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Low-dimensional concordances, Whitney towers and isotopies

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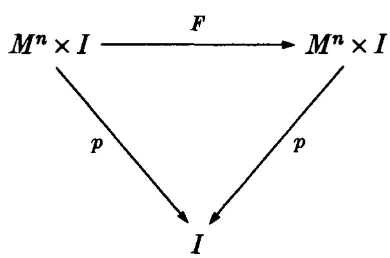
Introduction

Let M^n be a smooth closed n-dimensional manifold and let DIFF (M^n) be the group of diffeomorphisms of M^n. Two diffeomorphisms $f_0, f_1 \in \text{DIFF}(M^n)$ are said to be concordant (pseudo-isotopic) if there is a diffeomorphism $F \in \text{DIFF}(M^n \times I)$, where $I = [0,1]$, such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in M^n$.

The diffeomorphisms $f_0, f_1 \in \text{DIFF}(M^n)$ are said to be isotopic if there exists a diffeomorphism $F \in \text{DIFF}(M^n \times I)$ such that

(1) $F(x,i) = f_i(x)$ $(i = 0, 1)$.
(2) The diagram



is commutative.

Obviously isotopic diffeomorphisms are concordant. The converse is in general not true (see [**7**, **9**]) but there is the following celebrated result due to J. Cerf (see [**3**]).

Theorem (Cerf). *Let M^n be a smooth closed simply connected manifold with $n \geq 5$. Then concordant diffeomorphisms are isotopic.*

In this paper we consider the case $n = 4$. Our investigations were motivated by the following question of L. Siebenmann [**20**].

P.I. Is it true that every topological automorphism of the pair $(M^4 \times I, \partial(M^4 \times I))$ that fixes $M^4 \times \{0\}$ pointwise is topologically isotopic to the identity fixing $M^4 \times \{0\}$? In particular, is the restriction $M^4 \times \{1\} \to M^4 \times \{1\}$ topologically isotopic to the identity?

Our goal is to prove that the answer to this question is 'no'. More precisely, we show that on the 4-dimensional torus T^4 there are diffeomorphisms which are pseudo-isotopic but not isotopic (even topologically) to the identity. This result provides a negative answer to a problem about the existence of such homeomorphisms posed by A. Hatcher (see [**11**], problem 4·35). Our proof of the existence of homeomorphisms of T^4 which are pseudo-isotopic but not isotopic to the identity is a natural extension of an analogous result proved by Hatcher and Wagoner (see [**7**, **8**]) in the case of T^n $(n \geq 5)$. We will not be able to generalize all of Hatcher and Wagoner's results to dimension 4, but the generalization which we obtain will enable us to provide a negative answer to Siebenmann's question. We will follow Hatcher and Wagoner's considerations from [8] very closely and will only supply necessary changes which are required to make their arguments work in the 4-dimensional case. In fact, what we are going to show is that the only missing ingredient in dimension 4 was the Whitney Lemma (cf. [13]). A topological version of this lemma is now known (see [4–6]) and hence part of Hatcher and Wagoner's considerations can be extended to dimension four.

From the geometric point of view the proof of the existence of pseudo-isotopic but not isotopic homeomorphisms on T^4 which we present has a rather unpleasant feature. Namely, it uses an abstract and heavy algebraic machinery developed by Hatcher and Wagoner in [8]. But it seems to us that there is no alternative. (We do not know of any construction of homeomorphisms pseudo-isotopic to the identity but not isotopic to it which really avoids algebraic considerations; cf. [7], p. 9.) On the other hand, this algebraic approach can be used to provide a proof (certainly not the favourite one) of the following 4-dimensional version of Cerf's theorem (cf. [16, 18]):

THEOREM. *Let M^4 be a smooth closed simply connected 4-dimensional manifold. The concordant diffeomorphisms are topologically isotopic.*

This paper is self-contained in the sense that it contains the main line of Hatcher and Wagoner's considerations. Many details in our exposition are omitted; this is a consequence of the assumption of familiarity with [3] and [8].

2. *Concordances and Whitehead groups*

In this section we recall basic facts concerning concordances following Cerf's functional approach (see [3, 8]). We will also recall (following [15]) some definitions and constructions connected with algebraic K-theory.

Let $(M, \partial M)$ be a compact smooth manifold. A pseudo-isotopy of $(M, \partial M)$ is a diffeomorphism $F\colon (M,\partial M)\times I \to (M,\partial M)\times I$ such that $F|_{(M,\partial M)\times\{0\}}$ is the identity and $F|_{\partial M\times I}$ is an isotopy. Let $\mathscr{P} = \mathscr{P}(M,\partial M)$ denote the group of pseudo-isotopies (group multiplication = composition) equipped with the C^∞-topology. Consider the space $\mathscr{F}$ of C^∞-functions $f\colon (M,\partial M)\times I \to I$ such that $f(x,0) = 0$ and $f(x,1) = 1$ for all $x\in M$ and such that $f(x,t) = t$ for all $x \in \partial M$. Let $\mathscr{E} \subset \mathscr{F}$ be the subset consisting of those functions with no critical points and let $p\colon (M,\partial M)\times I \to I$ be the standard projection. The correspondence $g \rightsquigarrow pg$ induces a fibration

$$\mathscr{T} \hookrightarrow \mathscr{P} \to \mathscr{E}$$

with the fibre $\mathscr{T}$ being the space of isotopies of the identity of M. Now, because $\mathscr{T}$ is contractible, we have a homotopy equivalence

$$\mathscr{P} \to \mathscr{E}.$$

The contractibility of $\mathscr{T}$ implies

$$\pi_i(\mathscr{P}) \cong \pi_i(\mathscr{E}) = \pi_{i+1}(\mathscr{F}, \mathscr{E}, p).$$

Therefore, to show, for example, that $\pi_0(\mathscr{P}) = 0$ it is enough to show that each path in $\mathscr{F}$ joining p with pg can be deformed, keeping endpoints fixed, to a path lying in $\mathscr{E}$. Note (cf. [3]) that $\pi_0(\mathscr{P}) = 0$ says 'pseudo-isotopy implies isotopy'. The space $\mathscr{F}$ has a natural stratification (see [3]):

$$\mathscr{F} = \mathscr{F}^0 \cup \mathscr{F}^1 \cup \mathscr{F}^2 \cup \ldots \cup \mathscr{F}^\infty,$$

where each $\mathscr{F}^k$ has codimension k in $\mathscr{F}$. It was shown in [3] that each path in $\mathscr{F}$ (rel. $\mathscr{E}$) can be approximated (rel. endpoints) by a 'generic' path lying in $\mathscr{F}^0 \cup \mathscr{F}^1$ such that this path is transverse to $\mathscr{F}^1$.

For each path $f_t, t \in [0,1]$ (path = 1-parameter family of maps) one defines the graphic of f_t as the subset of $I \times I$ consisting of all pairs (t, u) such that u is a critical value of f_t. For example, the graphics (death and birth), Fig. 1, show the lines which are images of critical points of index i and $i+1$.

Now following [3] and [8] we introduce the gradient-like vector fields.

By a gradient-like vector field we mean a triple (η, f, μ) where $f: M \times I \to I$ is an element in $\mathscr{F}$ and η is a vector field on $M \times I$ that is gradient-like for f with respect to the Riemannian metric μ on $M \times I$. (Gradient-like means that (1) $df(x).\eta(x) > 0$ whenever x is not a critical point of f and (2) near each critical point $\eta(x) = \operatorname{grad} f(x)$.)

Let $\hat{\mathscr{F}}$ denote the space of gradient-like vector fields and let $\hat{\mathscr{E}} \subset \hat{\mathscr{F}}$ be the subspace consisting of the triples (η, f, μ), where f has no critical points. The importance of gradient-like vector fields lies in the fact that there is a homotopy equivalence of pairs

$$(\mathscr{F}, \mathscr{E}) \simeq (\hat{\mathscr{F}}, \hat{\mathscr{E}}).$$

In particular $\pi_1(\mathscr{F}, \mathscr{E}, p) \cong \pi_1(\hat{\mathscr{F}}, \hat{\mathscr{E}}, \hat{p})$.

Let $q \in M \times I$ be an isolated critical point of f. Let ϕ_t be the one-parameter family of diffeomorphisms generated by η. We define the stable set of q as

$$W(q) = \{x \in M \times I \mid \lim_{t \to \infty} \phi_t(x) = q\}$$

and the unstable set of q as

$$W^*(q) = \{x \in M \times I \mid \lim_{t \to -\infty} \phi_t(x) = q\}.$$

Now we recall some basic facts concerning algebraic K-theory (cf. [15]).

Let Λ be a ring. Denote by $[x, y]$ the commutator of x and y.

Definition 2·1. For $n \geqslant 3$ the Steinberg group $\mathrm{St}(n, \Lambda)$ is the group defined by generators x_{ij}^λ subject to the relations:

(1) $x_{ij}^\lambda x_{ij}^\mu = x_{ij}^{\lambda+\mu}$,
(2) $[x_{ij}^\lambda, x_{jl}^\mu] = x_{il}^{\lambda\mu} \quad (i \neq l)$,
(3) $[x_{ij}^\lambda, x_{kl}^\mu] = 1 \quad (j \neq k, i \neq l)$.

Now let $e_{ij}^\lambda \in GL(n, \Lambda)$ denote the elementary matrix with entry λ in the (i,j)th place. Let $E(\Lambda) \subset GL(\Lambda)$ be the subgroup generated by all elementary matrices. There is a canonical homomorphism

$$\phi: \mathrm{St}(n, \Lambda) \to GL(n, \Lambda)$$

given by

$$\phi(x_{ij}^\lambda) = e_{ij}^\lambda,$$

and consequently there is a homomorphism

$$\phi: \mathrm{St}(\Lambda) \to GL(\Lambda)$$

with $\phi(\mathrm{St}(\Lambda)) \subset E(\Lambda)$, where $\mathrm{St}(\Lambda) = \lim_{n \to \infty} \mathrm{St}(n, \Lambda)$.

Definition 2·2. The kernel of the homomorphism $\phi: \mathrm{St}(\Lambda) \to GL(\Lambda)$ is called $K_2(\Lambda)$.

In fact, we will be interested in the Whitehead group $Wh_2(\Lambda)$ rather than in $K_2(\Lambda)$. Therefore, we recall its definition.

Fig. 1

Let $\Lambda = Z[\pi_1(M)]$ and let $W(\pm\pi_1(M)) \subset \mathrm{St}\,(Z[\pi_1(M)])$ be the subgroup generated by the words $w_{ij}^{\pm g}$, where

$$w_{ij}^{\pm g} = x_{ij}^{\pm g}\, x_{ji}^{\pm g^{-1}}\, x_{ij}^{\pm g} \quad \text{for} \quad g \in \pi_1(M).$$

$$Wh_2(\pi_1(M)) = K_2(Z[\pi_1(M)])/K_2(Z[\pi_1(M)]) \cap W(\pm\pi_1(M)).$$

We refer to [**8**] for more information concerning Wh_2.

We close this section by introducing the group $Wh_1(\pi_1(M); Z_2 \times \pi_2(M))$.

Let $(Z_2 \times \pi_2(M))[\pi_1(M)]$ denote the additive group of finite formal sums $\sum \alpha_i \sigma_i$, where $\alpha_i \in Z_2 \times \pi_2(M)$ and $\sigma_i \in \pi_1(M)$. Let $(\beta.1, \alpha\sigma - \alpha^\tau \tau\sigma\tau^{-1})$ be the additive subgroup of $(Z_2 \times \pi_2(M))[\pi_1(M)]$ generated by the elements $\beta.1$ and $\alpha\sigma - \alpha^\tau\tau\sigma\tau^{-1}$, where $\alpha, \beta \in Z_2 \times \pi_2(M), \sigma, \tau \in \pi_1(M), 1 \in \pi_1(M)$ is the identity, and α^τ denotes τ acting trivially on the Z_2 component of α and in the usual way on the π_2-factor. Now

$$Wh_1(\pi_1(M); Z_2 \times \pi_2(M)) = \frac{(Z_2 \times \pi_2(M))[\pi_1(M)]}{(\beta.1, \alpha\sigma - \alpha^\tau\tau\sigma\tau^{-1})}.$$

Observe that if $\pi_2(M) = 0$ and $\pi_1(M)$ is abelian, then

$$Wh_1(\pi_1(M); Z_2) = Z_2[\pi_1(M)]/Z_2(1).$$

Example 2·3 (cf. [**7**]). Let T^n be an n-dimensional torus. Then

$$Wh_1(\pi_1(T^n); Z_2) \cong Z_2[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]/Z_2(1) \cong Z_2 \oplus Z_2 \oplus \ldots \cong Z_2^\infty.$$

3. *Concordance implies isotopy*

In this section we show that concordant diffeomorphisms $f_0, f_1 \in \mathrm{DIFF}\,(M^4)$ are topologically isotopic when M^4 is simply connected.

Let $g \in \hat{\mathscr{F}}, g: M^4 \times I \to I$, be a nice Morse function satisfying the following condition:

(*) g has exactly r critical points $p_1, \ldots, p_r$ of index i and r critical points $q_1, \ldots, q_r$ of index $i+1$.

Let $\mathscr{A} \subset \hat{\mathscr{F}}$ be the space of pairs (f, η), where f satisfies (*) (here μ is omitted and we write simply (f, η) instead of (f, η, μ)). Let $\mathscr{B} \subset \mathscr{A}$ be the subspace of pairs (f, η) such that if p_i and p_j are critical points of f then

$$W(p_i) \cap W^*(p_j) = \varnothing \qquad \text{for} \quad p_i \neq p_j,$$

and the same holds for the critical points $q_1, \ldots, q_r$.

Now it is known (see [**8**], p. 80) that if $r \geqslant 3$ and $i = 2$ (or $i = 3$ by the symmetry), there is a bijection

$$\Delta_i \colon \mathrm{St}\,(r, Z[\pi_1(M^4 \times I)]) \to \pi_1(\mathscr{A}, \mathscr{B}, (f_0, \eta_0)),$$

where $(f_0, \eta_0) \in \mathscr{B}$ is a base point.

The map Δ can be described briefly as follows (cf. [**8**]).

Let $(h, \zeta) \in \mathscr{B}$ be given and let $p_1, \dots, p_r$ be (ordered) critical points of h. Assume that the p_j $(j = 1, \dots, r)$ are connected to a base point by paths γ_j and choose an orientation for each stable manifold $W(p_j)$. This gives a basis $\epsilon_1, \dots, \epsilon_r$ for the naturally determined chain complex C_2. (C_2 is considered as a $Z[\pi_1(M)]$-module.) Suppose that p_α, p_β are two critical points of h with $h(p_\alpha) > h(p_\beta)$. Let $x^\lambda_{\alpha\beta}$ be the Steinberg generator with $\lambda \in Z[\pi_1(M^4)]$. Under the above conditions there is a well-defined path $x^\lambda_{\alpha\beta}.(h, \xi) = (h_t, \xi_t)$, $t \in [0, 1]$ such that the stable manifolds of ξ_1 determine the basis $\epsilon_1, \dots, \epsilon_\alpha + \lambda\epsilon_\beta, \dots, \epsilon_r$ of C_2. Now the map Δ_i is defined as follows.

Let $z \in \mathrm{St}\,(r, Z[\pi_1(M^4 \times I)])$ be represented by the word

$$z = \prod_{j=1}^{m} z_j,$$

where each z_j is of the form $x^{\lambda_j}_{\alpha_j\beta_j}$ for $\lambda_j \in Z[\pi_1(M^4 \times I)]$. Then $\Delta_i(z) = z_m \cdot (z_{m-1} \cdots (z_1(f_0, \eta_0))$.

It is proved in [**8**] that Δ_i is well defined and indeed is a bijection. In fact there exists a map

$$\chi_i \colon \pi_1(\mathscr{A}, \mathscr{B}, (f_0, \eta_0)) \to \mathrm{St}\,(r, Z[\pi_1(M^4 \times I)]),$$

which is the inverse of the map Δ_i (see [**8**], p. 91). The map $\Delta_i \colon \mathrm{St}\,(r, Z[\pi_1(M^4 \times I)]) \to$-$\pi_1(\mathscr{A}, \mathscr{B}, (f_0, \eta_0))$ forms a step in the construction of a homomorphism

$$\Sigma \colon \pi_0(\mathscr{P}) \to Wh_2(\pi_1(M^4)).$$

The general construction of this homomorphism requires rather heavy algebraic machinery (see [**8**]). In our case one can give a reasonably simple description of this homomorphism by using the following geometric result (see [**8**], p. 184).

(**) Each one-parameter family (f_t, η_t), $f_t \colon M^4 \times I \to I$ can be deformed (rel. endpoints) to a family whose non-degenerate critical points are of index 2 and 3.

One can also assume (cf. [**8**]) that the gradient-like vector field η_t subordinated to f_t is in general position and that all birth-death points of f_t are independent.

Remark 3·1. The above result implies that the graphic of f_t looks as shown in Fig. 2. Here is the description of $\Sigma \colon \pi_0(\mathscr{P}) \to Wh_2(\pi_1(M^4))$.

Suppose that all critical points of index 3 are ordered and that stable manifolds of these points are oriented. Choose paths to a fixed basepoint for the arcs of critical points of index 3. Let the algebraic 3/2 intersection number of a birth pair be $+1 \in Z[\pi_1(M^4)]$. This gives preferred choices for the arcs of critical points of index 2. In all t-slices which do not contain 3/3 or 2/2 intersections we have the algebraic intersection matrix in $GL(Z[\pi_1(M^4)])$. Near $t = 0$ this algebraic 3/2 intersection matrix is the identity matrix. Now as t passes a 2/2 (3/3) intersection the matrix changes by right or (left) multiplication by an elementary matrix e^σ_{jk} for some

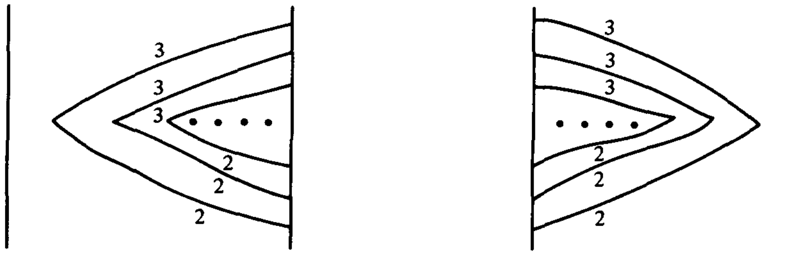


Fig. 2

$\sigma \in \pm\pi_1(M^4)$ and for some j and k. Near $t = 1$ the matrix is of the form $P.D =$ (permutation).(diagonal with entries in $\pm\pi_1(M^4) \subset Z[\pi_1(M^4)]$). Therefore one has

$$P.D = \prod_t e^{\sigma_t}_{j_t k_t}.$$

On the other hand (cf. [**8**], p. 107) $P.D$ can be represented as

$$P.D = \prod_s e^{\tau_s}_{p_s q_s} e^{-\tau_s^{-1}}_{q_s p_s} e^{\tau_s}_{p_s q_s}$$

for some $\tau_s \in \pi_1(M^4)$. Consequently the element

$$\Pi = \Big(\prod_t x^{\sigma_t}_{j_t k_t}\Big)\Big(\prod_s x^{\tau_s}_{p_s q_s} x^{-\tau_s^{-1}}_{q_s p_s} x^{\tau_s}_{p_s q_s}\Big)^{-1}$$

in $\mathrm{St}\,(Z[\pi_1(M^4)])$ lies in $K_2 Z[\pi_1(M^4)]$. It turns out that the image $-[\Pi]$ of Π in $Wh_2(\pi_1(M^4))$ is the invariant Σ.

Now let M^4 be a closed, *four-dimensional smooth manifold with* $\pi_1(M^4)$ poly-(finite or cyclic).

PROPOSITION 3·2. *If topological isotopies are allowed, then the map*

$$\Sigma : \pi_0(\mathscr{P}) \rightarrow Wh_2(\pi_1(M^4))$$

is surjective.

Sketch of proof. The proof of Proposition 3·2 is essentially the same as the proof of theorem 2 in [**8**], p. 213. Namely, let $z \in Wh_2(\pi_1(M^4))$ be represented by a word $\Pi x^{\lambda_t}_{\alpha_t \beta_t}$ in $K_2 Z[\pi_1(M^4)]$. By using the homomorphism Δ, or, better, by using its inverse χ, and by using the property (**), one gets a path (f_t, η_t), let us say for $0 \leqslant t \leqslant \frac{3}{4}$, from the standard projection to $(f_{\frac{3}{4}}, \eta_{\frac{3}{4}})$, where f_t has only critical points of index 2 and 3 and

$$\chi_3(f_{\frac{3}{4}}, \eta_{\frac{3}{4}}) = (\Pi x^{\lambda_t}_{\alpha_t \beta_t})^{-1}.$$

The pair $(f_{\frac{3}{4}}, \eta_{\frac{3}{4}})$ detects in a standard way a chain complex $C(f_{\frac{3}{4}}, \eta_{\frac{3}{4}}) = \{C_i, \partial_i\}$, and from the description of the boundary homomorphism (see [**8**], p. 99) it follows directly that $\partial_3 : C_3 \rightarrow C_2$ is the identity. Now by proceeding analogously as in Freedman's 5-dimensional (topological) s-cobordism theorem (see [**5**, **6**]) we can cancel (topologically) all critical points of index 2 and index 3 of $f_{\frac{3}{4}}$ without introducing new 2/2 or 3/3 intersections. Now the family (f_t, η_t), $t \in [0, \frac{3}{4}]$ can be extended to a family (f_t, η_t), $t \in [0, 1]$ with $f_1 \in \mathscr{E}$. By the construction $\Sigma[f_1] = z \in Wh_2(\pi_1(M^4))$. (Let us recall (cf. [**8**], p. 128) that $\Sigma[f_1]$ can be described as $\Sigma[f_1] = (\chi_3[f_1])^{-1} = (\chi_3(f_{\frac{3}{4}}))^{-1} = z \in Wh_2(\pi_1(M^4))$.)

Now we show how the above algebraic machinery can be used to give a proof of the 4-dimensional version of Cerf's result. As mentioned in the Introduction, this proof is not the best one and it is sketched here as an example of an application of algebra. The other reason for sketching this proof is that it involves an argument that also appears in the proof of the existence of pseudo-isotopic but not isotopic homeomorphisms on T^4.

Different proofs of the 4-dimensional version of Cerf's theorem were given by B. Perron [**16**] and F. Quinn [**18**].

THEOREM 3·3. *Let M^4 be a closed simply connected four-dimensional smooth manifold and let $\mathscr{P}_{\mathrm{TOP}}$ be the space of topological concordances of M^4 (compact-open topology). Let $j:\mathscr{P}\to\mathscr{P}_{\mathrm{TOP}}$ be the natural forgetful map. Then*

$$\pi_0(j(\mathscr{P}))=0.$$

Consequently, pseudo-isotopic diffeomorphisms of M^4 are topologically isotopic.

Sketch of proof. Let (f_t,η_t) be an element in $(\hat{\mathscr{F}},\hat{\mathscr{E}})$. We show that any such path can be deformed (rel. endpoints) to a path in $\hat{\mathscr{E}}$. Because in the deformation process we will use topological isotopies, our conclusion will be weaker than $\pi_0(\mathscr{P})=0$. Since $\pi_1(M^4)=0$ then $Wh_2(\pi_1(M^4))=0$ and by using proposition 3 in [**8**], p. 214, we know that the path (f_t,η_t) can be deformed (smoothly) to have the graphic shown in Fig. 3. This graphic can be deformed (topologically) to the graphic shown in Fig. 4 by repeating the procedure shown in Fig. 5. In order to apply this procedure we need a version of the swallowtail lemma (cf. [**3**, **8**]). Namely, we would like to replace the left-hand side in Fig. 6 by the right-hand side. The proof of the analogous higher-dimensional result relies heavily on the Whitney lemma (see [**3**]). It turns out that the 4-dimensional version of the Whitney lemma is applicable here, but this is rather far from being obvious. Therefore at this point we accept the 4-dimensional version of the swallowtail lemma and postpone until the end of the proof the description of the modifications required to make the 4-dimensional version of the Whitney lemma applicable. Actually there are two other places in the proof of Theorem 3·3 where the 4-dimensional Whitney cancellation procedure is used. In both these cases we take for granted that it works well; of course the modifications described at the end of the proof are necessary.

We can assume (see [**8**]) that the family f_t is ordered, and now we have only one critical point of index 2 and one critical point of index 3. This gives a 1-parameter handle decomposition of $M^4\times I$ with only one 2-handle and one 3-handle. Let N be the middle level manifold lying above the 2-handle and below the 3-handle. Let $\bar{S}^2$ be the attaching sphere of the 3-handle in N and S^2 the belt sphere of the 2-handle in N (i.e. if a is a critical point of index 2 and b is a critical point of index 3, then $\bar{S}^2=N\cap W(b)$ and $S^2=N\cap W^*(a)$).

Because

$$N-\bar{S}^2\simeq M^4-\text{circle}\simeq N-S^2$$

we infer that

$$\pi_1(N-\bar{S}^2)\cong\pi_1(N-S^2)=0.$$

It is easy to see (cf. [**23**]) that the simple-connectivity of N implies that the normal bundles $v(\bar{S}^2,N)$, $v(S^2,N)$ are trivial. To eliminate the graphic shown in Fig. 7 it is enough to show (see [**8**], p. 172, proposition 1·1) that the 3/2 intersections consist of one point in each t-slice, $t\in[0,1]$. Consider the global 3/2 intersection T. We have $T=\bar{S}^2\times I\cap S^2\times I\subset N\times I$, and therefore we infer that T is a one-dimensional compact manifold. We know that $\partial T:=$ two points, hence T consists of one D^1-component and let us say r S^1-components $S^1_1,S^1_2,\dots,S^1_r$. Choose paths from a base

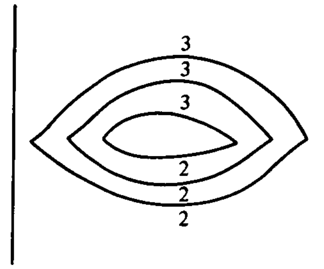


Fig. 3

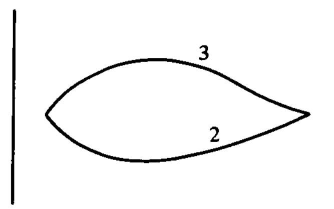


Fig. 4

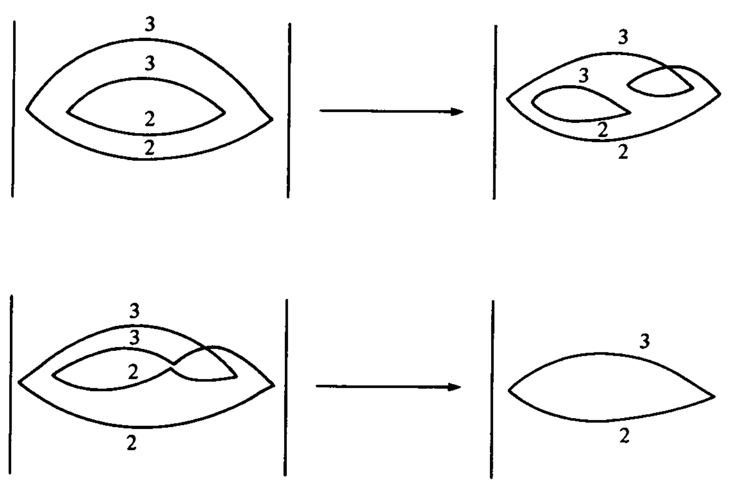


Fig. 5

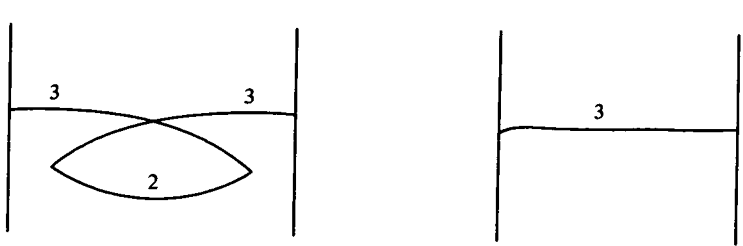


Fig. 6

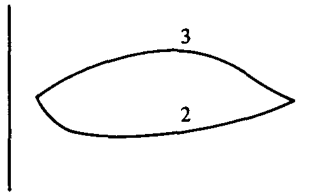


Fig. 7

point in N to $\bar{S}^2 \times I$ and $S^2 \times I$ and orientations for $\bar{S}^2 \times I$ and $S^2 \times I$ such that the algebraic intersection number

$$(\bar{S}^2 \times I).(S^2 \times I) := \bar{S}^2 . S^2 \in Z[\pi_1(N)] \cong Z$$

is equal to $+1$.

Each point of transverse intersection of T with a t-slice also defines the algebraic intersection number (cf. [**8**], pp. 221–222) in $\pm\pi_1(M^4) \subset Z[\pi_1(M^4)]$. In fact in our case for all components of T this number is equal to 1. Now, proceeding as in the proof of corollary 3 in [**3**], p. 36, one can show that an isotopy of $S^2 \times I \to N$ can be taken to be an embedding. The proof given in [**3**] remains valid in dimension four if we use the fact that $S^2 \hookrightarrow N$ has a trivial normal bundle. Therefore any isotopy of T in $S^2 \times I$ fixing $\partial(S^2 \times I)$ can be realized by an isotopy of the $S^2 \times I$ (rel. $\partial(S^2 \times I)$). Consequently we can join by surgery all components $S_1^1, S_2^1, \ldots, S_r^1$, of T into a single component. Namely, if S_1^1 intersects a t-slice transversely, it does so in at least two points with algebraic intersection numbers 1 and -1. We can suppose (by isotoping if necessary) that S_2^1 also intersects the same t-slice. Consider two intersection points, one in S_1^1 with the intersection number $+1$ and the second in S_2^1 with the intersection number -1. Now by using the Whitney cancellation process (with the modifications which are described below) one obtains an isotopy which joins S_2^1 and S_1^1. In fact, we can apply this procedure to the D^1-component as well, so we can assume that T consists of one D^1-component. Now consider the remaining D^1-component of T. This component is an arc which, in general, may be knotted in $S^2 \times I$. But [**3**], p. 39, tells us how to change this arc to an unknotted one by a sequence of embedded surgeries. Once more this is the place where the Whitney procedure is used (again after the modifications). Assuming that the D^1-component is unknotted, one cancels all 3/2 intersection points by an application of the already mentioned proposition 1·1 in [**8**], p. 172.

Now, following Quinn, we describe briefly some necessary modifications connected with the use of the 4-dimensional Whitney lemma. The detailed description of these modifications is contained in [**18**].

In all places where the Whitney cancellation process was mentioned, it was used to simplify the 1-parameter family of handlebodies by using the isotopy of the middle level manifold. In higher dimensions such an isotopy is detected by Whitney discs (see [**3**, **8**]). In dimension 4 two different kinds of Whitney discs are involved, namely Whitney discs coming from the Whitney moves and those coming from the finger moves (see [**6**, **18**]). The generic situation we are dealing with is the following:

Let N be the middle level manifold lying above the 2-handle and below the 3-handle. Let $\bar{S}^2$ be the attaching sphere of the 3-handle in N and S^2, the belt sphere of the 2-handle in N. The algebraic intersection of S^2 with $\bar{S}^2$ is equal to 1, and we have two collections $\{V\}$, $\{W\}$ of Whitney discs which eliminate all but one of the $S^2\bar{S}^2$ intersections. Let us agree that the collection $\{V\}$ came from finger moves and $\{W\}$

came from Whitney moves. The $\{V\}$ intersects S^2 in disjointly embedded arcs with endpoints given by intersections $S^2\bar{S}^2$ (the same for $\{W\}$). Therefore the union of these intersections is given by a collection of immersed circles and one arc (or one isolated point).

The extra difficulty in dimension 4 is connected with the fact that the discs V and W may intersect. In higher dimensions this problem does not arise because by general position argument we can always separate two Whitney discs. This possibility of separation of the Whitney discs is crucial because it allows us to simplify the 1-parameter family of handlebodies just by performing Whitney moves. The ideal situation in dimension 4 would be to have all the Whitney discs disjoint; then we could just repeat the higher-dimensional arguments essentially without any alterations. Unfortunately, in general there is no hope that we can make V and W disjoint; however, we can deform the generic situation described above to a situation which is simple enough so that the cancellation procedure can be applied.

Now we describe the necessary simplifications which are obtained by deforming the isotopy of the middle-level manifold through isotopies. The first step in this process is to simplify the intersections between $V \cup W$ and $S^2 \cup \bar{S}^2$. Namely, one can arrange that $V \cup W$ intersects S^2 and $\bar{S}^2$ in single embedded arcs. This is achieved by using the 'sum operation'. In this operation two V discs are cut apart and next reassembled by using an embedded square S, with two edges on V discs: one edge on S^2 and one on $\bar{S}^2$. The model is shown in Fig. 8.

The new V discs are obtained by cutting the given ones along the boundary edges of S and gluing in two parallel copies of S. If the interior of S is disjoint from S^2, $\bar{S}^2$ and all V discs, then the new discs are embedded Whitney discs and hence provide a new isotopy. In the above situation the edges of the square are given by properly chosen arcs which join $+1$ and -1 endpoints. (Note: the existence of such an embedded square is not automatic; it requires proof.) The next step in the simplification of the situation is to deform the situation to one with single V and W discs such that $\partial V \cup \partial W$ intersects $S^2 \cup \bar{S}^2$ in embedded arcs. The idea here is to push $\bar{S}^2$ across the W discs. The V discs are joined together and a single embedded disc $\tilde{V}$ is produced. Note that there is an inverse operation, namely the finger moves of the image of $\bar{S}^2$ along the arcs $A_j \subset \tilde{V}$, where the V discs were joined. It is clear that this operation recovers all W discs (Fig. 9).

It turns out that with some care the discs V can be recovered as well. Now the disc $\tilde{V}$ is used to deform the situation to a situation where one less V disc is recovered (and hence one less W disc). By repeating this procedure one finishes with only one V disc and one W disc. The final step in the simplification is to observe that if the intersections of $S^2\bar{S}^2$ have single V and W discs which intersect along embedded arcs, then the existence of an embedded transverse sphere V^t for V disjoint from $S^2 \cup \bar{S}^2 \cup W$ leads to the cancellation of all intersections. We recall that the transverse sphere for V is a framed 2-sphere which intersects V in exactly one point. To find V^t is in fact not difficult. One starts with a small linking sphere to the boundary arc of V in S^2 and modifies it to obtain V^t. The more serious difficulty is proving that the existence of V^t suffices for the cancellation of intersection points. The proof of this fact relies on a trick similar to one used by K. Igusa for 5-dimensional manifolds. (Let $\bar{S}^2$ be the attaching sphere of the 3-handle and S^2 the belt sphere of the 2-handle in the middle level manifold N for a given 1-parameter family of handlebodies. The

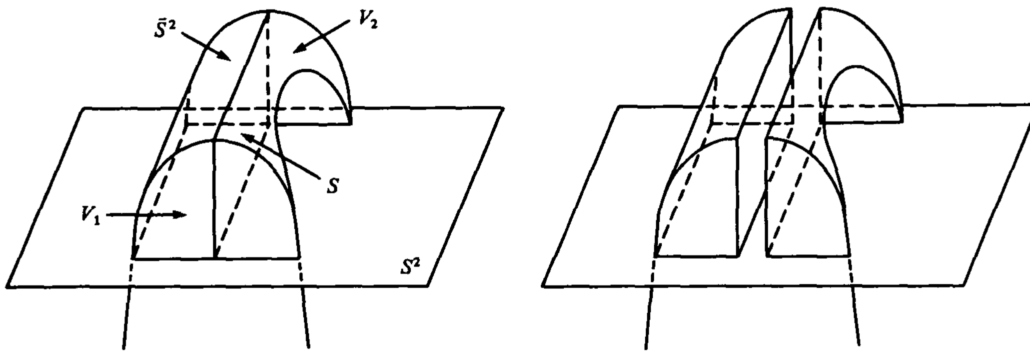


Fig. 8

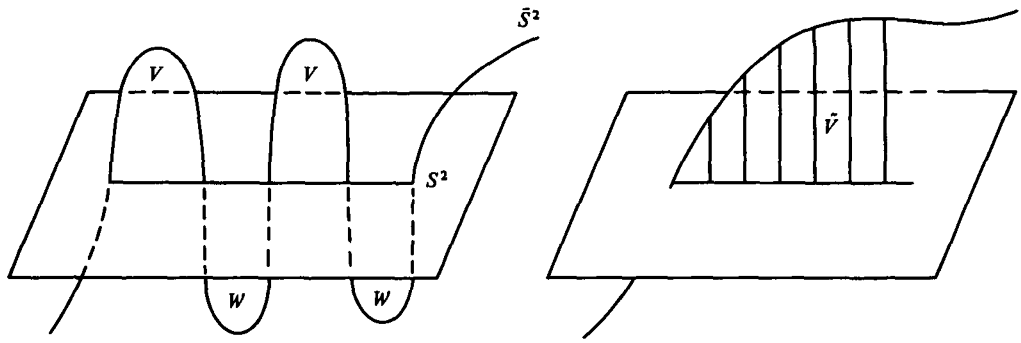


Fig. 9

trick referred to tells us how to deform the family so that the belt sphere of the 2-handle is switched from $S^2 \times \{0\}$ to $S^2 \times \{1\}$ in $S^2 \times I$.) With the above modifications, the 4-dimensional Whitney cancellation procedure replaces the higher-dimensional one perfectly well. Therefore the sketched proof of Theorem 3·3 can be carried over to dimension four as well.

Remark 3·4. Theorem 3·3 is definitely false when M^4 is not simply connected. For example, let $L^3(q)$ be a 3-dimensional lens space with $\pi_1(L^3(q)) \cong Z_q$, q a prime number and $q \geqslant 5$. Let $M^4 = L^3(q) \times S^1$. Then $Wh_2(\pi_1(M^4)) = Wh(Z_q \times Z) \neq 0$ (see [**21**]). By Proposition 3·2, there is a surjection $\Sigma : \pi_0(j(\mathscr{P})) \to Wh_2(\pi_1(M^4))$ and hence $\pi_0(j(\mathscr{P})) \neq 0$. (In fact, $\pi_0(j(\mathscr{P}))$ is infinite.)

4. *Pseudo-isotopic homeomorphisms on* T^4

In this section we show that on the 4-dimensional torus T^4 there exist homeomorphisms which are pseudo-isotopic to the identity but not isotopic to it. This provides a positive solution to a problem posed by A. Hatcher and simultaneously it answers Siebenmann's question negatively.

Let $\mathscr{D} \subset \pi_0(\mathscr{E})$ be the set of all $[f] \in \pi_0(\mathscr{E})$ such that f can be connected to the standard projection p by a path having the graphic shown in Fig. 10. It is not difficult to see (cf. [**8**], p. 211) that $\mathscr{D}$ is a subgroup. Now, following [**8**], we define a homomorphism

$$\Theta : \mathscr{D} \to Wh_1(\pi_1(M^4); Z_2).$$

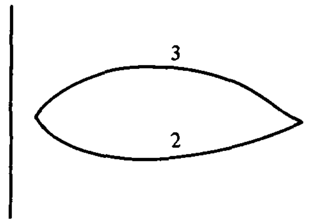


Fig. 10

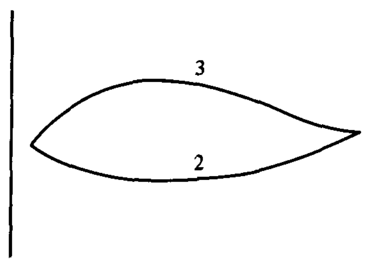


Fig. 11

Let $[f] \in \mathscr{D}$ so that it has the graphic shown in Fig. 11. In the proof of Theorem 3·3 it was shown how such a graphic detects a finite number of circles, say $S^1_1, \dots, S^1_r$. Every such S^1_j $(j = 1, \dots, r)$ yields (by ignoring sign) a well defined 'algebraic intersection number' $\sigma_j \in \pi_1(M^4)$ (cf. [**8**]). The above graphic also yields an element in Z_2 which is a framed bordism class in $\Omega^{\mathrm{fr}}_1 \cong Z_2$. Namely, it is the bordism class of $S^1_j \subset \overline{S}^2 \times I \subset S^3$ with normal bundle $v(S^1_j, \overline{S}^2 \times I) \cong v(\overline{S}^2 \times I, N \times I)|_{S^1_j}$ framed by the canonical framing of $v(\overline{S}^2 \times I; N \times I)$ detected by the fact that $\overline{S}^2$ is the attaching sphere of a 3-handle.

PROPOSITION 4·1. *If topological isotopies are allowed and $\pi_1(M^4)$ is poly-(finite or cyclic), $\pi_2(M) = 0$, then*

$$\Theta: \mathscr{D} \to Wh_1(\pi_1(M^4); Z_2)$$

is onto.

Proof. Let $b\sigma \in Z_2[\pi_1(M^4)]$ be an arbitrary element. Let $f: M^4 \times I \to I$ be a Morse function with only 2 critical points, of index 2 and 3, and with transverse intersection consisting of 3 points, p_1, p_σ, $p_{-\sigma}$, in an intermediate level manifold N. The intersection numbers are 1, σ, $-\sigma \in \pm\pi_1(M^4) \subset Z[\pi_1(M^4)]$. We join p_σ, $p_{-\sigma}$ by two arcs $C_1 \subset S^2$ and $C_2 \subset \overline{S}^2$ (here S^2 is the belt sphere of a 2-handle (in N) and $\overline{S}^2$ is an attaching sphere of a 3-handle). Let W be the Whitney tower associated to these two points. (It surely exists in our case.) Inside W we can find a Whitney disc D^2 which enables us to cancel (topologically) the intersection points σ and $-\sigma$.

In the construction of a Whitney disc D^2 an important role is played by a framing condition. Namely, one requires that the section s of the normal bundle $v(D^2, N)$, which on the Whitney circle $C_1 \cup C_2 = \partial D^2$ is given by

$s|_{C_2} :=$ oriented normal to $C_2 \subset \overline{S}^2$,

$s|_{C_1} :=$ oriented complement to the 3-plane bundle $(\tau(D^2) \oplus \tau(S^2))|_{C_1}$,

extends to a global section $s: D^2 \to v(D^2, N)$.

(Here $v(D^2, N)$ means the topological normal bundle. It exists by [**5**].) In higher codimensions (i.e. for $n-i \geqslant 3$) the Z_2 part of the $Z_2[\pi_1(M^4)]$ obstruction is realized by reframing the bundle over $C_1 \cup C_2$ (this bundle consists of vectors which over C_1 are normal to D^2 and normal to S^n and which over C_2 are normal to D^2 and tangent to S^{n-i}, cf. [**13**, **8**]). In codimension two (our case) the choice of the Whitney disc can be altered by re-choosing the section s over $C_1 \subset S^2$ (keeping it normal to D^2 along C_1, cf. [**8**], p. 228). Such a choice is classified by an element in $\pi_1(S^1) \cong Z$ and stably by an element in $\pi_1(O(\infty)) \cong Z_2$. Now, by repeating the construction of the Whitney tower, we can produce a Whitney disc with the correct framing with respect to the altered section of $v(D^2, N)|_{C_1 \cup C_2}$. Consequently, we can cancel the intersection points p_σ, $p_{-\sigma}$ in different ways according to different choices of an element b in $\pi_1(O(\infty)) \cong Z_2$. The cancellation process provides a one-parameter family f_t with the graphic shown in Fig. 12 and with one S^1 component of 3/2 intersections. As in [**8**], the invariant of this S^1 is $b\sigma \in Z_2[\pi_1(M^4)]$. By iterating this procedure one constructs several components of 3/2 intersections which one can add and thus obtain an arbitrary element in $Z_2[\pi_1(M^4)]$.

Remark 4·2. In the proof of Proposition 4·1 we used the fact that there is a Morse function on $M^4 \times I$ with two critical points, one of index 2 and one of index 3, and with given geometric intersection number. Alternatively (in the language of handle decomposition), there is a handle decomposition of $M^4 \times I$ with one 2-handle and one 3-handle and with given intersection number. Usually in such constructions one assumes (cf. [**14**], p. 398) $\dim M \geqslant 5$. But [**19**], p. 90, provides a construction which works when $\dim M = 4$ as well. Note that there is no difference between the PL and smooth considerations because, for a smooth manifold M, $\pi_0(\mathscr{P}(M)) \cong \pi_0(\mathscr{P}_{PL}(M))$ (see [**1**], p. 41, cf. [**2**]).

Remark 4·3. Actually one has a destabilized invariant in $Z[\pi_1(M^4)]$ instead of in $Z_2[\pi_1(M^4)]$. The above result shows that every element in $Z[\pi_1(M^4)]$ is realizable by a one-parameter family with the graphic shown in Fig. 13.

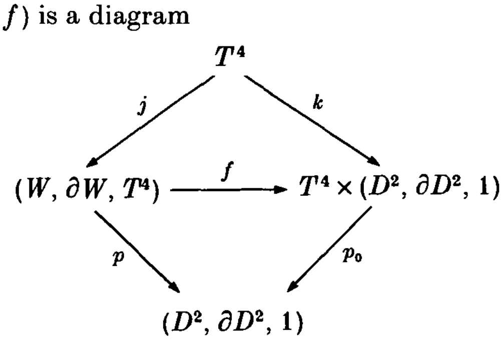
Now we show how the surjection

$$\Theta : \mathscr{D} \to Wh_1(\pi_1 T^4); Z_2)$$

yields the existence of diffeomorphisms on T^4 which are pseudo-isotopic to the identity but not isotopic to it.

First we recall the cobordism group $\pi_0(\mathscr{B}(T^4))$. This group was defined in [**9**] and we refer to [**9**] for more information. The brief description of $\pi_0(\mathscr{B}(T^4))$ goes as follows:

An object $\alpha = (W(\alpha), f)$ is a diagram



satisfying the following conditions:

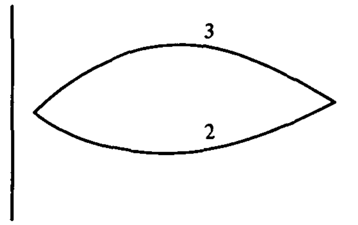


Fig. 12

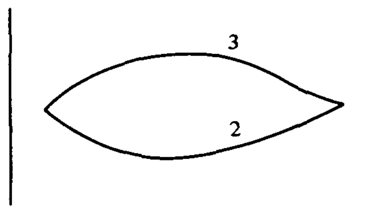


Fig. 13

(1) 1 is the base point of D^2, and the upper triangle is commutative with j, a homeomorphism, and k the standard identification of T^4 with $T^4 \times 1$;

(2) f is a simple homotopy equivalence;

(3) the lower triangle is commutative, with p_0 the projection onto the second factor, and $T^4 \xrightarrow{j} \partial W \xrightarrow{p} \partial D^2$ is a topological fibration.

Two such objects α_1, α_2 are cobordant if we have

$$\begin{array}{ccc} & T^4 \times I & \\ {\scriptstyle j \times id} \swarrow & & \searrow {\scriptstyle k \times id} \\ (U, V, T^4 \times I) & \xrightarrow{F} & T^4 \times (D^2, \partial D^2, 1) \times I \\ {\scriptstyle p} \searrow & & \swarrow {\scriptstyle p_0 \times id} \\ & (D^2, D^2, I) \times I & \end{array}$$

satisfying the following conditions:

(1) the upper triangle is commutative, with U an s-cobordism between $W(\alpha_1)$ and $W(\alpha_2)$ and V an s-cobordism between $\partial W(\alpha_1)$ and $\partial W(\alpha_2)$;

(2) the lower triangle is commutative such that $T^4 \to V \to \partial D^2 \times I$ is a topological fibration;

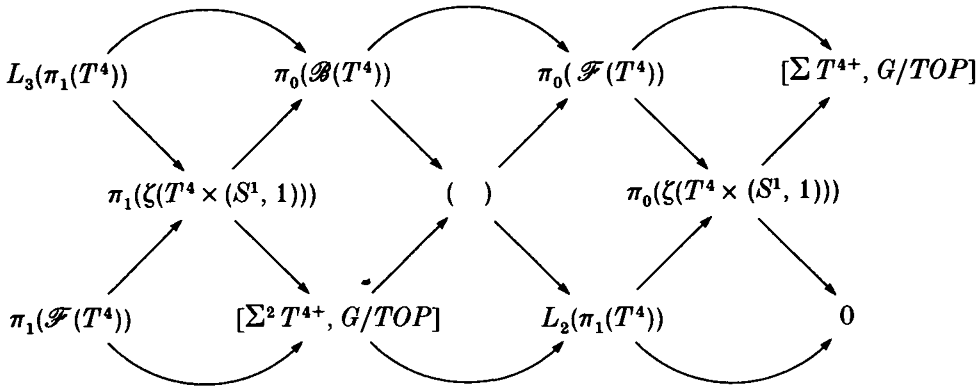
(3) $F|_{W(\alpha_i)\,(i=1,2)}$ induces the objects α_i $(i = 1, 2)$.

Now let $\text{Aut}\,(T^4) = \{f: T^4 \to T^4 \text{ such that } f \text{ is a simple homotopy equivalence}\}$.

Denote by $\mathscr{F}(T^4)$ the fibre in the fibration

$$\text{TOP}\,(T^4) \to \text{Aut}\,(T^4).$$

Let $\zeta(T^4 \times (S^1, 1))$ be a space of simple homotopy TOP structures of $T^4 \times S^1$ which are standard on $T^4 \times 1$ (cf. [**24**]). It turns out (cf. [**9**]) that the cobordism group $\pi_0(\mathscr{B}(T^4))$ fits into a braid of groups. Here is a piece of this braid. (Note that 4-dimensional topological surgery works for manifolds with poly-(finite or cyclic) groups; in particular there is a long Wall–Sullivan exact sequence.)



It is known (see [**24**]; cf. [**5**]) that $L_3(\pi_1(T^4)) \to \pi_1(\zeta(T^4 \times (S^1, 1)$ is onto, and hence

$$\pi_1(\zeta(T^4 \times (S^1, 1))) \to \pi_0(\mathscr{B}(T^4))$$

is trivial. Thus we obtain the short exact sequence

$$1 \to \pi_0(\mathscr{B}(T^4)) \to \pi_0(\mathscr{F}(T^4)) \to \pi_0(\zeta(T^4 \times (S^1, 1))) \to 1.$$

Since $\pi_0(\zeta(T^4 \times (S^1, 1))) = 0$ (cf. [**24**, **12**]), we have $\pi_0(\mathscr{B}(T^4)) \cong \pi_0(\mathscr{F}(T^4))$. The torus T^4 is a $K(Z^4, 1)$-space, and because the map $\mathrm{TOP}\,(T^4) \to \mathrm{Aut}\,(T^4)$ has a left homotopy inverse we get the split exact sequence

$$(*) \qquad 1 \to \pi_0(\mathscr{B}(T^4)) \to \pi_0(\mathrm{TOP}\,(T^4)) \to GL\,(4, Z) \to 1.$$

Let us now consider the group $\pi_0(\mathscr{B}(T^4))$. There is another description of this group (see [**1**, **7**, **10**]), namely

$$\pi_0(\mathscr{B}(T^4)) = \pi_0(\Omega(\widetilde{\mathrm{TOP}}\,(T^4)/\mathrm{TOP}\,(T^4))) = \pi_1(\widetilde{\mathrm{TOP}}\,(T^4)/\mathrm{TOP}\,(T^4)).$$

Here $\widetilde{\mathrm{TOP}}\,(T^4)$ is the space of block autohomeomorphisms of T^4. It follows directly (from the first or from the second description) that there is a natural homomorphism

$$\pi_0(\mathscr{P}_{\mathrm{TOP}}(T^4)) \to \pi_0(\mathscr{B}(T^4)).$$

We claim that this homomorphism is non-trivial. For, let $f: T^4 \times I \to T^4 \times I$ represents an element in the kernel. It is shown in [**10**] (lemma 2·6, p. 140) that there is an $F \in \pi_0(\mathscr{P}_{\mathrm{TOP}}(T^4 \times I))$ such that $F|_{M \times I \times \{1\}} = f$.

Now, because $T^4 \times I$ is 5-dimensional, we have (see [**1**])

$$\pi_0(\mathscr{P}_{\mathrm{TOP}}(T^4 \times I)) = \pi_0(\mathscr{P}(T^4 \times I)).$$

Consequently we can take $F \in \pi_0(\mathscr{P}(T^4 \times I))$. We know (see [**7**]) that

$$\pi_0(\mathscr{P}(T^4 \times I)) = Wh_1(Z^4; Z^2).$$

Geometrically, $\pi_0(\mathscr{P}(T^4 \times I))$ is represented by the subgroup $\mathscr{D} \subset \pi_0(\mathscr{P}(T^4 \times I))$ (cf. [**8**]) consisting of pseudo-isotopies with graphic given by Fig. 14. (Note that $Wh_2(Z^4) = 0$.) By using Proposition 4·1 one obtains that the suspension map S sends $(j(\mathscr{D})) \subset \pi_0(\mathscr{P}_{\mathrm{TOP}}(T^4))$ onto $\mathscr{D} \subset \pi_0(\mathscr{P}(T^4 \times I))$. In particular, $F = S(g)$ for $g \in (j(\mathscr{D})) \subset \pi_0(\mathscr{P}_{\mathrm{TOP}}(T^4))$. From this it follows that $f = g + \bar{g}$, where $^-: \mathscr{P}_{\mathrm{TOP}}(T^4) \to \mathscr{P}_{\mathrm{TOP}}(T^4)$ is the standard involution inducing the involution $^-$ on the π_0-level. Consequently one has a non-trivial homomorphism

$$[(j(\mathscr{D}))] \subset \pi_0(\mathscr{P}_{\mathrm{TOP}}(T^4))/\{x + \bar{x}\} \to \pi_0(\mathscr{B}(T^4)).$$

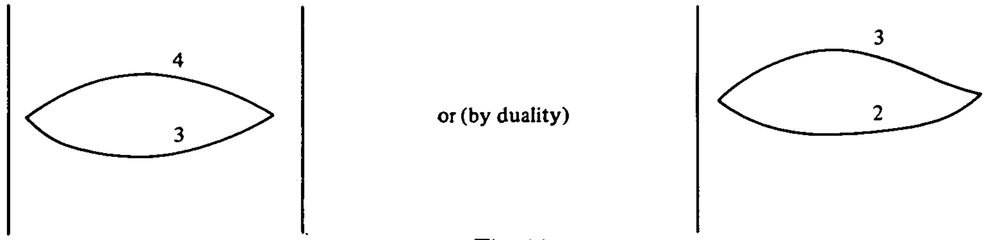


Fig. 14

(Note that $[(j(\mathscr{D}))]$ surjects onto $Z_2[t_1, \ldots, t_4]/Z_2(1)$, which is certainly non-trivial.) We recall that the involution on $Z_2[t_1, t_1^{-1}, \ldots, t_4, t_4^{-1}]/Z_2(1)$ is given (see [8]) by $\bar{t}_i = t_i^{-1}$. Now the image of $[(j(\mathscr{D}))]$ in $\pi_0(\mathscr{B}(T^4))$ (non-trivial), and consequently the image of $[(j(\mathscr{D}))]$ in $\pi_0(\mathrm{TOP}(T^4))$ (by using the exact sequence (*)), corresponds to the diffeomorphisms of T^4 which are pseudo-isotopic to the identity, but not isotopic (even topologically) to it.

Remark 4·4. The dimension 4 is the lowest dimension for the torus (and probably the lowest dimension at all) to support pseudo-isotopic but not isotopic homeomorphisms (cf. [**22**]).

Remark 4·5. L. Siebenmann has shown to the author [**20**] how to conclude from Theorem 3·3 by using Quinn's results (see [**17**]) that pseudo-isotopic homeomorphisms of closed simply connected 4-dimensional topological manifolds are topologically isotopic.

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