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Microbundles and Bundles

II. Semisimplicial Theory

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0. Introduction and Notations

The main result in Part II is

Theorem 1. *There exists an exact sequence*

$$\to \pi_j(\mathrm{PL}_n) \to \pi_j(S^{n-1}) \oplus \pi_{j-1}(C_{n-1}) \to \pi_{j-1}(\mathrm{PL}_{n-1}) \to \pi_{j-1}(\mathrm{PL}_n),$$

where PL_n is Milnor's semi simplicial (s.s.) *structural group for n-dimensional PL-microbundles (see [9]) and C_{n-1} is the s.s. group of concordances of PL-homeomorphisms of S^{n-1} and the identity.*

Applications of this theorem will be given in a forthcoming part III. Here we content ourselves to remark the obvious comparison between this sequence and the homotopy sequence for the fibration $0_{n-1} \subset 0_n \to S^{n-1}$, 0_n the orthogonal group. It is the fact that C_{n-1} is not contractible that leads to the difference between PL-bundles and orthogonal bundles. In order to define PL_n and C_{n-1} more explicitly we introduce some *notations*, also for later use. $\mathscr{H}(X)$ is the s.s. group complex whose k-simplices are PL-bundle isomorphisms $f\colon \Delta_k \times X \to \Delta_k \times X$ with base space Δ_k and fibre a locally finite simplicial complex X. $\mathscr{H}_A(X)$ is the subgroup leaving every point of $A \subset X$ fixed (If A consists of one point p or two points p and q, we write $\mathscr{H}_p(X)$ and $\mathscr{H}_{p,q}(X)$ respectively.) $\mathscr{H}^{N(p)}(X)$ is the subgroup of $\mathscr{H}(X)$ of those elements that have some neighbourhood (not necessarily the same neighbourhood for different simplices) of $p \in X$ pointwise fixed. Then $C_{n-1} = \mathscr{H}_{\partial_0}(S^{n-1} \times I)$, $\partial_0 = S^{n-1} \times 0$. $\mathscr{E}(A, X)$ is the s.s. complex whose k-simplices are bundle monomorphisms $f\colon \Delta_k \times A \to \Delta_k \times X$. In analogy with the above cases, $\mathscr{E}_0(N(0), \boldsymbol{R}^n)$ is the s.s. complex of bundle monomorphisms $f\colon \Delta_k \times N(0) \to \Delta_k \times \boldsymbol{R}^n$ having fixed $0 \in \boldsymbol{R}^n$, for some neighbourhood $N(0)$ (again not necessarily the same neighbourhood for different simplices). etc.

By definition, PL_n is the quotient complex of $\mathscr{E}_0(N(0), R^n)$ obtained by identifying two monomorphisms $f_i\colon \Delta_k \times N_i(0) \to \Delta_k \times R^n$, $i = 1, 2$, if they agree on $\Delta_k \times N_3(0)$, $N_3(0) \subset N_1(0) \cap N_2(0)$. PL_n is a s.s. group complex because inverses exist.

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Finally ΠL_n is defined to be $\mathscr{H}_0(D^n)$ with

$$D^n = \{t=(t_1, \dots, t_n) \in \boldsymbol{R}^n \mid |t_i| \leqq 1 \text{ all } i\}.$$

The definition of the homotopy groups of a s.s. complex is recalled in section 3A.

The main *technique* will consist in the use of s.s. fibre spaces (as in HIRSCH [3]). In section 1 fibre spaces are used in order to prove homotopy equivalence between certain complexes or spaces. In section 2 we obtain from the important diagram (11) an exact sequence which is up to weak homotopy the exact sequence of some Serre fibration:

$$\to \pi_j(\mathrm{PL}_n) \to \pi_j(S^{n-1}) \oplus \pi_j(\mathscr{B}_n) \to \pi_{j-1}(\mathrm{PL}_{n-1}) \to \pi_{j-1}(\mathrm{PL}_n) \to . \qquad (12)$$

In section 3 we take care of some lemmas needed and used in the earlier sections, in particular in order to obtain the main theorem from the sequence (12).

An important rôle is played throughout by the *covering isotopy theorem of* HUDSON [5]. We now recall this theorem and deduce consequences for later use. First we need some *definitions.*

We shall be concerned with PL-maps of a compact PL m-manifold M in a PL q-manifold Q which is not necessarily compact. The boundary of M will be denoted by ∂M etc. Let

$$I^n = \{t=(t_1, \dots, t_n) \in \boldsymbol{R}^n \mid 0 \leqq t_i \leqq 1 \text{ for all } i\}.$$

A *PL n-isotopy* of M in Q is a PL embedding $i\colon I^n \times M \to I^n \times Q$ which commutes with projection onto the first factor. We denote: $i((t,x)) = (t, i_t(x))$. An *ambient PL n-isotopy* of Q is a PL homeomorphism $h\colon I^n \times Q \to I^n \times Q$ which commutes with projection onto the first factor and such that $h_0\colon Q \to Q$ is the identity. The index 0 refers to the point $t=(0, \dots, 0)$. In this case if $i_0\colon M \to Q$ is an embedding then i, defined by $i((t,x)) = (t, h_t\, i_0(x))$, is a PL n-isotopy of M in Q. i is called a restriction of the ambient isotopy h. h is called an *ambient isotopy of i.*

A PL n-isotopy i of M in Q is called *proper* if $i_t^{-1}(\partial Q) = \partial M$, for all $t \in I^n$. It is called *locally trivial* if, for every point $(t,x) \in I^n \times M$, there are closed neighbourhoods V of t in I^n and U of x in M, and a PL embedding $\alpha\colon V \times (U \times D^{q-n}) \to I^n \times Q$ which commutes with projection into the first factor, with restriction $i = (\alpha \mid V \times (U \times 0))$ on identifying $V \times (U \times 0)$ with $V \times U$.

Theorem of Hudson. *Let $i\colon I^n \times M \to I^n \times Q$ be any proper PL n-isotopy of the compact M^m in Q^q. If i is locally trivial, then there exists an ambient PL n-isotopy h of i. If $q-m \geqq 3$ then i is locally trivial.*

If $i_t\colon M \to Q$ is constant in t on ∂M, then h_t may be taken constant in t on ∂Q. (In particular this applies if ∂M is empty.)

Remark. As I^n and the n-simplex Δ_n are PL-homeomorphic, we may replace throughout I^n by Δ_n and the theorem remains valid.

Lemma 0.1. Let $D^q_\varepsilon = \{x \in \boldsymbol{R}^q \,|\, |x_i| \leqq \varepsilon, i=1, \ldots, q\}$. Assume $\varepsilon < 1$, and let $f\colon D^q_\varepsilon \to R^q$ be an orientation preserving embedding with $f(0)=0$ and $f(D^q_\varepsilon) \subset \operatorname{Int} D^q (D^q = D^q_1)$. Then f is ambient isotopic to the inclusion, by an ambient 1-isotopy which is fixed in a) zero and b) the closure of the complement of D^q. The same holds with $\boldsymbol{R}^q$ replaced by S^q.

Proof. By the regular neighborhood theorem [7], an ambient isotopy h_t, $0 \leqq t \leqq 1$, of $\boldsymbol{R}^q$ exists satisfying a) and b) with $h_1 f$ sending D^q_ε onto itself. Since any PL-homeomorphism of a sphere which is orientation preserving is isotopic to the identity, and this isotopy may be extended to a product neighborhood of ∂D^q_ε in $\boldsymbol{R}^q$, there is by Hudson's Theorem an ambient isotopy$=h_{1+t}$, $0 \leqq t \leqq 1$ of $\boldsymbol{R}^q$ satisfying a) and b) with $h_2 f$ the identity on ∂D^q_ε. Finally, by the Alexander trick ([6] see part I), there is an ambient isotopy h_{2+t}, $0 \leqq t \leqq 1$, satisfying a) and b), with $h_3 f$ the identity on D^q_ε. $h_{3\,t}$, $0 \leqq t \leqq 1$ is the required ambient isotopy of $\boldsymbol{R}^n$.

Lemma 0.2. Let $f\colon \Delta_n \times \boldsymbol{R}^q \to \Delta_n \times \boldsymbol{R}^q$ be an orientation preserving bundle monomorphism preserving the zero section. Then there exists a bundle isomorphism $g\colon \Delta_n \times D^q \to \Delta_n \times D^q$, which agrees with f in a neighborhood of the zero section, and is the identity on $\Delta_n \times \partial D^q$.

Proof. Let ε be sufficiently small so that $f(\Delta_n \times D^q_\varepsilon) \subset \Delta_n \times \operatorname{Int} D^q$. Let $M = \partial D^q_\varepsilon$. Let $i = f \,|\, \Delta_n \times M\colon \Delta_n \times M \to \Delta_n \times D^q$. By Lemma 0.1, we may assume f_0 (and thus i_0) is the inclusion. (I.e., if h is the homeomorphism of $\boldsymbol{R}^q$ into itself such that $h f_0$ is the inclusion, then replace f by $(1_{\Delta_n} \times h) f$ and the resultant g by $(1_{\Delta_n} \times h \,|\, D^q)^{-1} g$.) Now i is locally trivial, because it extends to a product neighborhood of M in D^q. By Hudson's Theorem there exists an ambient PL n-isotopy h of D^q, fixed on ∂D^q, extending i. Since $i_t(M)$ separates D^q, h preserves the notions of inside and outside. For $t \in \Delta_n$ we define

$$g_t = \begin{cases} h_t & \text{on the outside of } M \text{, and on } M \\ f_t & \text{on the inside of } M. \end{cases}$$

Then g is the desired bundle isomorphism.

1. Examination of Some Principal s.s. Fibre Bundles

1. We begin by exhibiting some principal s.s. fibre bundles.

Definition 1.1. (Compare [1]). Let $r\colon E \to B$ be a morphism of s.s. complexes, and let $\mathscr{J}$ be a *s.s.* group acting on E. If

a) r is onto and

b) $\mathscr{J}$ acts freely and transitively on $p^{-1}(b)$ for all $b \in B$, then p is called a *principal* s.s. *bundle with group* $\mathscr{J}$.

Example:

Lemma 1.1. *For every connected manifold M^n without boundary and $x \in M^n$, the natural restriction map r: $\mathscr{H}(M^n) \to \mathscr{E}(x, M^n)$ is a principal s.s. bundle with group $\mathscr{H}_x(M^n)$.*

Proof. If f is a k-simplex of $\mathscr{H}(M^n)$ and g a k-simplex of $\mathscr{H}_x(M^n)$, then g acts on f by composition $g: f \to fg$. Then b) is immediately deduced. It remains to prove a). We take a simplex f: $\Delta_p \times x \to \Delta_p \times M^n$ in $\mathscr{E}(x, M^n)$. Because M^n is connected there exists for any $x, x' \in M^n$ a homeomorphism of M^n onto itself, which carries x' onto x. Applying this to the first vertex $e_0 \times x$ of $\Delta_p \times x$ and its image $f(e_0 \times x) = e_0 \times x'$, and taking the corresponding element h: $\Delta_p \times M^n \to \Delta_p \times M^n$, constant in the variable in Δ_p, we obtain hf: $\Delta_p \times x \to \Delta_p \times M^n$, which sends $e_0 \times x$ onto $e_0 \times x$. Hence we may just as well assume this property for f: $f(e_0 \times x) = e_0 \times x$. Next we show that f: $\Delta \hat{p} \times x \to \Delta_p \times M$ is a locally trivial isotopy. For any $t \in \Delta_p$ there exists a closed neighborhood V of t and an open neighborhood W in M such that

$$f(V \times x) \subset \Delta_p \times W \subset \Delta_p \times M$$

with W PL-homeomorphic to $\boldsymbol{R}^n$. Identify W with $\boldsymbol{R}^n$. The embedding

$$\alpha\colon\ V \times (x \times D^n) \to V \times \boldsymbol{R}^n (= V \times W \subset V \times M)$$

defined by

$$\alpha(t, x, u) = \big(t, \beta(x) + u\big)$$

with

$$f(t, x) = \big(t, \beta(x)\big)$$

proves the local triviality at $t \in \Delta_p$.

Finally we apply the theorem of Hudson which says that f is the restriction of some bundle isomorphism $\tilde{f}$: $\Delta_p \times M^n \to \Delta_p \times M^n$. Consequently r is onto.

Lemma 1.2. *The following are principal s.s. bundles (the term on the left being the fibre (=group) of the bundle).*

a) $\mathscr{H}_{p,q}(S^n) \subset \mathscr{H}_q(S^n) \xrightarrow{r} \mathscr{E}(p, S^n - q)$.

b) $\mathscr{H}_q(S^n) \subset \mathscr{H}(S^n) \xrightarrow{r} \mathscr{E}(q, S^n)$.

c) $\mathscr{H}_0(\boldsymbol{R}^n) \subset \mathscr{H}(\boldsymbol{R}^n) \xrightarrow{r} \mathscr{E}(0, \boldsymbol{R}^n)$.

d) $\mathscr{H}_{0,p}(\boldsymbol{R}^n) \subset \mathscr{H}_0(\boldsymbol{R}^n) \xrightarrow{r} \mathscr{E}(p, \boldsymbol{R}^n - 0)$.

e) $\mathscr{H}_q^{N(p)}(S^n) \subset \mathscr{H}_{p,q}(S^n) \xrightarrow{\gamma} PL_n$.

f) $\mathscr{H}^{N(0)}(\boldsymbol{R}^n) \subset \mathscr{H}_0(\boldsymbol{R}^n) \xrightarrow{\gamma} PL_n$.

g) $\mathscr{H}^{N(0)}(D^n) \subset \mathscr{H}_0(D^n) = \Pi L_n \xrightarrow{\gamma} PL_n$.

h) $\mathscr{H}^{N(0)}(D^n) \subset \mathscr{H}_{0,\partial}(D^n) \xrightarrow{\gamma} SPL_n$.

i) $\mathscr{H}_{0,\partial}(D^n) \subset \mathscr{H}_0(D^n) \xrightarrow{\partial} \mathscr{H}(S^{n-1})$.

In h) and i)

1) ∂ as a subscript means the boundary $\partial D^n = S^{n-1} \subset D^n$.

2) ∂ as a map means restriction to the boundary.

3) SPL_n is the subgroup of PL_n, of orientation preserving germs of monomorphisms.

In each case r is the obvious restriction map under given fixed identifications of $(S^n - q, p)$ with $(R^n, 0)$ and of $(D^n - \partial, 0)$ with $(R^n, 0)$, where in the last case the identification is the identity on some neighborhood of $0 \in D^n \subset R^n$. Analogously γ is the "restriction" to germs (that is to some nonfixed neighborhood).

Proof. a) b) c) and d) are special cases of lemma 1.1. In all cases part b) of definition 1.1 is immediate, so that only the part a), "onto", remains.

From onto in e) follows onto in f), because under the given identifications there is inclusion and commutativity in the triangle:

$$\begin{tikzcd} \mathscr{H}_{p,q}(S^n) \arrow[d] \arrow[dr, "\gamma"] & \\ \mathscr{H}_0(R^n) \arrow[r, "\gamma"'] & \mathrm{PL}_n . \end{tikzcd}$$

Analogously from onto in h) follows onto in g) and from that again onto in f). But onto in h) is expressed in lemma 0.2. And onto in e) is expressed in the last part of lemma 0.1.

There remains the proof of onto in i) which we will give now. It is sufficient to construct a cross-section ρ of ∂. This is a morphism

$$\rho \colon \mathscr{H}(S^{n-1}) \to \mathscr{H}_0(D^n)$$

such that $\partial \rho \colon \mathscr{H}(S^{n-1}) \to \mathscr{H}(S^{n-1})$ is identity: $\partial \rho = 1$.

As in part I, section 3, ρ is defined with the help of a PL-version of the Alexander trick:

a) Define $\rho(f)$ on any zero simplex $f \colon S^{n-1} \to S^{n-1}$ as the cone of f.

b) Assume inductively that ρ has been defined on j-simplices, $j < k$, so as to commute with face and degeneracy operators, ∂_i and s_i, respectively.

c) Let $f \colon \Delta_k \times S^{n-1} \to \Delta_k \times S^{n-1}$ be a k-simplex of $\mathscr{H}(S^{n-1})$.

If $f = s_i g$, define $\rho(f) = s_i \rho(g)$. If f is non-degenerate, define $\rho(f)$ over $\partial_i \Delta_k$ by $\rho(\partial_i f)$, and define $\rho(f)$ on $\Delta_k \times \partial D^n$ by f. Then $\rho(f)$ is defined on $\partial(\Delta_k \times D^n)$. Now let $v = (b, 0)$, b the barycenter of Δ_k, 0 the center of D^n. Define $\rho(f)$ to be the identity on v. Extend linearly over the join lines $v * x$, $x \in \partial(\Delta_k \times D^n)$. Then $\rho(f)$ is a bundle isomorphism, which extends f and satisfies $\partial_i \rho(f)=\rho(\partial_i f)$. This completes the definition of ρ. It is obvious that ρ has the required property $\partial\rho=1$.

Remark. Since all group complexes are Kan complexes and the total space of a principal s.s. bundle is a Kan complex if and only if the base space is a Kan complex (see [*1*]), all the complexes in lemma 1.2 are Kan complexes. Hence homotopy groups are defined and all the bundles have exact homotopy sequences. As base point we always take the s. s.-o-simplex which is an identity map or germ of map.

In order to prove that many of the maps in these fibre bundles are homotopy equivalences we first prove.

Lemma 1.3. *For all j, $\pi_j(\mathscr{E}(x, X)$ is naturally isomorphic to $\pi_j(X, x)$).*

Proof. Write $\mathscr{S}(X)$ for the singular complex of X, and $\mathscr{S}_{\mathrm{PL}}(X)$ for the subcomplex of PL-singular simplices. Define a s.s. map $\varphi: \mathscr{E}(x, X)\to \mathscr{S}_{\mathrm{PL}}(X)$ by assigning to f: $\Delta_k=\Delta_k\times x\to\Delta_k\times X$, defined by, say $f(u, x)=(u, g(u))$, the map $\varphi(f)$: $\Delta_k\to X$ such that $\varphi(f)(u)=g(u)$. Then φ is an isomorphism of s.s. complexes.

The simplicial approximation theorem implies that i: $\mathscr{S}_{\mathrm{PL}}(X)\to\mathscr{S}(X)$ induces isomorphisms on all homotopy groups. But we have the natural isomorphism $\pi_j(\mathscr{S}(X), x)\cong\pi_j(X, x)$. Consequently

Lemma 1.4. *The following have trivial homotopy groups, and hence are contractible:*

$$\text{a) } \mathscr{E}(0, \boldsymbol{R}^n), \qquad \text{b) } \mathscr{E}(p, S^n-q).$$

In section 3A we will prove moreover

Lemma 1.5. *The following are contractible:*

$$\text{c) } \mathscr{H}_{0,\partial}(D^n), \qquad \text{d) } \mathscr{H}_q^{N(p)}(S^n), \qquad \text{e) } \mathscr{H}^{N(0)}(\boldsymbol{R}^n).$$

Applying lemmas 1.7 and 1.5 to the bundles of lemma 1.2 we obtain

Lemma 1.6. *The following are homotopy equivalences*

a) i: $\mathscr{H}_{p,q}(S^n)\to\mathscr{H}_q(S^n)$.

c) i: $\mathscr{H}_0(\boldsymbol{R}^n)\to\mathscr{H}(\boldsymbol{R}^n)$.

e) γ: $\mathscr{H}_{p,q}(S^n)\to\mathrm{PL}_n$.

f) γ: $\mathscr{H}_0(R^n)\to\mathrm{PL}_n$.

i) ∂: $\mathscr{H}_0(D^n)\to\mathscr{H}(S^{n-1})$.

Moreover we have the isomorphism

$$\Delta:\ \pi_{j+1}(\mathrm{PL}_n)\xrightarrow{\cong}\pi_j(\mathscr{H}_\partial^{N(0)}(D^n)), \qquad j\geqq 0,$$

where Δ is the boundary homomorphism of the homotopy sequence of lemma 1.2 h). Lemma 1.6 implies in Particular (see part I):

Theorem 2. *There is a one-to-one correspondence between equivalence classes of:*

1) *n-dimensional microbundles.*

2) *S^n bundles with a given cross-section.*

3) *$\boldsymbol{R}^n$ bundles with (or without) a given cross-section.*

Lemma 1.6 i) together with $\partial\rho=1$ implies

Lemma 1.7. *$\rho\colon \mathscr{H}(S^{n-1})\to\mathscr{H}_0(D^n)$ is a homotopy equivalence.*

We need two further maps ρ_1 and ρ_2 which are related to ρ.

Definition of ρ_1. $\mathscr{H}(S^{n-1})\to\mathscr{H}_{0,\infty}(S^n)$: By our definitions we may identify $S^n=\partial I^{n+1}$ and $D^n=I^n$. Take two copies of D^n, denoted D_0^n and D_∞^n. Identify ∂I^{n+1} with $D_0^n\cup S^{n-1}\times I\cup D_\infty^n$ in the obvious fashion. Now define $\rho_1(f)$ as $\rho(f)$ in D_0^n as well as in and D_∞^n, and as $f\times 1$ in $S^{n-1}\times I$.

Definition of ρ_2. $\mathscr{H}(S^{n-1})\to\mathscr{H}_0(\boldsymbol{R}^n)$: Identify $(\boldsymbol{R}^n,0)$ with $(S^n-\infty,0)$ by a PL homeomorphism, so that $D^n\subset\boldsymbol{R}^n$ gets identified to $D_0^n\subset S^n$, and $(x_1,\dots,x_{n-1},t)\in\boldsymbol{R}^n$ gets identified to $(x_1,\dots,x_{n-1},t-1)\in D_0^{n-1}\times I\subset S^{n-1}\times I$ for $|x_i|<\delta$, $i=1,\dots,n-1$, $1>\delta>0$, and $1\leqq t\leqq 2$. Then ρ_2 is defined by restriction of ρ_1. ρ_2 has the properties:

a) $\rho_2(f)\,|\,\Delta_k\times D^n=\rho(f)$.

b) $\rho_2(f)=f\times 1$ in a neighborhood of $(\overbrace{0,\dots,0}^{n-1},\tfrac{3}{2})\in\boldsymbol{R}^n$.

From the construction we see that ρ_2 is one to one and it can therefore be considered as an inclusion. Moreover

Lemma 1.8. *The following diagram is commutative*

$$\begin{tikzcd}
\mathscr{H}(S^{n-1}) \arrow[r, "\rho"] \arrow[d, "\rho_2"] & \mathscr{H}_0(D^n) \arrow[d, "\gamma"] \\
\mathscr{H}_0(\boldsymbol{R}^n) \arrow[r, "\gamma_1"] & \mathrm{PL}_n
\end{tikzcd}$$

and ρ (lemma 1.7) *and* γ_1 (lemma 1.6f) *are homotopy equivalences.*

For later use we include

Lemma 1.9. Let

$$\begin{tikzcd}
P \arrow[r, "\alpha"] \arrow[d, "\gamma"] & Q \arrow[d, "\delta"] \\
R \arrow[r, "\beta"] & S
\end{tikzcd}$$

be a commutative diagram of inclusions of spaces and suppose

$$\pi_j(\alpha)\xrightarrow{\cong}\pi_j(\beta) \qquad \text{for all } j.$$

Then

$$\pi_j(\gamma) \xrightarrow{\cong} \pi_j(\delta) \qquad \text{for all } i.$$

Here $\pi_j(\alpha) = \pi_j(Q, P)$, etc.

Proof. From the exact sequences of the triples (S, Q, P) and (S, R, P) we get the commutative diagram.

$$\begin{tikzcd} & \pi_i(R,P) \arrow[d] \arrow[dr] & \\ \pi_i(Q,P) \arrow[r] \arrow[dr, "\cong"'] & \pi_i(S,P) \arrow[r] \arrow[d] & \pi_i(S,Q) \\ & \pi_i(S,R). & \end{tikzcd}$$

Then the exact vertical and horizontal sequences are seen to split, and the conclusion follows.

2. Geometric Realizations and Proof of the Main Theorem

Write $|\mathscr{S}|$ for the geometric realization of a s.s. complex $\mathscr{S}$. $|\mathscr{S}|$ is a CW-complex. If $\mathscr{S}$ is a Kan complex, then $\pi_i(\mathscr{S})$ is naturally isomorphic to $\pi_i(|\mathscr{S}|)$. Furthermore from results in [*1*] and [*8*] follows:

Lemma 2.1. *If*

$$F \subset E \xrightarrow{p} B$$

is a s.s. fibre space, then

$$|F| \subset |E| \xrightarrow{p} |B|$$

is a topological fibre space satisfying the covering homotopy property for finite complexes. (We use the same symbol for the induced map of geometric realizations.) *If B is a Kan complex, and hence E also, then the homotopy sequences of the s.s. fibre space and its geometric realization are naturally isomorphic.*

A *map*

$$E \xrightarrow{p} B$$

of topological spaces satisfying the covering homotopy property (CHP) for all spaces is called a *Serre fibration.* Chosing one fibre $F = p^{-1}(b_0)$, $b_0 \in B$, it is represented by the short sequence of maps

$$F \xrightarrow{i} E \xrightarrow{p} B$$

where i is injection.

A short sequence of maps of topological spaces

$$F \xrightarrow{i} E \xrightarrow{p} B$$

is called a *weak homotopy Serre fibration* (whS fibration) if there exists a Serre fibration

$$\tilde{F} \subset \tilde{E} \xrightarrow{\tilde{p}} B$$

and a homotopy commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle j_F} & & \downarrow{\scriptstyle j_E} & & \downarrow{\scriptstyle =} \\ \tilde{F} & \subset & \tilde{E} & \xrightarrow{\tilde{p}} & B \end{array}$$

such that j_E is a homotopy equivalence and j_F a weak homotopy equivalence.

From the homotopy sequence of the Serre fibration follows immediately *the homotopy sequence of a w.h. Serre fibration* of topological spaces

$$\pi_j(F) \xrightarrow{i_*} \pi_j(E) \xrightarrow{p_*} \pi_j(B) \xrightarrow{\Delta} \pi_{j-1}(F) \longrightarrow . \tag{1}$$

A short sequence of *semi simplicial maps of Kan complexes:*

$$F \xrightarrow{i} E \xrightarrow{p} B \tag{2}$$

is called a w.h. Serre fibration, if the corresponding sequence of geometrical realisations

$$|F| \xrightarrow{i} |E| \xrightarrow{p} |B| \qquad \text{is w.h. Serre}. \tag{3}$$

By lemma 2.1 the exact sequence of (3) is isomorphic to the well defined exact sequence (1) of (2).

Construction of Serre. Given a map of topological spaces $p\colon E \to B$, Serre constructs a Serre fibration as follows:

2.4. Let $\tilde{E} = \{(x, f) \subset E \times B^I \mid p(x) = f(0)\}$, and define $\tilde{p}\colon \tilde{E} \to B$ by $\tilde{p}(x,f) = f(1)$, and let $\tilde{F} = \tilde{p}^{-1}(b_0)$, b_0 a base point in B. Then

$$\tilde{F} \subset \tilde{E} \xrightarrow{\tilde{p}} B$$

is a fibre space satisfying the CHP for all spaces. Let $j\colon E \to \tilde{E}$ be given by $j(x) = (x, f_b)$, where $p(x) = b$ and $f_b(I) = b$. Then j identifies E with a deformation retract of $\tilde{E}$, and

$$\begin{array}{ccccc} p^{-1}(b_0) & \subset & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j} & & \updownarrow{\scriptstyle 1} \\ \tilde{F} & \subset & \tilde{E} & \xrightarrow{\tilde{p}} & B. \end{array}$$

(Warning: j_1 need not be a weak homotopy equivalence here!)

Lemma 2.2. *Let $p: (E, F) \to (B, b_0)$ be a map of pairs of topological spaces inducing isomorphisms on relative homotopy groups. Then*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

is a weak homotopy Serre fibration.

Proof. In the above diagram of the Serre construction we replace $p^{-1}(b_0)$ by $F \subset p^{-1}(b_0)$, and obtain the commutative diagram

$$\begin{tikzcd} F \subset E \arrow[d, "j_F"'] & E \arrow[r] \arrow[d, "j_E"] & B \arrow[d, "="] \\ \tilde{F} \subset \tilde{E} & \tilde{E} \arrow[r] & B \end{tikzcd}$$

with j_E a homotopy equivalence. We now have a diagram with exact rows and isomorphisms as indicated:

$$\begin{tikzcd} \pi_{i+1}(E) \arrow[r] \arrow[d, "\cong"] & \pi_{i+1}(E,F) \arrow[r] \arrow[d, "\cong"] & \pi_i(F) \arrow[r] \arrow[d] & \pi_i(E) \arrow[r] \arrow[d, "\cong"] & \pi_i(E,F) \arrow[d, "\cong"] \\ \pi_{i+1}(\tilde{E}) \arrow[r] & \pi_{i+1}(B) \arrow[r] & \pi_i(\tilde{F}) \arrow[r] & \pi_i(\tilde{E}) \arrow[r] & \pi_i(B). \end{tikzcd}$$

Then the unknown vertical arrow also represents an isomorphism by the five lemma for all i, and j_F is a weak homotopy equivalence. The conclusion of the lemma follows. With lemma 2.1 we conclude immediately to the analogous s.s. lemma:

Lemma 2.2 s.s. *Let $p: (E, F) \to (B, b_0)$ be a map of pairs of Kan complexes inducing isomorphisms on relative homotopy groups. Then*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

is a weak homotopy Serre fibration, and the exact sequence (1) *holds.*

Discussion and proof of a main lemma. We study three diagrams which will be combined into one big diagram.

First we have the commutative diagram of s.s. complexes

$$\begin{tikzcd} \mathscr{H}^{N(0)}(D^n) \arrow[r, "="] \arrow[d] & \mathscr{H}^{N(0)}(D^n) \arrow[d] \\ \mathscr{H}_0(D^n) = \Pi L_n \arrow[r, "\tilde{\mu}"] \arrow[d, "\gamma"] & \mathscr{U}_n \arrow[d, "p"] \\ \mathrm{PL}_n \arrow[r, "\mu"] & \mathscr{B}_n \end{tikzcd} \tag{4}$$

where γ is the principal s.s. bundle with group $\mathscr{H}^{N(0)}(D^n)$ of lemma 1.2g), p is the corresponding universal bundle with base $\mathscr{B}_n$, and μ is the classifying map.

The exact sequence of the fibre bundle p is

$$\longrightarrow \pi_j(\mathscr{U}) \longrightarrow \pi_j(\mathscr{B}_n) \longrightarrow \pi_{j-1}\,\mathscr{H}^{N(0)}(D^n) \longrightarrow \pi_{j-1}(\mathscr{U}_n) \longrightarrow .$$

As $\pi_j(\mathscr{U}_n)=0$ for all j, we get a natural isomorphism

$$\pi_j(\mathscr{B}_n) \xrightarrow{\Delta} \pi_{j-1}(\mathscr{H}^{N(0)}(D^n)). \tag{5}$$

The exact sequence of the fibre bundle γ is

$$\pi_j(\Pi L_n) \longrightarrow \pi_j(\mathrm{PL}_n) \longrightarrow \pi_{j-1}\,\mathscr{H}^{N(0)}(D^n) \longrightarrow \pi_{j-1}(\Pi L_n).$$

Or, in view of (5):

$$\pi_j(\Pi L_n) \longrightarrow \pi_j(\mathrm{PL}_n) \longrightarrow \pi_j(\mathscr{B}_n) \longrightarrow \pi_{j-1}(\Pi L_n). \tag{6}$$

We can also consider the exact sequence of the map γ (which up to homotopy equivalence is an inclusion of s.s. complexes).

$$\pi_j(\Pi L_n) \longrightarrow \pi_j(P\,L_n) \longrightarrow \pi_j(\gamma) \longrightarrow \pi_{j-1}(\Pi L_n)$$

which coincides with (6), hence

$$\pi_j(\gamma) \cong \pi_j(\mathscr{B}_n). \tag{7}$$

Next we recall the diagram of lemma 1.8:

$$\begin{array}{ccc} H(S^{n-1}) & \xrightarrow{\rho} & H_0(D^n) \\ \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \gamma} \\ H_0(\boldsymbol{R}^n) & \xrightarrow{\gamma_1} & \mathrm{PL}_n . \end{array} \tag{8}$$

Because ρ and γ_1 are homotopy equivalences we have for all j: $\pi_j(\rho_2)=\pi_j(\gamma)=\pi_j(\mathscr{B})$. The third diagram has two principal s.s. fibre bundles (lemma 1.2b) and d)) as rows:

$$\begin{array}{ccccc} \mathscr{H}_q(S^{n-1}) & \xrightarrow{i'} & \mathscr{H}(S^{n-1}) & \xrightarrow{r'} & \mathscr{E}(q, S^{n-1}) \\ \downarrow{\scriptstyle \rho_2'} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \sigma} \\ \mathscr{H}_{0,q}(\boldsymbol{R}^n) & \xrightarrow{i} & \mathscr{H}_0(\boldsymbol{R}^n) & \xrightarrow{r} & \mathscr{E}(q, \boldsymbol{R}^n - 0) \end{array} \tag{9}$$

where $p_2'=\rho_2\,|\,\mathscr{H}_q(S^{n-1})$, i, i' inclusions.

The groups

$$\pi_j(i') \cong \pi_j\,\mathscr{E}(q, S^{n-1}) \cong \pi_j(S^{n-1}) \qquad \text{(lemma 1.3)}$$

and

$$\pi_j(i) \cong \pi_j\,\mathscr{E}(q, R^n - 0) \cong \pi_j(S^{n-1})$$

are isomorphic for all j by virtue of the vertical maps in (9). Then by the s.s. version of lemma 1.9 there is an isomorphism

$$\pi_j(\rho_2') \xrightarrow{\cong} \pi_j(\rho_2)$$

by virtue of the horizontal maps in (9).

We combine the above diagrams into the commutative diagram

$$\begin{tikzcd} \mathscr{H}_q(S^{n-1}) \arrow[r, "i'"] \arrow[d, "\rho_2'"] & \mathscr{H}(S^{n-1}) \arrow[r, "\rho"] \arrow[d, "\rho_2"] & \Pi L_n \arrow[r, "\tilde{\mu}"] \arrow[d, "\gamma"] & \mathscr{U}_n \arrow[d, "p"] \\ \mathscr{H}_{0,q}(R^n) \arrow[r, "i"] & \mathscr{H}_0(R^n) \arrow[r, "\gamma_1"] & \mathrm{PL}_n \arrow[r, "\mu"] & \mathscr{B}_n . \end{tikzcd} \tag{10}$$

i', i, ρ_2', ρ_2 are one to one and can be considered as inclusions. ρ and γ_1 are homotopy equivalences. $\mathscr{U}$ is homotopy trivial. Hence for example $p\tilde{\mu}\rho$ is a homotopy trivial map. Then because ρ_2 is an inclusion, $\mu\gamma_1\colon \mathscr{H}_0(R^n) \to \mathscr{B}_n$ is homotopic to a map μ_1 which sends $\mathscr{H}(S^{n-1}) = \rho_2 \mathscr{H}(S^{n-1}) \subset \mathscr{H}_0(R^n)$ into $b_0 \in \mathscr{B}_n$. With this new map we have the commutative diagram

$$\begin{tikzcd} \mathscr{H}_q(S^{n-1}) \arrow[r, "i'"] \arrow[d, "\rho_2'"] & \mathscr{H}(S^{n-1}) \arrow[r, "\mu_1"] \arrow[d, "\rho_2"] & b_0 \arrow[d] \\ \mathscr{H}_{0,q}(R^n) \arrow[r, "i"] \arrow[d] & \mathscr{H}_0(R^n) \arrow[r, "\mu_1"] \arrow[d, "r"] \arrow[dr, "r\times\mu_1"] & \mathscr{B}_n \arrow[d, "q\times id"] \\ q \arrow[r] & \mathscr{E}(q, R^n-0) \arrow[r, "id\times b_0"] & \mathscr{E}(q, R^n-0)\times B_n . \end{tikzcd} \tag{11}$$

Main Lemma 2.3. *The main diagonal of* (11)

$$\mathscr{H}_q(S^{n-1}) \xrightarrow{\rho_2 i'} \mathscr{H}_0(R^n) \xrightarrow{r\times\mu_1} \mathscr{E}(q, R^n-0)\times\mathscr{B}_n \tag{12}$$

is a weak homotopy Serre fibration.

Proof. $r\times\mu_1$ defines a mapping of triples of s.s. complexes:

$$[\mathscr{H}_0(R^n), \mathscr{H}(S^{n-1}), \mathscr{H}_q(S^{n-1})]$$
$$\xrightarrow{r\times\mu_1} [\mathscr{E}(q, R^n-0)\times\mathscr{B}_n, \mathscr{E}(q, R^n-0)\times b_0, q\times b_0].$$

Then there is a natural homomorphism of the exact sequence of homotopy groups of the first triple into that of the second. Some of these homomorphisms of homotopy groups of pairs are:

I.
$$\begin{tikzcd} \pi_j[\mathscr{H}(S^{n-1}), \mathscr{H}_q(S^{n-1})] \arrow[r] \arrow[d, equal] & \pi_j[\mathscr{E}(q, R^n-0)\times b_0, q\times b_0] \arrow[d, "\cong"] \\ \pi_j(i') & \pi_j(S^{n-1}) . \end{tikzcd}$$

But this is an isomorphism for all j (see (8)).

$$\text{II.}\qquad \begin{array}{ccc} \pi_j[\mathscr{H}_0(R^n), \mathscr{H}(S^{n-1})] & \longrightarrow & \pi_j[\mathscr{E}(q, R^n-0)\times B_n, \mathscr{E}(q, R^n-0)\times b_0] \\ \Vert & & \downarrow\cong \\ \pi_j(\rho_2) & & \pi_j(\mathscr{B}_n, b_0)\,. \end{array}$$

But this is an isomorphism (see (8)) for all j.

Then the remaining homomorphisms are also isomorphisms by the five lemma (for all j):

$$\text{III.}\quad \pi_j[\mathscr{H}_0(R^n), \mathscr{H}(S^{n-1})] \xrightarrow{r\times\mu_1} \pi_j(\mathscr{E}(q, R^n-0)\times\mathscr{B}_n, q\times b_0)\,.$$

Consequently

$$\pi_j(\rho_2 i') \xrightarrow{\cong} \pi_j(\mathscr{E}(q, R^n-0)\times\mathscr{B}_n)$$

and the main lemma follows by applying lemma 2.2 s.s.

The exact sequence of (12) is

$$\pi_j\mathscr{H}_q(S^{n-1}) \xrightarrow{\rho_2 i'} \pi_j\mathscr{H}_0(R^n)$$
$$\xrightarrow{r\times\mu_1} \pi_j(\mathscr{E}(q, R^n-0)\times\mathscr{B}_n) \longrightarrow \pi_{j-1}\mathscr{H}_q(S^{n-1})\,.$$

Or, with the homotopy equivalences of lemma 1.6

$$\pi_j(\mathrm{PL}_{n-1}) \longrightarrow \pi_j(\mathrm{PL}_n) \longrightarrow \pi_j(S^{n-1}) \oplus \pi_j(\mathscr{B}_n) \longrightarrow \pi_{j-1}(\mathrm{PL}_{n-1})\,. \qquad (12)$$

From the exact sequence of the universal bundle

$$p\colon \mathscr{U}_n \to \mathscr{B}_n \quad \text{follows} \quad \pi_j(\mathscr{B}_n) \cong \pi_{j-1}(\mathscr{H}^{N(0)}(D^n))\,.$$

The main theorem 1 now follows from (12) with

Lemma 2.4. *There is a homotopy equivalence*

$$\lambda\colon\ C_{n-1} \longrightarrow \mathscr{H}^{N(0)}(D^n)\,.$$

This will be proved in section 3B. Observe that in (6) we now can replace $\pi_j(\mathscr{B}_n)$ by $\pi_{j-1}(C_{n-1})$.

The map which gives rise to the homomorphism $\pi_j(\mathrm{PL}_{n-1}) \to \pi_j(\mathrm{PL}_n)$ has been defined as a composition of a number of homotopy equivalences and the map $\rho_2 i'$. We will now prove that this map is homotopy equivalent to the natural embedding

$$\mathrm{PL}_{n-1} \xrightarrow{i} \mathrm{PL}_n\,.$$

Hence in the theorem and in (12) the homomorphism is induced by i. This is useful when we want to define the direct limit $\mathrm{PL} = \lim \mathrm{PL}_n$ in part III.

Lemma 2.5.

$$\begin{tikzcd}
& \mathcal{H}_q(S^{n-1}) \arrow[r, "i'"] \arrow[dd, "\bar{r}"] & \mathcal{H}(S^{n-1}) \arrow[dd, "\rho_2"] \\
\mathcal{H}_{p,q}(S^{n-1}) \arrow[ur] \arrow[dr, "r"'] & & \\
& \mathcal{H}_0(R^{n-1}) \arrow[r, "i"] \arrow[d, "\gamma"] & \mathcal{H}_0(R^n) \arrow[d, "\gamma"] \\
& \mathrm{PL}_{n-1} \arrow[r, "i"] & \mathrm{PL}_n
\end{tikzcd}$$

homotopy commutes, where

$$\bar{r}\colon \mathcal{H}_q(S^{n-1}) \overset{i^{-1}}{\approx} \mathcal{H}_{p,q}(S^{n-1}) \xrightarrow{r} \mathcal{H}_0(R^{n-1}),$$

i^{-1} a homotopy inverse of

$$i\colon \mathcal{H}_{p,q}(S^{n-1}) \to \mathcal{H}_q(S^{n-1}).$$

Proof. Since the bottom square commutes, and γ is a homotopy equivalence, it suffices to prove:

$$\gamma\rho_2 i'\colon \mathcal{H}_{p,q}(S^{n-1}) \xrightarrow{i'} \mathcal{H}(S^{n-1}) \xrightarrow{\rho_2} \mathcal{H}_0(R^n) \xrightarrow{\gamma} \mathrm{PL}_n$$

and

$$\gamma i r\colon \mathcal{H}_{p,q}(S^{n-1}) \xrightarrow{r} \mathcal{H}_0(R^{n-1}) \xrightarrow{i} \mathcal{H}_0(R^n) \xrightarrow{\gamma} \mathrm{PL}_n$$

are homotopic.

Referring to the definition of ρ_2, if we identify $q\in S^{n-1}$ with $0\in D_0^{n-1}$, and also with $(0, \ldots, 0, \frac{3}{2})\in R^n$, then

$$\rho_2' = \rho_2 i'\colon \mathcal{H}_{p,q}(S^{n-1}) \to \mathcal{H}_{0,q}(R^n) \subset \mathcal{H}_0(R^n);$$

and $\rho_2'(f) = f\times 1$ in a neighborhood of q.

Let $T_x^c\colon \mathcal{H}(R^n)\to\mathcal{H}(R^n)$ be given by $T_x^c(f)=(1\times T_x)f(1\times T_x)$, $T_x\colon R^n\to R^n$ translation by $x\in R^n$. Then T_x^c is homotopic to the identity, and $T_x^c\colon \mathcal{H}_x(R^n)\to\mathcal{H}_0(R^n)$. Since $\mathcal{H}_0(R^n)\to\mathcal{H}(R^n)$ is a homotopy equivalence,

$$\begin{tikzcd}
\mathcal{H}_{0,x}(R^n) \arrow[r, "i_0"] \arrow[d, "i_x"] & \mathcal{H}_0(R^n) \\
\mathcal{H}_x(R^n) \arrow[ur, "T_x^c"'] &
\end{tikzcd}$$

homotopy commutes.

Thus $\rho_2 i$ is homotopic to a map $\tilde{\rho}_2$ such that $\tilde{\rho}_2(f)=f\times 1$ in a neighborhood of zero, and hence $\gamma\tilde{\rho}_2=\gamma i r$.

Remark. T_x^c is actually homotopic to the identity through homomorphisms and consequently one may show that the diagram of lemma 2.5 with the groups replaced by their universal base spaces and induced maps is homotopy commutative.

By the same arguments as in the proof of the main lemma one may show

Proposition 2.6.

$$\mathrm{PL}_{n-1} \xrightarrow{\rho_2'(\gamma)^{-1}} \mathscr{H}_{0,q}(R^n) \xrightarrow{\mu_1 i} \mathscr{B}_n$$

is a weak homotopy Serre fibration.

This will be needed in part III.

3A. Homotopy Groups of $\mathscr{H}(X)$

Proof of Lemma 1.5.

Let S^{k-1} be the semi-simplicial complex generated by the simplicial complex $\partial\Delta_k$ by throwing in the degeneracies. The homotopy group $\pi_k(K)$ of a Kan complex K is the group of s.s. homotopy classes of maps of S^{k-1} into K. A homotopy class is represented by a s.s. map φ: $S^{k-1} \to K$; and such a map is determined by choosing a simplex of K for each non-degenerate simplex of $\partial\Delta_k$ so that they fit together along the faces.

For $\mathscr{H}(X)$, φ: $S^{k-1} \to \mathscr{H}(X)$ is therefore equivalent to a bundle isomorphism f: $\partial\Delta_k \times X \to \partial\Delta_k \times X$. Further, φ is homotopic to zero, if and only if f extends to a bundle isomorphism g: $\Delta_k \times X \to \Delta_k \times X$. Similar considerations hold for various subgroups of $\mathscr{H}(X)$.

Proof of Lemma 1.5.

1. $\mathscr{H}_{0,\partial}(D^n)$ *is contractible.*

We must show that the homotopy groups are trivial: If f: $(\partial\Delta_k) \times D^n \to (\partial\Delta_k) \times D^n$ is a bundle isomorphism which is the identity on $(\partial\Delta_k) \times \partial D^n$, then f can be extended to a bundle isomorphism g: $\Delta_k \times D^n \to \Delta_k \times D^n$ which is the identity on $\Delta_k \times \partial D^n$.

Now f can be extended to $\Delta_k \times \partial D^n$ as the identity, and hence f is defined on $\partial(\Delta_k \times D^n)$. Let b be the barycenter of Δ_k, then $\Delta_k \times D^n$ is the cone over $\partial(\Delta_k \times D^n)$ with vertex $b \times 0$. Define g on $b \times 0$ as the identity, and set $g=f$ on $\partial(\Delta_k \times D^n)$. Then extend g linearly to all of $\Delta_k \times D^n$. Then g commutes with projection and preserves the zero section since f does; i.e., g is a bundles isomorphism which extends f and is the identity on $\Delta_k \times \partial D^n$.

2. $\mathscr{H}_q^{N(p)}(S^n)$ *is contractible.*

A $(k-1)$-dimensional homotopy class is represented by a bundle isomorphism f: $(\partial\Delta_k) \times S^n \to (\partial\Delta_k) \times S^n$ which leaves $(\partial\Delta_k) \times q$ fixed, and for each simplex $\Delta_1 \subset \partial\Delta_k$, $f|\Delta_1 \times S^n$: $\Delta_1 \times S^n \to \Delta_1 \times S^n$ is the identity on some neighborhood, possibly different for each simplex, of p. Since only a finite number of simplices are involved, there is a *common* neighborhood of p which is left fixed by all of them. Choose a disc D_p^n about p, so that f is the identity on $(\partial \Delta_k)\times D_p^n$. Let D_q^n be the closure of the complement of D_p^n in S^n. Then $f|(\partial\Delta_k)\times D_q^n$ is the identity on $(\partial\Delta_k)\times q$ and $(\partial\Delta_k)\times\partial D_q^n$. Hence by (1) above, $f|(\partial\Delta_k)\times D_q^n$ is extendable to $\Delta_k\times D_q^n$ with the same properties. Extending f over $\Delta_k\times D_p^n$ as the identity, f is extendable to g: $\Delta_k\times S^n\to\Delta_k\times S^n$ such that g is the identity on q and in a neighborhood of p. But this means that the homotopy class is trivial.

3. $\mathscr{H}^{N(0)}(\boldsymbol{R}^n)$ *is contractible.*

By the same argument as in (2), there is a disc D_0^n so that the bundle isomorphism f: $(\partial\Delta_k)\times\boldsymbol{R}^n\to(\partial\Delta_k)\times\boldsymbol{R}^n$ is the identity on $(\partial\Delta_k)\times D_0^n$. Let $\hat{D}^n$ be the closure of the complement of D_0^n in R^n. $\hat{D}^n$ is a disc with its center ∞ removed. $f|(\partial\Delta_k)\times\hat{D}^n$ is the identity on $\partial\Delta_k\times\partial\hat{D}^n$. By the same argument as in (1), $f|(\partial\Delta_k)\times\hat{D}^n$ may be extended to $(\Delta_k\times\hat{D}^n)\cup(b\times\infty)$, so as to be the identity on $\Delta_k\times\partial D^n$ and on $b\times\infty$. Hence we can throw away the point $b\times\infty$, and extending f to $\Delta_k\times D_0^n$ as the identity, we get an extension g: $\Delta_k\times\boldsymbol{R}^n\to\Delta_k\times\boldsymbol{R}^n$ which is the identity on a neighborhood of the zero section. Hence the homotopy class represented by f is trivial, and $\mathscr{H}^{N(0)}(\boldsymbol{R}^n)$ is contractible.

3B. The Homotopy Equivalence λ

A map λ: $C_{n-1}\to\mathscr{H}^{N(0)}(D^n)$ is defined as follows:

Let f: $\Delta_k\times(S^{n-1}\times I)\to\Delta_k\times(S^{n-1}\times I)$ be in C_{n-1}. By identifying $S^{n-1}\times I$ to a fixed product neighborhood V of ∂D^n in D^n, and extending f to D^n-V as the identity; we may identify C_{n-1} with $\mathscr{H}_N(D^n)$, $N=\overline{D^n-V}$. Then λ is the morphism $C_{n-1}\equiv\mathscr{H}_N(D^n)\subset\mathscr{H}^{N(0)}(D^n)$ of group complexes.

Lemma 3.1. λ: $C_{n-1}\to\mathscr{H}^{N(0)}(D^n)$ *is a homotopy equivalence.*

We will show that the map λ: $C_{n-1}\to\mathscr{H}^{N(0)}(D^n)$ induces an isomorphism on homotopy groups. Then λ is a weak homotopy equi valence hence a homotopy equivalence.

λ is epimorphic (onto). Let f: $\partial\Delta_k\times D^n\to\partial\Delta_k\times D^n$ represent a homotopy class of $\mathscr{H}^{N(0)}(D^n)$. Just as in 3A one can show that there exist a fixed disc neighborhood N' of zero in D^n such that f is the identity on $\partial\Delta_k\times N'$. On the other hand, C_{n-1} has been identified to bundle isomorphisms f which are the identity on a disc neighborhood N.

Let h: $D^n\times I\to D^n\times I$ be an ambient isotopy sending N to N' which is fixed in a neighborhood of the boundary and in a neighborhood of zero. Then $H=(1_{\partial\Delta_k}\times h^{-1})\circ(f\times 1_I)\circ(1_{\partial\Delta_k}\times h)$ is an isotopy of f in $\mathscr{H}^{N(0)}(D^n)$ to a bundle isomorphism g in C_{n-1}, and represents the same homotopy class as f. I.e., λ is onto.

λ is monomorphic. Let $f: \partial \Delta_k \times D^n \to \partial \Delta_k \times D^n$, $f | \partial \Delta_k \times N$ the identity, represents a homotopy class in C_{n-1}, and suppose f is homotopic to zero in $\mathscr{H}^{N(0)}(D^n)$. Then f may be extended to $g: \Delta_k \times D^n \to \Delta_k \times D^n$, $g \in \mathscr{H}^{N(0)}(D^n)$. As above, it follows that $g | \Delta_k \times N'$ is the identity for some neighborhood N' of zero. We take for N' a disc neighborhood in N. Then we may take h as above such that $h_t(N) \subset N'$. Let $H = (1_{\Delta_k} \times h^{-1}) \circ (g \times 1_I) \circ (1_{\Delta_k} \times h)$. Then $H_1 | \partial \Delta_k \times D^n$ is homotopic to f in C_{n-1} and extends to the bundle isomorphism H_1 over Δ_k in C_{n-1}. Hence f is trivial in C_{n-1}, and λ is monomorphic.

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