# Knot Concordance

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# References

This talk is based on work of Tim Cochran, Kent Orr and Peter Teichner:

Knot Concordance, Whitney Towers and $L^2$ Signatures. Annals 157(2003) 433-519, and;

Structure in the Classical Knot Concordance Group; Comment. Math. Helv., pages 105 - 123, 2004.

and lecture notes of Peter Teichner, which Julia Collins and I typed up, which can be found online at:

http://www.maths.ed.ac.uk/~s0783888/sliceknots2.pdf.

# Definition

## Definition

A knot is a locally flat embedding of $S^1 \subset S^3$. It can be oriented in one of two ways. An embedding is locally flat if it is locally homeomorphic to a ball-arc pair.

## Definition

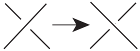
An oriented knot $K \colon S^1 \subset S^3$ is a *slice knot* if there is a locally flat embedded oriented disk $D^2 \subset D^4$, $\partial D^4 = S^3$, with $\partial D^2 = K$.

The knot $-K$ is given by reversing the orientation of both the string and the ambient space.

Two knots $K_1$ and $K_2$ are *concordant* if $K_1 \sharp -K_2$ is slice. Equivalently, if $K_1 \sqcup -K_2 = \partial(S^1 \times I) \subset S^3 \times I$.

Why is this interesting?

• Only the unknot bounds a disk in $S^3$.
However, every knot in $S^3$ can be unknotted in $S^4$, and therefore bounds a disc: there is no codimension 3 knotting (in this dimension).
To see this we need to use both $D^4$ hemispheres of $S^4$. Why?



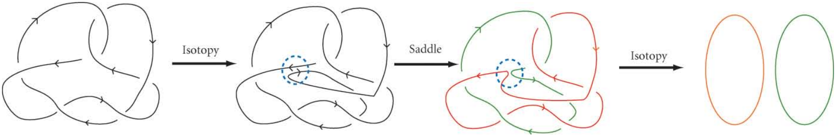
We can cross strands past one another using the 4th dimension. What if we only have one side? i.e. Only $D^4$? Some knots are slice and some are not.

# Motivation

• Forming the quotient of the set of (isotopy classes of) knots by slice knots makes knots into a group, $\mathcal{C}$, under connected sum.

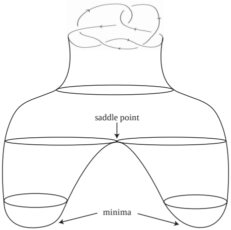
• Looking for a slice disk is related to the Whitney Trick, which works in high-dimensions, but in 4-dimensions is difficult - it is the centrepiece of Freedman's classification of 4-manifolds.
Recall Casson's Finger moves which kill commutators to get the Whitney trick to work;
Knot Groups, and the groups of Knot Concordances, are typically not solvable.

A Slice Movie:

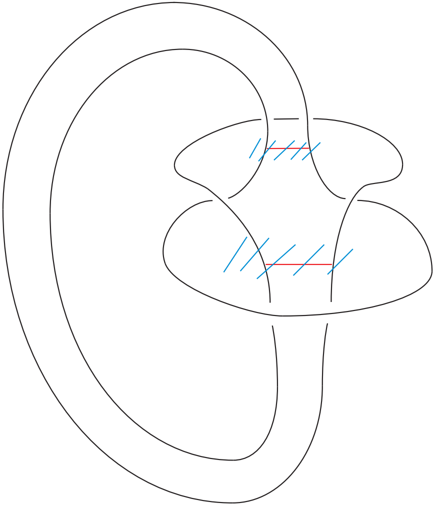



(Thanks to Julia Collins for this picture)

Here is a schematic of the resulting disc in $D^4$:

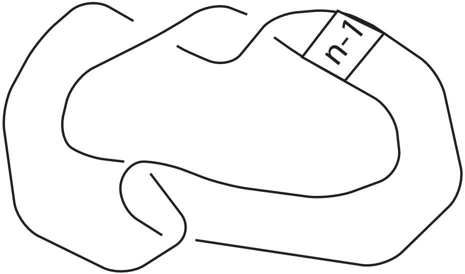


A ribbon knot, Trefoil ♯ (− Trefoil).



# Examples

Take the knot in the first example, but change the number of twists.




This is called the *n*th Twist Knot.
$n = 0 \rightarrow$ Unknot; $n = 2 \rightarrow$ Stevedore's knot $6_1$, the slice knot from two slides before. $n = -1 \rightarrow$ Trefoil knot $3_1$;
$n = 1 \rightarrow$ Figure Eight knot $4_1$.

# Casson-Gordon

**Theorem (Casson-Gordon (Slice Knots in Dimension 3, 1978))**

*The knots $K_n$ are slice only for $n = 0, 2$.*

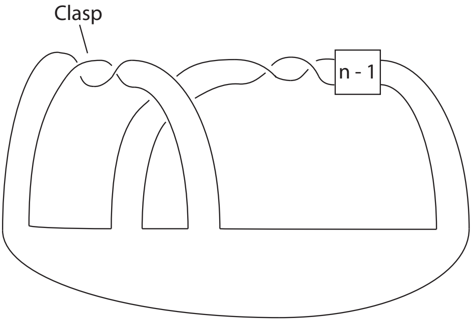
The goal of this talk is to explain the Cochran-Orr-Teichner proof of this result, which is motivated by work of Gilmer and D.Cooper.

The idea of the proof is to try to cut a Seifert surface up to make it a disc.

**Definition**

A Seifert Surface is a compact, connected, oriented surface $F$ embedded in $S^3$ with $\partial F = K$.

Here is a Seifert Surface for the twist knots. It has genus 1.



# Seifert Form

## Definition

The Seifert form on a Seifert Surface $F$ is a pairing:

$$V\colon H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$$

which is defined by:

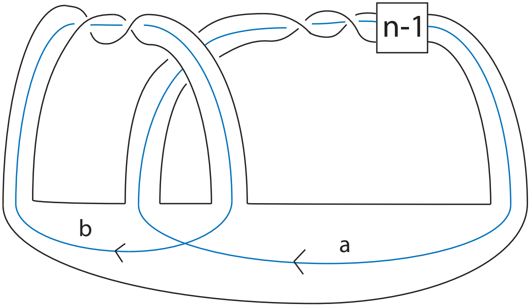
$$(x, y) \mapsto \mathrm{lk}(x^+, y)$$

where lk is the linking number in $S^3$ and $x^+$ is the push off of $x$ along a positive normal direction to $F$ - both $F$ and $S^3$ are oriented so this makes sense.

# Seifert Form - Example

With respect to the basis of curves shown the Seifert Surface is given by:

$$V = \begin{pmatrix} n & 1 \\ 0 & -1 \end{pmatrix}$$



# Knot Exterior and Zero-Surgery

We will return to the Seifert form after proving an important characterisation of slice knots.

## Definition

We define the knot exterior $X$:

$$X := \mathrm{cl}(S^3 \setminus (K \times D^2)),$$

and the zero-surgery:

$$\begin{aligned} M_K &:= \text{0-framed surgery on } K \subset S^3 \\ &= \mathrm{cl}(S^3 \setminus (K \times D^2)) \cup_{S^1 \times S^1} D^2 \times S^1 \\ &= X \cup D^2 \times S^1 \end{aligned}$$

# A Characterisation of Slice Knots

**Theorem**

*A knot $K$ is slice if and only $M_K$ is the boundary of a 4-manifold $W$ such that:*

(i) $H_1(M_K;\mathbb{Z}) \cong H_1(W;\mathbb{Z}) \cong \mathbb{Z}$ *with the isomorphism induced by the inclusion* $M_K \subseteq W$;

(ii) $H_2(W) = 0$, *so* $H_*(W;\mathbb{Z}) \cong H_*(S^1;\mathbb{Z})$;

(iii) $\pi_1(W)$ *is normally generated by the meridian of the knot.*

# Proof of Characterisation

Proof: Suppose $K$ is slice, and let $W$ be the exterior of the slice disc $\Delta$ in $D^4$.

$$W := \text{cl}(D^4 \setminus \Delta \times D^2).$$

Then $\partial W = M_K$. Now do Mayer-Vietoris on $D^4 = W \cup_{S^1 \times D^2} D^2 \times D^2$ with $\mathbb{Z}$ coefficients:

$$H_{i+1}(D^4) \xrightarrow{\partial} H_i(S^1 \times D^2) \to H_i(W) \oplus H_i(D^2 \times D^2) \to H_i(D^4)$$

which, for $i \geq 1$ yields:

$$0 \to H_i(S^1) \to H_i(W) \to 0$$

Similarly the Seifert-Van Kampen theorem can be used to show the statement about $\pi_1(W)$.

# Proof of Characterisation

Now suppose that $W$ is such that $\partial W = M_K$ and $W$ satisfies $H_*(W) \cong H_*(S^1)$ and $\pi_1(W)$ is normally generated by a meridian. Form a manifold $B$ by gluing $W$ to $D^2 \times D^2$ along $D^2 \times S^1$. Then $\partial B = S^3$ and

$$H_*(B) \cong H_*(D^4); \;\; \pi_1(B) \cong \{1\}.$$

So $K$ is slice in a homotopy 4-ball.

# h-Cobordism Theorem

**Theorem (Smale, 1960 and Freedman, 1983)**

*Let $(W^{n+1}; M^n, N^n)$ be an h-cobordism: that is a cobordism from $M$ to $N$ with $M \hookrightarrow W$ and $N \hookrightarrow W$ homotopy equivalences, and all manifolds simply-connected. If $n \geq 4$ then $W$ is homeomorphic to $M \times [0, 1]$.*

Smale proved this for diffeomorphisms for $n \geq 5$; Freedman added the case $n = 4$ in the topological category.
The crucial step in the proof of this is getting the Whitney Trick to work.

# Proof of Characterisation

We continue with the proof: we want to show that our homotopy 4-ball $B$ is homeomorphic to $D^4$. Attach $B \cup_{S^3} D^4$.
This is a homotopy $S^4$, which by the h-cobordism theorem is homeomorphic to $S^4$. We can then remove the image of $D^4$ to get a homeomorphism of $B$ to $D^4$ so that $K$ is slice in $D^4$.

# Seifert Form Metaboliser

Now we know $\mathbb{Z} \cong H_1(W;\mathbb{Z}) \cong H^1(W;\mathbb{Z}) \cong [W, S^1]$, pick a generator $f\colon W \to S^1$.

$f^{-1}(\{1\}) = N$, a 3-manifold with $\partial N = F \cup_K \Delta \subset \partial W = M_K$.

**Theorem**

$$P := \ker(H_1(F \cup \Delta;\mathbb{Q}) \to H_1(N;\mathbb{Q}))$$

*is a metaboliser for the Seifert form i.e.* $P = P^{\perp}$, $P$ *is a half-rank direct summand such that* $V(x,y) = 0$ *for all* $x, y \in H_1(F \cup \Delta;\mathbb{Q})$.

Use rational coefficients to remove problems with potential torsion in $H_1(N;\mathbb{Z})$, or use:

$$Q := \{x \in H_1(F \cup \Delta;\mathbb{Z}) \,|\, nx \in P \text{ for some } n \in \mathbb{Z} \setminus \{0\}\}.$$

# Seifert Form Metaboliser

**Theorem**

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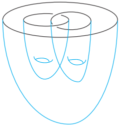
Idea of proof:

Main Idea: Linking in $S^3$ = Intersections of bounding surfaces in $D^4$.



Now, moving $x \to x^+$ moves a bounding surface off $N$: whence no intersections in $D^4$.

# Algebraically Slice

**Definition**

A knot $K$ is said to be algebraically slice if there is a Seifert Surface $F$, and a basis for $H_1(F;\mathbb{Z})$ such that $V$ is represented by a matrix of the form:

$$V = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$$

for some block matrices $A, B, C$, with $C = C^T$ and $A - B^T$ invertible. A matrix of this form is called a null-concordant matrix.

Every such Seifert form is realised as the Seifert form of some knot.

# Algebraically Slice

We have just shown that $K$ slice $\Rightarrow$ $K$ is algebraically slice.

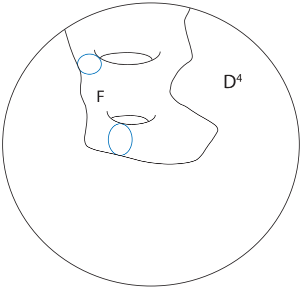
We can use the algebraic condition to obstruct slice-ness. e.g. for $\omega \in S^1 \subset \mathbb{C}$, calculate the $\omega$-*signature*:

$$\sigma((1-\omega)V + (1-\overline{\omega})V^T).$$

For each $\omega$ not a root of the Alexander polynomial $\Delta(t) := \det(tV - V^T)$ this gives a well-defined homomorphism $\mathcal{C} \to \mathbb{Z}$.
We can show using this that many of the Twist Knots are not slice: all those with $4n+1$ not a square number.
But the rest are algebraically slice, and we need some "higher order" obstructions.

# Ambient Surgery

Take $F \subset S^3$ and push it into $D^4$:



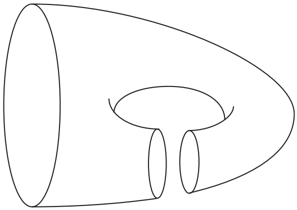

$P =$ half basis of curves with zero linking numbers.
Hope: they bound disjointly embedded framed discs in $D^4$;then we can surger surface $F$ to a disc $D^2 = \Delta$ *ambiently*.

## Definition

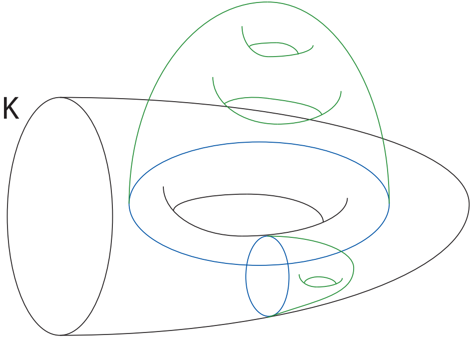
Let $g \colon S^r \times D^{n-r} \hookrightarrow M^n$ be a framed embedding of a sphere in an $n$-dimensional manifold. The effect of surgery along $g$ is:

$$M' := \mathrm{cl}(M \setminus g(S^r \times D^{n-r})) \cup_{S^r \times S^{n-r-1}} D^{r+1} \times S^{n-r-1}.$$

Here we would like to apply this to $F \subset D^4$ with $n = 2$ and $r = 1$, with the embedding of $D^2 \times S^0$ at the end also required to be embedded in $D^4$.



In general however, the curves will not bound discs, merely surfaces. This motivates the construction of a grope:




This is a grope of height 2, shown with a boring embedding.

# Gropes

The question then iterates: the algebraic slice condition obstructs the existence of zero-linking curves on the Seifert surface, which obstructs the existence of the second stage surfaces.

Once the second stage surfaces exist, we can then look for zero-linking curves on *these* surfaces. We then ask whether these bound discs, or perhaps just surfaces, and so on.

**Definition**

We say that a knot $K$ is $(n)$-grope solvable if $K$ bounds a framed grope of height $n$ in $D^4$; $n \in \frac{1}{2}\mathbb{N} \cup \{0\}$.
Half-integers correspond to just a half-basis of curves bounding surfaces on the penultimate level.

# Obstructions

We need a second level obstruction theory to show that the algebraically slice twist knots are not slice.
The first example of this was the work of Casson and Gordon.

Problem: we would have to investigate all possible gropes of a certain height.
We need an approach which does not depend on such choices so we can have an algebraic obstruction.

# Aside: High Dimensional Knots

An $n$-knot is an embedding $S^n \subset S^{n+2}$.

**Theorem (Kervaire, 1965 and Levine, 1969)**

*An even dimensional knot is always slice. An odd dimensional knot with $n \geq 3$ is slice if and only if it is algebraically slice.*
*The group of high, odd dimensional knots is therefore isomorphic to the group of Seifert matrices, with block sum as the addition, modulo null-concordant matrices, which is itself isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$.*

Kervaire proved this for even-dimensional knots and Levine did the odd-dimensional case.

Why?

• The dimension count works so that the Whitney trick works: the surgeries can be embedded disks, so that the geometry corresponds to the algebra.

• Every high-dimensional knot is concordant to a knot with $\pi_1(S^{n+2} \setminus K) \cong \mathbb{Z}$, so the algebra is more manageable.

• By contrast, in the classical case of $S^1 \subset S^3$ there is no avoiding the fundamental group - by Dehn's Lemma the unknot is detected by $\pi_1(X) \cong \mathbb{Z}$.

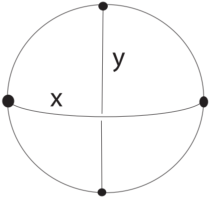
We now define some intrinsic algebraic invariants: linking forms.

Let $M^3$ be a 3-manifold with $H_1(M;\mathbb{Z})$ finite. We define:

$$L\colon H_1(M) \times H_1(M) \to \mathbb{Q}/\mathbb{Z}; \text{ by: } (x,y) \mapsto \frac{z \cdot y}{n}$$

where $\partial z = nx$ for some $n \in \mathbb{Z} \setminus \{0\}$.
e.g. $M = \mathbb{RP}^3$; $H_1(\mathbb{RP}^3) \cong \mathbb{Z}_2$

$x = y \neq 0$ in $H_1(\mathbb{RP}^3)$. Consider $\mathbb{RP}^3$ as $\dfrac{D^3}{a \sim -a \,|\, a \in \partial D^3}$




$L(x,y) = \frac{1}{2}$.

# Knot Exterior and Zero-Surgery

We recall the notation:

**Definition**

We define the knot exterior $X$:

$$X := \mathrm{cl}(S^3 \setminus (K \times D^2)),$$

and the zero-surgery:

$$\begin{aligned} M_K &:= \text{0-framed surgery on } K \subset S^3 \\ &= \mathrm{cl}(S^3 \setminus (K \times D^2)) \cup_{S^1 \times S^1} D^2 \times S^1 \\ &= X \cup D^2 \times S^1 \end{aligned}$$

# The Blanchfield Form

$$\begin{array}{ccc} \widetilde{M_K} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ M_K & \longrightarrow & S^1 \end{array}$$

Infinite cyclic covering of $M_K$.

The rational Alexander module $H_1(\widetilde{M_K};\mathbb{Q}) \cong H_1(M_K;\mathbb{Q}[\mathbb{Z}])$ (using $\rho\colon \pi_1(M_K) \to \mathbb{Z}$), as a module over the ring of Laurent polynomials, $\mathbb{Q}[t,t^{-1}] = \mathbb{Q}[\mathbb{Z}]$, is torsion.

Using the quotient field $\mathbb{Q}(\mathbb{Z})$ of $\mathbb{Q}[\mathbb{Z}]$, there is a linking pairing analogous to the $\mathbb{Q}/\mathbb{Z}$ form just described:

$$\mathrm{Bl}\colon H_1(M_K;\mathbb{Q}[\mathbb{Z}]) \times H_1(M_K;\mathbb{Q}[\mathbb{Z}]) \to \frac{\mathbb{Q}(\mathbb{Z})}{\mathbb{Q}[\mathbb{Z}]};$$

$$\mathrm{Bl}(x,y) = \frac{\sum_{i=-\infty}^{\infty} \langle z \cdot t^i y \rangle t^i}{\Delta(t)} \mod \mathbb{Q}[\mathbb{Z}];$$

where $\partial z = \Delta(t)x$ for some $\Delta(t) \in \mathbb{Q}[\mathbb{Z}]$.

# The Blanchfield Form

Advantages of the Blanchfield form: it does not depend on a Seifert surface, and it can be defined for more interesting covering spaces, i.e. those with a larger group of deck transformations than $\mathbb{Z}$, if we have the group ring of the deck transformations yields coefficients which have a suitable non-commutative localisation.

# The Blanchfield Form

The Blanchfield form is non-singular. To see this, it is defined using the following isomorphisms:

$$H_1(M_K;\mathbb{Q}[\mathbb{Z}]) \xrightarrow{\simeq} H^2(M_K;\mathbb{Q}[\mathbb{Z}]) \xrightarrow{\simeq} H^1(M_K;\mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}])$$

$$\xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{Q}[\mathbb{Z}]}(H_1(M_K;\mathbb{Q}[\mathbb{Z}]),\mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}])$$

given by Poincaré Duality, the Bockstein connecting homomorphism associated to the short exact sequence

$$0 \to \mathbb{Q}[\mathbb{Z}] \to \mathbb{Q}(\mathbb{Z}) \to \frac{\mathbb{Q}(\mathbb{Z})}{\mathbb{Q}[\mathbb{Z}]} \to 0,$$

and a Universal coefficient isomorphism ($\mathbb{Q}[\mathbb{Z}]$ is a PID).

So the Blanchfield form is indeed non-singular.

# The Blanchfield Form

**Theorem**

*Let $K$ be a slice knot. Then* $P := \ker(H_1(M_K;\mathbb{Q}[\mathbb{Z}]) \to H_1(W;\mathbb{Q}[\mathbb{Z}]))$ *is a metaboliser of the Blanchfield form; i.e.*

$$P = P^{\perp} = \{x \in H_1(M_K;\mathbb{Q}[\mathbb{Z}]) \,|\, \mathrm{Bl}(x,y) = 0 \textit{ for all } y \in P.\}$$

Proof: $H_*(W;\mathbb{Z}) \cong H_*(S^1;\mathbb{Z})$. This implies that $H_i(W;\mathbb{Q}(\mathbb{Z})) \cong 0$, for $i = 1, 2$, or that $H_i(W;\mathbb{Q}[\mathbb{Z}])$ is torsion, since $\mathbb{Q}(\mathbb{Z})$ is flat.$\mathbb{Q}[\mathbb{Z}]$ is a PID and $\mathbb{Z}$ is a torsion-free abelian group: modules over $\mathbb{Q}[\mathbb{Z}]$ are a sum of torsion and free parts. Free parts would map to non-zero homology in $H_i(W;\mathbb{Q})$ under augmentation on the chain level.

# The Blanchfield Form

For the rest of this proof coefficients are understood to be in $\mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$

$$H_2(W) \to H_2(W, M_K) \to H_1(M_K)$$

Since the outer two are torsion so is the middle one. There are therefore similarly defined, non-singular, relative linking pairings:

$$\beta\colon H_2(W, M_K) \times H_1(W) \to \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]$$

We will show that $P \subseteq P^{\perp}$. In what follows we use shorthand: $\bullet^{\wedge} = \mathrm{Hom}_{\mathbb{Q}[\mathbb{Z}]}(\bullet, \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}])$.

$$\begin{array}{ccccc} H_2(W, M_K) & \xrightarrow{\partial} & H_1(M_K) & \xrightarrow{i_*} & H_1(W) \\ \downarrow \beta & & \downarrow \mathrm{Bl} & & \downarrow \beta \\ H_1(W)^{\wedge} & \xrightarrow[i^{\wedge}]{} & H_1(M_K)^{\wedge} & \longrightarrow & H_2(W, M_K)^{\wedge} \end{array}$$

The vertical maps are all isomorphisms.
Let $P = \ker i_*$. Let $x \in P$, then $x = \partial y$. $Bl(x) = i^{\wedge}(\beta)(y)$ so for $p \in P$, $Bl(x)(p) = i^{\wedge}(\beta)(y)(p) = \beta(y)(i_*(p)) = \beta(y)(0) = 0$. So $P \subseteq P^{\perp}$ as claimed.
The other inclusion is also a consequence of duality and non-singularity but we skip it.

# The Blanchfield Form

As was shown in Levine's Knot Modules paper, this all works for $\mathbb{Z}[\mathbb{Z}]$ coefficients rather than $\mathbb{Q}[\mathbb{Z}]$, but the arguments are harder and do not generalise easily to other covering spaces.

**Theorem**

*A knot $K$ is algebraically slice iff its integral Blanchfield form is metabolic iff it bounds a grope of height 2.5 in $D^4$.*

So far we have used abelian representations, associated with the infinite cyclic or universal abelian cover of the zero-surgery.

$$\rho\colon \pi_1(M_K) \to \frac{\pi_1(M_K)}{\pi_1(M_K)^{(1)}} \cong \mathbb{Z} =: \Gamma_0$$

We obtain higher order invariants by considering the homology of metabelian covers. More discerning coefficients see higher order information.

# Metabelian Representations

**Definition**

For any group $G$ define the derived series as iterated commutator subgroups.

$$G^{(0)} := G; \;\; G^{(n)} := [G^{(n-1)}, G^{(n-1)}].$$

A meridian of a band of a Seifert surface lives in $\pi_1^{(1)}(M_K)$, so we would expect to need at least metabelian representations to detect knotting in the bands of the Seifert surface.

# Metabelian Representations

$$\rho\colon \pi_1(M_K) \to \frac{\pi_1(M_K)}{\pi_1(M_K)^{(2)}} \cong \frac{\pi_1(M_K)}{\pi_1(M_K)^{(1)}} \ltimes \frac{\pi_1(M_K)^{(1)}}{\pi_1(M_K)^{(2)}} \cong \mathbb{Z} \ltimes H_1(M_K;\mathbb{Z}[\mathbb{Z}])$$

$$\to \mathbb{Z} \ltimes H_1(M_K;\mathbb{Q}[\mathbb{Z}]) \xrightarrow{\mathrm{Bl}(p,\bullet)} \mathbb{Z} \ltimes \frac{\mathbb{Q}(\mathbb{Z})}{\mathbb{Q}[\mathbb{Z}]} =: \Gamma_1$$

with a choice of $p \in H_1(M_K;\mathbb{Q}[\mathbb{Z}])$.
Suppose that $p \in P$ - a metaboliser for Bl and $\ker i_*$ - then $\rho$ extends over $W$: a putative slice disc complement (replace $M_K$ by $W$ and Bl by $\beta$, $p \in H_2(W, M;\mathbb{Q}[\mathbb{Z}])$ in the above).

$$\begin{array}{ccc} \pi_1(M) & \xrightarrow{\rho} & \mathbb{Z} \ltimes \frac{\mathbb{Q}(\mathbb{Z})}{\mathbb{Q}[\mathbb{Z}]} \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle =} \\ \pi_1(W) & \xrightarrow{\widetilde{\rho}} & \mathbb{Z} \ltimes \frac{\mathbb{Q}(\mathbb{Z})}{\mathbb{Q}[\mathbb{Z}]} \end{array}$$

# Concordance Obstructions

We can now form the homology groups $H_*(W; \mathbb{Z}\Gamma_1)$ for a 4-manifold $W$ with $P$ a metaboliser for the Blanchfield form, which we hope to change to a slice disc complement.

Key Idea: We work backwards in a sense; we start with a slice disc: $X \cup D^2 \times S^1$, and a 4-manifold $W$ with the right $H_1$.

We then seek to calculate obstructions to doing surgery on $W$ to kill $H_2(W; \mathbb{Z})$: we would then have a slice disc complement.

However, 2-dimensional homology is typically represented by immersed spheres, or by embedded surfaces, but not by embedded spheres.

We look for algebraic obstructions: the surfaces can look like spheres to the intersection form on

$$H_2(W;\mathbb{Z}[\pi_1(W)/\pi_1(W)^{(2)}]) \xrightarrow{\mathbb{Z}\Gamma_1 \otimes_{\widetilde{\rho}}} H_2(W;\mathbb{Z}\Gamma_1)$$

This is good because we have little idea what $\pi_1(W)$ might be, since the 4-manifold is variable, but $\Gamma_1$ is fixed.

# Localisation

**Theorem**

$\Gamma_1$ *satisfies the Ore condition:*

$$\forall\, r, s \in \Gamma_1, s \neq 0, \exists a, b \text{ such that } rb = sa,\ b \neq 0$$

*We can therefore define non-commutative fractions of elements of* $\mathbb{Z}\Gamma_1$ *using the equivalence relation* $s\backslash r \sim a/b$*; to obtain the skew field* $\mathcal{K} = (\mathbb{Z}\Gamma_1 - \{0\})^{-1}\mathbb{Z}\Gamma_1$*, which inverts formally all non-zero elements of* $\mathbb{Z}\Gamma_1$*.*

# Localisation

Then $H_2(W;\mathcal{K})$ is a free module, and since $H_*(M_K;\mathcal{K}) \cong 0$, the intersection form

$$\lambda\colon H_2(W;\mathcal{K}) \times H_2(W;\mathcal{K}) \to \mathcal{K}$$

is non-singular.

Localising coefficients is like killing the boundary; the boundary constitutes the failure of the non-localised intersection form to be non-singular.
We want to be able to detect the algebraic obstruction given by the Witt class of this intersection form.

# $L^2$-signatures

## Theorem

*For a 3-manifold $M$ with a representation $\phi\colon \pi_1(M) \to \Gamma$ of its fundamental group, there is an invariant $\widetilde{\sigma}^{(2)}(M, \phi) \in \mathbb{R}$, which detects the Witt class of the intersection form over $\mathcal{K}$ of a 4-manifold $W$ with boundary $M$, over which $\phi$ extends. Moreover it is independent of the choice of 4-manifold.*
*If there is a subgroup $\langle m \rangle \cong \mathbb{Z} \leq \Gamma$ such that the intersection form $\lambda(m)$ only depends on elements in this subgroup, then*

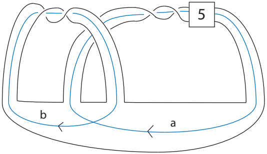
$$\widetilde{\sigma}^{(2)}(M, \phi) = \int_{\omega \in S^1 \subset \mathbb{C}} \sigma(\lambda(\omega)) \in \mathbb{R}$$

*with the $\sigma$ in the integrand denoting the usual $\mathbb{Z}$-valued signature of a Hermitian operator on $\mathbb{C}^n$.*

# Explicit 4-manifolds

It is unfortunately too much for this talk to expand further on the beautiful theory of $L^2$-signatures. We shall use their independence of the choice of 4-manifold to make calculations using a specific 4-manifold.

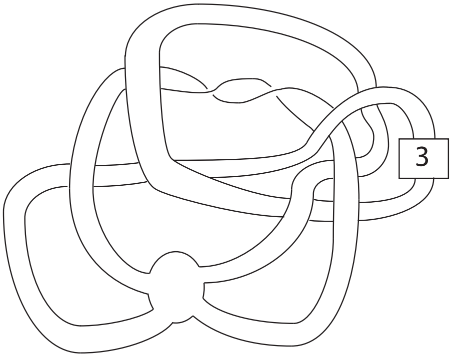
We do an example. Let $K$ be the twist knot with $n = 6$.




The curve $(-1, 2)$ has zero self-linking and so generates a metaboliser for the Seifert form:

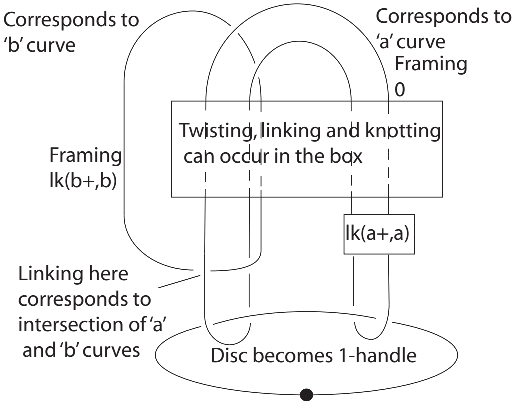
$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ -2 & -1 \end{pmatrix}$$

The knot and the Seifert surface can be isotoped so that the zero-linking curve is one of the bands.



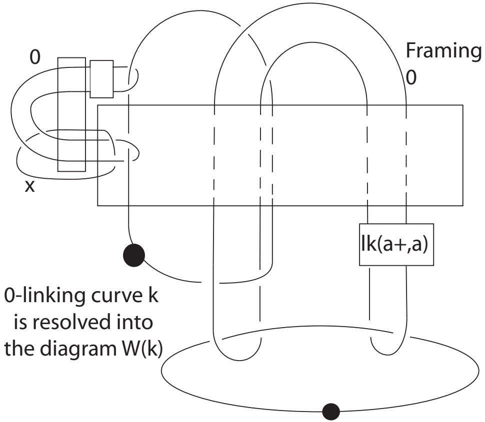

Note that the zero linking curve is knotted as a trefoil.

Given a Seifert surface of a genus one knot $J$, here is a picture of a 4-manifold, $W(J)$, using the Kirby notation for handle attaching.



# Explicit 4-manifolds

To obstruct the algebraically slice twist knots from being slice we need the intersection form of $W_0$ to measure the knotting of the zero linking curve on $F$. For this, we modify $W(K)$, in a construction reminiscent of gropes, to obtain $W_0$:



$W_0$ has the following properties.

- $\partial W_0 = M_K$; This can be seen with some handle slides and surgering 1-handles to 2-handles.

- $H_1(M_K;\mathbb{Q}[\mathbb{Z}]) \cong \frac{\mathbb{Q}[t,t^{-1}]}{(13-6(t+t^{-1}))}$.
  $13 - 6t - 6t^{-1} = (3t-2)(3t^{-1}-2)$,so metabolisers for the Blanchfield form are all multiples of $3t-2$, or of $3t^{-1}-2$. These correspond to zero linking curves on the Seifert surface for $K$, which give metabolisers for the Seifert form. These homology classes must vanish in $W_0$ so that the representation $\rho$ extends to $\pi_1(W)$.

# Explicit 4-manifolds

- $H_2(W_0; \mathbb{Z}\Gamma)$ is generated by the 2-handles which correspond to the Seifert surface for the zero linking curve.
- The intersection form on $H_2(W_0; \mathbb{Z}\Gamma)$ is given by the matrix:

$$h := \begin{pmatrix} 1 & -1 \\ -1 & (m-1)(m^{-1}-1) \end{pmatrix}$$

which, integrating the signatures for $m = \omega \in S^1$, gives rise to the $L^2$-signature $-\frac{4}{3} \neq 0$, which is the same as the integral around $S^1$ of the $\omega$-signatures of the trefoil knot; $h$ is congruent to $(1-m)V' + (1-m^{-1})V'^T$, where $V'$ is the Seifert matrix of the trefoil.

# Final Obstruction Theorem

It turns out then, that the integral of the classical $\omega$-signatures of the zero linking curve on the Seifert surface provides an obstruction to slicing the knot.

**Theorem (Cochran-Orr-Teichner, 2003)**

*Let $K$ be a slice knot with a genus 1 Seifert surface $F$. Then there exists a homologically essential simple closed curve $J \subset F$ with $\mathrm{lk}(J, J) = 0$ and*

$$\int_{\omega \in S^1 \subset \mathbb{C}} \sigma_\omega(J) = 0.$$

For all the algebraically slice twist knots, the zero linking curves are torus knots (of which the trefoil is a special case), which all have the integral above non-zero, completing the COT proof of the Casson-Gordon result.