

(8.5.1) at least one of the components of the boundary $\partial H_i$ is not contractible in $\partial R$.

For $i = 0$ this assertion is valid by Lemma (3.8), and moreover, each component of the surface $H_0$ is admissible by Lemma (4.2). Now let our assertions be valid for $i \leqslant k$ and $k < n$. Obviously each component of the surface $H_{k+1}$ is a regular surface in R. Since reconstruction along the boundary obviously carries a disk with holes into one or two disks with holes, each component of the surface $H_{k+1}$ is a disk with holes. Further, since each component of the surface $H_k$ is admissible, it follows from (8.5.1) that at least two components of the boundary $\partial H_k$ are not contractible in $\partial R$. Hence, by (8.4.1) the image $\operatorname{Im} h_{k+1}$ does not intersect some component S of the boundary $\partial H_k$ which is not contractible in $\partial R$. It is clear that S is also a component of the boundary $\partial H_{k+1}$ and that all components of the boundary $\partial H_{k+1}$ which are not contractible in $\partial R$ are isotopic with S. Since S is neither a parallel nor a meridian (by the inductive hypothesis), no component of the boundary $\partial H_{k+1}$ is either a parallel or a meridian. To justify the inductive step it remains for us to verify that each component of the surface $H_{k+1}$ either has exactly two components of the boundary which are not contractible in $\partial R$, or has no such component.

We shall prove this. For a regular surface F in R we shall denote by n(F) the number of components of the boundary $\partial F$ which are not contractible in $\partial R$. Let G be a component of the surface $H_k$ which intersects $\mathrm{Im}\, h_{k+1}$, and let G' be the result of reconstruction of the surface G by means of $h_k$. Since each component of the surface $H_{k+1}$ is obviously either a component of the surface $H_k$ or a component of the surface G', it suffices for us to show that n(G") = 0 or 2 for any component G" of the surface G'.

If $\mathrm{Im}\, h_{k+1}$ intersects two components of the boundary $\partial G$, then obviously the surface G' is connected. Here if $\mathrm{Im}\, h_k$ intersects two components which are not contractible in $\partial R$, then n(G') = 0, while in the opposite case n(G') = n(G) = 0 or 2 (by the inductive hypothesis).

Now let $\mathrm{Im}\, h_{k+1}$ intersect only one component S of the boundary $\partial G$ and let $S_0$, $S_1$ be components of the boundary $\partial G'$ obtained [by some Morse reconstruction; cf. (6.1)] from S. In this case obviously G' consists of two components $G_0$ and $G_1$ containing $S_0$ and $S_1$, respectively. We note that $n(G_i) \neq 1$, i = 0, 1. In fact, if $\partial G_i$ has exactly one component which is not contractible in $\partial R$, then this component is contractible in R (since $G_i$ is a disk with holes) and hence is a meridian, which contradicts what has been proved already. If S is contractible in $\partial R$, then obviously the circles $S_0$ and $S_1$ are either both contractible in $\partial R$, or are both not contractible. Thus in this case either n(G') = n(G), or n(G') = n(G) + 2, and hence n(G') = 0, 2, or 4. Combining this with the fact that $n(G') = n(G_0) + n(G_1)$ and $n(G_0)$, $n(G_1) \neq 1$, we get that $n(G_i) = 0$ or 2, i = 0, 1. It remains to consider the case when S is not contractible in $\partial R$. Let S' be a second component of the boundary $\partial G$ which is not contractible in $\partial R$ (it exists, since $H_k$, and thus G, is an admissible surface). Since $k < n$, one has $\mathrm{Im}\, h_{k+1} \cap S' = \emptyset$. If we cut R along S', then we get some ring C. The circle S divides C into two rings (since S is isotopic with S' in $\partial R$). It is clear that $h_{k+1}(S^1_- \times I)$ lies entirely in one of these rings. Since $S_0 \cup S_1$ is obtained from S by Morse reconstruction by means of the handle $h_{k+1}(S^1_- \times I)$, it is evident that one of the circles $S_0$, $S_1$ is contractible in $\partial R$, and the other is not. Thus n(G') = n(G) = 0 or 2. Combining this, as above, with the fact that $n(G') = n(G_0) + n(G_1)$ and $n(G_0)$, $n(G_1) \neq 1$, we get that $n(G_i) = 0$ or 2, i = 0, 1.

This completes the justification of the inductive step. Thus, we can conclude that each component of the surface $H_i$ with i = 0,...,n is an admissible surface.

(8.6) Now we show that each component of the surface $H_m$ is admissible. Since $H_m$ is isotopic with H, this completes the proof of the basic lemma.

If m = n, then everything is already proved. Let m > n. Let G be some component of the surface $H_n$, and let the images of the imbeddings $h_{n+1},\ldots,h_l$ intersect with G, and those of the imbeddings $h_{1+l},\ldots,h_m$ not (this can alreays be achieved by changing the indexing of the imbeddings). Let G' be obtained from G by simultaneous realization of the reconstructions by means of $h_{n+1},\ldots,h_l$. It is clear that G' is the union of several components of the surface $H_m$. By (8.5), G is an admissible surface. In view of (8.4.2), Lemma (4.4) is applicable to all pairs of imbeddings $h_i | \mathcal{D}^2_+ \times 0$, $h_j | \mathcal{D}^2_+ \times 0$, $n+1 \leq i, j \leq l$. It follows from this lemma that the images $\mathrm{Im}\, h_{n+1},\ldots,\mathrm{Im}\, h_l$ lie on the side of G. Since in addition each of these images intersects two components of the boundary $\partial G$ which are incontractible in ( )* (and there are always two such components), it follows that all components of the boundary $\partial G'$ are contractible in $\partial R$. Since the surface G' is obviously regular and is a union of several disks with holes, it is evident that each component of the surface G' is an admissible surface.

Since each component of the surface $H_m$ is obtained from some component of the surface $H_n$, we see that each component of the surface $H_m$ is admissible. This concludes the proof of the Basic Lemma.

## 9. Proof of Theorem 2

(9.1) We recall [cf. (8.1)], that for $u \in (0,1]$ we denote by $P_u$ the image of the product $T \times [0, u]$ in P, and by $P_u$ we denote $Cl(Q \setminus P_u)$. By $T_u$ we denote the torus $\partial R_u$.

If L is a surface in Q which is in general position with K, then for all sufficiently small u, the torus $T_u$ is transverse to L. This follows from Lemma (5.4).

*A symbol was missing in the Russian original — Publisher.

(9.2) To prove Theorem 2 it suffices for us to show that $\pi_n(E_0(K,Q), E_0(K,P)) = 0$ for all $n$. Let the family of imbeddings $\{f_t: K \to Q\}_{t \in \mathscr{D}^n}$ represent some element of $\pi_n(E_0(K,Q), E_0(K, P))$. We need to show that this element is equal to zero. Using Lemma (5.2), we can assume that all surfaces $f_t(K)$, $t \in \mathscr{D}^n$ are in general position with K. Moreover, we can assume that $f_t(K) \subset \operatorname{int} P$ for $t \in \partial\mathscr{D}^n$. We set $K_t = f_t(K)$, $t \in \mathscr{D}^n$.

Let $t \in \mathscr{D}^n$. Since $K_t$ is in general position with K, one has that $K_t$ is transverse to $T_u$ for all sufficiently small u, say for $u \leq u_n(t) < 1$ [cf. (9.1)]. It is clear that all pairs $(R_u, R_u \cap K_t)$ with $u \leq u_n(t)$ are diffeomorphic. We choose a closed spherical neighborhood $U_t^n$ of the point t in $\mathscr{D}^n$, such that $K_s$ is transverse to $T_{u_n}(t)$ for $s \in \mathcal{U}_t^n$.

The collection of neighborhoods $\mathcal{U}_t^n$, $t \in \mathscr{D}^n$ forms a covering of the ball $\mathscr{D}^n$. We choose a triangulation $\tau_n$ of the ball $\mathscr{D}^n$ subordinate to this covering, i.e., such that each closed simplex $\sigma$ of $\tau_n$ lies in the neighborhood $U^n_{t(\sigma)}$ for some point $t(\sigma) \in \mathscr{D}^n$. We set $u(\sigma) = u_n(t(\sigma))$ and we denote by $\Delta_n$ the collection of (closed) n-dimensional simplices of the triangulation $_n$.

Let $X_{n-1}$ be the $(n-1)$-dimensional skeleton of the triangulation $\tau_n$. Let $t \in X_{n-1}$. We choose a number $u_{n-1}(t)$ such that $0 < u_n(t) < u(\sigma)$ for $t \in \sigma \in \Delta^n$ and $K_t$ is transverse to $T_u$ for $u \leq u_{n-1}(t)$. We choose a closed spherical neighborhood $U_t^{n-1}$ of the point t in $\mathscr{D}^n$, such that $K_s$ is transverse to $T_{u_{n-1}(t)}$ for $s \in \mathcal{U}_t^{n-1}$.

The triangulation $\tau_n$ induces a triangulation of the skeleton $X_{n-1}$. We refine this triangulation to a triangulation $\tau_{n-1}$ of the skeleton $X_{n-1}$ having the following property: for each $(n-1)$-dimensional simplex $\sigma$ of the triangulation $\tau_n$ the restriction of the triangulation $\tau_{n-1}$ to $\sigma$ is subordinate to the covering $\{\mathcal{U}_t^{n-1}\}_{t \in \sigma}$ of the simplex $\sigma$. Thus, for each simplex $\sigma$ of $\tau_{n-1}$ one can find a point $t(\sigma) \in X_{n-1}$ such that $\sigma \subset \mathcal{U}^{n-1}_{t(\sigma)}$ and if $\sigma$ lies in the $(n-1)$-dimensional simplex $\sigma'$ of the triangulation $\tau_n$, then $t(\sigma) \in \sigma'$. We set $u(\sigma) = u_{n-1}(t(\sigma))$. Let $\Delta_{n-1}$ be the collection of $(n-1)$-dimensional simplices of the triangulation $\tau_{n-1}$. It is clear that if $\sigma \in \Delta_{n-1}$, $\sigma' \in \Delta_n$ and $\sigma \subset \sigma'$, then $u(\sigma) < u(\sigma')$.

Now we consider the $(n-2)$-skeleton $X_{n-2}$ of the triangulation $\tau_{n-1}$ and with it we make the analogous construction (to that made with $X_{n-1}$); now $\tau_{n-1}$ will play the role of $\tau_n$. Then we make the analogous construction with the $(n-3)$-dimensional skeleton $X_{n-3}$ of the triangulation $\tau_{n-2}$ obtained, etc. As a result, we get a sequence of triangulations $\tau_n, \ldots, \tau_0$ such that $\tau_{i-1}$ is a triangulation of the $(i-1)$-dimensional skeleton $X_{i-1}$ of the triangulation $\tau_i$ $(i = n, \ldots, 1)$. We denote by $\Delta_i$ the collection of i-dimensional simplices of the triangulation $\tau_i$, and by $\Delta$ the union $\Delta_n \cup \ldots \cup \Delta_0$. For each simplex $\sigma \in \Delta$ there are defined a number $u(\sigma)$ and a point $t(\sigma)$. Here if $s \in \mathcal{U}^i_{t(\sigma)}$, then $K_s$ is transverse to $T_{u(\sigma)}$ and if $u \leq u(\sigma)$, then $T_u$ is transverse to $K_{t(\sigma)}$. Moreover, if $\sigma \in \Delta_{i-1}$, $\sigma' \in \Delta_i$ and $\sigma \subset \sigma'$, then $t(\sigma) \in \sigma'$ and $u(\sigma) < u(\sigma')$. For $\sigma \in \Delta_i$ we set $\mathcal{U}_\sigma = \mathcal{U}^i_\sigma$, $T_\sigma = T_{u(\sigma)}$, $R_\sigma = R_{u(\sigma)}$ and $P_\sigma = P_{u(\sigma)}$.

(9.3) LEMMA. For $t \in \mathcal{U}_\sigma$ each component of the surface $K_t \cap R_\sigma$ is an admissible surface in $R_\sigma$.

Proof. Firstly, $K_t$ is transverse to $T_\sigma$ for $t \in \mathcal{U}_\sigma$ and hence all pairs $(R_\sigma, K_t \cap R_\sigma)$ with $t \in \mathcal{U}_\sigma$ are diffeomorphic. In particular, these pairs are diffeomorphic with the pair $(R_\sigma, K_{t(\sigma)} \cap R_\sigma)$. Further, $K_{t(\sigma)}$ is transverse to $T_u$ for $u \leq u(\sigma)$ and hence all pairs $(R_u, K_{t(\sigma)} \cap R_u)$ with $u \leq u(\sigma)$, in their own right, are diffeomorphic. Thus for $t \in \mathcal{U}_\sigma$ and $u \leq u(\sigma)$ the pairs $(R_\sigma, K_t \cap R_\sigma)$ and $(R_u, K_{t(\sigma)} \cap R_u)$ are diffeomorphic (we recall that $R_\sigma = R_{u(\sigma)}$). By the Basic Lemma (8.2), for sufficiently small u, each component of the surface $K_{t(\sigma)} \cap R_u$ is an admissible surface in $R_u$. Hence also each component of the subspace $K_t \cap R_\sigma$ for $t \in \mathcal{U}_\sigma$ is an admissible surface in $R_\sigma$.

(9.4) Let $t \in \mathfrak{D}^n$. For $\sigma \in \Delta$ such that $t \in \mathcal{U}_\sigma$ we denote by $\mathcal{Y}_{t\sigma}$ the collection of components of the intersection $K_t \cap T_\sigma$ bounding a disk in $T_\sigma$. We set $\mathcal{Y}_t = \bigcup_\sigma \mathcal{Y}_{t\sigma}$. By Lemma (3.4), each circle from $\mathcal{Y}_t$ bounds a disk in $K_t$ also. Hence there is a standard isotopy $\{K_{ts}\}_{s \in I}$, fixing the components of the intersection $K_t \cap T_\sigma$ belonging to $\mathcal{Y}_t$ (cf., e.g., [8, Sec. 1] or [19, Chap. 2, Paragraph 5.2]). In more detail, for $S \in \mathcal{Y}_{t\sigma}$ we denote by $\mathfrak{D}_T(S)$ and $\mathfrak{D}_K(S)$ disks bounded by the circle S in $T_\sigma$ and $K_t$, respectively. If the disk $\mathfrak{D}_T(S_0)$ with $S_0 \in \mathcal{Y}_{t\sigma}$ is minimal with respect to inclusion among disks $\mathfrak{D}_T(S)$ with $S \in \mathcal{Y}_{t\sigma}$, then the union $\mathfrak{D}_T(S_0) \cup \mathfrak{D}_K(S_0)$ is obviously a sphere. Since Q is irreducible, this sphere bounds a ball in Q. This ball defines an isotopy carrying $\mathfrak{D}_K(S_0)$ into $\mathfrak{D}_T(S_0)$. Extending this isotopy slightly farther, we make the image of the disk $\mathfrak{D}_K(S_0)$ encounter the disk $\mathfrak{D}_T(S_0)$ and thus we get an isotopy of the Klein bottle $K_t$, fixing the component $S_0$ of the intersection with $T_\sigma$. Successively realizing several such isotopies, we get an isotopy $\{K_{ts}\}_{s \in I}$ such that $K_{t0} = K_t$ and $K_{t1} \cap T_\sigma$ has no component bounding a disk in $T_\sigma$ (for $t \in \mathcal{U}_\sigma$). Obviously if $K_t \subset \operatorname{int} P$, for example, if $t \in \partial \mathfrak{D}^n$, then all of this isotopy takes place in int P, i.e., $K_{ts} \subset \operatorname{int} P$ for all $s \in I$.

We look at what happens with the components of the surface $K_t \cap R_\sigma$ (for $t \in \mathcal{U}_\sigma$) under the isotopy described. Each such component is an admissible surface by (9.3). If it is a nonessential [cf. (4.1)] admissible surface, then it obviously disappears under the isotopy. Now if it is an essential admissible surface, then as a result of the isotopy it becomes some new admissible surface. The boundary of this new admissible surface is the union of those components of the boundary of the original surface (components of the surface $K_t \cap R_\sigma$) which do not bound a disk in $T_\sigma$. Thus the boundary of this new surface consists of two components and the surface itself is a ring.

On the other hand, obviously each component of the surface $K_{t1} \cap R_\sigma$ comes from some component of the surface $K_t \cap R_\sigma$. In particular, all components of the surface $K_{t1} \cap R_\sigma$ are admissible.

(9.5) Now we want to put the isotopies $\{K_{ts}\}_{s \in I}$ of the preceding paragraph together into a continuous family $\{K_{ts}\}_{t \in \mathfrak{D}^n, s \in I}$. For this we use Hatcher's construction from [8, Sec. 1]. We already have almost everything we need to apply this construction. It is only necessary to note that we obviously can enlarge each ball $U_\sigma$ slightly and get a ball $\overline{U}_\sigma$ such that $K_t$ is transverse to $T_\sigma$ for $t \in \overline{\mathcal{U}}_\sigma$ and $\mathcal{U}_\sigma \subset \operatorname{int} \overline{\mathcal{U}}_\sigma$. We apply Hatcher's construction to the family $\{K_t\}_{t \in \mathfrak{D}^n}$, the coverings $\{\mathcal{U}_\sigma\}_{\sigma \in \Delta}$, $\{\overline{\mathcal{U}}_\sigma\}_{\sigma \in \Delta}$, the tori $T_\sigma$ and the family $\{\mathcal{Y}_t\}_{t \in \mathfrak{D}^n}$ in the roles, respectively, of the family $\{M_t\}_{t \in \mathfrak{D}^k}$, the coverings $\{B_i'\}, \{B_i\}$, the slices $N_i$, and the family $\{\mathcal{C}_t\}$. Hatcher's construction (recounted in [8, Sec. 1] in the simplest special case) works in our situation, requiring only one essential change: since we are working in the differential category, instead of Alexander's method one needs to use the Smale conjecture, proved by Hatcher [10, 11]. As a result we get a family $\{K_{ts}\}_{t \in \mathfrak{D}^n, s \in I}$ such that: $K_{t0} = K_t$ for all $t \in \mathfrak{D}^n$; $K_{t1} \cap T_\sigma$ has no component bounding a disk in $T_\sigma$ for $t \in \mathcal{U}_\sigma$, $\sigma \in \Delta$; each component of the surface $K_{t1} \cap R_\sigma$ is an admissible surface in $R_\sigma$ and simultaneously a ring (for $t \in \mathcal{U}_\sigma$); $K_{ts} \subset \operatorname{int} P$ for $t \in \partial \mathfrak{D}^n$.

(9.6) Let $t \in \mathcal{U}_\sigma$. Let $A_1, \ldots, A_l$ be the collection of components of the surface $K_{t1} \cap R_\sigma$. Since each surface $A_i$ is admissible [cf. (9.5)], the manifolds $W(A_i) = W_{R_\sigma}(A_i)$ are defined. By Lemma (4.5), for any i, j, either $W(A_i) \cap W(A_j) = \emptyset$, or $W(A_i) \subset W(A_j)$, or $W(A_j) \subset W(A_i)$. From which it follows that the manifold $W = W(A_1) \cup \ldots \cup W(A_l)$ is a trivial cobordism between $W \cap \partial R_\sigma$ and $\operatorname{Cl}(\partial W \setminus \partial R_\sigma)$. It follows from this, in its own right, that the manifold $C_{t\sigma}=P_\sigma \cup W$ is a tubular neighborhood of the Klein bottle K in Q. Obviously $K_{t_1} \subset C_{t\sigma}$. Moreover, if $t \in \partial \mathfrak{D}^n$, then $C_{t\sigma} \subset \operatorname{int} P$ (since $K_{t_1} \subset \operatorname{int} P$).

Now we want to get from the tubular neighborhoods $C_{t\sigma}$ a continuous family $\{C_t\}_{t\in\mathfrak{D}^n}$ of tubular neighborhoods of the Klein bottle K in Q, such that $K_{t_1} \subset C_t$ for all $t \in \mathfrak{D}^n$ and $C_t = P$ for $t \in \partial\mathfrak{D}^n$. Paragraphs (9.7)-(9.9) are devoted to this problem.

(9.7) LEMMA. Let $t \in \mathcal{U}_\sigma \cap \mathcal{U}_\tau$, while $\sigma \subset \tau$, $\sigma \neq \tau$. Then $C_{t\sigma} \subset C_{t\tau}$.

Proof. Let C' be a slightly enlarged tubular neighborhood of $C_{t\tau}$ such that $C_{t\tau} \subset \operatorname{int} C'$. Obviously, it suffices to prove that $C_{t\sigma} \subset C'$ for any such C'. Since $\sigma \subset \tau$ one has $u(\sigma) <$ $u(\sigma)$ and hence $P_\sigma \subset P_\tau$; in particular, $P_\sigma \subset C'$. Let $A_1, \ldots, A_l$ be the components of the surface $K_{t_1} \cap R_\sigma$, as in (9.6). Since $P_\sigma \subset C'$, it remains for us to show that $W(A_i) \subset C'$ for all i. Since $\partial W(A_i) \subset (K_{t_1} \cup T_\sigma)$, one has $\partial W(A_i) \subset \operatorname{int} C'$. Hence either $W(A_i) \subset \operatorname{int} C'$, or $W(A_i) \supset \partial C'$. However the inclusion $W(A_i) \supset \partial C'$ is impossible: since $A_i$ is an admissible ring, the inclusion homomorphism $\pi_1(W(A_i)) \to \pi_1(R_\sigma)$ is not an epimorphism; on the other hand, the inclusion homomorphism $\pi_1(\partial C') \to \pi_1(R_\tau)$, and with it the inclusion homomorphism $\pi_1(\partial C') \to \pi_1(R_\sigma)$, is obviously an epimorphism. Thus $W(A_i) \subset \operatorname{int} C'$, which completes the proof.

In view of this lemma the existence of the family $\{C_t\}_{t\in\mathfrak{D}^n}$ needed is essentially obvious. The reader who does not doubt this can skip Paragraphs (9.8) and (9.9), which are devoted to the formal proof.

(9.8) LEMMA. Let $X_0$ and $X_1$ be two polyhedra, where $X_0 \supset X_1$, and let $\{C_t^0\}_{t\in X_0}$ and $\{C_t^1\}_{t\in X_1}$ be two families of tubular neighborhoods of the Klein bottole K in Q. If $\operatorname{int} C_t^1 \supset C_t^0$ for all $t \in X_1$, then there exists a family $\{C_t\}_{t\in X_0}$ of tubular neighborhoods such that $C_t = C_t^1$ for $t \in X_1$ and $\operatorname{int} C_t \supset C_t^0$ for $t \in X_0$.

[It is clear that the pair (Q, K) can be replaced by a pair consisting of an arbitrary manifold and a compact submanifold of it.]

Proof. If we cut Q and $C_t^0$ along K, then $C_t^0$ becomes a collar of the boundary of the manifold obtained by cutting Q. Applying to the family of collars obtained in this way the uniqueness theorem for collars (cf., e.g., [18, Essay 1, Appendix A]) we see that the family $\{C_t^0\}_{t\in X_0}$ is isotopic with the constant family. Applying the collar uniqueness theorem again, we can reduce the lemma to the case of constant families $\{C_t^0\}_{t\in X_0}$ and $\{C_t^1\}_{t\in X_1}$. In this case the lemma obviously follows from the fact that $C_t^1 \setminus \operatorname{int} C_t^0$ is diffeomorphic with $\partial C_t^0 \times I$ (which, in its own right, is a consequence of the collar uniqueness theorem).

(9.9) This paragraph completes the proof of the existence of a family $\{C_t\}_{t\in\mathfrak{D}^n}$ of tubular neighborhoods such that $K_{t_1} \subset C_t$ for all $t \in \mathfrak{D}^n$ and $C_t = P$ for $t \in \partial\mathfrak{D}^n$.

It is well known (cf., e.g., [23]) that with each triangulation of a piecewise-linear manifold there is connected a subdivision of this manifold into handles in which to a simplex of dimension i there corresponds a handle of index i. We need, in the case of the ball $\mathfrak{D}^n$, a small generalization of this construction, in which to handles there correspond simplices of $\Delta$ not lying in $\partial\mathfrak{D}^n$. In more detail, we set $\Delta' = \{\sigma \in \Delta : \sigma \not\subset \partial\mathfrak{D}^n\}$. We need a subdivision into handles $\mathfrak{D}^n = C \cup \bigcup_{\sigma\in\Delta'} H_\sigma$ of the ball $\mathfrak{D}^n$ with respect to the boundary $\partial\mathfrak{D}^n$, in which C is a collar of the boundary $\partial\mathfrak{D}^n$ and to each simplex $\sigma \in \Delta'$ corresponds a handle $H_\sigma$ of index $\dim\sigma$, where $H_\sigma \subset \mathcal{U}_\sigma$. Such a subdivision into handles can be constructed by induction on $\dim\sigma$: first as C one takes a sufficiently small collar of the boundary $\partial\mathfrak{D}^n$ and as $H_\sigma$ with $\dim\sigma = 0$ one takes sufficiently small balls containing $\sigma$; then these balls

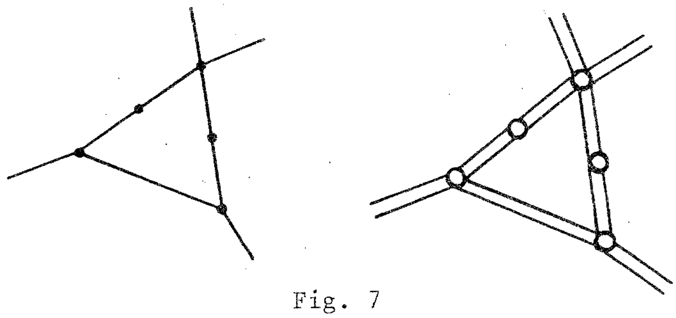


Fig. 7

are combined with one another and with C by sufficiently small handles $H_\sigma$ of index 1, going along one-dimensional simplices $\sigma\in\Delta'$ ; etc. Cf. Fig. 7. Since $C_{t\tau}\subset \operatorname{int} P$ for $t\in\partial\mathfrak{D}^n\cap\mathcal{U}_\tau$, $\tau\in\Delta$, the collar C can be chosen so small that $C_{t\tau}\subset\operatorname{int} P$ for $t\in C\cap\mathcal{U}_\tau$, $\tau\in\Delta$.

We set $F_i = C\cup\bigcup_{\dim\sigma\geq i} H_\sigma$; $i=n,\dots,0$. We define $C_t$ for $t\in F_n$ as follows: for $t\in C$ we set $C_t$ = P; for $t\in H_\sigma$ with $\dim\sigma=n$ we take as $C_t$ a slight enlargement of $C_{t\sigma}$, so that $\operatorname{int} C_t\supset C_{t\sigma}$. Let us assume that the neighborhoods $C_t$ are already constructed for $t\in F_i$ and have the following property: $\operatorname{int} C_t\supset C_{t\sigma}$ for $t\in H_\sigma$, $\sigma\in\Delta'$, $\dim\sigma\geq i$. Let $\tau\in\Delta'$, $\dim\tau=i-1$. Then $\operatorname{int} C_t\supset C_{t\tau}$ for $t\in F_i\cap H_\tau$. In fact, if $t\in C\cap H_\tau$, then $\operatorname{int} C_t=\operatorname{int} P\supset C_{t\tau}$. Now if $t\in H_\sigma\cap H_\tau$ with $\dim\sigma\geq i$, then $\sigma\supset\tau$ (by virtue of the construction of the handle decomposition) and by Lemma (9.7) $C_{t\tau}\subset C_{t\sigma}\subset\operatorname{int} C_t$. Thanks to this we can use Lemma (9.8) and extend the family $\{C_t\}$ to $H_\tau$ so that $\operatorname{int} C_t\supset C_{t\tau}$ for $t\in H_\tau$. Doing the same thing with the remaining simplices of dimension i — 1, we extend the family $\{C_t\}$ to $F_{i-1}$ so that $\operatorname{int} C_t\supset C_{t\sigma}$ for $t\in F_i$ and $t\in H_\sigma$, $\sigma\in\Delta'$, $\dim\sigma\geq i-1$. Applying (descending) induction on i, we get a family $\{C_t\}_{t\in\mathfrak{D}^n}$, such that $\operatorname{int} C_t\supset C_{t\sigma}$ for $t\in H_\sigma$. Moreover, $C_t=P$ for $t\in C$, in particular, for $t\in\partial\mathfrak{D}^n$. Since $C_{t\sigma}\supset K_{t1}$ for $t\in H_\sigma$ (we recall that $H_\sigma\subset\mathcal{U}_\sigma$), one has $C_t\supset K_{t1}$ for all t. Thus we have gotten the family needed.

(9.10) Now we can complete the proof of Theorem 2.

Cutting Q and $C_t$ along K, we turn $C_t$ into a collar of the boundary of the manifold obtained by cutting Q [as in the proof of Lemma (9.8)]. From the parametric theorem on uniqueness of collars (cf., e.g., [18, Essay 1, Appendix A]), it follows that the space of collars is weakly homotopy equivalent with a point. Hence the family $\{C_t\}_{t\in\mathfrak{D}^n}$ can be deformed into a constant family, all of whose terms coincide with P, while the deformation can be chosen to be the identity on $\partial\mathfrak{D}^n$. The deformation of the family of tubular neighborhoods induces a deformation of the family $\{K_{t1}\}_{t\in\mathfrak{D}^n}$ of Klein bottles. Thus we get a family $\{K_{ts}\}_{t\in\mathfrak{D}^n, s\in[1,2]}$ of Klein bottles in Q such that $K_{t2}\subset P$ for all t.

Let $S_m(K, Q)$ be the space of Klein bottles in Q. The canonical map $E(K, Q)\to S_m(K, Q)$ is a locally trivial fibration with fiber Diff (K). Applying the covering homotopy theorem to this fibration, we see that the family $\{K_{ts}\}_{t\in\mathfrak{D}^n, s\in[0,2]}$ of Klein bottles lifts to a family of imbeddings $\{f_{ts}: K\to Q\}_{t\in\mathfrak{D}^n, s\in[0,2]}$, such that $f_{t0} = f_t$ for all t [we recall that $\{f_t\}_{t\in\mathfrak{D}^n}$ is introduced in (9.2)]. Since obviously $f_{t2}(K)\subset P$ for all t, it is evident that the element of the group $\pi_n(E_0(K,Q), E_0(K,P))$ represented by the family $\{f_t\}_{t\in\mathfrak{D}^n}$ is equal to zero. Thus Theorem 2 is proved.

## 10. Proof of Theorem 1

(10.1) The proof of Theorem 1 is based on Theorem 2 and the calculation of the homotopy type of spaces of diffeomorphisms of certain Waldhausen manifolds. The results on Waldhausen manifolds we need are the following.

For a smooth manifold M we denote by $\mathcal{Diff}(M,\partial)$ the space of diffeomorphisms $M \to M$ which are the identity on the boundary and by $G(M, \partial)$ the space of homotopy equivalences $M \to M$ which are the identity on $\partial M$. If M is a Waldhausen manifold, then the inclusion $\mathcal{Diff}(M,\partial) \hookrightarrow G(M,\partial)$ is a homotopy equivalence. This follows from results of [14] and the validity of the "Smale conjecture" [10, 11]. Instead of [14] one can use the corresponding piecewise-linear results of Hatcher [8] and the PL/Diff-comparison theorem of Burghelea—Lashof—Morley [4]. If M is a Waldhausen manifold with nonempty boundary, $\partial M \neq \emptyset$, then the components of the space $G(M, \partial)$, and hence the components of the space $\mathcal{Diff}(M,\partial)$, are contractible. This follows from elementary obstruction theory and the fact that Waldhausen manifolds are aspherical.

We need these results for $M = \mathcal{D}^2 \times \mathcal{S}^1, T \times I$ and P. Obviously $\mathcal{D}^2 \times \mathcal{S}^1$ and $T \times T$ are Waldhausen manifolds. As to P, it is obviously irreducible, does not contain two-sided projective planes (since it is orientable), and the image of the surface $\pi^{-1}(\alpha) \times I$ under the canonical map $T \times I \to P$ [we recall that $\alpha$ is defined in (3.1), and $\pi$ in (1.2)] is obviously a nonshrinkable surface in P. Thus P is also a Waldhausen manifold.

The space $\mathcal{Diff}(\mathcal{D}^2 \times \mathcal{S}^1, \partial)$ is connected. This follows from the fact that the space $G(\mathcal{D}^2 \times \mathcal{S}^1, \partial)$ is connected, which in its own right follows from obstruction theory. Thus the space $\mathcal{Diff}(\mathcal{D}^2 \times \mathcal{S}^1, \partial)$ is contractible.

We shall list the components of the space $\mathcal{Diff}(T \times I, \partial)$. Let $\varkappa \in T$. We note that a homotopy equivalence $T \times I \to T \times I$, which is the identity on $\partial(T \times I)$, is determined, up to a homotopy bound on $\partial(T \times I)$, by its restriction to the segment $\varkappa \times I$, considered up to a homotopy bound at the ends. This follows from elementary obstruction theory, since $T \times I$ is an aspherical space and has a cellular subdivision in which the only one-dimensinal cell, not lying in $\partial(T \times I)$, is $\varkappa \times I$. The projection $T \times I \to T$ identifies the homotopy classes considered (of maps of segment $\varkappa \times T$) with elements of the group $\pi_1(T, \varkappa)$. Since the inclusion $\mathcal{Diff}(T \times I, \partial) \hookrightarrow G(T \times I, \partial)$ is a homotopy equivalence, we get a one-to-one correspondence between the components of the space $\mathcal{Diff}(T \times I, \partial)$ and the elements of the group $\pi_1(T, \varkappa)$.

(10.2) Let R be the solid torus from (2.4) and let $\tau_0 : \partial R \to \partial R$ be the involution which permutes the sheets of the covering $\partial R \to K$ of (2.4). We need the space $\mathcal{D}_0$ of diffeomorphisms $R \to R$ which coincide on $\partial R$ with either $id_{\partial R}$ or $\tau_0$. Obviously $\mathcal{D}_0$ is a (topological) group and hence all components of the space $\mathcal{D}_0$ are homeomorphic. Moreover, since R is diffeomorphic with $\mathcal{D}^2 \times \mathcal{S}^1$, one has that $\mathcal{D}_0$ consists of two components, each of which is homeomorphic with $\mathcal{Diff}(\mathcal{D}^2 \times \mathcal{S}^1, \partial)$ and hence is contractible.

Suppose further $\tau_1 : T \to T$ is the involution permuting the sheets of the covering $\pi : T \to K$ of (1.2). We need along with $\mathcal{D}_0$ the space $\mathcal{D}_1$ of diffeomorphisms $T \times I \to T \times I$, which coincide on $T \times 1$ with $id_{T\times 1}$ and on $T \times 0$ with either $id_{T\times 0}$ or $\tau_1$. Just as in the case of $\mathcal{D}_0$, all components of the space $\mathcal{D}_1$ are homeomorphic, while those of them which contain $id_{T\times I}$, coincide with the components of the space $\mathcal{Diff}(T \times I, \partial)$ which contain $id_{T\times I}$.

Now we shall list components of the space $\mathcal{D}_1$. Just as in the case of diffeomorphisms from $\mathcal{Diff}(T, I, \partial)$, a diffeomorphism from $\mathcal{D}_1$ is determined up to a homotopy bound on $\partial(T \times I)$ [and hence, up to an isotopy bound on $\partial(T \times I)$] by its restriction to the segment $\varkappa \times I$ [cf. (10.1)], considered up to a homotopy bound at the ends. The composition $T \times I \xrightarrow{pr} T \xrightarrow{\pi} K$ identifies the homotopy classes considered with elements of the group $\pi_1(K)$.

Obviously under these identifications of $\pi_0(\mathcal{Diff}(T \times I, \partial))$ with $\pi_1(T)$ and of $\pi_0(\mathcal{D}_1)$ with $\pi_1(K)$ the inclusion $\pi_0(\mathcal{Diff}(T \times I, \partial)) \hookrightarrow \pi_0(\mathcal{D}_1)$ goes into the homomorphism $\pi_* : \pi_1(T) \to \pi_1(K)$.

(10.3) Now we can turn directly to the proof of Theorem 1.

We consider the canonical map $\mathcal{Diff}_0(Q) \to E_0(K, Q) : f \mapsto f|K$. As is well known, it is a locally trivial fibration. Its fiber over the inclusion $K \hookrightarrow Q$ can be identified naturally with $\mathcal{D}_0$ (since the result of cutting the manifold Q along K can be identified naturally with R). If $E_0(K, Q)$ is homotopy equivalent with a circle, then since $\mathcal{D}_0$ consists of two contractible components, $\mathcal{D}iff_0(Q)$ is homotopy equivalent with the connected two-sheeted covering of the circle, i.e., is homotopy equivalent with a circle. On the other hand, by Theorem 2, the spaces $E_0(K, Q)$ and $E_0(K, P)$ are homotopy equivalent. Thus, it suffices for us to prove that $E_0(K, P)$ is homotopy equivalent with a circle.

We consider the canonical map $\mathcal{D}iff_0(P, \partial) \to E_0(K, P): f \mapsto f|K$. It is a locally trivial fibration, like the map $\mathcal{D}iff_0(Q) \to E_0(K, Q)$. Its fiber over the inclusion can be identified naturally with the union of some components of the space $\mathcal{D}_1$ (since the result of cutting the manifold P along K can be identified naturally with $T \times I$).

We shall clarify which components these are. Let $l$ be the image in P of the union $\varkappa \times I \cup \tau_1(\varkappa) \times I$. It is clear that $l$ is a segment with ends on $\partial P$. Maps $l \to P$, which coincide on $\partial l$ with the inclusion, considered up to a homotopy bound on $\partial l$, can be identified with elements of the group $\pi_1(K)$ (with the help of the natural map $P \to K$). Let $f \in \mathcal{D}_1$ and let f lie in the component of the space $\mathcal{D}_1$ corresponding to the element $a^r b^s$ of the group $\pi_1(K)$ [each element of the group $\pi_1(K)$ can be written in this form]. Let $\bar{f}: P \to P$ be a diffeomorphism corresponding to f. One can verify directly that the segment $\bar{f}(l)$ lies in the homotopy class $a^r b^s a^r b^{-s} \in \pi_1(K)$. Since $a^r b^s a^r b^{-s} \neq 1$ for $r \neq 0$, one has for $r \neq 0$ that the diffeomorphism $\bar{f}$ is not homotopic (bound on $\partial P$) with idp. On the other hand, if $r = 0$, then $\bar{f}$ is isotopic with idp. We shall show this. The torus T can be identified in a natural way with the standard torus $S^1 \times S^1$ so that the generators $a$ and $b^2$ of the group $\pi_1(T)$ will be represented by the circles $S^1 \times 1$ and $1 \times S^1$, and the involution $\tau_1$ will be defined by the complex formula $(x, y) \to (\bar{x}, -y)$. Using this identification, we define the isotopy $\{i_t^s: T \to T\}_{t \in I}$ by the complex formula $i_t^s(x, y) = (x, e^{\pi i s t} y)$. Obviously this isotopy covers some isotopy $\{j_t^s: K \to K\}_{t \in I}$, where $j_1^s = \mathrm{id}_K$ [as is known, $\mathcal{D}iff_0(K)$ is contractible to the circle $\{j_t^1 : t \in I\}$]. We extend the isotopy $\{j_t^s\}_{t \in I}$ to some isotopy $\{h_t^s: P \to P\}_{t \in I}$ fixed on $\partial P$ and starting with $h_0^s = $ idp. The diffeomorphism $h_1^s$ induces a diffeomorphism $T \times I \to T \times I$ which, as one can verify directly, lies in the component of the space $\mathcal{D}_1$ corresponding to $b^s$. Thus, this diffeomorphism is isotopic with $\bar{f}$ (we recall that now $r = 0$) and $\bar{f}$ is isotopic with idp. Thus, the components of the space $\mathcal{D}_1$ of interest to us correspond to elements of the infinite cyclic subgroup of the group $\pi_1(K)$ generated by b.

Combining this fact with the fact that $\mathcal{D}iff_0(P, \partial)$ and the components of the space $\mathcal{D}_1$ are contractible, and using the homotopy sequence of the fibration $\mathcal{D}iff_0(P, \partial) \to E_0(K, P)$, we get that $E_0(K, P)$ is homotopy equivalent with a circle. In addition $E_0(K, P)$ is contractible onto the circle $\{in \circ j_t^1 : t \in I\}$, where $in: K \hookrightarrow P$ is the inclusion.

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# QUADRATIC FORMS OF CLOSED MANIFOLDS

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In this paper it is shown that for any integral-valued unimodular quadratic form and any number n of the form 8k + 4 (where $k \geqslant 1$), there exists a smooth closed n-dimensional manifold with this quadratic form. The proof is based on the construction (with the help of the "plumbing" construction) of smooth closed three-connected eight-dimensional manifolds with given form.

1. The terminology of the paper is differential-topological. All manifolds are assumed to be compact and oriented. By the quadratic form (or simply form) of a 4k-dimensional manifold W is meant the form on the group $H_{2k}(W)/\mathrm{Tors}H_{2k}(W)$ defined by the intersection index.

THEOREM 1. Let $\Sigma$ be a seven-dimensional homotopy sphere. Any unimodular integer-valued quadratic form of rank no less than three can be realized as the form of a three-connected eight-dimensional manifold whose boundary is diffeomorphic with $\Sigma$.

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