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Pseudoisotopies and the Bökstedt trace

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Abstract In this paper we study the group of isotopy classes of pseudoisotopies of a compact manifold of high dimension. We set up Igusa's exact sequence from the point of view of A-theory and use the Bökstedt trace map to detect elements coming from the Wh_1^+-term. The main result relates the non-vanishing of $\pi_0\mathcal{P}(M)$ to a curious question in group theory.

Keywords K-theory of spaces · Psudoisotopy theory · Trace maps · Igusa exact sequence

Mathematics Subject Classification (2000) 19D10 · 57R55

1 Introduction

Let $\mathcal{P}(M)$ be the space of smooth pseudoisotopies of a compact differentiable manifold M. This space has been extensively studied and has played an important rôle in topology over the last decades. However, many problems remain, and one important motivation for this paper is to try to answer the following question:

Question 1.1 *Is $\pi_0\mathcal{P}(M)$ nontrivial if $\pi_1 M$ is nontrivial?*

The history of this question goes back to the important work by Hatcher and Wagoner [5, especially part II, by Hatcher]. There it was claimed that if M has dimension at least 6, $\pi_0\mathcal{P}(M)$ is isomorphic to the direct sum of two abelian groups $Wh_2(\pi_1 M)$ and $Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M)$, and it is easy to see that the latter is nontrivial if $\pi_1 M$ is nontrivial.

However, it turned out that Hatcher's work depended on assumptions that are not always fulfilled, and in [6] Igusa corrected his result by constructing an exact sequence

$$K_3(\mathbb{Z}[\pi_1 M]) \xrightarrow{\chi} Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M) \to \pi_0\mathcal{P}(M) \xrightarrow{\lambda} Wh_2(\pi_1 M) \to 0 \qquad (1)$$

Dedicated to Bruce Williams.

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which is valid when $\dim M \geqslant 7$. He also gave examples to show that χ is nonzero in general.

$Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M)$ is very easy to compute (unlike $Wh_2(\pi_1 M)$), and for pseudoisotopy the important question is how much of this term actually survives to $\pi_0\mathcal{P}(M)$—in other words: *what is the cokernel of* χ?

It is clear that in general the cokernel can be very large. A simple, but striking example is when $\pi_1 M = \mathbb{Z}$. Then $K_3(\mathbb{Z}[\pi_1 M]) \cong \mathbb{Z}/48 \oplus \mathbb{Z}/2$, and $Wh_1^+(\mathbb{Z}; \mathbb{Z}/2)$ is infinitely generated. Therefore an immediate consequence of (1) is that $\pi_0\mathcal{P}(M)$ is also infinitely generated. (In fact, an infinite sum of $\mathbb{Z}/2$'s.)

It is hard to compute coker χ in general, but one might hope to do so in special cases. In this paper, the main idea is to compose the map $Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M) \to \pi_0 P(M)$ with a map derived from the Bökstedt trace map from $A(M)$. The resulting map can be computed and used to get bounds on coker χ.

Our main result on Question 1.1 relates it to a curious problem in group theory. In Sect. 5 we prove

Corollary 5.7 *If* $\dim M \geqslant 7$ *and* $\pi_1 M$ *has an element which is not conjugate to its square, then* $\pi_0\mathcal{P}(M)$ *is non-trivial.*

(In fact, this is true also in dimensions 5 and 6, see Remark 5.11 at the end of the paper.) It may be that all finitely presented groups have such elements; as far as I know, no counterexample is known. See also Remark 5.8. If so, Question 1.1 has a positive answer for every compact manifold M of dimension at least 5.

We also give more explicit computations. For example, if $\pi_1 M$ is free or free abelian and $\pi_2 M = 0$, then $Wh_1^+(\pi_1 M; \mathbb{Z}/2) \to \pi_0\mathcal{P}(M)$ is an isomorphism.

Igusa constructed the sequence (1) as an extension of Hatcher and Wagoner's work [5]. However, after Waldhausen's work the natural approach to pseudoisotopy is through A-theory and Igusa's stability theorem. Then, if 0 is in in the stability range for M, computing $\pi_0\mathcal{P}(M)$ is essentially the same as computing $\pi_2 A(M)$. We will see that (1) can be derived from the fibration sequence one gets by comparing $A(M)$ to $K(\mathbb{Z}[\pi_1 M])$.

In Sect. 2 we set up Igusa's sequence from this point of view. In fact, it turns out that we get almost for free an extension of (1) one step to the left: if $\dim M$ is high enough (current status is $\geqslant 9$) there is an exact sequence

$$\begin{aligned} \pi_1\mathcal{P}(M) \to Wh_3(\pi_1 M) &\xrightarrow{\chi} \\ &\to Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M) \to \pi_0\mathcal{P}(M) \to Wh_2(\pi_1 M) \to 0 \end{aligned} \tag{2}$$

($Wh_3(\pi)$ is a certain quotient of $K_3(\mathbb{Z}[\pi])$; see Definition 2.6. Hence the kernel of χ tells us something about $\pi_1\mathcal{P}(M)$. (2) was also constructed by Igusa, but only when $\pi_2 M = 0$, and by completely different methods (see [7]).

For the proof we need Theorem 2.4, which says that the "monomial" homomorphism $\pi_2^S(B\pi_+) \to K_2(\mathbb{Z}[\pi])$ is (split) injective. This result is certainly known to K-theory experts, but since I have not found a reference in the literature, a proof is given in Sect. 3.

The map $Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M) \to \pi_0\mathcal{P}(M)$ comes essentially from the inclusion of '1×1-matrices' into $\widehat{GL}(Q(M))$. It turns out that a large part of its image can be detected using Bökstedt's trace map $A(M) \to THH(M) \simeq \Omega^\infty S^\infty(\Lambda M_+)$. This is the theme of the rest of the paper. The basic constructions and proofs are given in Sect. 4, and the application to $\pi_0\mathcal{P}(M)$ is in Sect. 5.

2 Waldhausen's results and Igusa's sequence

If Y is a topological space, we let Y_+ be Y with an extra basepoint added. Set $\widetilde{Q}(Y) = \Omega^\infty S^\infty(Y)$ and $Q(Y) = \widetilde{Q}(Y_+)$. If X is a connected space, the definition of Waldhausen's algebraic K-theory of X that we shall use is

$$A(X) = \Omega B\left(\coprod_n B\widehat{GL}_n(Q(GX))\right)$$

(group completion with respect to direct sum of matrices), where GX is a topological group model for the loop space ΩX [11].

One of the main results of the theory is that

$$A(X) \simeq Wh^{\mathrm{Diff}}(X) \times Q(X),$$

where $Wh^{\mathrm{Diff}}(X)$ is a space, functorial in X, such that if X is a manifold, then $\pi_{k+2} Wh^{\mathrm{Diff}}(X) \cong \pi_k(\mathcal{P}(X \times I^l))$ for l large. The map from $Q(X)$ to $A(X)$ is given by inclusion of the permutation matrices with entries in $GX \subset Q(GX)$ into $\widehat{GL}(Q(X))$, and then group completing:

$$Q(X) \simeq \Omega B\left(\coprod_n B(\Sigma_n \wr GX)\right) \to \Omega B\left(\coprod_n B\widehat{GL}_n(Q(GX))\right) = A(X)$$

Remark 2.1 If X is connected this map is an isomorphism on π_0, which is canonically isomorphic to $\mathbb{Z}$.

From now on, we set $\pi = \pi_1 X$. Let $K(R) = \Omega B\left(\coprod_n BGL_n(R)\right)$ be the (free module) K-theory space of a ring R, and denote by $\mathcal{F}$ be the homotopy fiber of the "linearization" map $A(X) \to K(\mathbb{Z}[\pi])$. Then there is a composed mapping $Q(B\pi) \to A(B\pi) \to K(\mathbb{Z}[\pi])$, and we have a map of fibrations up to homotopy

$$\begin{array}{ccccc} \Omega\widetilde{Q}(B\pi/X) & \longrightarrow & Q(X) & \longrightarrow & Q(B\pi) \\ \downarrow{\scriptstyle \phi_{rel}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \mathcal{F} & \longrightarrow & A(X) & \longrightarrow & K(\mathbb{Z}[\pi]), \end{array} \tag{3}$$

where $B\pi/X$ is the homotopy cofibre of $X \to B\pi$.

Recall that $Wh_2(\pi) = \operatorname{coker}(\pi_2^S(B\pi_+) \to K_2(\mathbb{Z}[\pi])$, and define (for the moment) $W_3(\pi) = \operatorname{coker}(\pi_3^S(B\pi_+) \to K_3(\mathbb{Z}[\pi]))$. Then we have the following commutative diagram, with the two leftmost columns and all rows exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_3^S(X_+) & \longrightarrow & \pi_3 A(X) & \longrightarrow & \pi_3 Wh^{\mathrm{Diff}}(X) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \pi_3^S(B\pi_+) & \longrightarrow & K_3(\mathbb{Z}[\pi]) & \longrightarrow & W_3(\pi) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
 & & \pi_3^S(B\pi/X) & \xrightarrow{\phi_{rel}} & \pi_2\mathcal{F} & \longrightarrow & \mathrm{coker}\,(\phi_{rel}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & \pi_2^S(X_+) & \xrightarrow{\phi} & \pi_2 A(X) & \longrightarrow & \pi_2 Wh^{\mathrm{Diff}}(X) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\lambda} & & \\
 & & \pi_2^S(B\pi_+) & \xrightarrow{\phi_\pi} & K_2(\mathbb{Z}[\pi]) & \longrightarrow & Wh_2(\pi) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

(The surjectivity of β follows from [11, Proposition 1.1].) We shall refer to this as the "basic diagram."

The group $\pi_2\mathcal{F}$ may be computed using the approach of [11, Proposition 1.2], applied to the 1-connected map of rings up to homotopy

$$Q(GX) \to \pi_0 Q(GX) \cong \mathbb{Z}[\pi].$$

(Waldhausen works in the setting of simplicial rings, but his method works equally well in our situation.) All components of $Q(GX)$ are homotopy equivalent, and since $\mathbb{Z}[\pi]$ is discrete, all homotopy fibers are equivalent to the component $Q_0(GX)$ of the trivial element. Then Waldhausen's calculation gives

$$\pi_2\mathcal{F} \cong HH_0(\mathbb{Z}[\pi], \pi_1 Q_0(GX)).$$

The two-sided action of π, hence also of $\mathbb{Z}[\pi]$, on $\pi_1 Q_0(GX)$ is induced from (two-sided) multiplication on GX by representatives of elements of $\pi = \pi_0(GX)$.

A simple calculation with the Atiyah–Hirzebruch spectral sequence yields

$$
\begin{aligned}
\pi_1 Q_0(GX) &\cong \pi_1^S(GX_+)\\
&\cong H_1(GX; \pi_0^S) \oplus H_0(GX; \pi_1^S)\\
&\cong \Sigma_{\gamma\in\pi}\pi_1(GX_\gamma) \oplus \Sigma_{\gamma\in\pi}\mathbb{Z}/2 \qquad (4)
\end{aligned}
$$

In this formula, the actions of π are the obvious ones. (Note that $\pi_1(GX_\gamma)$ is abelian, so choice of basepoint does not matter.) However, now we identify each $\pi_1(GX_\gamma)$ with $\pi_1(GX_e)$ ($e \in GX$ is the identity element) using *right* multiplication with γ^{-1}, and we obtain an isomorphism

$$\Sigma_{\gamma\in\pi}\pi_1(GX_\gamma) \cong \Sigma_{\gamma\in\pi}\pi_1(GX_e)\gamma \cong \pi_1(GX_e)[\pi].$$

On the $\pi_1(GX_e)$-summand denoted $\pi_1(GX_e)\gamma$, right action of an element $\tau \in \pi$ is gotten by the composition

$$\pi_1(GX_e)\gamma \xrightarrow{\cdot\gamma} \pi_1(GX_\gamma) \xrightarrow{\cdot\tau} \pi_1(GX_{\gamma\tau}) \xrightarrow{\cdot(\gamma\tau)^{-1}} \pi_1(GX_e)\gamma\tau$$

i.e. if $\alpha \in \pi_1(GX_e)$, then $(\alpha\gamma)\cdot\tau = \alpha\gamma\tau$. On the other hand, left action by τ is given by the composition

$$\pi_1(GX_e)\gamma \xrightarrow{\cdot\gamma} \pi_1(GX_\gamma) \xrightarrow{\tau\cdot} \pi_1(GX_{\tau\gamma}) \xrightarrow{\cdot(\tau\gamma)^{-1}} \pi_1(GX_e)\tau\gamma.$$

Thus, $\tau\cdot(\alpha\gamma) = (\tau\alpha\tau^{-1})\,\tau\gamma$. But via the the canonical isomorphism $\pi_1(GX_e) \cong \pi_2(X)$, the action $(\tau,\alpha) \mapsto \tau\alpha\tau^{-1}$ corresponds to the standard action $(\tau,\beta) \mapsto \beta^\tau$ of $\pi_1(X)$ on $\pi_2(X)$.

On the other summand of $\pi_1 Q_0(GX)$—$\Sigma_{\gamma\in\pi}\mathbb{Z}/2 \cong \mathbb{Z}/2[\pi]$—the actions are ordinary right and left multiplication on π.

Now recall that Hochschild homology of a group ring $\mathbb{Z}[\pi]$ with coefficients in a bimodule B can be identified with homology of the group π with coefficients in B considered as a π-module by conjugation. Using this, we can now write

$$\pi_2\mathcal{F} \cong H_0(\pi; (\pi_2 X \oplus \mathbb{Z}/2)[\pi]),$$

where the π-action now is given by

$$\tau\cdot(\beta,\epsilon)\gamma = (\beta^\tau,\epsilon)\tau\gamma\tau^{-1},$$

where $(\beta,\epsilon) \in (\pi_2 X \oplus \mathbb{Z}/2)$ and $\gamma \in \pi$.

Remark 2.2 Let $Q_1(GX) \subset Q(GX)$ denote the component of the map

$$S^n = S^n \wedge e_+ \subset S^n \wedge (GX_+),$$

which is the identity element in $Q(GX)$. Then $Q_1(GX)$ is a submonoid of $\widehat{GL}_1(Q(GX))$, and the induced map $BQ_1(GX) \to A(X)$ clearly lifts to a map $BQ_1(GX) \to \mathcal{F}$. In fact, we can think of $Q_1(GX)$ as $1 + Q_0(GX)$, and then $\pi_2 BQ_1(GX)$ is identified with $\pi_1 Q_0(GX)$ in the calculations above. Thus we see that $\pi_2\mathcal{F}$ is represented by "units" in $Q(GX)$.

We next want to compute coker (ϕ_{rel}). This will follow from

Lemma 2.3 $\pi_3^S(B\pi/X) \cong H_0(\pi;\pi_2 X)$*, and* ϕ_{rel} *is the inclusion of the summand* $H_0(\pi;(\pi_2 X)[1])$.

Proof $B\pi/X$ is 2-connected, hence we have isomorphisms

$$\pi_3^S(B\pi/X) \cong \widetilde{H}_3(B\pi/X) \cong H_3(B\pi, X),$$

and by the Hurewicz theorem the last group is isomorphic to the quotient of $\pi_3(B\pi, X)$ by the action of $\pi_1 X = \pi$, i.e. $H_0(\pi;\pi_3(B\pi,X))$. But $\pi_3(B\pi,X) \cong \pi_2 X$.

To prove the statement about ϕ_{rel} we consider the diagram

$$\begin{array}{ccc} G_0X & \xrightarrow{\subset} & Q_1(GX) \\ \downarrow{\scriptstyle\subset} & & \downarrow{\scriptstyle\subset} \\ GX & \xrightarrow{\subset} & \widehat{GL}_1(Q(GX)) \\ \downarrow & & \downarrow \\ \pi & \xrightarrow{\subset} & GL_1(\mathbb{Z}[\pi]) \end{array}$$

where G_0X is the component of the trivial loop; i.e. $G_0X \simeq \Omega\widetilde{X}$. Taking classifying spaces, this maps into diagram (3). But then we see that both $\pi_2\Omega Q(B\pi/X)$ and the '1'-component of $\pi_2\mathcal{F}$ are identified with the same quotient of $\pi_2 BG_0X = \pi_2\widetilde{X} \cong \pi_2 X$. □

The following theorem ought to be well known, but I have not found a reference. Since the result has independent interest, a proof is given in Sect. 3.

Theorem 2.4 *$\phi_\pi : \pi_2^S(B\pi_+) \to K_2(\mathbb{Z}[\pi])$ is split injective if π is finitely generated.*

By standard diagram chasing (or, consider the basic diagram as a short exact sequence of complexes and take homology) we then get

Corollary 2.5 *All three columns in the basic diagram are exact, and*

$$\operatorname{coker}(\phi_{rel}) \cong H_0(\pi; (\pi_2 X \oplus \mathbb{Z}/2)[\pi])/H_0(\pi; \pi_2 X)$$

However, we can do slightly better than this by observing that since the basic diagram is functorial, the diagram for X = one point sits inside as a direct summand. In $\pi_2\mathcal{F}$ this is $H_0(1; \mathbb{Z}/2[1]) \cong \mathbb{Z}/2$, and $\delta : K_3(\mathbb{Z}) \to \mathbb{Z}/2$ is surjective (see e.g. [13, Corollary 3.7. + Remark]). Hence it induces an isomorphism $\delta : W_3(1) \to \mathbb{Z}/2$.

Definition 2.6 $Wh_3(\pi) = K_3(\mathbb{Z}[\pi])/(K_3(\mathbb{Z}) + \operatorname{im} \pi_3^S(B\pi_+))$
$Wh_1^+(\pi_1 X; \mathbb{Z}/2 \oplus \pi_2 X) = H_0(\pi; (\pi_2 X \oplus \mathbb{Z}/2)[\pi])/H_0(\pi; (\pi_2 X \oplus \mathbb{Z}/2)[1])$

Corollary 2.7 (*Igusa's exact sequence*) *There is a functorial exact sequence*

$$\begin{aligned}\pi_3 Wh^{\text{Diff}}(X) \to Wh_3(\pi_1 X) \xrightarrow{\chi} \\ \to Wh_1^+(\pi_1 X; \mathbb{Z}/2 \oplus \pi_2 X) \to \pi_2 Wh^{\text{Diff}}(X) \to Wh_2(\pi_1 X) \to 0. \quad (5)\end{aligned}$$

From this, (2) follows if 1 is in the stable range for M, i.e. $\pi_i \mathcal{P}(M) \cong \pi_{i+2} Wh^{\text{Diff}}(M)$ for $i = 0, 1$.

3 Proof of Theorem 2.4

Let $\widetilde{K}_*(\mathbb{Z}[\pi]) = \ker(K_*(\mathbb{Z}[\pi]) \to K_*(\mathbb{Z}))$ be reduced K-theory. Since $\pi_2^S = \pi_2^S(*_+) \cong K_2(\mathbb{Z})$, it suffices to prove that $\pi_2^S(B\pi) \to \widetilde{K}_2(\mathbb{Z}[\pi])$ is a split injection.

We first compute $\pi_2^S(B\pi)$.

Lemma 3.1 *For connected X there is a natural, split exact sequence*

$$0 \to H_1(X; \mathbb{Z}/2) \to \pi_2^S(X) \to H_2(X) \to 0 \quad (6)$$

Proof From the E^2-term of the Atiyah–Hirzebruch spectral sequence for $\pi_*^S(X)$ we get an exact sequence

$$\pi_3^S(X) \xrightarrow{\eta} H_3(X) \xrightarrow{d_2} H_1(X; \mathbb{Z}/2) \to \pi_2^S(X) \to H_2(X) \to 0.$$

But η is surjective, since H_3 is representable by orientable manifolds, and all orientable 3-manifolds are parallelizable. Hence d_2 must be trivial, and we have the exact sequence (6). To prove that it splits, we make use of the naturality of the sequence.

Recall that $H_1(X; \mathbb{Z}/2) \cong H_1(B\pi; \mathbb{Z}/2) \cong \pi/[\pi, \pi] \otimes \mathbb{Z}/2$, which we shall denote by $\pi/2$. If π is finitely generated, this is a finite product of $\mathbb{Z}/2$'s. Observe also that for $\pi = \mathbb{Z}/2$ the exact sequence reduces to an isomorphism $H_1(\mathbb{Z}/2; \mathbb{Z}/2) \cong \pi_2^S(B(\mathbb{Z}/2))$.

Assume now given an isomorphism $\alpha : \pi/2 \cong (\mathbb{Z}/2)^n$ with components α_i. By naturality we then get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(X;\mathbb{Z}/2) & \longrightarrow & \pi_2^S(X) & \longrightarrow & H_2(X) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1(B\pi;\mathbb{Z}/2) & \longrightarrow & \pi_2^S(B\pi) & \longrightarrow & H_2(B\pi) & \longrightarrow & 0 \\
 & & \Pi\alpha_{i*}\downarrow & & \Pi B\alpha_{i*}\downarrow & & \downarrow & & \\
0 & \longrightarrow & (H_1(\mathbb{Z}/2;\mathbb{Z}/2))^n & \xrightarrow{\cong} & (\pi_2^S(B(\mathbb{Z}/2)))^n & \longrightarrow & 0 & &
\end{array}$$

The splitting is now obvious. □

Remark 3.2 (i) The splitting is not natural, but depends on α. However, given a map $f : X \to X'$ we may chose isomorphisms α and α' such that the induced homomorhism $f_* : \pi/2 \to \pi'/2$ looks like $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$. Then $f_* : \pi_2^S(X) \to \pi_2^S(X')$ will preserve the splittings.

(ii) The *unreduced* theory $\pi_2^S(X_+)$ has a natural splitting as a direct sum of $H_0(X; \pi_2^S)$ and a term given by the exact sequence (6).

(iii) It is easy to extend to more general π, e.g. such that $\pi/2$ is a countable sum of $\mathbb{Z}/2$'s.

(iv) *Terminology*: The map $\pi_2^S(X) \to H_2(X)$ in (6) can be thought of as evaluation on a fundamental class. For convenience we shall generally use the name *Hurewicz homomorphism* for such homomorphisms. The main examples are maps like $\pi_k(X) \to \pi_k^S(X_+) \to H_k(X)$.

Next we want to compare the maps in (6) with K-theory. First we study the composition $\phi \circ \iota : H_1(\pi; \mathbb{Z}/2) \cong \pi/2 \xrightarrow{\iota} \pi_2^S(B\pi) \xrightarrow{\phi} \widetilde{K}_2(\mathbb{Z}[\pi])$:

Lemma 3.3 *$\phi \circ \iota$ injects onto a direct summand.*

Proof For this, we again exploit functoriality. Let $\phi : \pi \to \mathbb{Z}/2$ be a homomorphism, and consider the commutative diagram

$$\begin{array}{ccccc}
\pi/2 & \longrightarrow & \pi_2^S(B\pi) & \longrightarrow & \widetilde{K}_2(\mathbb{Z}[\pi]) \\
\downarrow\phi & & \downarrow\phi_* & & \downarrow \\
\mathbb{Z}/2 & \xrightarrow{\cong} & \pi_2^S(B(\mathbb{Z}/2)) & \longrightarrow & \widetilde{K}_2(\mathbb{Z}[\mathbb{Z}/2]).
\end{array}$$

$\pi_2^S(B(\mathbb{Z}/2)) \to \widetilde{K}_2(\mathbb{Z}[\mathbb{Z}/2])$ is an isomorphism by [4]. Let $\mathcal{B}$ be a basis for the $\mathbb{Z}/2$-vector space $\pi/2$. Then $\pi/2 \cong \oplus_{\mathcal{B}}\mathbb{Z}/2$, and we let $\phi_b : \pi \to \mathbb{Z}/2,\ b \in \mathcal{B}$ be the projections. Taking products, we get the diagram

$$\begin{array}{ccccc}
\pi/2 & \longrightarrow & \pi_2^S(B\pi) & \longrightarrow & \widetilde{K}_2(\mathbb{Z}[\pi]) \\
\Pi_b\phi_b\downarrow\cong & & \downarrow\Pi_b(\phi_b)_* & & \downarrow \\
\oplus_b\mathbb{Z}/2 & \xrightarrow{\cong} & \oplus_b\pi_2^S(B(\mathbb{Z}/2)) & \xrightarrow{\cong} & \oplus_b\widetilde{K}_2(\mathbb{Z}[\mathbb{Z}/2]).
\end{array}$$

The splitting is obvious from this. □

To complete the proof of Theorem 2.4, we need one more ingredient:

Lemma 3.4 *There is a natural (in π) transformation $\tau : K_n(\mathbb{Z}[\pi]) \to H_n(B\pi)$ such that the composition*

$$\pi_n^S(B\pi_+) \xrightarrow{\phi_\pi} K_n(\mathbb{Z}[\pi]) \xrightarrow{\tau} H_n(B\pi)$$

is the Hurewicz homomorphism.

Proof This is a consequence of results of Waldhausen [12, Appendix, Propositions 6.1 and 6.2]. From that point of view, Lemma 3.4 is a K-theory analogue of the splitting of the map $Q(X) \to A(X)$. □

Proof of Theorem 2.4 Let ρ be a left inverse of $\phi \circ \iota$ as provided by Lemma 3.3. Then $h : \pi_2^S(B\pi) \to H_2(B\pi)$ has a splitting (right inverse) σ which is well defined by the formula $\sigma(y) = \hat{y} - \iota\rho\phi(\hat{y})$,

Now we can define a left inverse g of ϕ by

$$g(z) = \iota\rho(z) + \sigma\tau(z).$$

If $z = \phi(x)$, we can chose $\widehat{\tau(z)} = x$ by Lemma 3.4. Therefore

$$g(\phi(x)) = \iota\rho\phi(x) + x - \iota\rho\phi(x) = x.$$

□

Remark 3.5 Lemma 3.4 can obviously be used to construct nontrivial elements of $K_n(\mathbb{Z}[\pi])$. For example, the surjectivity of $\pi_3^S(B\pi_+) \to H_3(B\pi)$ implies:

Corollary 3.6 *The homomorphism $D' : K_3(\mathbb{Z}[\pi]) \to H_3(B\pi)$ is surjective.*

Take for example $\pi = \mathbb{Z}/n$. Then it follows that there is a surjective homomorphism

$$K_3(\mathbb{Z}[\mathbb{Z}/n]) \to \mathbb{Z}/n.$$

A similar application of Lemma 3.4 to higher K-theory is to a result of Dennis, saying that if π is a group with vanishing homology below dimension n, then $D' : K_n(\mathbb{Z}[\pi]) \to H_n(B\pi)$ is surjective ([6]). (For $n = 3$, this is a special case of Corollary 3.6.) Dennis' result follows because by the Atiyah–Hirzebruch spectral sequence $\pi_n^S(B\pi_+) \to H_n(B\pi)$ now will be surjective.

4 The Bökstedt trace on $\pi_2\mathcal{F}$

From the formulation of Waldhausen's calculation of $\pi_2\mathcal{F}$ in Sect. 2, it follows that we can think of the homomorphism $\pi_2\mathcal{F} \to \pi_2 A(X)$ as induced by the inclusion

$$BQ_1(GX) \to B\widehat{GL}_1(Q(GX)) \to A(X).$$

From Remark 2.1 it follows that the image of this map is contained in the 1-component $A_1(X)$.

Recall now the Bökstedt trace map, which in the case of A-theory is an infinite loop map

$$\tau : A(X) \to Q(\Lambda X),$$

where ΛX is the free loop space $\mathrm{Map}(S^1, X)$ (see [2,3]). It is a consequence of Remark 2.1 and Lemma 5.1 below that τ maps the component $A_1(X)$ into the component $Q_1(\Lambda X)$ of the map $S^n = S^n \wedge (*_+) \subset S^n \wedge (\Lambda X_+)$, where the basepoint in ΛX is the trivial loop at the basepoint of X.

In this section we shall consider the composition

$$\pi_k BQ_1(GX) \to \pi_k A_1(X) \to \pi_k Q_1(\Lambda X).$$

and compute it completely for $k = 2$.

It will be convenient in the following to adjust the notation slightly, and write $G = GX$, $X \simeq BG$. Then Bökstedt's trace is induced by composition of maps of the homotopy type of

$$B\widehat{GL}_n(QG) \xrightarrow{\iota} \Lambda B\widehat{GL}_n(QG) \xrightarrow{\tau_n'} Q(\Lambda BG)$$

where ι as before is the inclusion of the constant loops. τ_n' is a more complicated construction, involving Morita equivalence (for $n > 1$) and replacing (stably) compositions with smash products. Taking restriction to $BQ_1G \subset B\widehat{GL}_1(QG)$, we see that the maps we need to compute are

$$BQ_1G \xrightarrow{\iota} \Lambda BQ_1G \xrightarrow{\tau_1'} Q_1(\Lambda BG).$$

Hence Morita equivalence does not enter, and the crucial property of τ_1' for our computation is that it is equivariant with respect to the natural S^1-actions. Thus we have a diagram

$$\begin{array}{ccc} S^1 \times Q_1G & = & S^1 \times Q_1G \\ \downarrow & & \downarrow \\ S^1 \times \Lambda BQ_1G & \xrightarrow{S^1\times\tau_1'} & S^1 \times Q_1(\Lambda BG) \\ \downarrow & & \downarrow \\ \Lambda BQ_1G & \xrightarrow{\tau_1'} & Q_1(\Lambda BG) \end{array}$$

(The top vertical maps are induced by the natural inclusions $Q_1G \subset \Omega BQ_1 \subset \Lambda BQ_1G$ and $G \subset \Omega BG \subset \Lambda BG$.) In fact, both actions leave the basepoints fixed, so we have a commutative diagram

$$\begin{array}{ccc} S^1_+ \wedge Q_1G & = & S^1_+ \wedge Q_1G \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \bar{\mu}} \\ \Lambda BQ_1G & \xrightarrow{\tau_1'} & Q_1(\Lambda BG) \end{array} \qquad (7)$$

First we consider the homomorphisms induced by μ on homotopy groups. More generally, observe that $\mu : S^1_+ \wedge Y \to \Lambda BY$ is defined for any monoid Y. To formulate the next result, we need the following construction. Assume Y connected, so that $\Omega BY \simeq Y$. Let $\eta : S^{k+1} \to S^k$ be the Hopf map, for $k \geqslant 2$. For $k > 2$ the composition $\pi_{k-1}Y \cong \pi_k BY \xrightarrow{\eta^*} \pi_{k+1}BY \cong \pi_k Y$ is also induced by a Hopf map, but we will be mostly interested in the map we get for $k = 2$, which we will also denote by η^*. (Note that this is not in general a homomorphism. Moreover, it depends on the delooping BY, i.e. on a particular monoid structure on Y.)

Lemma 4.1 *Let Y be a connected monoid, and let $\widetilde{\mathbb{Z}}[\pi_1 Y]$ be the additive group $\mathbb{Z}[\pi_1 Y]/\mathbb{Z}[1]$. Then there are natural isomorphisms*

$$\pi_2(S^1_+ \wedge Y) \cong \pi_2 Y \oplus \widetilde{\mathbb{Z}}[\pi_1 Y] \tag{8}$$
$$\pi_2(\Lambda BY) \cong \pi_2 Y \oplus \pi_1 Y, \tag{9}$$

such that μ_* is given by $\mu_*(a, g) = (a + \eta^*(g), g)$ for $a \in \pi_2 Y,\ g \in \pi_1 Y$. (Note that $\pi_1 Y$ is abelian.)

Proof Let $p : \widetilde{Y} \to Y$ be the universal covering space of Y. To prove (8), we use the isomorphisms

$$\pi_2(\mathrm{S}^1_+ \wedge Y) \cong \pi_2(\widetilde{\mathrm{S}^1_+ \wedge Y}) \cong H_2(\widetilde{\mathrm{S}^1_+ \wedge Y}).$$

Since $\mathrm{S}^1_+ \wedge Y = \mathrm{S}^1 \times Y/\mathrm{S}^1 \times * \simeq \mathrm{S}^1 \times Y \cup D^2 \times *$, $\widetilde{\mathrm{S}^1_+ \wedge Y}$ is weakly equivalent to $\mathrm{S}^1 \times \widetilde{Y} \cup D^2 \times p^{-1}(*)$. H_2 of this space can be computed with the Mayer–Vietoris sequence, and with obvious identifications we obtain a short exact sequence

$$0 \to \pi_2 Y \to \pi_2(\mathrm{S}^1_+ \wedge Y) \to I[\pi_1 Y] \to 0,$$

where $I[\pi_1 Y]$ is the augmentation ideal of $\mathbb{Z}[\pi_1 Y]$ (with its additive structure). The sequence clearly splits to the left, but we will need a natural splitting from the right. This is slightly more conveniently defined if we use the obvious identification $I[\pi_1 Y] \cong \widetilde{\mathbb{Z}}[\pi_1 Y]$. Then the splitting maps the free generator $g : \mathrm{S}^1 \to Y$ to the composition

$$g' : \mathrm{S}^2 \xrightarrow{/} \mathrm{S}^1_+ \wedge \mathrm{S}^1 \xrightarrow{1\wedge g} \mathrm{S}^1_+ \wedge Y.$$

Remark For $n, m \geqslant 1$ there are canonical homeomorphisms

$$\mathrm{S}^n_+ \wedge \mathrm{S}^m \cong \mathrm{S}^n \times \mathrm{S}^m/\mathrm{S}^n \times * \cong \mathrm{S}^n \times D^m/\mathrm{S}^n \times \mathrm{S}^{m-1},$$

and the latter can be identified with $\mathrm{S}^{n+m}/\mathrm{S}^{m-1}$ for a standard $\mathrm{S}^{m-1} \subset \mathrm{S}^{n+m}$. Hence there is a canonical homotopy class of identification maps $\mathrm{S}^{n+m} \to \mathrm{S}^n_+ \wedge \mathrm{S}^m$. Maps based on this construction will generically be labelled '/'.

Note also that combined with the natural inclusion $* \times \mathrm{S}^m \subset \mathrm{S}^n_+ \wedge \mathrm{S}^m$ this defines a homotopy equivalence $\mathrm{S}^{n+m} \vee \mathrm{S}^m \simeq \mathrm{S}^n_+ \wedge \mathrm{S}^m$.

(9) is a standard consequence of the homotopy sequence of the fibration $\Omega BY \to \Lambda BY \xrightarrow{\rho} BY$ with the section ι defined above (taking $x \in BY$ to the constant path in x). In fact, the formula has an obvious analogue for any π_n, $n \geqslant 2$, and the projection $\pi_n(\Lambda BY) \to \pi_n Y$ has the following explicit description in terms of the construction / above:

Let $f : \mathrm{S}^n \to \Lambda BY$ represent an element in $\pi_n(\Lambda BY)$. f has an adjoint map $\mathrm{S}^1_+ \wedge \mathrm{S}^n \to BY$, which may be composed with $\mathrm{S}^{n+1} \xrightarrow{/} \mathrm{S}^1_+ \wedge \mathrm{S}^n$ to produce an element in $\pi_{n+1} BY \cong \pi_n Y$.

We now calculate the map μ_*. Let $\nu : \Sigma Y \to BY$ be the adjoint of $Y \simeq \Omega BY$ (or inclusion of '1-skeleton'). Then the diagram

$$\begin{array}{ccccc} Y & \longrightarrow & \mathrm{S}^1_+ \wedge Y & \longrightarrow & \Sigma Y \\ \downarrow{\simeq} & & \downarrow{\mu} & & \downarrow{\nu} \\ \Omega BY & \longrightarrow & \Lambda BY & \xrightarrow{\rho} & BY \end{array}$$

is easily seen to be commutative, and it follows immediately that $\mu_*(a, 0) = (a, 0)$. It is also clear that if $[g] \in \pi_1 Y$ is a generator of the $\widetilde{\mathbb{Z}}[\pi_1 Y]$-summand, then the $\pi_1(Y)$-component of of $\mu_*[g]$ is again g. Hence it remains to identify the $\pi_2 Y$-component.

Think of g as a map $\mathrm{S}^1 \to \Omega BY$. By the description above, the component of $\mu_*[g]$ in $\pi_2 Y \cong \pi_3 BY$ is given by the composition

$$\mathrm{S}^3 \xrightarrow{/} \mathrm{S}^1_+ \wedge \mathrm{S}^2 \xrightarrow{/} \mathrm{S}^1_+ \wedge \mathrm{S}^1_+ \wedge \mathrm{S}^1 \xrightarrow{1 \wedge g} \mathrm{S}^1_+ \wedge \mathrm{S}^1_+ \wedge \Omega BY \xrightarrow{(s,t,\omega) \mapsto \omega(st)} BY.$$

Hence it factors through the composition

$$\mathrm{S}^3 \xrightarrow{/} \mathrm{S}^1_+ \wedge \mathrm{S}^2 \xrightarrow{/} \mathrm{S}^1_+ \wedge \mathrm{S}^1_+ \wedge \mathrm{S}^1 \xrightarrow{m_+ \wedge \mathrm{S}^1} \mathrm{S}^1_+ \wedge \mathrm{S}^1 \xrightarrow{c} \mathrm{S}^2, \tag{10}$$

where m is the product map on S^1 and c is the obvious collapse. The composition of the first two maps can be realized by collapsing two fibers of the Hopf map one point. But then it is easily seen that the composition of the first three maps is the same as the composition $\mathrm{S}^3 \xrightarrow{\eta} \mathrm{S}^2 \xrightarrow{/} \mathrm{S}^1_+ \wedge \mathrm{S}^1$, where / now identifies the images of these two fibers under η. Since $c \circ / \simeq \mathrm{id}_{\mathrm{S}^2}$, the composition (10) is homotopic to η. □

Now we go back to the case $Y = Q_1 G$. We want to compute $(\tau_1' \iota)_*(g)$ for $g \in \pi_1 Q_1 G$. But in terms of the decompositions in Lemma 4.1 we have $\iota_*(g) = (0, g) = \mu_*(-\eta^*(g), g)$. Hence it follows from diagram (7) that

$$(\tau_1' \iota)_*(g) = \bar{\mu}_*(-\eta^*(g), g) = \bar{\mu}''_*(g) - \bar{\mu}'_*(\eta^*(g)) \tag{11}$$

where

$$\bar{\mu}'_* : \pi_2 Q_1 G \to \pi_2 Q_1(\Lambda BG)\,, \quad \text{and}$$
$$\bar{\mu}''_* : \widetilde{\mathbb{Z}}[\pi_1 Q_1 G] \to \pi_2 Q_1(\Lambda BG)$$

are the two summands of $\bar{\mu}_*$.

$\bar{\mu}$ is the composition $\mathrm{S}^1_+ \wedge Q_1 G \to Q_1(\mathrm{S}^1_+ \wedge G) \xrightarrow{Q_1(\mu)} Q_1(\Lambda BG)$, where the first map is defined by $(t, f : \mathrm{S}^n \to \mathrm{S}^n \wedge (G_+)) \mapsto (f : \mathrm{S}^n \to \mathrm{S}^n \wedge (t \times G)_+)$. (This makes sense if the basepoint in $Q_1 G$ is the 'identity', represented by maps $\mathrm{S}^n \xrightarrow{=} \mathrm{S}^n \wedge (1_+)$, where 1 is the basepoint (unit) in G.) Using this, we see that

- $\bar{\mu}'_*$ is the homomorphism induced by the natural inclusion $\iota : G \to \Lambda BG$
- $\bar{\mu}''_* = Q(\mu)_* \circ \gamma$, where γ is the composed map $\widetilde{\mathbb{Z}}\pi_1 Q_1 G \to \pi_2(\mathrm{S}^1_+ \wedge Q_1 G) \to \pi_2 Q_1(\mathrm{S}^1_+ \wedge G)$.

For any space, $\pi_i Q_1 Y \cong \pi_i^S(Y_+)$. For $i = 2$ this was computed in Lemma 3.1 (cf. Remark 3.2(ii)), and it is also easy to see that

$$\pi_1^S(Y_+) \cong H_1(Y; \pi_0^S) \oplus H_0(Y; \pi_1^S) \quad \text{(naturally)}.$$

The following theorem describes $\bar{\mu}''_*$ more precisely:

Proposition 4.2 *$\bar{\mu}''_*$ is given on generators by the vertical composition in the following commutative diagram with split exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_0(G;\mathbb{Z}/2) & \longrightarrow & \pi_1 Q_1 G & \longrightarrow & H_1(G;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma'} & & \\
0 & \to & H_0(S^1_+\wedge G;\mathbb{Z}/2)\oplus H_1(S^1_+\wedge G;\mathbb{Z}/2) & \to & \pi_2 Q_1(S^1_+\wedge G) & \to & H_2(S^1_+\wedge G;\mathbb{Z}) & \to & 0 \\
 & & \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle Q\mu_*} & & \downarrow{\scriptstyle \mu_*} & & \\
0 & \to & H_0(\Lambda BG;\mathbb{Z}/2)\oplus H_1(\Lambda BG;\mathbb{Z}/2) & \to & \pi_2 Q_1(\Lambda BG) & \to & H_2(\Lambda BG;\mathbb{Z}) & \to & 0
\end{array}$$

The two maps labeled γ' are both given by cross product with the fundamental class of S^1.

Proof To identify γ', we use the isomorphisms $\pi_i Q_1 Y \cong \Omega_i^{fr}(Y)$. Then γ corresponds to the map $\Omega_1^{fr}(G) \to \Omega_2^{fr}(\mathrm{S}^1_+\wedge G)$ given by $(f : M \to G) \mapsto (\mathrm{S}^1 \times f : \mathrm{S}^1 \times M \to \mathrm{S}^1_+\wedge G)$. But this corresponds precisely to the cross product in homology. □

Note that the image of $\bar{\mu}''_*$ has no component in $H_0(\Lambda BG;\mathbb{Z}/2)$, which splits off naturally (cf. Remark 3.2(ii)).

5 Further calculations and applications to $\pi_0\mathcal{P}(M)$

Before we can apply these results to detect elements in $\pi_0\mathcal{P}(M)$, we need

Lemma 5.1 *The composition $Q(M) \overset{\phi}{\to} A(M) \overset{\tau}{\to} Q(\Lambda M)$ is $Q(\iota)$, where $\iota : M \subset \Lambda M$ is the 'trivial loop' embedding.*

Proof We know that τ and ϕ are infinite loop maps. Hence $\tau\circ\phi$ is determined by its restriction to $M \subset Q(M)$. But in the group completion model for $Q(M)$, this subset corresponds to 1×1-matrices with entries in GM. The result now follows from the description of τ in Sect. 4 above. □

Corollary 5.2 *The Bökstedt trace induces a map $Wh^{\mathrm{Diff}}(M) \to \widetilde{Q}(\Lambda M/M)$.*

In fact, we do not loose any information passing to $\pi_2^S(\Lambda M/M)$, since we already divided out everything that mapped to $\pi_2^S(M_+)$.

Our main goal is to say something about question 1.1 in the introduction: "Is $\pi_0\mathcal{P}(M)$ nontrivial if $\pi_1 M$ is nontrivial?" Let t be the composition

$$t : \pi_2 BQ_1(GX) \to \pi_2 A(X) \to \pi_2\widetilde{Q}(\Lambda X/X).$$

t factors through $Wh_1^+(\pi_1 X;\mathbb{Z}/2\oplus\pi_2 X) \to \pi_2 Wh^{\mathrm{Diff}}(X) \cong \pi_0\mathcal{P}(X)$, and we now ask the apparently stronger question:

Question 5.3 *Is t nontrivial whenever $\pi_1 X$ is nontrivial?*

When the question is asked in this generality, it suffices to consider the case where X is a $K(\pi, 1)$, i.e. when $G = \pi$ is discrete. In fact, $\pi_2 BQ_1(GX) \to \pi_2 BQ_1(\pi_1 X)$ is surjective, so by functoriality it follows that if t is nontrivial for $K(\pi_1 X, 1)$, it is also nontrivial for X.

Therefore we now specialize the discussion in Sect. 4 to the case G discrete. Formula (11) shows that what we need to do, is to compute $\bar{\mu}''_*(g)$ and $\bar{\mu}'_*(\eta^*(g))$ for $g \in \pi_1 Q_1 G \cong \mathbb{Z}/2[G]$.

For $\bar{\mu}''_*(g)$ we use the diagram in Proposition 4.2. The important part of this diagram is now

$$\begin{array}{ccc} H_0(G;\mathbb{Z}/2) & \xrightarrow{\cong} & \pi_1 Q_1 G \\ \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} \\ H_1(S^1_+ \wedge G;\mathbb{Z}/2) & \longrightarrow & \pi_2 Q_1(S^1_+ \wedge G) \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle Q\mu_*} \\ H_1(\Lambda BG;\mathbb{Z}/2) & \longrightarrow & \pi_2 Q_1(\Lambda BG) \end{array}$$

Choose a subset $G_c \subset G$ consisting of one element from each conjugacy class of G. Then $\Lambda BG \simeq \coprod_{g\in G_c} BC_g$, where C_g is the centralizer of g, and $\mu_*\gamma'$ maps every element hgh^{-1} in the conjugacy class of g to the class of g in $H_1(BC_g;\mathbb{Z}/2) \cong C_g/2$. This determines $\bar{\mu}''_*$ completely, and we see that $\bar{\mu}''_*(g)$ is trivial if and only if g is a square in C_g mod $[C_g, C_g]$.

Remark 5.4 Note that when G is discrete, the inclusion $BG \subset \Lambda BG$ maps BG by a homotopy equivalence to the component corresponding to the trivial conjugacy class. Hence $\Lambda BG/BG \simeq (\coprod_{g\in G'_c} BC_g)_+$, where $G'_c \subset G_c$ represents the *nontrivial* conjugacy classes of G. Since only the identity element of G maps into $H_1(BC_1;\mathbb{Z}/2) = H_1(BG;\mathbb{Z}/2)$ by $\bar{\mu}''_*$, we loose nothing by passing to $\pi_2\widetilde{Q}(\Lambda BG/BG)$.

$\bar{\mu}'_*$ is determined by the diagram

$$\begin{array}{ccc} H_0(G,\mathbb{Z}/2) & \xrightarrow{\cong} & \pi_2 Q_1 G \\ \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \bar{\mu}'_* = Q_1\iota_*} \\ H_0(\Lambda BG,\mathbb{Z}/2) & \longrightarrow & \pi_2 Q_1(\Lambda BG) \end{array} \tag{12}$$

We immediately notice that $\bar{\mu}'_*$ and $\bar{\mu}''_*$ have images in complementary summands of $\pi_2 Q_1(\Lambda BG)$, so if $\bar{\mu}''_*(g) \neq 0$ for some element g, we know that also $t(g) \neq 0$. Hence we have an affirmative answer to Question 5.3 for groups such that there exists an element which is not a square in its abelianized centralizer. This is certainly true for many groups, but not e.g. if all elements have odd order. As we shall see later, $\bar{\mu}'_*(\eta * (g))$ turns out to be a much sharper invariant.

$H_0(G,\mathbb{Z}/2) \cong \mathbb{Z}/2[G]$ and $H_0(\Lambda BG,\mathbb{Z}/2) \cong \mathbb{Z}/2\{G_c\}$—the free $\mathbb{Z}/2$-module on the set of conjugacy classes of G—and ι_* takes a group element to its conjugacy class. Hence, what remains is to determine what the map $\eta^* : \pi_1 Q_1 G \to \pi_2 Q_1 G \cong \mathbb{Z}/2[G]$ does to a generator $g \in \pi_1 Q_1 G \cong \mathbb{Z}/2[G]$.

To do this we need the following lemma, which translates the calculation of η^* into a homology calculation. Let more generally Y be a connected loop space, and consider the diagram

$$\begin{array}{ccc} \pi_1 Y & \xrightarrow[\cong]{h_1} & H_1(Y) \\ {\scriptstyle \eta^*}\downarrow & & \downarrow{\scriptstyle q} \\ \pi_2 Y & \xrightarrow{h_2} & H_2(Y) \end{array}$$

(The horizontal maps are Hurewicz homomorphisms and q is defined so that the diagram is commutative.) Recall that the H-space structure on Y induces a Pontrjagin product on H_*.

Lemma 5.5 *q is the square with respect to the Pontrjagin product. If Y is a double loop space, h_2 is (split) injective.*

Proof Let $\bar{\eta} : S^2 \to \Omega S^2$ be adjoint to the Hopf map $S^3 \to S^2$. For $g : S^1 \to Y$, $\eta^*(g)$ can be described as the composition $S^2 \xrightarrow{\bar{\eta}} \Omega S^2 \xrightarrow{\Omega\Sigma g} \Omega\Sigma Y \xrightarrow{\theta} Y$, hence $\eta^*(g) = \theta_* \Omega\Sigma g_*(\bar{\eta})$.

Let $\sigma_Z : Z \times Z \to \Omega\Sigma Z$ be the James construction on the space Z. Then σ_{S^1} and $\bar{\eta}$ both represent the generator of $H_2(\Omega S^2)$, and $\theta \circ \sigma_Y$ is the product in Y. The identification of q now follows by mapping $[\bar{\eta}] \in \pi_2(\Omega S^2)$ along the edges of the following diagram

$$\begin{array}{ccccc} \pi_2(\Omega S^2) & \xrightarrow[\cong]{h_2} & H_2(\Omega S^2) & \xleftarrow{\sigma_*} & H_2(S^1 \times S^1) \\ {\scriptstyle \Omega\Sigma g_*}\downarrow & & {\scriptstyle \Omega\Sigma g_*}\downarrow & & \downarrow{\scriptstyle (g\times g)_*} \\ \pi_2(\Omega\Sigma Y) & \xrightarrow{h_2} & H_2(\Omega\Sigma Y) & \xleftarrow{\sigma_*} & H_2(Y \times Y) \\ {\scriptstyle \theta_*}\downarrow & & {\scriptstyle \theta_*}\downarrow & & \downarrow{\scriptstyle \theta_*\sigma_*} \\ \pi_2(Y) & \xrightarrow{h_2} & H_2(Y) & = & H_2(Y) \end{array}$$

The injectivity of h_2 is proved in [1]. □

We want to apply this to $Q_1 G$ under the composition product, which in general can only be delooped once. However, $Q_1 G$ is homotopy equivalent to $Q_0 G$, which is an *infinite* loop space under the loop sum. Therefore h_2 is still an injection, so we can use the Pontrjagin product to compute η^* for $Q_1 G$ as well.

We need some more notation. Recall that QG is a "ring up to homotopy". We denote the (loop) sum by $*$ and the (composition) product by $\circ$.

$\pi_0 QG \cong \mathbb{Z}[G]$, and we let $Q_\lambda G$ be the component corresponding to $\lambda \in \mathbb{Z}[G]$. Addition (wrt. $*$) of an element in $Q_\lambda G$ then gives a homotopy equivalence $Q_0 G \simeq Q_\lambda G$.

$*$ and $\circ$ both give rise to Pontrjagin products in homology, and these will also be denoted $*$ and $\circ$. For example, if $[\lambda]$ is the canonical generator of $H_0(Q_\lambda G)$, the isomorphism $H_*(Q_0 G) \cong H_*(Q_\lambda G)$ is given by $*$-composition with $[\lambda]$.

Moreover, we have

$$H_k(Q_\lambda G) \otimes H_l(Q_\mu G) \begin{cases} \xrightarrow{*} H_{l+m}(Q_{\lambda+\mu} G) \\ \xrightarrow{\circ} H_{l+m}(Q_{\lambda\mu} G) \end{cases}$$

Note that $[1] \in H_0(Q_1 G)$ acts a unit for $\circ$ and $[0] \in H_0(Q_0 G)$ is a unit for $*$. Also, $[0] \circ w = w \circ [0] = 0$ for all w in positive degrees.

Since G is discrete, $Q_0 G \simeq \Pi'_{g\in G} Q_0 g$, where Π' means restricted product and the equivalence is given by loop sum. $Q_0 g = \lim_n (\Omega^n \Sigma^n (g_+)_0 \cong \lim_n (\Omega^n S^n)_0$, which has $\pi_1 \cong \pi_2 \cong H_1 \cong H_2 \cong \mathbb{Z}/2$. The generator of π_1 and H_1 is η, the generator of π_2 is η^2, and the generator of H_2 is $\eta \circ \eta = \eta * \eta$.

Similarly, each $H_1(Q_0 g) \cong \mathbb{Z}/2$ generated by a class denoted η_g, and we have $H_1(Q_0 G) \cong \oplus_{g\in G} H_1(Q_0 g)$.

Using the Künneth formula we see that $H_2(Q_0 G)$ is a $\mathbb{Z}/2$ vector space with generators $\eta_g^2 = \eta_g * \eta_g \in H_2(Q_0 g)$ and classes $\eta_g \otimes \eta_h = \eta_g * \eta_h$ for $g \neq h$. Moreover, $\eta_g \circ \eta_h = \eta_{gh}^2$.

We wish to compute $w \circ w$ for $w \in H_1(Q_1 G)$. Write $w = u * [1]$, where $u \in H_1(Q_0 G)$. The crucial tool is the following

Lemma 5.6 *Let $\Delta : H_*(QG) \to H_*(QG \times QG)$ be the map induced by the diagonal. If a, b and $c \in H_*(QG)$ and $\Delta(c) = \Sigma c' \otimes c''$ then*

(*i*) $(a * b) \circ c = \Sigma(-1)^{(\deg c')(\deg b)}(a \circ c') * (b \circ c'')$

(*ii*) $c \circ (a * b) = \Sigma(-1)^{(\deg a)(\deg c'')}(c' \circ a) * (c'' \circ b)$

Lemma 5.6 computes $(a * b) \circ c$ and $c \circ (a * b)$ if $\Delta(c) \in H_*(QG) \otimes H_*(QG) \subset H_*(QG \times QG)$. This is true for all c of degree 1 and 2, or always if we use field coefficients. Proofs are as in [9, Lemma 3.40(ii)] and [10, Theorem 2.2], with small modifications. Note, however, that [9, 3.40 (i)] does not generalize.

For w and u as above we have $\Delta w = [1] \otimes w + w \otimes [1]$ and $\Delta u = [0] \otimes u + u \otimes [0]$, and repeated use of Lemma 5.6 gives

$$\begin{aligned} w \circ w &= (u * [1]) \circ w \\ &= (u \circ [1]) * ([1] \circ w) + (u \circ w) * ([1] * [1]) \\ &= u * w + (u \circ w) * [1] \\ &= (u * u) * [1] + (u \circ w) * [1] \end{aligned}$$

Similarly,

$$\begin{aligned} u \circ w &= u \circ (u * [1]) \\ &= ([0] \circ u) * (u \circ [1]) + (u \circ u) * ([0] \circ [1]) \\ &= 0 * u + (u \circ u) * [0] = u \circ u \end{aligned}$$

(All elements have order two, so the signs disappear.)

Hence we have

$$w \circ w = (u * u + u \circ u) * [1]. \tag{13}$$

We are now ready to read off η^*, interpreted as the leftmost vertical map in the following diagram:

$$\begin{array}{ccccc} H_0(G, \mathbb{Z}/2) & \xrightarrow{\cong} & \pi_1 Q_1 G & \xrightarrow{\cong} & H_1(Q_1 G) \\ \downarrow{\scriptstyle \eta^*} & & \downarrow{\scriptstyle \eta^*} & & \downarrow{\scriptstyle q} \\ H_0(G, \mathbb{Z}/2) & \xrightarrow{\cong} & \pi_2 Q_1 G & \longrightarrow & H_2(Q_1 G) \end{array}$$

A generator g of the upper left group $H_0(G, \mathbb{Z}/2)$ corresponds to the generator $\eta_g * [1] \in H_1(Q_1 G)$ via the isomorphisms at the top of the diagram, and we have just seen that the squaring map q maps this to

$$(\eta_g * \eta_g + \eta_g \circ \eta_g) * [1] = \eta_g^2 * [1] + \eta_{g^2}^2 * [1]$$

But this is the image of $g + g^2 \in \mathbb{Z}/2[G] \cong H_0(G, \mathbb{Z}/2)$ under the (injective) bottom horizontal map. Going all the way to $H_0(\Lambda BG; \mathbb{Z}/2) \cong \mathbb{Z}/2\{G_c\}$, we get

$$\bar{\mu}'_*(\eta^*(g)) = \langle g \rangle + \langle g^2 \rangle,$$

where $\langle g \rangle$ denotes the conjugacy class of g.

As in Remark 5.4 we also here see that we do not loose anyhthing bu passing to $\pi_2 \widetilde{Q}(\Lambda BG/BG)$. The only way $\bar{\mu}'_*(\eta^*(g))$ can be trivial is if g and g^2 are conjugate. Thus we have the following answer to Question 5.3 (and hence also Question 1.1):

Corollary 5.7 *If* $\dim M \geqslant 7$ *and* $\pi_1 M$ *has the property that there exists a nontrivial element which is not conjugate to its square, then* $\pi_0 \mathcal{P}(M)$ *is non-trivial.*

Remark 5.8 It appears to be unknown whether every finitely presented group has this property. Suppose G is a counterexample, i.e. a nontrivial group such that every element is conjugate to its square. Then we observe:

(i) If g has order two, then $< g > \neq < g^2 > = 1$, so G can have no element of order two.
(ii) If $hgh^{-1} = g^2$, then $g = [h, g]$, so G has to be perfect. Hence, for example, by (i) G cannot be finite. In fact, since every homomorphic image of G has to have the same property, it cannot even admit a nontrivial homomorphism to a finite group.
(iii) Elements of the center of G are not conjugate to any other elements, so the center has to be trivial.

We conclude with examples showing that the methods developed above also can be used to obtain quantitative results. The first is for G free or free abelian. Here we even get complete computation in many cases. Is this case $g \mapsto g + g^2$ induces an *injection* $\widetilde{\mathbb{Z}/2}\{G_c\} \rightarrow \widetilde{\mathbb{Z}/2}\{G_c\}$, hence $Wh_1^+(G; \mathbb{Z}/2) \rightarrow \pi_2 Wh^{\text{Diff}}(M)$ is also injective if $\pi_1(M) \cong G$. But for such groups $Wh_2(G) \cong 0$, so we get

Corollary 5.9 *Assume* M *is a manifold of dimension* $\geqslant 7$ *with free or free abelian fundamental group and with trivial first* k*-invariant.*
Then $\pi_0 \mathcal{P}(M) \cong Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M)$.

For our second example, let X be a connected space, with $G = \Omega X$ as before. We have $\pi_0 G \cong \pi_1 X$, and $\pi_1(G, g) = \pi_1 \Omega_g X \cong \pi_2 X$, where $\Omega_g X$ denotes the component of $g \in \pi_1 X$. Hence

$$H_1(G; \mathbb{Z}) = \oplus_{\pi_1 X} H_1(\Omega_g X) \cong \oplus_{\pi_1 X} \pi_2 X.$$

From our calculations we know that the trace map

$$\tau_* : \pi_1 Q_1 G \rightarrow \pi_2 B Q_1 G \rightarrow \pi_2 Q_1(\Lambda BG)$$

has a component $H_1(G; \mathbb{Z}) \xrightarrow{\tau_*} H_2(\Lambda X; \mathbb{Z})$. ΛX has one connected component $\Lambda_g X$ for each conjugacy class $<g>$ in $\pi_1 X$, and τ_* maps $H_1(\Omega_g X)$ to $H_2(\Lambda_g X)$.

Instead of computing $H_2(\Lambda_g X)$ in general, we will here be content with the following observation, which is sufficient for many applications. The evaluation map $\rho : \Lambda X \rightarrow X$ splits into a sum of components $\rho_g : \Lambda_g X \rightarrow X$, and we may consider the composition

$$\oplus_{\pi_1 X} \pi_2 X \cong H_1(G; \mathbb{Z}) \xrightarrow{\tau_*} H_2(\Lambda X : \mathbb{Z}) \xrightarrow{\rho_*} \oplus_{<\pi_1 X>} H_2(X; \mathbb{Z}).$$

It is then straightforward to check that the following is true:

Lemma 5.10 *This composition is the Hurewicz homomorphism on each summand.*

Hence elements mapped nontrivially by the Hurewicz homomorphism h_2 give rise to elements mapped nontrivially by $\alpha : \pi_2 \mathcal{F} \rightarrow \pi_2 A(X)$. Note that the kernel of the $h_2(X)$ is determined precisely by the first k-invariant $k_1(X)$ of X:

Let $\pi_2'X = H_0(\pi_1 X; \pi_2 X)$, i.e. $\pi_2 X$ divided by the action of $\pi_1 X$. Then the Hurewicz homomorphism factors through $h' : \pi_2' X \to H_2(X)$, and there is an exact sequence

$$H_3(X) \xrightarrow{p_*} H_3(B\pi_1 X) \xrightarrow{k_1(X)_*} \pi_2' X \xrightarrow{\overline{h}_2} H_2(X) \xrightarrow{p_*} H_2(B\pi_1 X) \to 0$$

(From the Atiyah-Hirzebruch spectral sequence for the fibration $\widetilde{X} \to X \xrightarrow{p} B\pi_1 X$.) This should be compared to Igusa's counterexamples [6], which depended on the non-vanishing of the first k-invariant in an essential way.

Remark 5.11 We end with some remarks on applications in lower dimensions. In principle one can use the trace map to try to detect elements of $\pi_0\mathcal{P}(M)$ for any M, but the problem is to construct candidates for such elements. To define $\chi : Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M) \to \pi_0\mathcal{P}(M)$ in general one needs $\dim M \geqslant 5$, and under certain conditions it turns out that parts of this map can be defined even in dimensions 3 and 4. See [8], which expoits the observation that $Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M)$ is represented by units (1×1-matrices). Using this, on gets for instance that Corollary 5.7 is true in dimension $\geqslant 5$.

Other examples: if M^3 is a reducible 3-manifold with at least one $S^1 \times S^2$-summand, $\pi_0\mathcal{P}(M)$ has elements that can be detected by Lemma 5.10. In dimension 4 one can also construct elements coming from $\mathbb{Z}/2[\pi]$, provided M contains an immersed sphere with normal bundle with Euler characteristic 1.

In contrast, observe that the maps $\pi_i\mathcal{P}(M) \to Wh_{i+2}(\pi_1 M)$ can be defined for all M and i, but they are very difficult to compute. However, this means that all maps in the sequence (2) can be defined if the dimension of M is at least five. One might conjecture that the whole sequence is exact if $\dim M \geqslant 6$, and removing $\pi_1\mathcal{P}(M)$ it should be exact if $\dim M \geqslant 5$.

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