Isotopy classes of diffeomorphisms of (k-1)-connected almost-parallelizable 2k-manifolds

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§ 1 Results

The group of isotopy classes of orientation preserving diffeomorphisms on a closed oriented differentiable manifold M is denoted by $\pi_0 \mathrm{Diff}(M)$; the group of pseudo isotopy classes is denoted by $\tilde{\pi}_0 \mathrm{Diff}(M)$. In this paper we will compute $\pi_0 \mathrm{Diff}(M)$ for M a closed differentiable (k-1)-connected almost-parallelizable 2k-manifold in terms of exact sequences for $k \geq 3$, and classify elements in $\tilde{\pi}_0 \mathrm{Diff}(M)$ for any simply-connected closed differentiable 4-manifold.

In the following M stands for a closed differentiable (k-1)-connected almost-parallelizable 2k-manifold if $k \geq 3$ and a simply-connected manifold if $k=2$. To describe our results we need some invariants. We denote by Aut $H_k(M)$ the group of automorphisms of $H_k(M) := H_k(M;\mathbb{Z})$ preserving the intersection form on M and (for $k \geq 3$) commuting with the function $\alpha : H_k(M) \longrightarrow \pi_{k-1}(SO(k))$, which assigns to $x \in H_k(M)$ the classifying map of the normal bundle of an embedded sphere representing x. As the induced map in homology of any orientation preserving diffeomorphism lies in Aut $H_k(M)$, we obtain a homomorphism

$$\pi_0 \mathrm{Diff}(M) \longrightarrow \mathrm{Aut}\, H_k(M), \; [f] \longmapsto f_*.$$

We denote the kernel of this map by $\pi_0 S\,\mathrm{Diff}(M)$.

Our next invariant is defined for elements [f] in $\pi_0 S\,\mathrm{Diff}(M)$. It assigns to [f] a homomorphism $H_k(M) \longrightarrow S\pi_k(SO(k))$, where S is the map $\pi_k(SO(k)) \longrightarrow \pi_k(SO(k+1))$ induced by inclusion. If $x \in H_k(M)$ is represented by an embedded sphere $S^k \subset M$ we can assume that $f|_{S^k} = \mathrm{Id}$. As the stable normal bundle of S^k in M is trivial the operation of f on $\nu(S^k) \oplus 1$ given by the differential of f corresponds to an element of $\pi_k SO(k+1)$. It is obvious that this element lies in the image of $\pi_k(SO(k)) \longrightarrow \pi_k(SO(k+1))$.

[+]This work was begun in Bonn in 1976 and was partially supported by the Sonderforschungsbereich (SFB 4o). It was finished during a stay in Aarhus (Denmark) in 1978. I would like to thank the University of Aarhus for the invitation and the stimulating atmosphere there.

Lemma 1: The construction above leads to a well defined homomorphism

$$\chi : \pi_0 S\,\mathrm{Diff}(M) \longrightarrow \mathrm{Hom}(H_k(M),\, S\pi_k(SO(k))).$$

The proof of this Lemma for $k>3$ is contained in the papers of Wall ([19]; [20], Lemma 2 3), the case k=3 follows from Lemma 2 below. I want to repeat here the warning of Wall that it is not obvious that χ and similar invariants are well defined. The difficult point is to show that $\chi(f)$ depends only on the isotopy class of f.

From the work of Kervaire ([5]) one can easily deduce the following list for $S\pi_k(SO(k))$ for $k>2$ and $k \neq 6$:

k mod 8	0	1	2	3	4	5	6	7
$S\pi_k(SO(k))$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$

For k=6 we have $S\pi_6(SO(6)) = 0$.

Thus, for k=3 mod 4 we can identify $\mathrm{Hom}(H_k(M), S\pi_k(SO(k))) = \mathrm{Hom}(H_k(M),\mathbb{Z})$ with $H^k(M)$. In this case we can describe $\chi(f)$ by an invariant defined by Browder using the Pontrjagin class of the mapping torus $M_f = I\times M_{(o,x)\sim(1,f(x))}$ ([2]). The definition is as follows. We consider the map $c : M_f \longrightarrow M_{f/\{0\}\times M} = \Sigma M^+$ From the Wang sequence we know that $i^* : H^k(M_f) \longrightarrow H^k(M)$ is surjective, if $f_* = \mathrm{Id}$. Thus we obtain an isomorphism $c^* : H^{k+1}(\Sigma M^+) \longrightarrow H^{k+1}(M_f)$. The invariant $p'(f) \in H^k(M)$ is defined as the image of the inverse suspension isomorphism applied to $c^{*-1}(p_{(k+1)/4}(M(f))$. It is not difficult to see that $f \longmapsto p'(f)$ is a homomorphism. It is related to $\chi(f)$ in the following way.

If $x \in H_k(M)$ is represented by an embedded sphere S^k and $f|_{S^k} = \mathrm{Id}$ then $S^1\times S^k/_{\{1\}\times S^k}$ represents the image of x in $H_{k+1}(\Sigma M^+)$ under the suspension isomorphism. We denote it by y. Now we consider the stable vector bundle E over $S^1\times S^k/_{\{1\}\times S^k}$ classified by $\chi(f)(x)$. By the classification of vector bundles over spheres we know that the Kronecker product $\langle p(E), [S^1\times S^k/_{\{1\}} \times S^k]\rangle = \pm a_{(k+1)/4}((k-1)/2)!\,\chi(f)(x)$, where $a_m = 2$ for m odd and 1 for m even. But it is obvious that $(c|_{S^1\times S^k})^*(E)$ is equal to the restriction of the stable tangent bundle of M_f to $S^1\times S^k$. Thus $\langle p'(f), y\rangle = \pm a_{(k+1)/4}((k-1)/2)!\,\chi(f)(x)$. This implies:

Lemma 2: If k=3 mod 4

$$p'(f) = \pm a_{(k+1)/4}((k-1)/2)!\,\chi(f),$$

where $a_m = 2$ for m odd and 1 for m even.

Now we are ready to formulate our results.

Theorem 1: For a simply-connected closed differentiable 4-manifold the homomorphism

$$\tilde{\pi}_0 \mathrm{Diff}(M) \longrightarrow \mathrm{Aut}\, H_2(M), \; [f] \longmapsto f_*$$

is injective.

Remarks:

1) This result is completely analogous to the classification of isotopy classes of diffeomorphisms of an oriented connected surface by the operation of the diffeomorphism on the fundamental group.

2) It seems very hard to determine the image of the map $\tilde{\pi}_0 \mathrm{Diff}(M) \longrightarrow \mathrm{Aut}\, H_2(M)$. Wall has shown that it is surjective if M is of the form $M = N \# S^2 \times S^2$ and the intersection form is indefinite or has rank ≤ 8 ([18]).

In the following examples it is obvious that the map is surjective and we obtain the following results:

$$\tilde{\pi}_0 \mathrm{Diff}(S^2 \times S^2) = \mathbb{Z}_4$$

$$\tilde{\pi}_0 \mathrm{Diff}(P_2\mathbb{C} \# \overline{P_2\mathbb{C}}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$\tilde{\pi}_0 \mathrm{Diff}(\underbrace{P_2\mathbb{C} \# \dots \# P_2\mathbb{C}}_{k}) = O(k;\mathbb{Z}).$$

$O(k;\mathbb{Z})$ is the group of matrices with exactly one coefficient ± 1 in each row and column and zero otherwise. It is the group of automorphisms of a k-dimensional cube. For instance: $\tilde{\pi}_0 \mathrm{Diff}(P_2\mathbb{C}) = \mathbb{Z}_2$; $\tilde{\pi}_0 \mathrm{Diff}(P_2\mathbb{C} \# P_2\mathbb{C}) = D_8$, the dihedral group.

$$\tilde{\pi}_0 \mathrm{Diff}(\underbrace{P_2\mathbb{C} \# \dots \# P_2\mathbb{C}}_{k} \# \underbrace{\overline{P_2\mathbb{C}} \# \dots \# \overline{P_2\mathbb{C}}}_{l}) \subset O(k,l;\mathbb{Z})$$

the group of integer matrices preserving the form $\begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix}$.

Theorem 2: $k \geq 3$. The following sequences are exact:

$$0 \longrightarrow \pi_0 S\,\mathrm{Diff}(M) \longrightarrow \pi_0 \mathrm{Diff}(M) \longrightarrow \mathrm{Aut}\, H_k(M) \longrightarrow 0$$

$$0 \longrightarrow \Theta_{2k+1}/\Sigma_M \longrightarrow \pi_0 S\,\mathrm{Diff}(M) \xrightarrow{\chi} \mathrm{Hom}(H_k(M), S\pi_k(SO(k))) \longrightarrow 0$$

Σ_M is an element in the group of (2k+1)-dimensional homotopy spheres Θ_{2k+1} of order 2 and depending only on M.

Σ_M can be described as follows. We consider the embedding of $S^1 \times D^{2k}$ into $S^1 \times M$ obtained from a product embedding twisted by the nontrivial element of $\pi_1 SO(2k) = \mathbb{Z}_2$. Then we replace $S^1 \times M$ by a homotopy sphere Σ_M by a sequence of surgeries first killing the fundamental group with this embedding and then killing the k-th homotopy by arbitrary surgeries.

The map $\Theta_{2k+1}/\Sigma_M \longrightarrow \pi_0 S\,\mathrm{Diff}(M)$ is induced by the following map $\Theta_{2k+1} \to \pi_0 S\mathrm{Diff}(M)$. Consider $\Sigma \in \Theta_{2k+1}$ as $D^{2k+1} \cup_f D^{2k+1}$ and assume that f is the identity on a neighbourhood of the lower hemisphere $D^{2k}_- \subset S^{2k}$. Then we get an element of $\pi_0 S\,\mathrm{Diff}(M)$ by the diffeomorphism on M which is the identity outside an embedded disk in M and is equal to $f|_{D^{2k}_+}$ on this disk.

Part of this result is contained in the work of Wall ([19], [20]) where he computes the group of pseudo isotopy classes of diffeomorphisms of M minus an open embedded disk. Complete results were known for the case $M = S^k \times S^k$ and $k \geq 4$ (compare [17]) and for homotopy spheres. Then Σ_M coincides with the $\gamma(M)$ of [21].

To complete our computation we have to determine Σ_M. First, I state some properties of Σ_M.

Lemma 3:

a) $\Sigma_{M \# N} = \Sigma_M + \Sigma_N$

b) If M bounds a framed manifold then $\Sigma_M = 0$

c) If M is a homotpy sphere we get Σ_M from the Milnor-Munkres-Novikov pairing $\Theta_{2k} \times \pi_1(SO) \to \Theta_{2k+1}$ as the image of (M, η) where $\eta \in \pi_1(SO)$ is the nontrivial element.

Σ_M is closely related to the following diffeomorphism on M. We consider an embedding of $2\cdot D^{2k}$ into M and a differentiable map $\alpha : [1,2] \longrightarrow SO(2k)$ which maps a small neighbourhood of the boundary to the identity matrix and represents the nontrivial element in $\pi_1(SO(2k))$. Then we get a diffeomorphism f_α of M by taking the identity on D^{2k} and outside $2\cdot D^{2k}$ and by mapping $x \in 2\cdot D^{2k} \setminus D^{2k}$ to $\alpha(|x|)\cdot x$.

Lemma 4: $\Sigma_M = 0 \Longleftrightarrow f_\alpha$ is isotopic to the identity rel. D^{2k}.

To formulate our main result about Σ_M we have to distinguish between the case where M can be framed and the case where it cannot. Under our assumptions M can automatically be framed if $k \neq 0 \bmod 4$ and in the case $k = 0 \bmod 4$ it can be framed if and only if the signature $\tau(M)$ vanishes.

We identify a framed manifold $[M,\beta] \in \Omega_{2k}^{fr}$ by the Pontrjagin-Thom construction with the corresponding element in π_{2k}^s. We denote the map $\Theta_n \longrightarrow \operatorname{cok} J_n$ by T ([6]) and the projection map $\pi_n^s \longrightarrow \operatorname{cok} J_n$ by P.

Theorem 3:

a) If M is an s-parallelizable manifold then

$T(\Sigma_M) = P(\eta \circ [M,\beta])$

where β is any framing on M. $\circ$ denotes the composition map in the stable homotopy groups.

b) If $\tau(M^{4n}) = s\,\sigma_n, s \neq 0$, where $\sigma_n/8$ is the order of bP_{4n}, then

$T(\Sigma_M) \subset P([s\varrho, 2, \eta])$

where ϱ is the element of order 2 in im J_{4n-1} and $[s\varrho, 2, \eta]$ denotes the Toda bracket.

c) If M is an s-parallelizable manifold then

$$\Sigma_M = 0 \Longleftrightarrow \begin{cases} \text{there exists a framing } \beta \text{ on M such that} & \\ \eta \circ [M,\beta] = 0 & k \text{ odd or } k \text{ even an } bP_{2k+2} = 0 \,; \\ \text{there exists a framing } \beta \text{ on M such that } \eta \circ [M,\beta] \in \text{im } J & \\ \text{and an invariant } \alpha(M) \in \mathbb{Z}_2 \text{ vanishes} & k \text{ even and } bP_{2k+2} \neq 0 \end{cases}$$

$\alpha(M)$ is defined as the Arf invariant of Σ_M. It is only defined if the first condition is fulfilled. For then we will show that $\Sigma_M \in bP_{2k+2}$.
Especially it follows that for k odd $\Sigma_M = 0 \Leftrightarrow \Sigma_M \in bP_{2K+2}$. This extends a result of Levine ([21], Prop.8).

Remark: I have no example where $\mathfrak{a}(M) \neq 0$. Thus it may be that the condition $\mathfrak{a}(M) = 0$ can be omitted.

Now, we will discuss some consequences of our theorems. First we will give some examples where Σ_M is nonzero. In the case of stably parallelizable manifolds we can use Toda's tables ([15]) to get complete information about Σ_M in low dimensions. As $\eta\circ\mu_{8k+2} \neq 0$ ([1]) we get, furthermore, a series of examples in higher dimensions. This completes the computations of ([21], 16).

Corollary 1: Notations as in Toda's tables ([15]). If M is a framed manifold which represents one of the following elements in π_*^S then Σ_M is nonzero

$$\bar{\nu}, \varepsilon; \eta\circ\mu, \eta\circ\mu + \beta_1; \kappa, \kappa + \sigma^2; \eta^*, \eta^* + \eta\rho; \eta\circ\bar{\mu}, \eta\circ\bar{\mu} + \nu^*$$

For all other framed manifolds of dim ≤ 18 Σ_M is zero.
If M^{8k+2} is a framed manifold representing μ_{8k+2} then Σ_M is nonzero.

In the case of non s-parallelizable manifolds we get examples of M with nonzero Σ_M in dim 8k. For Adams has proved that $e_c\,[\rho, 2, \eta]$ is nonzero for all elements of this Toda bracket, where $\rho \in J_{8k-1}$ is the element of order 2 ([1], 11.1). But ([1], 7.19) implies that no element of $[\rho, 2, \eta]$ is contained in im J_{8k+1}. Thus $\Sigma_M \neq 0$, if M^{8k} has signature $(2r+1)\,\sigma_{2k}$.

Corollary 2: If the signature of M^{8k} is an odd multiple of σ_{2k} then

$$\Sigma_M \neq 0.$$

From these examples we can see that in most cases $\pi_0 \mathrm{Diff}(M)$ depends on the differentiable structure on M. This was known in some dimensions for a sphere ([11]). But our examples show that this is the case for all highly connected s-parallelizable 8k+2-dim manifolds. For if M is such a manifold with $\Sigma_M = 0$ then we can change the differentiable structure on M by replacing M by the connected sum of M with a framed homotopy sphere representing μ_{8k+2}. By Lemma 3 and Corollary 1 we know that for M with this differentiable structure Σ_M is nonzero. Thus $\pi_0 \mathrm{Diff}(M)$ has changed. On the other hand on every M there exists a differentiable structure such that $\Sigma_M = 0$. For if Σ_M is nonzero we know that M is framed bordant to a homotopy sphere N. By Lemma 3 we know that $\Sigma_M = \Sigma_N$ and that $\Sigma_{M \# (-N)} = 0$.

Corollary 3: For every highly connected s-parallelizable 8k+2-manifold M the group $\pi_0 \mathrm{Diff}(M)$ depends on the differentiable structure on M.

The proofs of our results are very much in the spirit of Kervaire-Milnor's work on homotopy spheres and are based on direct surgery arguments. They make no use of the general machinery of surgery as developed by Browder, Novikov, Sullivan, Wall. This machinery leads to very interesting informations about the rational homotopy type of Diff(M) ([14] ; [16]; compare the report of Burghelea at this conference). But it seems hard to get complete information from it. I want to indicate this very briefly.

For a 1-connected manifold M^n of dim $\geqslant 5$ the general surgery theory gives the following information ([17]). There are exact sequences:

$$\begin{array}{ccccccc} 0 \longrightarrow & bP_{n+2} & \longrightarrow & \mathcal{S}(M \times I, M \times \dot{I}) & \longrightarrow & [\Sigma M, G/O] \\ & & & \downarrow \emptyset & & \\ & & & \pi_0 \mathrm{Diff}(M)^{\pi} & = & \text{isotopy classes of diffeomorphisms homotopic to Id} \\ & & & \downarrow & & \\ & & & 0 & & \end{array}$$

It seems that for highly connected almost parallelizable manifolds $\pi_0 \mathrm{Diff}(M)^{\pi} = \pi_0 S\,\mathrm{Diff}(M)$. The difficulties in applying these sequences to the computation of $\pi_0 \mathrm{Diff}(M)$ are 1) the computation of $[\Sigma M, G/O]$ and with it of $\mathcal{S}(M \times I, M \times \dot{I})$ and 2) the computation of Ker $\emptyset$. I have no idea how to solve especially the last problem. Perhaps the knowledge of the results for sufficiently many examples would suggest the solution. The present paper could be understood as a first step into this direction.

§ 2 Proofs

Before we give the proof of theorem 1 we formulate a general criterion for the problem, which diffeomorphism on the boundary of a 1-connected manifold can be extended for the interior and spezialize it to the problem of existence of pseudo-isotopies.

Proposition 1: (compare [3], 2.3 ;[9], Lemma 7) Let N be a 1-connected manifold of diemsnion≥ 5 and f an orientation preserving diffeomorphism of ∂N. f can be extended to a diffeomorphism on N if and only if the twisted double $N \cup_f - N$ bounds a 1-connected manifold W such that all relative homotopy groups $\pi_k(W,N)$ and $\pi_k(W, -N)$ are zero, where N and -N mean the two embeddings of N into $N \cup_f -N$.

Proof: If we introduce corners along the boundary of a tubular neighbourhood of ∂N into $N \cup_f -N$ we see that W is a relative h-cobordism between $(N, \partial N)$ and $(-N, \partial N)$. Then the proposition is a standard application of the relative h-cobordism theorem ([12]).

If we spezialize this proposition to the case where N is equal to $M \times I$, M a 1-connected manifold of dimension≥ 4, and consider diffeomorphisms of $\partial N = M + (-M)$ of the form f + Id we obtain the following criterion for the existence of pseudo-isotopies between f and Id. For dim $M \geq 5$ we get the existence of isotopies using the deep result of Cerf ([4]).

Proposition 2: Let M be a 1-connected manifold of dimension ≥ 4. An orientation preserving diffeomorphism of M is pseudo-isotopic (isotopic, if dim $M \geq 5$) to Id if and only if the mapping torus $M_f = I \times M/_{(0,x)\sim(1,f(x))}$ bounds a 1-connected manifold W with $\pi_k(W, M) = \{0\}$ for all k.

Remark: The conditions of Proposition 1 can be reformulated as: M_f is h-cobordant to $M \times S^1$.

Proof of Theorem 1: We consider an orientation preserving diffeomorphism $f : M \longrightarrow M$ of a simply-connected closed differentiable 4-manifold with $f_* : H_2(M) \longrightarrow H_2(M)$ the identity. All we have to do is to construct a 6-manifold W with the conditions of Proposition 2. The idea is to start with an arbitrary manifold W bounding M_f and to modify this manifold by surgery in the interior of W until the properties are fulfilled. But in general this does not work, for we can only do surgery if we can represent homology classes by embedded spheres with trivial normal bundle. As we are in the oriented case each embedded 1-sphere has trivial normal bundle and each bundle over S^3 is trivial, so the only problem arises at embedded 2-spheres. But the normal bundle of an embedded 2-sphere is trivial if and only if the Stiefel Whitney class w_2 is zero. So there is no problem if W is a spin-manifold. We will see that we can choose W as a spin-manifold if M is a spin-manifold and that we don't need any condition for W if M is not a spin-manifold.

Using this idea we first have to check that for a diffeomorphism $f : M \longrightarrow M$ with $f_* = Id$ the mapping torus M_f bounds an oriented 6-manifold W which can be chosen as a spin-manifold if M admits a spin structure. As $f_* = Id$ the Wang sequence shows that the inclusion induces an isomorphism $H^2(M_f) \longrightarrow H^2(M)$. Thus if M admits a spin structure, which means $w_2(M) = 0$, then M_f admits one. But the bordism group of 5-dimensional spin-manifolds is zero ([13]), so M_f bounds a spin-manifold W.

If M admits no spin structure, we want to show that M_f bounds an oriented 6-manifold W (without any additional condition). The only obstruction for this is the Stiefel Whitney number $w_2(M_f)w_3(M_f)$. But by a formula of Lusztig, Milnor and Peterson:
$w_2(M_f)w_3(M_f) = \dim H_2(M_f;\mathbb{Q}) + \dim H_4(M_f;\mathbb{Q}) - \dim H_2(M_f;\mathbb{Z}_2) - \dim H_4(M_f;\mathbb{Z}_2) \mod 2,$
the mod 2 difference of the semicharacteristics with coefficients in $\mathbb{Q}$ and $\mathbb{Z}_2$ resp. ([10]). But as M is simply-connected and $f_* = Id$ the Wang sequence shows that $H_*(M_f)$ is torsion free. Thus the semicharacteristics with coefficients in $\mathbb{Q}$ and $\mathbb{Z}_2$ are the same and $w_2(M_f)w_3(M_f) = 0$.
Now we want to do surgery on W to kill $\pi_1(W)$ and $\pi_i(W,M)$ for $i \geq 2$, which is equivalent to killing $\pi_1(W)$ and $H_i(W,M)$ for all i. It is well known that we can kill $\pi_1(W)$ by a sequence of surgeries and can do this in such a manner that the resulting simply-connected manifold is a spin-manifold if W was. We denote this simply connected manifold again by W.

The next step is to kill $H_2(W,M)$. As $H_2(W) \longrightarrow H_2(W,M)$ is surjective we can represent an element x of $H_2(W,M)$ by $\bar{x} \in H_2(W)$. As $\pi_2(W) \cong H_2(W)$ we can represent $\bar{x}$ by an embedded $S^2 \hookrightarrow W$. This sphere has trivial normal bundle if and only if the Kronecker product $\langle w_2(W), \bar{x} \rangle$ is zero. If M admits a spin structure we have supposed that W has one and so $w_2(W) = 0$. If M admits no spin structure there exists $z \in H_2(M)$ with $\langle w_2(M), z \rangle \neq 0$. If $\langle w_2(W), \bar{x} \rangle \neq 0$ we replace $\bar{x}$ by $\bar{x} + i_* z$, i the inclusion $M \longrightarrow W$. In $H_2(W,M)$ the element $\bar{x} + i_* z$ again represents x, but $\langle w_2(W), \bar{x} + i_* z \rangle = 0$.
So we can represent each element x of $H_2(W,M)$ by an embedded sphere $S^2 \hookrightarrow W$ with trivial normal bundle. Surgery with this S^2 kills x and so we can kill $H_2(W,M)$ by a sequence of surgeries giving a simply-connected manifold, again denoted by W with $H_2(W,M) = \{0\}$.

Now we come to the final step namely killing $H_3(W,M)$. If we can do this we are finished for by Poincaré duality

$$H_k(W,M) \cong H^{6-k}(W, \partial W - M) \cong H^{6-k}(W,M) .$$

Again from Poincaré duality and the universal coefficient theorem it follows that $H_3(W,M)$ is torsion free.

To see how to kill $H_3(W,M)$ we consider the following situation. Let $x \in H_3(W,M)$ be a primitive element representable by an embedded sphere $S^3 \hookrightarrow W$. This sphere has trivial normal bundle. Now an easy generalization of a standard argument of surgery theory (compare [6]) shows that if we do surgery with this embedded sphere the resulting manifold W' is again simply-connected, $H_2(W',M) = \{0\}$ and $H_3(W',M) = H_3(W,M)/\mathbb{Z}x + \mathbb{Z}y$ where y is an element of $H_3(W,M)$ such that the intersection number of the embedded sphere S^3 with y is 1.

This shows that we can kill $H_3(W,M)$ by a sequence of surgeries if there exists a direct summand U in $H_3(W,M)$ with the following properties:

1.) $\dim U = \frac{1}{2} \dim H_3(W,M)$
2.) each $x \in U$ can be represented by an embedded sphere $S^3 \hookrightarrow W$
3.) for $x, y \in U$ the intersection number $x \circ y$ vanishes.

Then we choose a basis of $H_3(W,M)$ of the from $x_1, \ldots, x_k$, $y_1, \ldots, y_k$ such that $x_1, \ldots, x_k$ is a basis of U and $x_i \circ y_i = 1$ for all i. But by condition 2.) we can represent each x_i by an embedded sphere $S_i^3 \hookrightarrow W$ and condition 3.) allows us to choose these embeddings disjointly. According to the considerations above it follows that we can kill $H_3(W,M)$ by a sequence of surgeries with S^3_i.

To show that such a subspace $U \subset H_3(W,M)$ exists we first compute the dimension of $H_3(W,M)$. We consider the following exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & H_4(W,\partial W) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & H_3(\partial W) & & & & & & \\
 & & \downarrow i_* & & & & & & \\
0 & \longrightarrow & H_3(W) & \xrightarrow{k_*} & H_3(W,M) & \longrightarrow & H_2(M) & \longrightarrow & H_2(W) \longrightarrow 0 \\
 & & \downarrow j_* & & & & & & \\
 & & H_3(W,\partial W) & & & & & &
\end{array}$$

The zero at the top results from the fact that the map $H_4(W) \longrightarrow H_4(W, \partial W)$ is the Poincaré dual of $H^2(W, \partial W) \longrightarrow H^2(W)$ which factorizes through $H^2(W,M) = \{0\}$.
From these exact sequences it follows:

$\dim H_3(W,M) = \dim H_3(W) + \dim H_2(M) - \dim H_2(W)$

$\qquad = \operatorname{rank} j_* + \operatorname{rank} i_* + \dim H_2(M) - \dim H_2(W)$.

But $\operatorname{rank} i_* = \dim H_3(\partial W) - \dim H_4(W, \partial W)$ and $\dim H_3(\partial W) = \dim H_2(M)$ by the Wang sequence and $\dim H_4(W, \partial W) = \dim H_2(W)$ by Poincaré duality. So $\dim H_2(M) - \dim H_2(W)$ $= \operatorname{rank} i_*$ and we have:

$\dim H_3(W,M) = \operatorname{rank} j_* + 2 \operatorname{rank} i_*$.

As $H_3(W,M)$ is torsion free, the same holds for $H_3(W)$. We decompose $H_3(W)$ into subspaces $S \oplus V$ such that $\operatorname{im} i_* \subset S$ and $\dim S = \operatorname{rank} i_*$. From this it follows that for $x \in S$ and $y \in H_3(W)$ the intersection number $x \circ y$ vanishes. Furthermore it follows that $\dim V = \operatorname{rank} j_* =$ rank of the intersection form on W. The restriction of the intersection form to V is non-degenerate and as this form is antisymmetric there exists a direct summand T of V such that $\dim T = \frac{1}{2} \dim V$ and the intersection form vanishes on T. Thus $U = k_*(S \oplus T)$ is a direct summand in $H_3(W,M)$, of dimension $\frac{1}{2} \dim H_3(W,M)$, on which the intersection form vanishes.

To show that U fulfils condition 2.) we consider the following commutative diagramm:

$$\begin{array}{ccccccc}
 & & \pi_3(W) & \longrightarrow & \pi_3(W,M) & \longrightarrow & \pi_2(M) \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & H_3(W) & \xrightarrow{k_*} & H_3(W,M) & \longrightarrow & H_2(M)
\end{array}$$

It shows that $\pi_3(W) \longrightarrow H_3(W)$ is surjective and so we can represent each $x \in U$ by an embedded sphere $S^3 \hookrightarrow W$.

Thus we have shown that a subspace $U \subset H_3(W,M)$ with the desired properties exists and this brings the proof of Theorem 1 to an end.

The proof of Theorem 2 splits into two parts. First, we compute $\tilde{\pi}_0\mathrm{Diff}(M \text{ rel } D^{2k})$, the group of pseudo-isotopy classes of diffeomorphisms leaving an embedded disk D^{2k} fixed. This is easier than the computation of $\pi_0\mathrm{Diff}(M)$. But $\pi_0\mathrm{Diff}(M)$ can be expressed as a quotient of $\tilde{\pi}_0\mathrm{Diff}(M \text{ rel } D^{2k})$ and this leads to the proof of theorem 2.

Proposition 3: $k \geq 3$. The following sequences are exact:

$$0 \longrightarrow \tilde{\pi}_0\, S\,\mathrm{Diff}(M \text{ rel } D^{2k}) \longrightarrow \tilde{\pi}_0\mathrm{Diff}(M \text{ rel } D^{2k}) \longrightarrow \mathrm{Aut}\, H_k(M) \longrightarrow 0$$
$$0 \longrightarrow \theta_{2k+1} \longrightarrow \tilde{\pi}_0\, S\,\mathrm{Diff}(M \text{ rel } D^{2k}) \longrightarrow \mathrm{Hom}(H_k(M), S\pi_k(SO(k))) \longrightarrow 0$$

The maps are defined as in Theorem 1.

Proof: We denote the manifold obtained from M by removing a disk disjoint from D^{2k} by N. Wall has shown that every element of Aut $H_k(N)$ = Aut $H_k(M)$ can be realized by a diffeomorphism on N rel D^{2k} ([19], Lemma 1o). This follws rather easily using a handle decomposition of N. A similar argument shows that every element of Hom $(H_k(N), S\pi_k(SO(k)))$ = Hom $(H_k(M), S\pi_k(SO(k)))$ can be realized by an element of S Diff(N rel D^{2k}). Thus the sequences would be exact on the right-hand side if every diffeomorphism on N could be extended to a diffeomorphism on M and this is equivalent to the fact that the restriction of any diffeomorphism of N to $\partial N = S^{2k-1}$ beeing isotopic to Id. But if we identify the restriction of diffeomorphisms of N to ∂N with the inertia group of M we see from the work of Kosinski that all diffeomorphisms of N can be extended to M ([7]).

To finish our proof we have to show that the homomorphism $\theta_{2k+1} \longrightarrow \tilde{\pi}_0 S\, \mathrm{Diff}(M \text{ rel } D^{2k})$ is injective and that its image is equal to the kernel of $\tilde{\pi}_0 S\, \mathrm{Diff}(M \text{ rel } D^{2k}) \longrightarrow \mathrm{Hom}(H_k(M), S\pi_k(SO(k)))$. We show this by constructing an inverse σ from this kernel to θ_{2k+1}.

The map σ is defined as follows. We fix embeddings $(S^k \times D^{k+1})_i \subset M \times (0,1)$, disjoint from D^{2k}, representing a basis of $H_k(M)$. Now, for a diffeomorphism $f \in \ker \tilde{\pi}_0 S\, \mathrm{Diff}(M \text{ rel } D^{2k}) \longrightarrow \mathrm{Hom}(H_k(M), S\pi_k(SO(k)))$ we take its mapping torus M_f. We want to kill $\pi_i(M_f)$ by a sequence of surgeries. We do this using the embedding $S^1 \times D^{2k} \subset M_f$, which exists since $f|_{D^{2k}} = \mathrm{Id}$, and the embeddings $(S^k \times D^{k+1})_i \subset M \times (0,1) \subset M_f$. From the work of Kervaire-Milnor ([6]) together with the fact that $H_k(M_f) = H_k(M)$ is torsion free it follows that the resulting manifold is a homotopy sphere which depends only on the pseudo-isotopy class of f rel D^{2k} and is denoted by $\sigma(f)$.

We get a bordism between M_f and $\sigma(f)$ by adding handles to $M_f \times I$ using the embeddings above. This bordism W is a k-connected manifold and its k+1-homology is isomorphic to $H_{k+1}(M_f)$ by inclusion. For our proof we need an additional property of this bordism, namely that all elements of $H_{k+1}(W)$ can be represented by embedded spheres with trivial normal bundle. I don't know whether this is already true for this bordism. But in any case we can get such a manifold by two surgeries on this bordism. First we do surgery with $S^1 \times D^{2k+1} \subset M_f \times (0,1)$ which is contained in our original bordism. The resulting manifold already has the desired property for H_{k+1}. For this we use that $\chi(f) = 0$. But its second homology is now equal to $\mathbb{Z}$ which can be killed by a second surgery.

We summarize the properties of the bordism W

1) W is k-connected

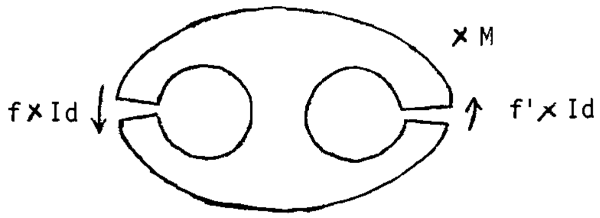
2) the inclusion $H_{k+1}(M_f) \longrightarrow H_{k+1}(W)$ induces an isomorphism and all elements of $H_{k+1}(W)$ can be represented by embedded spheres with trivial normal bundle. This implies that the signature of W is zero.

3) The embedding of $S^1 \times D^{2k}$ into M_f coming from the fact that $f|_{D^{2k}} = Id$ can be extended to an embedding of $D^2 \times D^{2k}$ into W meeting M_f transversally.

Remark: It's an easy exercise in elementary surgery to show that if W is any manifold with these properties and ∂W is equal to M_f and a homotopy sphere then this homotopy sphere is equal to $\sigma(f)$.

Remark: If M_f is a framed manifold and the embeddings above are compatible with the framing we get W as a framed manifold and in particular we get a framing on $\sigma(f)$ from the framing on M_f. We need this for the proof of theorem 3.

Now, we show that σ is a homomorphism. For diffeomorphisms f and f' in $\ker \pi_0 S\,\mathrm{Diff}(M \text{ rel } D^{2k}) \longrightarrow \mathrm{Hom}\,(H_k(M), S\pi_k(SO(k)))$ we consider manifolds W and W' as above. Let S denote the bordism between $M_f + M_{f'}$ and $M_{ff'}$ given by the fibration with fibre M over the twice punctured disk D^2 classified by f and f' as indicated in the following picture.



Now, we consider the manifold $S \cup W \cup W'$ with boundary consisting of $M_{ff'}$ and $\sigma(f) + \sigma(f')$. It follows again from a standard surgery argument that we can by a sequence of surgeries replace this manifold by one which fulfils the conditions above. Together with the first remark above this implies that $\sigma(ff') = \sigma(f) + \sigma(f')$.

σ is surjective. This follows from the fact that for a diffeomorphism f which is the image of a homotopy sphere Σ under the homomorphism $\theta_{2k+1} \longrightarrow \tilde{\pi}_0 S\,\mathrm{Diff}(M \text{ rel } D^{2k})$ it is known that $M_f = M \times S^1 \# \Sigma$ ([2], Lemma 1). This implies $\sigma(f) = \Sigma$.

We finish the proof by showing that σ is injective. If $\sigma(f) = S^{2k+1}$ we consider

$W \cup D^{2k+2}$, where W is as above a bordism between M_f and $\sigma(f)$. Then we attach to this manifold a handle along $S^1 \times D^{2k} \subset M_f$. The resulting 1-connected manifold $\tilde{W}$ has the following properties, which can be verified rather easily.

1) $\partial\tilde{W} = N \times I \cup_{f \cup Id} N \times I$, where $N = M - \mathring{D}^{2k}$ and $f \cup Id$ is the diffeomorphism on $\partial(N \times I) = N \cup N$ given by f and Id.

2) $H_2(\tilde{W}) \cong \mathbb{Z}$, generated by an embedded sphere with trivial normal bundle.
$H_i(\tilde{W}) = \{0\}$ for $2 < i \leq k$.

3) $H_{k+1}(W) \xrightarrow{\cong} H_{k+1}(\tilde{W})$ and we have an exact sequence
$0 \longrightarrow H_{k+1}(\tilde{W}) \longrightarrow H_{k+1}(\tilde{W}, N) \longrightarrow H_k(N) \longrightarrow 0$

As $H_{k+1}(\tilde{W})$ is a subspace of half dimension in $H_{k+1}(\tilde{W}, N)$ in which all elements can be represented by embedded spheres with trivial normal bundle we can kill $H_*(\tilde{W}, N)$ by a sequence of surgeries. Now, Proposition 1 implies that the diffeomorphism $f \cup Id$ on $\partial(N \times I)$ can be extended to $N \times I$. But this implies that $f|_N$ is pseudo-isotopic to Id rel $\partial N = S^{2k-1}$. Thus f is pseudo-isotopic to Id in Diff(M rel D^{2k}).

q.e.d.

To complete the computation of π_0Diff(M) we use the following exact sequence for a 1-connected manifold ([19], p.265):

$$\mathbb{Z}_2 = \pi_1(SO(2k)) \longrightarrow \tilde{\pi}_0\text{Diff}(M \text{ rel } D^{2k}) \longrightarrow \pi_0\text{Diff}(M) \longrightarrow 0$$

The homomorphism $\pi_1(SO(2k)) \longrightarrow \tilde{\pi}_0\text{Diff}(M \text{ rel } D^{2k})$ is defined as follows. We extend the embedding of D^{2k} into M to an embedding of $2 \cdot D^{2k}$ into M. For $\gamma: (I, \partial I) \longrightarrow (SO(2k), e)$ we define a diffeomorphism on M by the identity on D^{2k} and outside $2D^{2k}$ and by $x \longmapsto \gamma(|x|-1) \cdot x$ for $x \in 2D^{2k} - D^{2k}$.

It is obvious that this diffeomorphism is contained in ker $\tilde{\pi}_0$S Diff(M rel D^{2k}) $\longrightarrow$ Hom($H_k(M)$, S$\pi_k(SO(k))$). Thus we can apply σ to it. If γ is the nontrivial element in $\pi_1 SO(2k)$ we denote the image under σ of the corresponding diffeomorphism by Σ_M. Now, it is clear that Theorem 2 follows from Propositon 3 and the exact sequence above. Then the definition of Σ_M gives Lemma 4.

Remark: It is useful to have the following description of Σ_M. Let f be the diffeomorphism corresponding to the nontrivial element in $\pi_1(SO(2k))$. There is a diffeomorphism $M_f \longrightarrow S^1 \times M$ which is the identity outside $S^1 \times 2D^{2k}$ and whose restriction to $S^1 \times D^{2k}$ corresponds to the twisting by the nontrivial element $\gamma \in \pi_1\, SO(2k)$. Thus Σ_M can be obtained from $S^1 \times M$ by a sequence of surgeries starting with the embedding of $S^1 \times D^{2k}$ into $S^1 \times M$, which maps $(x,y) \mapsto (x, \gamma(x)\cdot y)$ and then killing $H_k(S^1 \times M)$ by arbitrary surgeries.

Now we come to the proof of Theorem 3.

Proof of Theorem 3: If M is a framed manifold with framing α we can obtain Σ_M by framed surgeries on $S^1 \times M$ with the product of the nontrivial framing on S^1 and the framing α on M. Then we obtain Σ_M as a framed manifold which is framed bordant to $\eta \circ [M,\alpha]$.
This gives the proof of theorem 3, a.

For the proof of part b) and c) we need the δ-invariant of a framed manifold ([8]). For a framed manifold (V^{4n-1}, α) there exists an $r>0$ such that $r(V,\alpha)$ bounds a framed manifold (W, β). $\delta(V,\alpha) := \frac{1}{r}\cdot \tau(W) \in \mathbb{Q}$. It can be considered as the defect of the signature theorem for any manifold bounding V, where we have to use relative characteristic classes with respect to α in the L-polynomial. We need the following properties of this invariant. If we fix a framing β on V then - with respect to this framing - the set of all homotopy classes of framings on V is euqal to $[V,SO]$. The following formula is true.

$$\delta(V, \gamma_1\cdot\gamma_2) = \delta(V,\gamma_1) + \delta(V,\gamma_2) - \delta(V,\beta)$$

where $\gamma_1, \gamma_2 \in [V,SO]$ and (V, γ_i) denotes the framed manifold corresponding to γ_i with respect to β . If we fix the restriction to S^{4n-1} of the framing of D^{4n} then $\delta : \pi_{4k-1}(SO) = \mathbb{Z} \longrightarrow \mathbb{Q}$ is an injective homomorphism. The framings on S^{4k-1} are classified by δ . The δ-invariant mod 1 is a framed bordism invariant and is equal to $\pm a_n \cdot 2^{2n+1}(2^{2n-1}-1)\cdot e_{\mathbb{R}}$, the real Adams invariant, where $a_n = \begin{cases} 1 \text{ for } n \text{ even} \\ 2 \text{ for } n \text{ odd.}\end{cases}$

For the proof of b) we consider a manifold M^{4n} with $\tau(M) = s\cdot\sigma_n$, $s \neq 0$. We consider a framing β on $M - \mathring{D}^{4n}$. The restriction of β to S^{4n-1} is a nontrivial element in $\pi_{4n-1}(SO)$, as the δ-invariant is equal to $\tau(M)$. Since im J has even order this element has even order. Thus there exists a framing $\tilde{\beta}$ on S^{4n-1} such that $2\tilde{\tilde{\beta}} = \beta|_{S^{4n-1}}$ regarded as elements in $\pi_{4n-1}(SO)$. From the correspondence between the δ-invariant and the real e-invariant it follows that the framed bordism class $[S^{4n-1}, \tilde{\beta}]$ is equal to $s \cdot \varsigma$, where ς is the element of order 2 in im J_{4n-1}.

Now we construct an element in the Toda bracket $[s \cdot \varsigma\ , 2, \eta]$ as follows. We consider the standard framed bordism between $2(S^{4n-1}, \tilde{\beta})$ and $(S^{4n-1}, \beta|_{S^{4n-1}})$ and glue the product of this manifold with (S^1, γ) to $(S^1 \times I \times S^{4n-1}, \gamma \times \mathrm{Id} \times \tilde{\beta})$ along $2(S^1 \times S^{4n-1})$ with an appropriate orientation preserving diffeomorphism to obtain a framed manifold (V, ζ) with boundary $(S^{4n-1}, \beta|_{S^{4n-1}})$. The union of (V, ζ) with $(S^1 \times (M - \mathring{D}^{4n}), \gamma \times \beta)$ along $S^1 \times S^{4n-1}$ is contained in $[s \cdot \varsigma, 2, \eta]$. To finish the proof we have to show that this manifold is framed bordant to Σ_M with a suitable framing.

We will show that (V, ζ) is framed bordant modulo boundary to a manifold which is diffeomorphic to $D^2 \times S^{4n-1}$ by a diffeomorphism which is equal to $(x,y) \overset{g}{\longmapsto} (x, \gamma(x) \cdot y)$ on the boundary. γ is the nontrivial element in $\pi_1(SO(4n))$. But this implies that our manifold above is framed bordant to $D^2 \times S^{4n-1} \cup_g S^1 \times (M - \mathring{D}^{4n})$ with some framing. Now, this manifold is obtained from $S^1 \times M$ by surgery with the embedding $(x,y) \mapsto (x, \gamma(x) \cdot y)$ and by the remark on p. 16 we can obtain Σ_M from it by a sequence of framed surgeries.

To show that (V, ζ) is framed bordant modulo boundary to a manifold diffeomorphic to $D^2 \times S^{4n-1}$ we do surgery on it. V has the following homology:

$H_1(V) \cong \mathbb{Z} \oplus \mathbb{Z}$; $H_2(V) \cong \mathbb{Z}$; $H_3(V) = \dots = H_{4n-2}(V) = \{0\}$; $H_{4n-1}(V) \cong H_{4n}(V) \cong \mathbb{Z}$.

Now, we kill $H_1(V)$ and $H_2(V)$ by framed surgery and obtain a framed manifold S with the desired properties. This can be seen as follows. We consider $\tilde{S} := S - \mathring{D}^2 \times S^{4n-1}$, where $D^2 \times S^{4n-1}$ is a tubular neighbourhood of an embedded $S^{4n-1} \subset \mathring{S}$ which is isotopic to $\{1\} \times S^{4n-1} \subset \partial(S) = S^1 \times S^{4n-1}$. $\tilde{S}$ fulfils the condition of the Browder-Levine fibration theorem ([3]). Thus the fibration $\partial\tilde{S} = S^1 \times S^{4n-1} + S^1 \times S^{4n-1} \longrightarrow S^1$ can be extended to a fibration $\tilde{S} \longrightarrow S^1$. From the homology of V it is easy to see that the fibre is a h-cobordism between S^{4n-1} and S^{4n-1}. Thus it is diffeomorphic to $S^{4n-1} \times I$. This implies that S is diffeomorphic to $D^2 \times S^{4n-1}$

by a diffeomorphism whose restriction to the boundary is given by an element of $\pi_1(SO(4n-1))$. But as the framing on ∂S given by $\gamma \times \beta|_{S^{4n-1}}$ can be extended to S, this must be the nontrivial element. This ends the proof of part b.

For the proof of part c we begin with the case k odd. Suppose $\Sigma_M = S^{2k+1}$. For a framing β on M we have shown in the second remark on p.14 that we can extend the framing $\gamma \times \beta$ on $S^1 \times M$ to a framing on W. We denote the restriction of this framing to $\Sigma_M = S^{2k+1}$ by ζ. We are done if ζ extends to D^{2k+2} and this is equivalent to $\delta(S^{2k+1},\zeta) = 0$. But $\delta(S^{2k+1},\zeta) = \delta(S^1 \times M, \gamma \times \beta)$ as $\tau(W) = 0$. Since γ considered as an element of $\pi_1(SO)$ has order 2 the formula for the δ-invariant above implies:

$$\delta(S^1 \times M, \tau \times \beta) = 2\cdot\delta(S^1 \times M, \gamma \times \beta) - \delta(S^1 \times M, \tau \times \beta)$$

where τ is the trivial framing on S^1. On the other hand $\delta(S^1 \times M, \tau \times \beta) = 0$, as $(S^1 \times M, \tau \times \beta)$ bounds the framed manifold $D^2 \times M$ with signature 0. Thus $\Sigma_M = 0$ implies $\eta \circ [M,\beta] = 0$.

If $\eta \circ [M,\beta] = 0$ then (Σ_M,ζ) bounds a framed manifold (V,ς). Thus $\Sigma_M \in bP_{2k+2}$ and is determined by the signature of V ([6]). But $\tau(V) = \delta(\Sigma_M,\zeta)$ and this is zero as shown above.

The case k even and $bP_{2k+2} = \{0\}$ can be seen in a similar but even simpler way. For the case k even and $bP_{2k+2} \neq \{0\}$ we first have to show that if $\eta \circ [M,\beta] = 0$ then $\Sigma_M \in bP_{2k+2}$. If $\eta \circ [M,\beta] = 0$ it follows that (Σ_M, γ) is framed bordant to zero. Thus Σ_M bounds a framed manifold. Now, the case k even and $bP_{2k+2} \neq 0$ follows as the cases above using in addition the fact that bP_{2k+2} is classified by the Arf invariant ([6]).

q.e.d.

<u>Proof of Lemma 3:</u> a) Let V be the standard bordism between M + N and M#N. We consider the manifold $S := W_M + W_N \cup S^1 \times V \cup W_{M\#N}$ where W_M is the bordism between $S^1 \times M$ and Σ_M as in the definition of Σ_M. We want by a sequence of surgeries to replace S by an h-cobordism between $\Sigma_M + \Sigma_N$ and $\Sigma_{M\#N}$.

S is 1-connected and has the following homology. $H_i(S) = \{0\}$ for $0 < i \leq k$ and $i \neq 2$. $H_2(S) \cong \mathbb{Z} \oplus \mathbb{Z}$. The second Stiefel-Whitney class $w_2(S)$ is zero. This follows from the fact that the product of the non-trivial spin-structure of S^1 with the spin-structure on M,N and $M\# N$ can be extended to W_M, W_N, $W_{M\# N}$ and $S^1\times V$. This gives a spin-structure on S. Thus all elements in $H_2(S)$ can be represented by embedded spheres with trivial normal bundle.

For $H_{k+1}(S)$ one obtains the following information from a Mayer-Vietoris sequence. There is an exact sequence

$$0 \longrightarrow H_{k+1}(S^1\times M) \oplus H_{k+1}(S^1\times N) \longrightarrow H_{k+1}(S) \longrightarrow H_k(S^1\times M) \oplus H_k(S^1\times N) \longrightarrow 0$$

As the map on the left side factorizes through W_M and W_N and all elements in $H_{k+1}(W_M)$ and $H_{k+1}(W_N)$ can be represented by embedded spheres with trivial normal bundle we get a subspace of half the dimension in $H_{k+1}(S)$ with the same property.

It is well known that these properties imply that we can replace S by a sequence of surgeries by a h-cobordism between $\Sigma_M + \Sigma_N$ and $\Sigma_{M\# N}$.

b) If M bounds a framed manifold V than Σ_M bounds the s-parallelizable manifold $S := W \cup S^1\times V$. Thus $\Sigma_M \in bP_{2k+2}$. If k is odd the vanishing of the signature of W and the Novikov-additivity imply that $\tau(S) = 0$. Thus $\Sigma_M = 0$ in this case.

If k is even we have to show that the Arf-invariant of Σ_M is zero. First we can assume that V is k-1-connected and that $H_k(V,M) = \{0\}$. This implies that $H_i(S) = \{0\}$ for $0 < i \leq k$ and $i \neq 2$ and that $H_2(S) \cong \mathbb{Z}$. A Mayer-Vietoris argument similar to that in a) shows that there is a direct summand in $H_{k+1}(S)$ of half the dimension in which all elements can be represented by spheres with trivial normal bundle. So the Arf-invariant of Σ_M vanishes.

c) This follows immediately from the definition of Σ_M and the geometric description of the Milnor-Munkres-Novikov pairing.

q.e.d.

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References

[1] J.F. Adams: On the group J(x) - IV, Topology 5, 21-71 (1966)

[2] W. Browder: Diffeomorphisms of 1-connected manifolds, Trans. A.M.S. 128, 155-163 (1967).

[3] W. Browder and J. Levine: Fibering manifolds over a circle, Comm. Math. Helv. 4o, 153-16o (1965/66)

[4] J. Cerf: The pseudo-isotopy theorem for simply connected differentiable manifolds, Manifolds Amsterdam, Springer Lecture notes 197, 76-82 (197o)

[5] M. Kervaire: Some nonstable homotopy groups of Lie groups, Illinois J. Math 4, 161-169 (196o)

[6] M. Kervaire and J. Milnor: Groups of homotopy spheres, Ann. of Math. 77, 5o4-537 (1963)

[7] A. Kosinski: On the inertia group of π-manifolds, Am. J. Math. 89 227-248 (1967)

[8] M. Kreck: Eine Invariante für stabil parallelisierte Manigfaltigkeiten, Bonner Math. Schriften Nr. 66 (1973)

[9] M. Kreck: Bordismusgruppen von Diffeomorphismen, Habilitationsschrift, Bonn (1976)

[1o] G. Lusztig, J. Milnor and F.P. Peterson: Semi-characteristics and cobordism, Topology 8, 357-36o (1969)

[11] H. Sato: Diffeomorphism groups of $S^p \times S^q$ and exotic spheres, Quart. J. Math. Oxford, 2o, 255-276 (1969)

[12] S. Smale: On the structure of manifolds, Amer. J. Math. B4, 3B7-399 (1962)

[13] R.E. Stong: Notes on cobordism theory. Math. notes, Princeton University Press (196B)

[14] D. Sullivan: Infinitesimal computations in topology, Publ. I.H.E.S. 47 Paris, 197B

[15] H. Toda: Composition methods in homotopy groups of spheres, Ann. of Math. Study 49, Princeton University Press, 1962

[16] E.C. Turner: Diffeomorphisms homotopic to the Identity, Trans. A.M.S 1B6, 4B9-49B (1973).

[17] E.C. Turner: A survey of diffeomorphism groups. Algebraic and geometrical methods in topology. Springer lecture notes 42B, 2oo-219 (1974)

[1B] C.T.C.Wall: Diffeomorphisms of 4-manifolds, J. London Math. Soc. 39, 131-14o (1964)

[19] C.T.C. Wall: Classification problems in differential topology II: Diffeomorphisms of handlebodies, Topology 2, 263-272 (1963)

[2o] C.T.C. Wall: Classification problems in differential topology III: Applications to special cases, Topology 3, 291-3o4 (1964)

[21] J. Levine: Inerta groups of manifolds and diffeomorphisms of spheres, Ann.J.Math.92, 243-258 (1970).