On the $Wh_1^+$-obstruction and Pseudo-isotopies of Manifolds of dimension 3 and 4.

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This is a preliminary report on a method to construct pseudo-isotopies that also works for many manifolds of dimension 3 and 4. For example, P(M) is not even finitely generated if M is a connected sum of at least two copies of $S^1 \times S^2$, and I conjecture that it is nontrivial for every 3-manifold with $\pi_2 \neq 0$, except possibly $S^1 \times S^2$.

Introduction. Let M be compact, connected, orientable manifold, and let P(M) be the space of pseudoisotopies of M. Recall from [1] [2] that there is an exact sequence (if dim(M) is $\geqslant 6$):

$$(1) \quad K_3(Z[\pi_1(M)]) \to Wh_1^+(\pi_1(M); Z_2 \oplus \pi_2(M)) \xrightarrow{\theta} \pi_0(P(M)) \xrightarrow{\Sigma} Wh_2(\pi_1(M)) \to 0 .$$

$\Sigma$ is defined also for low-dimensional manifolds, but so far there is no way of constructing nonzero elements in $\pi_2$(M) that can be detected this way, for M of dimension 3 or 4. Besides, $Wh_2$ is very difficult to compute, in general.

$\theta$ on the other hand, is only defined for high-dimensional manifolds Therefore we must suspend before we can detect anything, so the question we ask is: If dim(M) is 3 or 4, can we construct elements x in $\pi_0(P(M))$ such that $j_*(X)$ is in the image of $\theta$ for some $j_*: \pi_0(P(M)) \to \pi_0(P(M \times I^{\ell}))$ which is a combination of positive and negative suspensions?

If $j_*(x) = \theta(y)$, we say that x represents y. (Note that x may not be unique.)

Before stating the main theorem, let us recall the definition of $Wh_1^+$:

$$(2) \quad Wh_1^+(\pi_1(M); Z_2 \oplus \pi_2(M)) = (Z_2 \oplus \pi_2(M))[\pi_1(M)] / (\beta\cdot 1, \alpha\cdot\gamma - \alpha^{\tau}\tau\gamma\tau^{-1})$$

(Here $\pi_1(M)$ actstrivially on $Z_2$ and by the usual action on $\pi_2(M)$.)

Theorem 1. a) If $\dim(M) = 4$, $Wh_1^+(\pi_1(M); \pi_2(M))$ can be represented by elements in $\pi_0 P(M)$. If there is a map $f:S^2 \to M$ such that $f^*TM$ is non-trivial, then $Wh_1^+(\pi_1(M);Z_2)$ can also be represented.

b) If $\dim(M) = 3$, we can represent the subgroup generated by all elements of the form $[\beta\cdot\gamma]$, where $\beta$ is represented by an embedded sphere which does not intersect $\gamma$.

The proof of Theorem 1 is by a new interpretation of $\theta$. In [1], $Wh_1^+$ comes from a certain framed cobordism group, but from our point of view it derives from the fundamental group of a mapping space. This group we compute completely, and give very explicit description of its elements. Now the realization problem becomes the problem of realizing loops of mappings by loops of embeddings, and Theorem 1 tells to what extent this can be solved in dimension 3 and 4.

## 1. P(M) and deformation of handles.

In $M\times I$, we fix an embedded, cancelling pair of $(n-k,n-k+1)$-handles. The $(n-k)$-handle is represented by an embedding $F: (D^{k+1}\times D^{n-k}, D^{k+1}\times S^{n-k-1}) \to (M\times I, M\times 1)$, and we denote its image by $\mathcal{H}$. Let $W = \mathrm{cl}(M\times I-\mathcal{H})$, and let $f$ be the map $S^k\times D^{n-k} \to \partial W$ defined by restriction of $F$.

Composition with $f$ defines a fibration (over a union of components)

$$\mathrm{Diff}_+(W) \xrightarrow{p} \mathrm{Emb}(S^k\times D^{n-k}, \partial_+ W), \quad \text{where}$$

$\mathrm{Diff}_+$ means diffeomorphisms fixed on $M\times 0 \cup \partial M\times I$,

$\partial_+ W = \partial W - (M\times 0 \cup \partial M\times I)$

The fiber of $p$ is $\mathrm{Diff}_+(W, \mathrm{rel}(\mathrm{im}\, f))$, and extension by the identity over $\mathcal{H}$ defines an inclusion of this into $P(M)$. (There is, strictly speaking, a smoothing problem here, but this can easily be solved, e.g. by replacing $\mathrm{Diff}_+(W,\mathrm{rel}(\mathrm{im}\, f))$ by diffeomorphisms which are the identity near im f. This does not change the homotopy type.)

Hence we obtain a map

$$\Omega\, \mathrm{Emb}_f(S^k\times D^{n-k}, \partial_+ W) \to P(M),$$

and we let $\omega$ be the induced homomorphism

$$\omega : \pi_1(\mathrm{Emb}_f) \to \pi_0(P(M))$$

($\mathrm{Emb}_f$ is the component containing f.)

Let $C_f$ be the space of mappings $S^k \to W$ in the homotopy class of $f|S^k\times 0$. restriction gives an obvious map $\mathrm{Emb}_f \to C_f$, so we can compare $\pi_1(\mathrm{Emb}_f)$ to $\pi_1(C_f)$, which is much easier to compute.

Proposition 2. There is a homomorphism

$$\pi_1(C_f) \to (\mathbb{Z}_2 \oplus \pi_2(M))[\pi_1(M)] \,/\, (\pi_2(M)\cdot 1) \;\oplus \pi_{k+1}(M)$$

which is an isomorphism if $k \geq 3$, and surjective if $k \geq 2$.

The proof is deferred to the next section.

Note that $Wh_1^+(\pi_1(M); \mathbb{Z}_2 \oplus \pi_2(M))$ is a quotient of the second group in Prop.2, so we get a homomorphism

$$\sigma : \pi_1(\mathrm{Emb}_f) \to Wh_1^+(\pi_1(M); \mathbb{Z}_2 \oplus \pi_2(M)).$$

<u>Theorem 3.</u> If $2 \leq k \leq n-3$, $\omega = (-1)^k \theta \circ \sigma$, and $\sigma$ is surjective.

Proof of the factorization:

That such factorization exists, is almost obvious from the definitions. In fact, $\theta$ is defined by constructing a one-parameter family of functions on $M \times I$ with at most two critical points, and studying how the stable and unstable manifolds intersect in a level surface between the critical points. But given an element $g: S^k \times D^{n-k} \times I \to \partial_+ W$ in $\pi_1(\mathrm{Emb}_f)$, we can construct such a family as follows:

Let $V_t$, $t \in I$, be the manifold $W \cup_{g_t} D^{k+1} \times D^{n-k}$. Then there is an obvious diffeomorphism $V_0 \cong M \times I$, which can be included in a continuous family of diffeomorphisms $G_t : V_t \to M \times I$, rel $M \times 0$.

In fact, since $g_0 = g_1$, this way we obtain two diffeomorphisms between $V_1 = V_0$ and $M \times I$, and the difference is a pseudo-isotopy of $M$ which represents $\omega(g) \in \pi_0(P(M))$.

Now let $\varphi : M \times I \to I$ be a Morse function with one critical point of index $k$ in $W$, and one of index $k+1$ in $\mathcal{H}$, and with $\partial_+ W$ as a level surface between them. (Note that we do not require that $\varphi^{-1}(1) = M \times 1$, only that gradient lines should be transverse to $M \times 1$.) Then, clearly, $\varphi | W$ and $\varphi | \mathcal{H}$ also define a Morse function on $V_t$ for all $t$, and via $G_t$ we get a one-parameter family of Morse functions on $M \times I$. Cancel the critical points the same way for $t=0$ and $t=1$ to obtain a family of functions $\varphi_t$, with $\varphi_0$ and $\varphi_1$ nonsingular. By Cerf's functional approach to pseudo-isotopies, this defines an element in $\pi_0(P(M))$ which is the same as $\omega(g)$. But then it follows from [1] that it comes from an element in $Wh_1^+$. The proof that this element is $(-1)^k \sigma$ is postponed to section 3, as is the proof of the surjectivity of $\sigma$.

## 2. Proof of proposition 2.

Choose basepoints $*$ in $S^k$ and $W$ such that $* = f(*)$, and let $C_f^* \subset C_f$ be the subspace of basepoint- preserving maps. Then evaluation at $*$ defines a fibration $C_f \to W$ with fiber $C_f^*$, and we get a long, exact sequence

(3) $$\pi_{i+1}(W) \to \pi_i(C_f^*) \to \pi_i(C_f) \to \pi_i(W) \to \cdots$$

(We use $*$ and $f$ as base-points.) But $\pi_i(C_f^*)$ is isomorphic to $\pi_{i+k}(W)$, hence we obtain the sequence

(4) $$\pi_{i+1}(W) \xrightarrow{\delta} \pi_{i+k}(W) \to \pi_i(C_f) \to \pi_i(W) \to \cdots$$

But $\delta(\beta) = [\beta, f]$ (Whitehead product, see [3]), so we need to compute these products for $i = 1$ and $2$.

Now $W$ can also be thought of as $M \times I$ with a k-handle attached, hence there is a homotopy equivalence $W \simeq M \vee S^k$.

Lemma 4. $\pi_{k+1}(W) \cong \pi_{k+1}(M) \oplus (\mathbb{Z}_2 \oplus \pi_2(M))[\pi_1(M)]$ if $k \geq 3$.

Proof: (i) Assume first $\pi_1(M) = 0$, hence also $\pi_1(M \vee S^k) = 0$. The homotopy exact sequence for the pair $(M \times S^k, M \vee S^k)$ splits naturally into split, short exact sequences

(5) $$0 \to \pi_{k+2}(M \times S^k, M \vee S^k) \to \pi_{k+1}(M \vee S^k) \leftrightarrows \pi_{k+1}(M \times S^k) \to 0$$

But $H_j(M \times S^k, M \vee S^k) = \tilde{H}_j(\Sigma^k M) = \tilde{H}_{j-k}(M) = 0$ for $j-k < 2$.

Hence, by the Hurewicz theorem:

$$\pi_{k+2}(M \times S^k, M \vee S^k) = H_{k+2}(M \times S^k, M \vee S^k) = \tilde{H}_2(M) = \pi_2(M)$$

Thus $\pi_{k+1}(M \vee S^k) = \pi_{k+1}(M) \oplus \pi_{k+1}(S^k) \oplus \pi_2(M)$,

and the homomorphism $\pi_2(M) \to \pi_{k+1}(M \vee S^k)$ is given as follows: Let $h \in \pi_2(M)$ be represented by a map $(D^2, S^1) \to (M, *)$, also denoted by $h$, and let $c: (D^k, S^{k-1}) \to (S^k, *)$ be collapse of the boundary. The image of h in $\pi_{k+1}(M\vee S^k)$ is represented by the boundary of $h\times c\colon (D^2\times D^k, \partial(D^2\times D^k)) \to (M\times S^k, M\vee S^k)$. But this is clearly the same as $[h,S^k]$, where we also use $S^k$ to denote the embedding $*\vee S^k \subset M\vee S^k$.

(ii) Let now M be arbitrary. Then

$$\pi_{k+1}(M\vee S^k) \cong \pi_{k+1}(\widetilde{M\vee S^k}) \cong \pi_{k+1}(\tilde{M}\vee(\bigvee_{\pi_1(M)} S^k)) \cong \lim_{F} \pi_{k+1}(\tilde{M}\vee(\bigvee_{F} S^k)),$$

where the limit is taken over all finite subsets F of $\pi_1(M)$.
But from (i) we get by induction that

$$\pi_{k+1}(M\vee(\bigvee_F S^k)) \cong \pi_{k+1}(M) \oplus (\bigoplus_F (Z_2\oplus\pi_2(M))) \quad .$$

The result follows.
In fact, a specific isomorphism

$$\pi_{k+1}(M) \oplus (Z_2\oplus\pi_2(M))[\pi_1(M)] \to \pi_{k+1}(M\vee S^k)$$

is given by:

on $\pi_{k+1}(M)$: Induced by the inclusion $M\subset M\vee S^k$,
on $Z_2[\pi_1(M)]$ : $\eta\cdot\gamma \to (S^k\circ\eta)^\gamma$, where $\gamma\in\pi_1(M)$ and $\eta\in\pi_{k+1}(S^k)\cong\mathbb{Z}_2$
on $\pi_2(M)[\pi_1(M)]$ : $\beta\cdot\gamma \to [\beta^\gamma,(S^k)^\gamma]$ (Whitehead product) .

<u>Lemma 5.</u> $\pi_k(W) = \pi_k(M) \oplus \mathbb{Z}[\pi_1(M)]$ (for $k\geq 1$) .

Proof: Similar to the proof of lemma 4, using that if X is simply connected, then $\pi_k(X\vee S^k) = \pi_k(X)\oplus \mathbb{Z}$ . The homomorphism

$Z[\pi_1(M)] \to \pi_k(M\vee S^k)$ is given by $\gamma \to (S^k)^\gamma$.

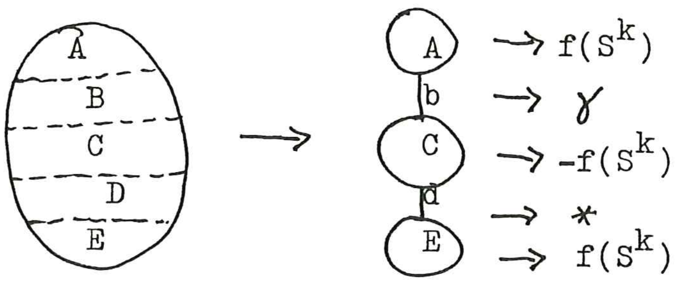
If $\gamma\in\pi_1(M)$ , $[\gamma, S^k] = (S^k)^\gamma - S^k$, hence it follows from lemma 5 that $\delta$ in (4) for i=0 is injective. Thus

$$\pi_1(C_f) = \operatorname{coker}(\delta\colon \pi_2(W) \to \pi_{k+1}(W)) ,$$

and the proposition for $k \geq 3$ follows from lemma 4.
We could try to make similar computations for $k=2$, but this is more complicated - mainly because now $\pi_2(W) \neq \pi_2(M)$. Instead, we only get a surjective map in lemma 4, and the kernel contains the image under $\delta$ of the "$\mathbb{Z}[\pi_1(M)]$"-part of $\pi_2(M \vee S^2)$ .(Cfr. lemma 5.)
Therefore we get a surjective map as in proposition 2.

Formulae (6) now provide a way of representing elements in $Wh_1^+(\pi_1(M);\ Z_2 \oplus \pi_2(M))$ by elements in $\pi_1(C_f)$. Here is an explicit description of an element representing $(\eta \oplus \beta)\gamma$, where $\eta \in \mathbb{Z}_2$, $\beta \in \pi_2(M)$ and $\gamma \in \pi_1(M)$:
Start with $f(S^k)$, and pull the basepoint along a path representing $\gamma$. When we are back again, we wrap a neighborhood of the basepoint around $f(S^k)$ such that we finally cover it twice, once with each orientation. At this point we think of the mapping as in the following picture:

(7)




Note that the two cylinders B and D are squeezed to lines b and d, which are mapped to $\gamma$ and the basepoint of $f(S^k)$, respectively.
Now let d move around a map representing $\beta$ and rotate E to generate the element $\eta \in \mathbb{Z}_2 \cong \pi_1(SO(k+1))$. When we are back to the situation of (7) again, we can reverse the first moves and get back to $f(S^k)$.

<u>Remark:</u> We also need to computethe image in $Wh_1^+$ of an element in $\pi_1(C_f)$. This we get by mapping to $M \vee S^k$ and considering the preimage of a point on $S^k$, different from the basepoint. This consists of a disjoint union of framed circles, which bound discs in $S^1 \times S^k$. Choosing paths from the basepoint to each component, the disc and the path map to an element of $\pi_2(M)[\pi_1(M)]$. The $\mathbb{Z}_2$-component comes from the induced framings.

## 3. End of proof of Theorem 2.

To finish the proof of the factorization, it now suffices to observe that Hatcher and Wagoner's method of associating a $Wh_1^+$ - invariant to the one- parameter family of functions $\varphi_t$ in section 1, gives precisely $(-1)^k\sigma(g)$. But this follows from a careful comparison of definitions, using the remark at the end of section 2. The sign serves the purpose of making the definition of $\theta$ independent of the index of the critical points.

Surjectivity of $\sigma$ will be proved in two steps. The first step is to approximate the representing family of maps constructed in section 2 by a family of embeddings into $\partial_+W$. In fact, most of the procedure can be done with embeddings only using general position. The only trouble is that after we have run the tube (D in figure (7)) around $\beta$ and rotated by the element $\eta$, the tube D may have a twist on it, thus preventing us to go back. But this twist can be thought of as an element of $\pi_1$(k-frames in (n-1)-space) $\cong \pi_1(SO(n-1)/SO(n-1-k))$ , which is 0 for $n-1-k \geq 2$, or $n-k \geq 3$. Therefore, in this codimension the twisting can be undone, and we can deform back via embeddings.

The next step is to thicken this loop of embeddings of $S^k$ in $\partial_+W$ to a loop of embeddings of $S^k \times D^{n-k}$. We can easily thicken to a path of such embeddings, and the problem is to decide if the end maps coincide. They differ by an element in $\pi_k(SO(n-k))$, and we have to show that this element is zero.

Consider the following diagram:

(8)
$$\begin{tikzcd}
 & SO(n-k)^{S^k} \arrow[r, "\cong"] \arrow[d] & SO(n-k)^{S^k} \arrow[d] & SO(n-k)^{D^{k+1}} \arrow[l] \arrow[d] \\
 & Emb(S^k \times D^{n-k}, \partial_+W) \arrow[r] \arrow[d] & Emb(S^k \times D^{n-k}, M \times I) \arrow[d] & Emb(D^{k+1} \times D^{n-k}, M \times I) \arrow[l] \arrow[d] \\
I \arrow[r, "\alpha"] \arrow[ur, dashed, "\tilde{\alpha}"] & Emb(S^k, \partial_+W) \arrow[r] & Emb(S^k, M \times I) & Emb(D^{k+1}, M \times I) \arrow[l]
\end{tikzcd}$$

where the lower vertical maps and the maps pointing to the left are fibrations. It follows that $\tilde{\alpha}(1) \in SO(n-k)^{S^k}$ is the restriction of an element in $SO(n-k)^{D^{k+1}}$, hence it is trivial up to homotopy.

## 4. Proof of Theorem 1. Applications.

Let $j_+(j_-)$ denote the 'positive' ('negative') suspension map $\pi_0(P(M)) \to \pi_0(P(M\times I))$. If $\dim(M)=n=4$, we then have the commutative diagram

(9)
$$\begin{array}{ccc} \pi_1(\mathrm{Emb}_f) & \xrightarrow{\omega} & \pi_0(P(M)) \\ {\scriptstyle (-1)^k\sigma}\downarrow & & \downarrow{\scriptstyle j_+} \\ Wh_1^+ & \xrightarrow{\theta} & \pi_0(P(M\times I)) \end{array}$$

for k = 2 or 3. Let now k=2. We shall prove that $\sigma$ maps onto the $\pi_2$-part of $Wh_1^+$, so we go back to the proof of the surjectivity of in higher dimensions. This time, we get into trouble both with the twisting and with the isotopy of the tube over the $\pi_2$-element. Let us study the isotopy first.

Claim: Every $\pi_2$-element can be represented by an immersed sphere.

Proof: Hirsch-Smale immersion theorem. In fact, all we need to do is to embed $TS^2$ linearly in $g^*TM$ for an arbitrary map $g:S^2 \to M$. But $g^*TM$ is classified by an element in $\pi_1(SO(4)) = \mathbb{Z}_2$. If it is trivial, it follows by stable triviality of $TS^2$. If not, $g^*TM \cong TCP^2|S^2$, which also has $TS^2$ naturally included. (Alternative proof: By general position, the only singularities are at isolated points, and there they look like a cone over a knot in $S^3$. But such knots also bound immersed discs in $D^4$.)

Using this, we can assume that our $\pi_2$-element is represented by an immersed sphere where the self-intersections occur at isolated points. But then we can move a small tube by an isotopy along the immersed sphere, only making sure that we cross the two points of a self-intersection at different times.

Then, what about the twisting? Well, going back to $g^*TM$, we see that we can always split off a trivial rank 2-bundle. If we keep the $S^1$-factor of the tube in these two directions, we can make sure that we come back without a twist. Hence, if we do not rotate, we can go back as before. This proves the first part of a).

To get the second part, observe that if there is such a map f, we can go around the sphere as before, but this time making sure that we do come back with a twist. But this can be undone by an appropriate rotation - hence we get an element of the $\mathbb{Z}_2$-component. By a combination of isotopies with and without twists, we can obtain all elements of $Wh_1^+(\pi_1(M); \mathbb{Z}_2 \oplus \pi_2(M))$. (*)

It remains to prove b). To this end we want to use k=1, but then we have not defined $\sigma$. We shall extend the definition to this case, using the idea of the 'negative' suspension.

Observe first that $\sigma$ is defined via maps $Emb_f \to C_f \to (M \vee S^k)^{S^k}$. Now define $j_-: (M\vee S^k)^{S^k} \longrightarrow (M\vee S^{k+1})^{S^{k+1}}$ by 'product with I, and collapse of $M\times I \cup S^k\times \partial I$ to M'. Using this with k=1, we obtain a homomorphism $\sigma': \pi_1(Emb_f) \longrightarrow Wh_1^+$

<u>Claim:</u> The diagram

(10)
$$\begin{array}{ccc} \pi_1(Emb_f) & \overset{\omega}{\rightarrow} & \pi_0(P(M)) \\ \sigma' \downarrow & & \downarrow j_+ j_- \\ Wh_1^+ & \underset{\theta}{\rightarrow} & \pi_0(P(M \cancel{\times} I^2)) \end{array}$$

is commutative.

In fact, this follows from the description of $\omega$ in terms of functions in section 1.

Now observe that the construction of the representing family of maps in prop.1 makes sense also for k=1, except for the rotation. The $\pi_2$-part of the construction for k=2 can actually be realized as the suspension (inthe above sense) of the one for k=1. Hence

(*) This idea is due to Igusa.

all of $Wh_1^+(\pi_1(M);\pi_2(M))$ can be realized by elements of $\pi_1(C_f)$ even for k=1. That the elements in theorem 1,b) can be realized by elements in $\pi_1(Emb_f)$ is now obvious.

Theorem 1 provides many candidates for nontrivial elements in $\pi_0(P(M))$, but so far little is known about the kernel of $\theta$. However, Igusa has shown that the $\pi_2$-part of $Wh_1^+$ injects if the first k-invariant of M is trivial. (See [2]). The only 3-manifolds having this property with $\pi_2 \neq 0$ are connected sums of $S^1 \times S^2$'s, and to get examples with elements of the type in theorem 1,b) we need at least two of them. But then we actually get an infinitely generated group this way - e.g. from the elements $\beta \cdot t^i$, $i \in Z$, where $\beta$ is the $S^2$ of one copy, and t is the generator of $\pi_1$ of the other copy.
In dimension 4, we similarly get examples of the type $S^1 \times S^1 \times S^2$, $(S^1)^4$, or connected sums of such. To get an example where all of $Wh_1^+(\pi_1(M); Z_2 \oplus \pi_2(M))$ can be realized and injects into $\pi_0(P(M))$, take $CP^2 \# (S^1)^4$.

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References:

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