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On the Symmetries of the Fake $\mathbb{C}P^2$

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In this paper we investigate the symmetries of the fake $\mathbb{C}P^2$. The fake $\mathbb{C}P^2$ is a closed topological manifold homotopy equivalent to $\mathbb{C}P^2$, but not homeomorphic to it. The fake $\mathbb{C}P^2$ is, in fact, nonsmoothable. This manifold was constructed by Freedman in [1] and was denoted by $\mathbb{C}h$ (for Chern manifold). This notation will be also used in this paper. The standard $\mathbb{C}P^2$ is strongly symmetric in the sense that it admits an S^1-action. It turns out that, though very similar to $\mathbb{C}P^2$ (i.e., homotopy equivalent), the manifold $\mathbb{C}h$ is much less symmetric. Our result concerning the symmetries of $\mathbb{C}h$ seems to be something of a surprise; namely,

Theorem. *For each odd prime $p>1$ there is a locally smoothable Z_p-action on $\mathbb{C}h$, but $\mathbb{C}h$ does not admit a nontrivial locally smoothable Z_2-action.*

It is worthwhile to note that the above theorem provides the negative answer (in the locally smoothable category) to the following problem in the theory of transformation groups [6, 8].

Existence of Circle Actions. Let M be a closed manifold which admits a Z_p-action for almost all primes p that are sufficiently large, does M admit a topological circle action?

Before we give a proof of the theorem, we recall briefly the construction of the fake $\mathbb{C}P^2$, $\mathbb{C}h$. Let K be the trefoil knot Fig. 1 which is embedded in $S^3=\partial D^4\subset D^4$.

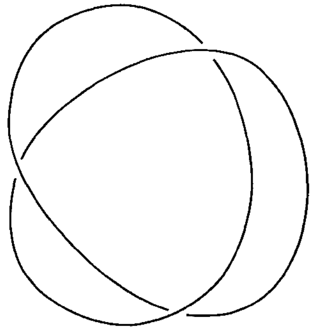

Fig. 1

The Dehn surgery on K with coefficient equal to one produces a four-dimensional manifold T with a boundary $\partial T = \Sigma^3$, where Σ^3 is the Poincaré homology 3-sphere. It was proved in [1] that every homology 3-sphere bounds a contractible topological manifold. Let V be such a manifold for Σ^3. Now the fake $\mathbb{C}P^2$ manifold $\mathbb{C}h$ is given by:

$$\mathbb{C}h = T \bigcup_{\Sigma^3} V .$$

It was proved in [1] that $\mathbb{C}h$ is nonsmoothable and in fact nonsmoothable even stably i.e. $\mathbb{C}h \times R$ is nonsmoothable.

In the following, we will not distinguish between the smooth and PL situations. Of course, we are justified in this because in dimensions $\leqq 6$ there is, in fact, no difference [3].

Proof of Theorem. First we show how to construct a locally smoothable Z_p-action on $\mathbb{C}h$ for each odd $p > 1$. We start from a smooth Z_p-action on $\mathbb{C}P^2$ which has 3 isolated fixed points. Here is one explicit example. In terms of homogeneous coordinates $(z_0 : z_1 : z_2)$ on $\mathbb{C}P^2$ the action takes the form:

$$t(z_0 : z_1 : z_2) = (t^{a_0} z_0 : t^{a_1} z_1 : t^{a_2} z_2) ,$$

where $t = \exp(2\pi i/p)$ and $a_i \not\equiv a_j (\operatorname{mod} p)$ for $i \neq j$. This action of Z_p has 3 isolated fixed points $p_0 = (1:0:0)$, $p_1 = (0:1:0)$, $p_2 = (0:0:1)$. Let X be a manifold obtained from $\mathbb{C}P^2$ by excluding small disjoint invariant open discs around these points, i.e.

$$X = \mathbb{C}P^2 - \bigcup_{i=0}^{2} \mathring{D}(p_i) .$$

The manifold X is simply connected manifold whose boundary ∂X is given by 3 copies of S^3. Let $p : (X, \partial X) \to (X, \partial X)/Z_p = (Y, \partial Y)$ be the projection onto the orbit space. The embedding theorem from [2] together with results in [7] enables us to do topological surgery on four-dimensional topological manifolds with poly-(finite or cyclic) fundamental groups. Consequently we have the following Wall-Sullivan exact sequence

$$0 = L_5^s(\pi_1(Y)) \to S_{\mathrm{TOP}}(Y, \partial Y) \xrightarrow{\tau} [Y, \partial Y; G/\mathrm{TOP}, *] \xrightarrow{\theta} L_4^s(\pi_1(Y)) .$$

What we shall do below is to construct a manifold $(N, \partial N)$ and a homotopy equivalence $h : (N, \partial N) \to (Y, \partial Y)$ such that $h|_{\partial N}$ is a homeomorphism. [The pair $((N, \partial N), h)$ represents an element in $S_{\mathrm{TOP}}(Y, \partial Y)$.] The construction of $(N, \partial N)$ will have one salient feature; namely, it will guarantee that $(N, \partial N)$ is stably nonsmoothable. Upon lifting h to universal covers, we obtain a stably nonsmoothable manifold $(\tilde{N}, \partial \tilde{N})$ which is homotopy equivalent to $(X, \partial X)$. Now, the Z_p-action on $\partial \tilde{N}$ is sufficiently nice to allow us to plumb discs along $\partial \tilde{N}$ so that

$$\tilde{N} \bigcup_{\partial \tilde{N}} \bigcup D^4$$

is a manifold homotopy equivalent to $\mathbb{C}P^2$. This manifold, however, cannot be homeomorphic to $\mathbb{C}P^2$ because it is stably nonsmoothable; hence it must be $\mathbb{C}h$. Of course it supports locally smoothable Z_p-action.

Now we show how to find the required manifold $(N, \partial N)$ and a homotopy equivalence $h:(N,\partial N)\to(Y,\partial Y)$. Consider the following sequence of fibrations which form the commutative diagram

$$\begin{array}{ccccccc} & & G/PL & \xrightarrow{\psi'} & G/\mathrm{TOP} & \searrow^{\bar{k}} & \\ \mathrm{TOP}/PL & \nearrow\searrow & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & K(Z_2,4) \\ & & BPL & \xrightarrow{\psi} & B\mathrm{TOP} & \nearrow_{k} & \\ & & \downarrow & & \downarrow & & \\ & & BG & \xrightarrow{\mathrm{id}} & BG & & \end{array}$$

where $k: B\mathrm{TOP}\to K(Z_2,4)$ is induced by the universal triangulation obstruction [3] and $\bar{k}=k\circ i_2$. All the spaces in the above diagram are H-spaces and all maps are H-maps [3]. Thus one obtains the following exact sequence of abelian groups,

$$[Y,\partial Y;\mathrm{TOP}/PL,*]\longrightarrow[Y,\partial Y;G/PL,*]\xrightarrow{\psi'_*}[Y,\partial Y;G/\mathrm{TOP},*]$$
$$\xrightarrow{\bar{k}_*}[Y,\partial Y;K(Z_2,4)]$$

which one can write in the form

$$H^3(Y,\partial Y;Z_2)\to[Y,\partial Y;G/PL,*]\to[Y,\partial Y;G/\mathrm{TOP},*]\to H^4(Y,\partial Y;Z_2).$$

The 4-stage in the Postnikov system for G/TOP is given by $K(Z_2,2)\times K(Z,4)$ [3] which implies

$$[Y,\partial Y;G/\mathrm{TOP},*]\approx H^2(Y,\partial Y;Z_2)\oplus H^4(Y,\partial Y;Z). \qquad (*)$$

The 4-stage for G/PL after localization at 2 is given [3] by

$$K(Z_2,2)\underset{\delta Sq^2}{\times}K(Z_{(2)},4).$$

This implies [3] the following:

$$[Y,\partial Y;G/\mathrm{TOP},*]/\psi'_*[Y,\partial Y;G/PL,*]=\bar{k}_*[Y,\partial Y;G/\mathrm{TOP},*]$$
$$=\mathrm{red}(H^4(Y,\partial Y;Z)+Sq^2H^2(Y,\partial Y;Z_2)$$

where the last sum forms a subgroup in $H^4(Y,\partial Y;Z_2)$ and $\mathrm{red}: H^4(Y,\partial Y;Z)\to H^4(Y,\partial Y;Z_2)$ is the reduction of coefficients.

Now let $g:(Y,\partial Y)\to(G/\mathrm{TOP},*)$ be a map which represents an element

$$(t,0)\in H^2(Y,\partial Y;Z_2)\oplus H^4(Y,\partial Y;Z)$$

in the decomposition $(*)$. Let $f:(M,\partial M)\to(Y,\partial Y)$ be a normal map which corresponds to g.

Claim. The surgery obstruction $\Theta(f)=\Theta(g)$ for f vanishes.

To see this, note that $[L^3;G/\mathrm{TOP}]=0$ for any lens space L^3. Therefore, we can suppose that

$$f|_{\partial M}:\partial M\to\partial Y \quad \text{is a homeomorphism}.$$

Let $\bar{W}$ be a closed Z_p-manifold obtained from $(\tilde{M},\partial\tilde{M})$ and $(\tilde{Y},\partial\tilde{Y})$ using the identification of their boundaries. Now $\theta(g)$ is just the multi-signature $\tilde{\sigma}(\bar{W})$ of $\bar{W}$ and, in fact [9] is the standard signature $\sigma(\bar{W})$ of $\bar{W}$ which is equal to $\sigma(\tilde{M})-\sigma(\tilde{Y})$.

The difference $\sigma(M)-\sigma(Y)$ can be described as follows [9]. Let $p_1 \in H^4(B\mathrm{TOP};Q)$ be the first Pontriagin class and let $p_1(Y)$ be the first Pontriagin class for the tangent bundle of Y. Denote by $p_1(G/\mathrm{TOP})$ the corresponding Pontriagin class induced by

$$i_2: G/\mathrm{TOP} \to B\mathrm{TOP}.$$

Let $L(Y)$ be the Hirzebruch polynomial for Y and let $L(G/\mathrm{TOP})$ be the Hirzebruch polynomial with respect to $p_1(G/\mathrm{TOP})$. Now

$$\sigma(M)-\sigma(Y)=\langle L(Y)\cdot g^*(L(G/\mathrm{TOP})); [Y,\partial Y]\rangle$$

and

$$L(Y)\cdot g^*L(G/\mathrm{TOP}) \in H^4(Y,\partial Y;Q);$$

hence $\sigma(M)-\sigma(Y)=0$ by our choice of

$$(t,0)\in H^2(Y,\partial Y,Z_2)\otimes H^4(Y,\partial Y;Z).$$

Thus the proof of the claim is complete.

Because the surgery obstruction $\Theta(g)$ is trivial then there exists an element $(N,\partial N;h)$ in $S_{\mathrm{TOP}}(Y,\partial Y)$ such that

$$\tau(N,\partial N;h)=(t,0).$$

Up to this point, the choice of t was irrelevant. We shall now choose a nontrivial t with the property that t^2 is also nontrivial in $H^4(Y,\partial Y;Z_2)$.

The action of Z_p on $H^*(X,\partial X;Z)$ is trivial therefore by the transfer argument [note that $(2,p)=1$] we infer that

$$H^*(Y,\partial Y;Z_2) \xrightarrow{p^*} H^*(X,\partial X;Z_2)$$

is an isomorphism. In particular, $H^2(X,\partial X;Z_2)\approx H^2(Y,\partial Y;Z_2)\approx Z_2$ and if $t\in H^2(Y,\partial Y;Z_2)$ is the nontrivial element, then $t^2=t\cup t\in H^4(Y,\partial Y;Z_2)$ is a nontrivial element too.

We recall that the Kirby-Siebenmann invariant of N, $k(N)\in H^4(N,\partial N;Z_2)\approx Z_2$ is the pull back, using the classifying map

$$f:(N,\partial N)\to(B\mathrm{TOP},*)$$

of the universal triangulation obstruction $k\in H^4(B\mathrm{TOP};Z_2)$. Also note that by the results of Taylor and Lashof [5] or Quinn [7] the Kirby-Siebenmann invariant $k(N)$ can be viewed as the obstruction to smoothing $N\times R \operatorname{rel} \partial N\times R$.

Now if $g:(Y,\partial Y)\to(G/\mathrm{TOP},*)$ is a map associated to $h:(N,\partial N)\to(Y,\partial Y)$ then $\langle t^2;[Y,\partial Y]\rangle \neq 0$ represents the obstruction to lift g through G/PL. But $g\in[Y,\partial Y;G/\mathrm{TOP};*]$ represents the difference between the stable normal bundles of $(N,\partial N)$ and $(Y,\partial Y)$. Therefore, the commutativity of the diagram

$$\begin{array}{ccc} G/\mathrm{TOP} & \xrightarrow{\bar{k}} & \\ \downarrow{\scriptstyle i_2} & & K(Z_2,4) \\ B\mathrm{TOP} & \xrightarrow{k} & \end{array}$$

implies that $\langle t^2;[Y,\partial Y]\rangle \neq 0$ represents the difference of the Kirby-Siebenmann invariants $k(N)-k(Y)$. Because $k(Y)=0$ then the invariant $k(N)$ must be nontrivial. This completes the first part of the proof of our theorem.

Now we show that $\mathbb{C}h$ does not admit a nontrivial locally smoothable Z_2-action. This was proved in [4], but in order that this paper be self-contained, we include a brief sketch of a proof here. This proof works for all 4-manifolds with nontrivial Kirby-Siebenmann invariant and with an orientation preserving involution. Because the signature $\sigma(\mathbb{C}h)$ of the fake $\mathbb{C}P^2$ is different from zero, then every involution must be orientation preserving. This implies that the fixed point set of such an involution would consist of isolated points and disjoint surfaces. Now, it can be shown that the surfaces would have equivariant tubular neighborhoods on which the involution is smooth. For this, one can use the existence of normal microbundles for surfaces in 4-manifolds (due to M. Freedman or D. Ruberman) or more simply, just raise the dimension by 1 by crossing with R (trivial involution on R) and apply the well-known existence theorem for normal microbundles in this case. The fact that $O(2)$ is a strong deformation retract of TOP(2) guarantees the existence of a smooth structure on these tubular neighborhoods. By excluding from $\mathbb{C}h$ the above tubular neighborhoods together with small invariant discs around fixed points, one would obtain a manifold with boundary $\mathbb{C}h$ with a fixed point free involution. This implies that the relative Kirby-Siebenmann invariant $k(\mathbb{C}h)$ is trivial. But on the other hand, from the construction, it follows directly that $k(\mathbb{C}H) \neq 0$, which yields a contradiction. For more details we refer to [4].

Remark. It follows directly from the proof of our theorem that $\mathbb{C}h$ admits a locally smoothable Z_k-action for all odd (not necessary prime) $k>1$.

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