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MICROBUNDLES AND SMOOTHING

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INTRODUCTION

IN THIS PAPER we concern ourselves with the following question. Suppose one has an unbounded combinatorial manifold K contained in an unbounded differentiable manifold M. When does there exist a piecewise differentiable isomorphism $h: M \to M$ such that $h(K)$ is a smooth submanifold of M. It is clearly necessary that K have a vector bundle neighborhood in M. Our main result asserts that if K is the homotopy type of a finite complex the converse is true.

This theorem can be thought of as a refined version of the fundamental theorem of smoothing theory due to Cairns–Hirsch. While the "classical" Cairns–Hirsch theorem is our takeoff point the generalization is not straightforward. Our basic tool is the theory of piecewise linear microbundles due to Milnor [5], and we assume familiarity with these notes. Aside from a strong form of the "classical" Cairns–Hirsch theorem due to Mazur [4] and the notes of Milnor, we utilize techniques and results in differential topology which have already appeared in print.

Chapter I is a bit technical, building up the necessary simplicial machinery. The most important results are in §3, yielding a functorial triangulation in the best possible sense of a vector bundle over a complex. This result appears to have applications beyond the considerations of this paper.

In Part II we apply the machinery of Part I to the problem of smoothing theory. §5 deals with stable tubular neighborhood theorems in the combinatorial category. Here the results are what one would expect and in fact more or less assumed by those working in the field. However, our proof is the first we know of in the general case. It is based on a careful analysis of a proof of Mazur for an important special case. §6 deals with results and application of results concerning the representability of the functor which assigns to each combinatorial manifold its set of smoothings. Here our machinery of Part I pays off in yielding a relatively straightforward and unique proof of a theorem on Γ_n due independently to Hirsch and Mazur; along with its generalizations announced by Mazur. Of particular interest may be a proof of a conjecture of Mazur that the set of smoothings of a differentiable manifold form in a natural way an abelian group. In §7 we conclude the paper by proving the theorem promised at the beginning of this introduction.

In a subsequent paper we will apply the machinery developed in this paper to the question of classifying imbeddings of homotopy n-spheres in S^{n+k}. In particular, we will give there a proof and elaboration of the exact sequences which we announced in Seattle in the Summer 1963.

We might remark that this line of research was inspired by hearing of the result of Hirsch that $\Gamma_n = \pi_n(PL/O)$ and trying to understand this result. Half-way through our own work we came upon the notes of Mazur [4] which considerably clarified and broadened our outlook as well as supplying some key ideas. We are also indebted to conversations with Hirsch and Mazur at the Seattle conference on Differential Topology.

PART I—SEMI-SIMPLICIAL BUNDLES

§1. PIECEWISE DIFFERENTIABLE VECTOR BUNDLES

We begin by constructing a semi-simplicial analogue of the orthogonal group O_n. This will turn out to be more suitable for comparison with Milnor's semi-simplicial group PL_n [5], and hence for comparing vector bundles with microbundles.

DEFINITION (1.1). *The c.s.s. group O_n.*

Let Δ_r be the standard r-simplex, and $\Delta_r \times R^n$ the product n-plane bundle over Δ_r. An r-simplex in O_n is a vector bundle map $f: \Delta_r \times R^n \to \Delta_r \times R^n$ over the identity, which is differentiable over each simplex σ of some rectilinear subdivision of Δ_r. Composition makes the set $O_n^{(r)}$ of r-simplexes into a group. Each monotone simplicial map $\lambda : \Delta_s \to \Delta_k$ defines a homomorphism $\lambda^{\#} : O_n^{(k)} \to O_n^{(s)}$ where $\lambda^{\#}f$ is the vector bundle map over the identity uniquely defined by the condition that

$$\begin{array}{ccc} \Delta_s \times R^n & \xrightarrow{\lambda f} & \Delta_s \times R^n \\ \downarrow{\scriptstyle \lambda \times I} & & \downarrow{\scriptstyle \lambda \times I} \\ \Delta_r \times R^n & \xrightarrow{f} & \Delta_r \times R^n \end{array}$$

commute.

It is easy to check that $\lambda^{\#}f$ is differentiable over each simplex of a rectilinear subdivision of Δ_s. Thus $O_n = \{0_n^{(r)}, \lambda^{\#}\}$ is a c.s.s. group complex.

In the remainder of this section, a complex (unless otherwise specified) will mean a countable, finite dimensional, locally finite, simplicial complex.

DEFINITION (1.2). *A piecewise differentiable structure on an n-plane bundle ξ over a complex K is a presentation of the total space $E(\xi)$ as the union of $\sigma \times R^n$, $\sigma \in K$, with the coordinate transformations from σ to $\partial_i\sigma$ being in $O_n^{(r)}$, where $\dim \sigma = r + 1$.*

A vector bundle, together with a piecewise differentiable structure will be called a *piecewise differentiable vector bundle* (*p.d.* bundle).

Remark. It follows from the definition, that $\exists$ a rectilinear subdivision K_1 of K, such that $\pi^{-1}(\sigma)$, $\sigma \in K_1$, has a well defined differentiable structure, induced from $\tau \times R^n$, where $\tau \in K$ is any simplex containing σ. We will say that ξ is *differentiable over* K_1.

DEFINITION (1.3). *Two p.d. bundles ξ_1 and ξ_2 over K are called equivalent if $\exists$ a map $\varphi : E(\xi_1) \to E(\xi_2)$ such that $\varphi|_{\pi_1^{-1}(\sigma)} : \pi_1^{-1}(\sigma) \to \pi_2^{-1}(\sigma)$ is in $O_n^{(r)}$, where dim $\sigma = r$, $r \in K$.*

THEOREM (1.1). *Two p.d. bundles ξ_1 and ξ_2 over K are equivalent if and only if they are equivalent as vector bundles.*

Before beginning the proof we need the following Lemma which follows immediately from the above remark and Definition (3).

LEMMA. *Two p.d. bundles ξ_1 and ξ_2 over K are equivalent if and only if $\exists$ a subdivision K_1 of K and a map $\varphi : E(\xi_1) \to E(\xi_2)$ satisfying:*

(a) *ξ_1 and ξ_2 are differentiable over K_1*

(b) *$\varphi_{|\pi_1^{-1}(\sigma)} : \pi_1^{-1}(\sigma) \to \pi_2^{-1}(\sigma)$ is a differentiable map of differentiable vector bundles, $\sigma \in K_1$.*

Proof of Theorem (1.1). The "only if" part is obvious. Now assume we have replaced K by a rectilinear subdivision over which ξ_1, ξ_2 are differentiable. Let φ^0 be any bundle map of $E(\xi_1) \to E(\xi_2)$ over the identity. Then φ_0 is differentiable over the vertices. Now suppose that φ^0 has been deformed through bundle maps to a bundle map $\varphi^{(r-1)}$ such that $\varphi^{(r-1)}|\pi_1^{-1}(\tau) \to \pi_2^{-1}(\tau)$ is differentiable, $\tau \in K^{(r-1)}$, ($K^{(r-1)}$, the $(r-1)$-skeleton of K). We will show that $\varphi^{(r-1)}$ may be deformed through bundle maps to a bundle map $\varphi^{(r)}$, such that

(a) $\varphi^{(r)}|\pi_1^{-1}(K^{(r-1)} = \varphi^{(r-1)}|\pi_1^{-1}(K^{(r-1)})$

(b) $\varphi^{(r)}$ is differentiable over the r-simplicies of K. The result will then follow by induction.

Let σ be an r-simplex. We will consider σ to be imbedded in standard fashion n E^r. Now $\varphi^{(r-1)}$ over σ is represented by a continuous map $f : \sigma \to O(n, R)$, which is differentiable on the boundary $\dot{\sigma}$ of σ. It is sufficient to deform f relative to the boundary to a differentiable map. For this will define a deformation of $\varphi^{(r-1)}$ on each r-simplex, leaving it fixed over $K^{(r-1)}$, and this deformation may be extended by the homotopy extension principle to a deformation of $\varphi^{(r-1)}$ over K through bundle maps.

Consequently, the theorem will follow if we can prove:

LEMMA. *Let $f : \sigma \to M$ be a continuous map of an r-simplex into a smooth closed manifold M, such that $f|_{\dot{\sigma}}$ is smooth, then f may be deformed relative to the boundary to a smooth map.*

Proof. Let $i : M \to E^n$ be an imbedding in a sufficiently high dimensional Euclidean space. Let U be a tubular neighborhood of M in E^n, and $r : U \to M$ a smooth retract. It will be sufficient to deform *if*$: \sigma \to U \subset E^n$, in U, to a smooth map, relative to $\dot{\sigma}$; since the retraction will throw this deformation back into M.

By R. Thom, [9], if $\dot{\sigma}$ may be extended to a smooth map g of some neighborhood of $\dot{\sigma}$ in E^r into U, and thus may be extended to a smooth map of σ into U (Thom proves a real-valued smooth function on $\dot{\sigma}$ may be extended to a smooth function on E^r; the above assertion follows immediately). Since $\dot{\sigma}$ has a product neighborhood in σ, it is easy to see that it may be deformed in U relative to $\dot{\sigma}$ to a map $\bar{g}$ which agrees with g on some sufficiently small neighborhood of $\dot{\sigma}$. But now by the approximation theorem, $\bar{g}$ is homotopic to a smooth map $h : \sigma \to U$, relative to a closed neighborhood of $\dot{\sigma}$ in σ. Thus we get a deformation of it to h, relative to $\dot{\sigma}$; and hence a deformation of $f = rif$ to the smooth map $rh : \sigma \to M$, relative to $\dot{\sigma}$.

This completes the proof of the Lemma and hence of Theorem (1.1).

THEOREM (1.2). *Every vector bundle over a complex is equivalent to one which admits a p.d. structure.*

Proof of Theorem (1.2). Imbed K combinatorially in E^n, n sufficiently large. Then if U is the interior of the second regular neighborhood of K in E^n, K is a deformation retract of U; $r : U \to K$ the retraction. Let $n = r^*(\xi)$ then $n|K$ is equivalent as a vector bundle to ξ. Since U has a natural differential structure, $E(n)$ can be given the structure of a differentiable vector bundle. Let K_1 be a sufficiently fine rectlinear subdivision of K, such that each simplex $\sigma \in K_1$ is contained in a coord. neigh. V_σ of n. Then the diff. coord. transformation from $V_{\delta_i\sigma}$ to V_σ restricts to a diff. coord. transformation from $\partial_i\sigma$ to σ, and makes $n|K$ a *p.d.* vector bundle, equivalent to ξ as a vector bundle.

This completes the proof of Theorem (1.2).

Theorem (1.1) and Theorem (1.2) give

THEOREM (1.3). *An equivalence class of vector bundles over a complex K determines a unique equivalence class of piecewise differentiable vector bundles over K, and conversely.*

Theorem (3) enables us to treat vector bundles semi-simplicially, the same way Milnor [5] treats *p.l.* micro-bundles.

DEFINITION (1.4). *Let ξ be a p.d. vector bundle over a finite complex K.* We define the *associated principal* (s.s.) bundle $\tilde{\xi}$ as follows; The base space $\tilde{K}$ is the c.s.s. complex consisting of monotone simplicial maps $f : \Delta_k \to K$; with $\lambda^\# : \tilde{K}^{(k)} \to \tilde{K}^{(e)}$ defined by $\lambda^\# f = f \circ \lambda$. A k-simplex of the total space P consists of

(1) a k-simplex $f \in \tilde{K}^{(k)}$

(2) a *p.d.* bundle equiv. $F : \Delta_k \times R^n \to f^*\xi$.

The functions $\lambda^\# : \tilde{P}^{(k)} \to \tilde{P}^{(l)}$ are defined by the formula $\lambda^\#(f, F) = (f \circ \lambda, \lambda^* F)$. The right translation function $\tilde{P} \times O_n \to \tilde{P}$ is just the operation of composing *p.d.* bundle equivalences. Since $O_n^{(k)}$ operates freely in $\tilde{P}^{(k)}$, with orbit set $\tilde{K}^{(k)}$; it follows that $\tilde{\xi}$ is a principal s.s. O_n-bundle.

LEMMA. *Two p.d. vector bundles ξ, η over a complex K are equivalent if and only if the associated c.s.s. principal bundles $\tilde{\xi}$, $\tilde{\eta}$ are equivalent.*

(Proof as in Milnor [5], p. 25]

LEMMA. *Let K be a complex. Any principal O_n-bundle π over $\tilde{K}$ is isomorphic to $\tilde{\xi}$ for some p.d. vector bundle ξ over K.*

Proof. (As in Milnor, [5], p. 26).

By Heller's classification theorem [1], we have

THEOREM (1.4). *The equivalence classes of n-dimensional p.d. vector bundles and hence (Theorem (1.3)) the equivalence classes of n-dimensional vector bundles over a complex K are in 1–1 correspondence with the homotopy classes of (s.s.) maps of $\tilde{K}$ into the base space of any universal (s.s.) bundle for O_n.*

In the remainder of this paper we assume familiarity with elementary piecewise linear micro-bundle theory, as contained in Milnor [5]. In particular, the definition and properties of the piecewise linear complex of n germs, PL_n, is assumed.

2. THE C.S.S. COMPLEX PD_n

DEFINITION (2.1). *A map $f: K \to M$ of a simplicial complex K into a smooth manifold M is called piecewise differentiable (p.d.), if f is differentiable on each simplex of a rectilinear subdivision of K.*

DEFINITION (2.2) PD_n. A k-simplex in PD_n is a germ of a topological microbundle equivalence $f: \Delta_k \times R^n \to \Delta_k \times R^n$ (i.e. f is fibrewise and preserves zero section), such that $f: \Delta_k \times R^n \to \Delta_k \times R^n \subset R^k \times R^n$ is a *p.d.* imbedding w.r.t. the product triangulation of $\Delta_k \times R^n$. The set of k-simplicies is denoted $PD_n^{(k)}$. For each monotone simplicial map $\lambda: \Delta_l \to \Delta_k$ define a function $\lambda^{\#}: PD_n^{(k)} \to PD_n^{(l)}$ by $\lambda^{\#}(f)$ is the germ of the topological microbundle equivalence $\lambda^{\#} f$ uniquely defined by the condition that

$$\begin{array}{ccc} \Delta_l \times R^n & \longrightarrow & \Delta_l \times R^n \\ {\scriptstyle \lambda \times I}\downarrow & & \downarrow{\scriptstyle \lambda \times I} \\ \Delta_k \times R^n & \longrightarrow & \Delta_k \times R^n \end{array} \qquad \text{commutes.}$$

It is easy to check that λf is a *p.d.* imbedding. Thus $PD_n = \{PD_n^{(k)}, \lambda^{\#}\}$ is a c.s.s. complex. It is also easy to check that it satisfies the Kan condition [7].

Although PD_n is not a group complex, it contains PL_n and we will see that the inclusion is a homotopy equivalence. PD_n has the advantage over PL_n, that O_n acts freely on it.

DEFINITION (2.3). *Consider K to be a rectilinear simplicial complex in some R^p, and let $f: K \to M$ be a p.d. map into a smooth manifold,* i.e. *f is differentiable on each simplex of a subdivision K_1 of K. Then a p.d. map*

$$f_x: St(x, K_1) \longrightarrow \tau(fx) \qquad \text{(the tangent space to } M \text{ at } fx)$$

is defined by $f_x(y) = (f|\sigma)_x(y)$, $x \in \sigma \in K_1^*$, $y \in \sigma \subset St(x, K_1)$, where $(f|\sigma)_x$ is the induced map on tangent spaces (and we identify the tangent space to σ at x with σ).

The map f is called *regular* if f_x is 1–1 (into) for every $x \in K$. It is called *non-singular* if it is a regular *p.d.* imbedding. A non-singular *p.d.* homeomorphism of K onto M is called a *smooth triangulation.*

LEMMA (Milnor [6]). *A p.d. imbedding $f: K \to M$ which is a homeomorphism onto, is non-singular (and hence a smooth triangulation).*

Remark. A map $f: \Delta_k \times R^n \to \Delta_k \times R^n$ representing an element of $PD_n^{(k)}$ is nonsingular.

In fact, since $\partial\Delta_k$ has a product neighborhood in R^k, f may be extended to a *p.d.* imbedding of $N \times R^n$ onto an open set in $N \times R^n$, where N is an open neighborhood of Δ_k in R^k. By the lemma, the extension of f, and hence f itself, is non-singular.

DEFINITION (2.4). *A piecewise differentiable isotopy between two p.d. imbeddings f_0, $f_1 : K \to M$ is a p.d. imbedding $F : K \times I \to M \times I$, such that $F_t : K \times t \to M \times t$ and $F|K \times 0 = f_0$, $F|K \times 1 = f_1$.*

Remark. If F is non-singular, F_t is non-singular.

LEMMA. *Let F be a p.d. isotopy between $f_0, f_1 : K \to M$, and G a p.d. isotopy between $f_1, f_2 : K \to M$, then $H : K \times I \to M \times I$, $H_t = F_{2t}$, $0 \leqq t \leqq 1/2$, $H_t = G_{2t-1}$, $1/2 \leqq t \leqq 1$, is a p.d. isotopy between f_0 and f_2.*

We need two theorems of Whitehead [11].

THEOREM (A). *Let $f : K \to R^q$ be a non-singular p.d. imbedding. Then there are maps $\xi, \eta : K \to R_+$ such that every (ξ, η)-approximation to f is non-singular.*

An (ξ, η)-approximation to f is a *p.d.* map $g : K \to R^q$, such that $|g(x) - f(x)| < \xi(x)$, $|g_x(y) - f_x(y)| \leqq \eta(x)\,|f_x(y)|$, $x \in K$, $x + y \in St(x, K_1)$, where K_1 is a rectilinear subdivision of K on the simplices of which both f, g are differentiable.

THEOREM (B).† *Let $f : K \to R^q$ be a non-singular p.d. imbedding, then for any $\xi, \eta : K \to R_+$, there exists a subdivision of K, such that the linear approximation to f is an (ξ, η)-approximation.*

The linear approximation of f w.r.t. the subdivision K_1 of K, is the *p.l.* map g defined by $g(\Sigma s_j a_j) = \Sigma s_j f(a_j)$, where $(a_0, \ldots, a_1)$ is an r-simplex in K_1, and s_j are the barycentric coordinates of a point in the r-simplex.

From Theorems (A) and (B) we have:

THEOREM (C). *Let $f : K \to R^q$ be a non-singular p.d. imbedding, let (ξ, η) be given, then there exists a rectilinear subdivision K_1 of K, such that the K_1 linear approximation g to f satisfies*

(a) g is a non-singular imbedding.
(b) g is an (ξ, η)-approximation to f.
(c) $F : K \times I \to R^q \times I$ given by $F(x, t) = (1 - t)f(x) + tg(x)$ is a *p.d.* isotopy.
(d) F is a non-singular imbedding (in $R^q \times R^1$).
(e) If f is *p.l.* on some subcomplex L of K, $F_t|_L = f$.

Proof. (a), (b), (c), (e) are immediate from Theorem (A) and (B). To see (d), note that F will be an (ξ', η')-approximation to $f \times 1$, if g is an (ξ, η)-approximation to f for sufficiently small (ξ, η). But $f \times 1$ is clearly non-singular.

THEOREM (2.1). *Let $f : \Delta_k \times R^n \to \Delta_k \times R^n$ be a topological micro-bundle equivalence, and a p.d. imbedding with respect to the product triangulation of $\Delta_k \times R^n$ (as a map into $R^k \times R^n$). Then there exists a non-singular p.d. isotopy*

$$F : I \times \Delta_k \times R^n \longrightarrow I \times \Delta_k \times R^n$$

† See proof of Theorem (9) of [11].

such that

(a) *F is a topological microbundle equivalence (over $I \times \Delta_k$).*

(b) *If f is p.l. in some subcomplex K of a rectilinear subdivision of $\Delta_k \times R^n$, then $F_t|_K = f$.*

(c) *$g = F_1$ is a p.l. microbundle equivalence (over Δ_k)*

$$g : \Delta_k \times R^n \longrightarrow \Delta_k \times R^n.$$

Proof. Since f is *p.l.* on $\Delta_k \times 0$ and K, it is enough to apply Theorem (C) and show that $\pi F_t = F_t \pi$, $\pi : \Delta_k \times R^n \to \Delta_k$ the projection. Let L be the rectilinear subdivision of the product triangulation of $\Delta_k \times R^n$ given by Theorem (C). Let $(a_0, \ldots, a_r)$ be an r-simplex in L. Then by assumption $\pi f(a_j) = f\pi a_j = \pi a_j$. Now let $(\sum s_j a_j)$, $\sum s_j = 1$, be an arbitrary point in the r-simplex. Then

$$F_t(\sum s_j a_j) = (1 - t) f(\sum_j s_j a_j) + t \sum_j s_j f(a_j)$$
$$\pi F_t(\sum s_j a_j) = (1 - t) f\pi(\sum_j s_j a_j) + t \sum s_j \pi a_j.$$

But since we have a rectilinear subdivision of the product triangulation of $\Delta_k \times R^n$, $\pi \sum s_j a_j = \sum s_j \pi a_j$, and $\pi F_t(\sum s_j a_j) = \pi(\sum s_j a_j) = F_t \pi(\sum s_j a_j)$.

THEOREM (2.2). *PL_n is a deformation retract of PD_n.*

Proof. We rely on Proposition (1) of Appendix A to Chapter I in [7].

PROPOSITION (Moore [7]). *If X and Y are s.s. complexes, then $f_0, f_1 : X \to Y$ are homotopic if and only if there exists functions $K_i : X^{(q)} \to Y^{(q+1)}$ defined for $i = 0, \ldots, q$, and all q such that*

(1) $\partial_0 k_0 = f_1$	(5) $\partial_i k_j = k_j \partial_{i-1}$, $i > j + 1$
(2) $\partial_{q+1} k_q = f_0$	(6) $s_i k_j = k_{j+1} s_i$, $i \leqq j$
(3) $\partial_i k_j = k_{j-1} \partial_i$, $i < j$	(7) $s_i k_j = k_j s_{i-1}$, $i > j$
(4) $\partial_{i+1} k_{j+1} = \partial_{j+1} k_j$,	

If A is a subcomplex of X, and $f_0|A = f_1|A$, then f_0 is homotopic to f_1 relative to A if and only if

$$k_i(\sigma) = f_0(s_i(\sigma)), \quad \text{for} \quad \sigma \in A.$$

We apply this to our problem in taking f_0 = identity on PD_n, f_1 = retraction of PD_n into $PL_n \subset PD_n$. We use the following:

LEMMA. *Let $\tau \in PD_n^{(R)}$, and $g_0 : \Delta_R \times R^n \to \Delta_R \times R^n$ be a p.d. imbedding representing τ. Then if $g : I \times \Delta_R \times R^n \to I \times \Delta_R \times R^n$ is a p.d. isotopy and a topological microbundle equivalence, such that $g|0 \times \Delta_R \times R^n = g_0$, there exist functions $k_i : L(\tau)^{(q)} \to PD_n{}^{(q+1)}$, satisfying (1)–(7) above, where $L(\tau)$ is the subcomplex of PD_n generated by σ, f_0 = identity, and $f_1(\lambda^\# \sigma) = \lambda^\#(g_1)$, λ any monotone simplicial map of $\Delta_q \to \Delta_R$.*

Proof. Subdivide $I \times \Delta_q$ into the $q + 1$ simplicies $(a_0, \ldots, a_{i-1} a_i, \ldots a_p)$. Define $k_i(\lambda^\#(\tau)) = (\lambda^\# \times 1 \circ g | \pi^{-1}(a_0, \ldots, a_{i-1} a_i, \ldots, a_p))$. We leave it to the reader to check properties (1)–(7).

In order to prove the theorem, we need to pick representatives g for each non-degenerate simplex τ. Then by Theorem (1), there exists a *p.d.* isotopy g from g_0 to a *p.l.* homeomorphism g_1, relative to any sub-complex on which g_0 is a *p.l.* homeomorphism. We must show that the isotopy given for the representative of τ and the representative of $\partial_i\tau$ agree in some neighborhood of the zero-section over $\partial_i\Delta_p$. To do this, we proceed by induction. We assume the isotopies have been defined for representatives of all $p-1$ simplexes to satisfy the above condition. This means that we have a well defined isotopy of g_0 on some neighborhood of the zero section above the complete boundary of Δ_p. Now there is a *p.l.* isotopy of $\Delta_p \times R^n$ into itself which is a deformation of the identity to a map into any prescribed neighborhood of the zero section, relative to any smaller neighborhood of the zero section. Assuming inductively that our isotopies are always constant for some initial and final interval in some neighborhood of the zero section; Theorem (1) and the above deformation, show that it agrees in some neighborhood of the zero section with the isotopies defined over the boundary, and satisfies the stationary condition on (possibly smaller) initial and final intervals. (Note that any *p.d.* isotopy above the boundary of Δ_p may be extended to a *p.d.* isotopy over Δ_p, since the boundary has a product neighborhood in Δ_p. Also once g_0 over the boundary of Δ_p has been deformed to a *p.l.* map, application of Theorem (1) leaves this part of the map unchanged.)

§3. ON TRIANGULATING VECTOR BUNDLES—EXISTENCE AND UNIQUENESS

PROPOSITION (3.1). *Let ξ_1, ξ_2 be differentiable vector bundles over a smooth manifold M. Let $f: K \to M$ be a smooth triangulation. Then if $\varphi : E(\xi_1) \to E(\xi_2)$ is a p.d. vector bundle map over the identity* (i.e. *φ is differentiable over each simplex of some rectilinear subdivision K_1 of K*), *φ is isotopic through p.d. bundle maps to a differentiable bundle map.*

Proof. Let P_1, P_2 be the total space of the assoc. princ. diff. bundle. Then φ induces $\Phi : P_1 \to P_2$, a bundle map diff. over each simplex of K_1. Now Φ is defined by a cross-section of the following bundle (G = group of orthogonal transformations of R^n):

$$\begin{array}{ccc} d^*(P_1 \times_G P_2) & \longrightarrow & P_1 \times_G P_2 \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad d \quad} & M \times M. \end{array}$$

This is a diff. bundle with fibre G and group $G \times G$. Consequently, it may be approx. by a diff. cross-section Ψ. Now put a Riemannian metric on G, invariant under the action of $G \times G$. Then if Ψ is sufficiently close to Φ, $\Psi(x)$ may be joined to $\Phi(x)$, $x \in M$, by a unique shortest geodesic, in the fibre over x. Deforming Φ to Ψ along the geodesics preserves differentiability, whenever Φ and Ψ are both diff; i.e. over each simplex of K_1. This defines a *p.d.* bundle deformation of φ to a diff. bundle map Ψ.

DEFINITION (3.1). *Triangulation of p.d. vector bundles. A* triangulation of a *p.d.* vector bundle ξ over a complex K, is a *p.l.* microbundle μ over K and a fibrewise map $\varphi : E(\mu) \to E(\xi)$, preserving the zero cross-section, and such that φ is a *p.d.* homeomorphism over each simplex $\sigma \in K$. We denote such a triangulation by the pair (μ, φ).

THEOREM (3.1). *Let ξ be a p.d. vector bundle over K, and (μ_i, φ_i), $i = 1, 2$ be triangulations* *of ξ. Then there exists a map of p.l. microbundles $\psi : E(\mu_1) \to E(\mu_2)$ such that*

$$\begin{tikzcd}
E(\mu_1) \arrow[d, "\psi"'] \arrow[dr, "\varphi_1"] & \\
 & E(\xi) \\
E(\mu_2) \arrow[ur, "\varphi_2"] &
\end{tikzcd}$$

commutes up to homotopy. (Note that $(\mu_1, \varphi_2\psi)$ is a triangulation of ξ, and the homotopy between $\varphi_2\psi$ and φ_1 is to be a triangulation at each stage.)

COROLLARY. *Let (μ_i, φ_i) be triangulations of p.d. vector bundles ξ_i, $i = 1, 2$ over rectilinear subdivisions of K. If $\lambda : E(\xi_1) \to E(\xi_2)$ is a p.d. bundle equivalence, there exists a microbundle equivalence $\psi : E(\mu_1) \to E(\mu_2)$ such that*

$$\begin{tikzcd}
E(\mu_1) \arrow[r] \arrow[d, "\psi"'] & E(\xi_1) \arrow[d, "\lambda"] \\
E(\mu_2) \arrow[r] & E(\xi_2)
\end{tikzcd}$$

commutes up to homotopy.
(Note that $(\mu_1, \lambda\varphi_1)$ and $(\mu_2, \varphi_2\psi)$ are triangulations of ξ_2, and the homotopy is to be a triangulation at each stage.)

Proof of Theorem (3.1). The idea of the proof is to deform φ_1 fibrewise and relative to the zero section to a *p.l.* homeomorphism ψ with respect to the triangulation (μ_2, φ_2) of $E(\xi)$. We do this by induction over the skeletons of K.

But first we may change $E(\mu_i)$, $i = 1, 2$ to equivalent microbundles, so that $E(\mu_i) = \cup\sigma \times R^n$, $\sigma \in K$; $(\sigma \times R^n)$ attached to $\partial_i\sigma \times R^n$ by a *p.l.h.* of $\sigma \times R^n$ over $\partial_i\sigma$ into $\partial_i\sigma \times R^n$.

Over each vertex $v \in K^{(0)}$, φ_1 is a *p.d.h.* of $(R^n, 0)$ into $(R^n, 0)$, and is isotopic through *p.d.* homeomorphisms relative to $0 \in R^n$, to a *p.l.* homeomorphism (see Theorem (2.1)) w.r.t. the triangulation induced by (μ_2, φ_2). This isotopy of $\varphi_1|\pi_1^{-1}(K^0)$ may be extended to a fibrewise isotopy of φ_1 on $E(\mu_1)$ relative to the zero section, *p.d.* over each simplex; by induction over the skeletons, using the fact that the boundary of each simplex has a product neighborhood in the simplex.

Now assume that φ_1 is a *p.l.* homeomorphism over the $(r-1)$-skeleton of K, and preserves the zero section; and let σ be an r-simplex of K. $\varphi_1|\pi_1^{-1}(\sigma)$ is a *p.d.h.*, which is *p.l.* over $\dot\sigma$ and *p.l.* on the zero section. By Theorem (2.1) we can deform $\varphi_1|\pi_1^{-1}(\sigma)$ fibrewise, through *p.d.* homeomorphisms, and relative to $\varphi_1|\pi_1^{-1}(\ )$ and the zero section, to a *p.l.h.* w.r.t. (μ_2, φ_2). This, having been done for each r-simplex, gives a *p.d.* isotopy of φ_1 over $K^{(r)}$ relative to $K^{(r-1)}$ and the zero section. This isotopy may now be extended to a *p.d.* isotopy of φ_1 relative to $\pi^{-1}(K^{(r-1)})$ and the zero section.

This completes the induction step and the proof of Theorem (3.1).

THEOREM (3.2). *There exists a unique function T, functorial in the base space, from equivalence classes of n-dimensional p.d. vector bundles (and hence ordinary vector bundles) to equivalence classes of n-dimensional p.l. microbundles, satisfying*

(a) *Given a p.d. vector bundle ξ; if $\mu \in T[\xi]$, there exists a map $\varphi : E(\mu) \to E(\xi)$, such that (μ, φ) is a triangulation of ξ.*

Proof. The uniqueness follows from Theorem (3.1). To show existence, we first define a map of BO_n into BPL_n.

Let PD_n be the c.s.s. complex defined previously; i.e. a k-simplex of PD_n is a fibre preserving *p.d.* homeomorphism

$$f: \Delta_k \times R^n \to \Delta_k \times R^n, f|_{\Delta_k \times O} = \text{identity. Then}$$

$PD_n = \{PD_n^{(k)}, \lambda^\#\}$. We recall that PD_n is not a group complex, but has the following properties:

(a) $O_n, PL_n \subset PD_n$

(b) $\pi_i(PL_n) \to \pi_i(PD_n)$ is an isomorphism, all i

(c) PL_n acts freely on the right of PD_n, O_n acts freely on the left of PD_n.

Let U_n be the total space of a universal s.s. bundle for PL_n. Then $\pi_i(U_n) = 0$, all i. Let PL_n act on both factors of $PD_n \times U_n$. Then $\pi_i((PD_n \times U_n)/PL_n) = 0$, all i; since $(PD_n \times U_n, PL_n, (PD_n \times U_n)/PL_n)$ is a principal s.s. bundle (and hence has an exact homotopy sequence), and $\pi_i(PL_n) \to \pi_i(PD_n \times U_n)$ is an isomorphism, all i.

Let O_n act on $PD_n \times U_n$ by acting on PD_n only. Then the action of O_n commutes with the action of PL_n, and O_n acts freely on $(PD_n \times U_n)/PL_n$. Hence $(PD_n \times U_n)/PL_n, O_n, O_n\backslash PD_n \times U_n/PL_n)$ is a universal s.s. bundle for O_n. Write $BO_n = O_n\backslash PD_n \times U_n/PL_n$, and $BPL_n = U_n/PL_n$. Then the projection $p_2 : PD_n \times U_n \to U_n$ induces a s.s. map $p : BO_n \to BPL_n$ by passage to the quotients. We note for future use

LEMMA (3.3). *$(BO_n, PD_n/O_n, BPL_n)$ is a s.s. fibre space.*

Now let ξ be a *p.d.* vector bundle over some rectilinear subdivision of K, and $\tilde{\xi}$ the associated principal s.s. bundle. Then $\tilde{\xi}$ is equivalent to a bundle induced by a s.s. map $f: \tilde{K} \to BO_n$. But $pf: \tilde{K} \to BPL_n$ induces a principal s.s. bundle over K with group PL_n. Let μ be the associated microbundle. We set $T[\xi] = [\mu]$. Since any two classifying maps are homotopic, T is well-defined.

We wish to define a map $\varphi : E(\mu) \to E(\xi)$ such that (μ, φ) is a triangulation of ξ. This is done as follows:

If $\sigma \in \tilde{K}^{(p)}$, choose $y_\sigma \in U_n^{(q)}$ which lies over $pf(\sigma)$. Further choose $x_\sigma \in PD_n^{(q)}$ so that $(x_\sigma, y_\sigma) \in PD_n \times U_n$, lies over $f(\sigma)$; and denote the class of (x_σ, y_σ) in $(PD_n \times U_n)/PL_n$ by $[x_\sigma, y_\sigma]$. Then

$$\partial_i y_\sigma = y_{\partial_i \sigma} \cdot F, \qquad F \in PL_n^{(q-1)},$$

and

$$\partial_i[x_\sigma, y_\sigma] = [\partial_i x_\sigma, \partial_i y_\sigma] = [x_{\partial_i \sigma}, y_{\partial_i \sigma}]G, \; G \in 0_n^{(q-1)},$$

where we define right action of G by

$$[x_{\partial_i\sigma}, y_{\partial_i\sigma}]G = [G^{-1}x_{\partial_i\sigma}, y_{\partial_i\sigma}].$$

By definition $E(\mu) = \bigcup_{\sigma\in K} \sigma \times R^n$, with $\sigma \times R^n$ identified with $\partial_i\sigma \times R^n$ by F, $E(\xi) = \bigcup_{\sigma\in K} \sigma \times R^n$, with $\sigma \times R^n$ identified to $\partial_i\sigma \times R^n$ by G. (Here $\sigma \in K$ is considered as the simplex in $\tilde{R}^{(q)}$ representing it.) To define φ, it is sufficient, in view of Theorem (1), to consider $E(\xi)$ to be the equivalent bundle above. Define φ by

$$\phi|\sigma \times R^n = x_\sigma, \qquad \sigma \in K.$$

We claim this is well-defined; in fact

$$G^{-1}x_{\partial_i\sigma} = \partial_i x_\sigma \cdot F^{-1},$$

since $[\partial_i x_\sigma, \partial_i y_\sigma] = [\partial_i x_\sigma, y_{\partial_i\sigma}F] = [\partial_i x_\sigma F^{-1}, y_{\partial_i\sigma}] = [G^{-1}x_{\partial_i\sigma}, y_{\partial_i\sigma}]$.
It follows that φ is well-defined, and (μ, φ) is a triangulation of ξ.

This completes the proof of Theorem (3.2).

§4. ON WHITNEY SUMS

Let $\alpha \in PD_k$, $\alpha : \Delta_0 \times R^k \to \Delta_0 \times R^k$ and $\gamma \in PD_n$, $\gamma : \Delta_0 \times R^n \to \Delta_0 \times R^n$ define the Whitney sum $\alpha \oplus \gamma \in PD_{k+n}$, $\alpha \oplus \gamma : \Delta_0 \times R^k \times R^n \to \Delta_0 \times R^k \times R^n$ by $\alpha \oplus \gamma(x, u, v) = (x, \alpha_1(x, u), \gamma_1(x, u))$ where $\alpha(x, u) = (x, \alpha_1(x, u)$, $\gamma(x, v) = (x, \gamma_1(x, v))$ then $\oplus$ is a semi simplicial map

$\oplus : PD_k \times PD_n \to PD_{k+n}$. By restriction, we get
$\oplus : PL_k \times PL_n \to PL_{k+n}$
$\oplus : O_k \times O_n \to O_{n+k}$.

By product action $O_k \times O_n$ acts on $PD_k \times PD_n$ and we have a commutative diagram

$$\begin{array}{ccc} (O_k \times O_n) \times (PD_k \times PD_n) & \longrightarrow & PD_k \times PD_n \\ \downarrow \oplus\times\oplus & & \downarrow \oplus \\ O_{k+n} \times PD_{k+n} & \longrightarrow & PD_{k+n}. \end{array}$$

Thus the above map passes to quotients

$$\oplus\ PD_k/O_k \times PD_n/O_n \longrightarrow PD_{k+n}/O_{k+n}.$$

We use [ ] to denote semi simplicial homotopy classes of maps.

Let K be a complex $\alpha_k : K \to PD_k$, $\alpha_n : K \to PD_n$ then the above operation induces a map $\alpha_k \oplus \alpha_n : K \to PD_{k+n}$. Notice $(\alpha_k \oplus \alpha_n) \oplus \alpha_e = \alpha_k \oplus (\alpha_n \oplus \alpha_e)$.

Homotopy commutativity of Whitney sums

(We use $\sim$ for homotopy equivalence of maps)

The vertices of O_n are just the general linear group GL_n. Let $w \in GL_n$ and $\overline{W}$ the smallest subcomplex of O_n containing w. For any complex there is a unique map $wK : K \to \bar{W}$. Since $\pi_0(O_n) = Z_2$, $w_1 K \sim w_2 K$ in O_n iff det. $w_1 \cdot$det. $w_2 > 0$. Since each $\bar{W} \subseteq PL_n$ we have a commutative diagram

$$\begin{tikzcd} & & PL_n \arrow[dr] & & \\ K \arrow[r, "wK"] & \bar{W} \arrow[ur] \arrow[dr] & & PD_n \arrow[r] & PD_n/O_n \\ & & O_n \arrow[ur] & & \end{tikzcd}$$

Now assume det. $w_1 \cdot$det. $w_2 > 0$. Since $w_1 K \sim w_2 K$ in O_n, $w_1 K \sim w_2 K$ in PD_n. Since $\imath : PL_n \to PD_n$ is a homotopy equivalence $w_1 K \sim w_2 K$ in PL_n. Let A_n be one of the complexes $O_n, PL_n, PD_n, PD_n/O_n$. Then using the group structure in PL_n, O_n and their actions on PD_n we have maps

$$\bar{W} \times A_n \longrightarrow A_n \qquad A_n \times \bar{W} \longrightarrow A_n.$$

Thus given a map $\alpha : K \to A_n$, we have products $wK \cdot \alpha : K \to A_n$ and $\alpha \cdot wK : K \to A_n$. Using the homotopy equivalence between $w_1 K$ and $w_2 K$ for $w_1, w_2 \in GL_n$ one has $w_1 K \cdot \alpha \sim w_2 K \cdot \alpha$ and $\alpha \cdot w_1 K \sim \alpha \cdot w_2 K$.

Consider $e_n \in GL_n$, the identity matrix, then $e_n K \cdot \alpha = \alpha = \alpha \cdot e_n K$. Thus for any $w \in GL_n$, det. $w > 0$, $wK \cdot \alpha \sim \alpha \sim \alpha \cdot wK$.

LEMMA (I). *Let* $\alpha_1 : K \to A_n$ $\alpha_2 : K \to A_k$.
Then $\alpha_1 \oplus \alpha_2 = w_2 K \cdot (\alpha_2 \oplus \alpha_1) \cdot w_1 K$, where $w_1 \in GL_{n+k}$ is defined by $w_1(e_i) = e_{i+k}$ for $i \leqq n$, $w_1(e_i) = e_{i-n}$ for $i > n$; $(e_1, \ldots, e_{n+k}$ canonical basis for $R^{n+k})$ and $w_2 = w_1^{-1}$.

Proof. By direct substitution in equation defining $\oplus$. A direct computation shows det. $w_1 > 0$ iff $n \cdot k = 0 (\text{mod } 2)$. Thus

COROLLARY (I). *If* $nk = 0$ (*mod* 2) *then* $\alpha_1 \oplus \alpha_2 \sim \alpha_2 \oplus \alpha_1$.

Let $t \in GL_{k+n}$ be defined by $t(e_i) = e_i$ for $i > 1$, $t(e_1) = -e_1$. Let $\bar{t} = t|R^k$, $R^k \leqq R^k \times R^n = R^{k+n}$. Using notation of Lemma (I).

COROLLARY (II). *If* $nk = 1$ (*mod* 2) *then* $\alpha_1 \oplus \alpha_2 \sim tK \cdot (\alpha_2 \oplus \alpha_1) \cdot tk = (\bar{t}K \cdot \alpha_2 \cdot \bar{t}K \oplus \alpha_1)$.

Since $tK \cdot e_k \cdot \bar{t}K = e_k$ for any k, we have combining Corollary (I, II).

COROLLARY (III). *For any k*

$$e_k K \oplus \alpha_1 \sim \alpha_1 \oplus e_k K.$$

Combining Corollary (I, II, III).

COROLLARY (IV). $\alpha_1 \oplus \alpha_2 \oplus e_1 K \sim \alpha_2 \oplus \alpha_1 \oplus e_1 K$.

Proof. If $nk = 0$ (mod 2) this follows from Corollary (I). If $nk = 1$ (mod 2) we have

$$\alpha_1 \oplus (\alpha_2 \oplus e_1 K) \overset{\text{Cor(I)}}{\sim} (\alpha_2 \oplus e_1 K) \oplus \alpha_1 \overset{\text{Cor(III)}}{\sim} \alpha_2 \oplus \alpha_1 \oplus e_1 K.$$

Consider the diagram

$$\begin{array}{ccc} A_m \times A_k & \xrightarrow{\oplus} & A_{m+k} \\ \downarrow \iota_r \quad \downarrow \iota_s & & \downarrow \iota_{r+s} \\ A_{n+r} \times A_{k+s} & \xrightarrow{\oplus} & A_{n+k+r+s} \end{array} \qquad \text{with } \iota_t(\alpha) = \alpha \oplus e_t\alpha.$$

This is not commutative, however, if $\alpha_1 : K \to A_n$ $\alpha_2 : K \to A_k$ then $\iota_r\alpha_1 = \alpha_1 \oplus e_rK$, $\iota_s\alpha_2 = \alpha_2 \oplus e_sK$. And

$$\iota_r\alpha_1 \oplus \iota_s\alpha_2 = (\alpha_1 \oplus e_rK \oplus \alpha_2 \oplus e_sK) \overset{\text{Cor (III)}}{\sim} \alpha_1 \oplus \alpha_2 \oplus e_rK \oplus e_sK = \alpha_1 \oplus \alpha_2 \oplus e_{r+s}K$$

while $\iota_{r+s}(\alpha_1 \oplus \alpha_2) = \alpha_1 \oplus \alpha_2 \oplus e_{r+s}K$.

COROLLARY (V). *Thus applying the functor $[K, \ldots]$ one gets a commutative diagram*

$$\begin{array}{ccc} [K, A_n] \times [K, A_k] & \xrightarrow{\oplus} & [K, A_{n+k}] \\ \downarrow \iota_r{}^* \quad \downarrow \iota_s{}^* & & \downarrow \iota_{r+s}{}^* \\ [K, A_{n+r}] \times [K, A_{k+s}] & \xrightarrow{\oplus} & [K, A_{n+k+r+s}]. \end{array}$$

The family (A_i, ι_s) is a directed system of complexes. Define

$$A = \underset{\overrightarrow{i}}{\lim}\, A_i.$$

We call a complex K, finite if it has only a finite number of non-degenerate simplicies. For any finite complex K an easy argument shows $[K, A] = \underset{\overrightarrow{i}}{\lim}\, [K, A_i]$. It immediately follows that this is true for any complex K of the same homotopy type as a finite complex. Such a complex will be said to be of finite homotopy type. By Corollary (IV, V) and above remark we have

THEOREM (4.1). *The Whitney sum induces on $[K, A]$ for any finite homotopy type complex K, the structure of an associative abelian monoid with two sided identity.*

We will see that the restriction that K be finite is unnecessary and that $[K, A]$ is actually a group. We first consider the case $A = O, PL$. Then A is actually a group under composition inherited from O_n, PL_n. We will see that the above operation comes from the group structure.

Let $\bar{\alpha}_1, \bar{\alpha}_2 \in [K, A]$, K finite. Let $\alpha_1 \in [K, A_n]$, $\alpha_2 \in [K, A_n]$ represent $\bar{\alpha}_1, \bar{\alpha}_2$. Then by Corollary (III), $\alpha_1 \oplus e_nK$ and $e_nK \oplus \alpha_2$ also represent $\bar{\alpha}_1, \bar{\alpha}_2$. Then $\bar{\alpha}_1 \cdot \bar{\alpha}_2$ is represented by $(\alpha_1 \oplus e_nK)\cdot(e_nK \oplus \alpha_2) = \alpha_1 \oplus \alpha_2$. Thus the two operations in $[K, A]$ agree.

COROLLARY (VII). *For $A = O, PL$ the Whitney sum on $[K, A]$ is induced from group multiplication and thus extends to arbitrary K. Further $[K, A]$ is an abelian group.*

Since $PL \to PD$ is a homotopy equivalence, the conclusion of Corollary (VII) applies to $[K, PD]$.

The following results on PD/O make use of some deeper information. The only use we make of them in the sequel is of Theorem (4.2) for complexes K of finite homotopy type, the proof of which for this case does not depend on any of the following assumptions.

ASSUMPTION (A). *Γ_n is finite for all $n \geqq 0$.*

A proof for the only doubtful case Γ_4 has been announced by Cerf but has not appeared. The other cases are dealt with in [8] and [3].

ASSUMPTION (B). $\pi_n(PD/O) = \Gamma_n$.

This will be proved in the sequel.

For any complex K, let K_n be the subcomplex generated by the n section of K. A complex K is called finite dimensional if $K = K_n$ for some n.

PROPOSITION (I). *Let K be finite dimensional.* Then there is a j such that $\iota^* : [K, PD_j/O_j] \to [K, PD/O]$ is onto.

Proof. Let M be a minimal subcomplex of PD/O [7]. Then since $\pi_n(PD/O)$ is finite for all n, M_n is finite for all n. Then $M_n \xrightarrow{\iota} PD/O$ factors back $M_n \xrightarrow{\gamma} PD_i/O_i \xrightarrow{\iota} PD/O$ up to homotopy. If K is finite dimensional, $\alpha \in [K, PD/O]$ comes from $\alpha' \in [K, M_n]$ for some n. Thus since diagram

$$\begin{tikzcd} {[K, M_n]} \arrow[r, "\gamma^*"] \arrow[rr, bend right, "\iota^*"'] & {[K, PD_i/O_i]} \arrow[r, "\iota^*"] & {[K, PD/O]} \end{tikzcd}$$

commutes, the proposition follows.

COROLLARY (VIII). *If K is finite dimension,* $[K, PD/O] = \lim\limits_{\substack{\longrightarrow \\ i}} [K, PD_i/O_i]$. Thus the Whitney sum operation extends to finite dimensional K.

Proof is easy from above proposition.

Remark. The following diagram

$$\begin{tikzcd} {[K, PD_n/O_n] \times [K, PD_k/O_k]} \arrow[r, "\oplus"] \arrow[d, "\zeta"] & {[K, PD_{n+k}/O_{n+k}]} \arrow[d, "\zeta"] \\ {[K, BO_n] \times [K, BO_k]} \arrow[r, "\oplus"] & {[K, BO_{n+k}]} \end{tikzcd}$$

commutes, where $\oplus$ on the bottom denotes the Whitney sum of bundles. The proof comes from defining the Whitney sum of bundles through co-ordinate functions.

THEOREM (4.2). *For any finite dimensional complex K the sequence*

$$[K, O] \xrightarrow{\mu} [K, PD] \xrightarrow{\lambda} [K, PD/O] \xrightarrow{\zeta} [K, BO] \xrightarrow{\rho} [K, BPL]$$

is an exact sequence of abelian groups.

Proof. That this is an exact sequence of basepointed sets is the usual property of fiber spaces applied to the fiber spaces

I $O \to PD \to PD/O$

II $PD/O \to BO \to BPL$

noticing that I is fibrations induced from II by inclusions $PD/O \to BO$.

That the maps are additive follows from definitions; [The fact that $\rho(\xi_1 \oplus \xi_2) = \rho(\xi_1) \oplus \rho(\xi_2)$, where $\oplus$ denotes usual Whitney sum, follows from uniqueness of triangulation Theorem (3.1)]; and from above remark.

That $[K, BO]$ and $[K, BPL]$ are abelian groups is known. It only remains then to show that $[K, PD/O]$ is actually a group, i.e. that inverses exist. Let $\alpha \in [K, PD/O]$ then $\zeta[\alpha] \in [K, BO]$ has an inverse $v \in [K, BO]$ since ρ is a group homomorphism $\rho(v) = 0$. Thus there is an $\alpha' \in [K, PD/O]$ with $\zeta(\alpha') = v$. Thus $\zeta(\alpha \oplus \alpha') = 0$ or there is a $\beta \in [K, PD]$ with $\lambda(\beta) = \alpha \oplus \alpha'$. Now β has a negative in $[K, PD]$ so that $\lambda(-\beta) + (\alpha + \alpha') = \lambda(\beta + -\beta) = 0$. Thus $\alpha' + \lambda(-\beta)$ is an inverse to α. Q.E.D.

THEOREM (4.3). *PD/O can be given the structure of an H-space such that for each finite dimensional K, the Whitney sum on $[K, PD/O]$ comes from this H-space structure on PD/O.*

Proof. Let M be a minimal subcomplex of PD/O. Then there exists a deformation retraction $r : PD/O \to M$. As noted previously, the Whitney sum is defined on $[M_n \times M_n, PD/O] \times [M_n \times M_n, PD/O] \to [M_n \times M_n, PD/O]$.

Let $\pi_i : M_n \times M_n \to PD/O$, $i = 1, 2$, be projections and let α_n represent $\pi_1 \oplus \pi_2 \in [M_n \times M_n, PD/O]$. Then $\alpha_{n+1}|M_n \times M_n \sim \alpha_n$. By homotopy extension theorem [7] ((PD/O is a Kan complex)), we can choose α_{n+1} with $\alpha_{n+1}|M_n \times M_n = \alpha_n$. Choosing inductively in this manner we have an $\alpha : M \times M \to PD/O$. The retraction of $PD/O \to M$ turns PD/O into an H-space with desired properties. Q.E.D.

PART II—BUNDLES AND SMOOTHING THEORY

§5. TUBULAR NEIGHBORHOOD THEOREM

THEOREM (5.1). *Given $K \times R^k \xrightarrow{\alpha} K \times R^k$ p.l. homeomorphism, $\alpha|K \times O =$ identity, K unbounded combinatorial manifold. Then*

$$\alpha \times 1_n : K \times R^k \times R^n \longrightarrow K \times R^k \times R^n,$$

n sufficiently large, is isotopic to a fibrewise map relative to $K \times O$.

The proof will be preceeded by some definitions:

Let K be a complex. Let $H_n(K)$ be set of germs of *p.l.* imbeddings $\alpha : K \times R^n \to K \times R^n$ with $\alpha|K \times O =$ identity. We will denote by α_2 the composite

$$K \times R^n \xrightarrow{\alpha} K \times R^n \xrightarrow{\pi_2} R^n.$$

Since K is countable and locally finite, the set of simplicial maps $K \to PL_n$, PL_n^K, is in 1-1 correspondence with the set of germs of $p.l.$ imbeddings $r : K \times R^n \to K \times R^n$ with $\gamma | K \times O =$ identity, and $\pi_1(\gamma(k, y)) = k$. Thus there is an injection $PL_n^K \to H_n(K)$ and we henceforth consider $P_n^k \subseteq H_n(K)$.

Let α represent an element of $H_n(K)$, γ an element of PL_n^K. Define $\alpha \oplus \gamma : K \times R^n \times R^s \to K \times R^n \times R^s$ by $\alpha \oplus \gamma : (k, x, y) = (\alpha(k, x), \gamma_2(k, y))$. If $1_n : K \times R^n \xrightarrow{\text{identity}} K \times R^n$ then $\alpha \oplus \gamma =$ composition $(1_n \oplus \gamma)(\alpha \oplus 1_r)$, and thus is an imbedding. Hence Whitney sum induces a pairing:

$$H_n(K) \times PL_r^K \overset{\oplus}{\longrightarrow} H_{n+r}(K).$$

Note that if γ, ρ represent elements in PL_n^K, PL_r^K : $\oplus$ is the Whitney sum defined in I, 4 and that

$$\alpha \oplus (\gamma \oplus \rho) = (\alpha \oplus \gamma) \oplus \rho.$$

Let α, α' represent elements of $H_n(K)$. Define $\alpha \overset{i}{\equiv} \alpha'$ if there is a t representing an element of $H_n(I \times K)$ such that $t|0 \times K \times R^n = \alpha$, $t|1 \times K \times R^n = \alpha'$. It is not difficult to see that if α, α' have the same germ, $\alpha \overset{i}{\equiv} \alpha'$, so that the relation induces an equivalence relation in $H_n(K)$, still denoted by $\overset{i}{\equiv}$. It is clear that if $\bar{\gamma}, \bar{\gamma}'$ are homotopic elements of PL^K that $\bar{\gamma} \overset{i}{\equiv} \bar{\gamma}'$ and further that if $\bar{\alpha}, \bar{\alpha} \in H_n(K)$, $\bar{\alpha} \overset{i}{\equiv} \bar{\alpha}'$ then $\bar{\alpha} \oplus \bar{\gamma} \overset{i}{\equiv} \bar{\alpha}' \oplus \bar{\gamma}'$.

Finally, define for $\bar{\alpha} \in H_n(K)$, $\bar{\bar{\alpha}} \in H_r(K)$, $\bar{\alpha} \overset{s}{\equiv} \bar{\bar{\alpha}}$ if there exist integers j, l with $\bar{\alpha} \oplus 1_j \equiv {}^{-} \oplus 1_l$. It then follows that the equivalence classes in $H(K) = \bigcup_{n=0}^{\infty} H^n(K)$ under relation $\equiv$ are acted on by elements of $[K, PL]$ (homotopy classes of simplicial maps).

We also make use of the property that if $\bar{\alpha} \overset{s}{\equiv} {}^{-}$ then $\bar{\alpha}\bar{\imath} \overset{s}{\equiv} \bar{\bar{\alpha}}\bar{\imath}$; $\bar{\alpha}, \bar{\bar{\alpha}}, \bar{\imath} \in H_n(K)$.

Lemma. *For $\bar{\alpha} \in H_n(K)$, $\bar{\gamma} \in PL_n^K$, $\bar{\gamma}\bar{\alpha} \overset{s}{\equiv} \bar{\alpha} \oplus \bar{\gamma}$.*

Proof. Letting α, γ represent $\bar{\alpha}$, $\bar{\gamma}$, we have $\alpha \oplus \gamma = (1_n \oplus \gamma)(\alpha \oplus 1_n)$. By Corollary III, p. 23, $1_n \oplus \gamma = \gamma \oplus 1_n$ in $[K, PL]$.

Thus $(1_n \oplus \gamma)(\alpha \oplus 1_n) \overset{s}{\equiv} (\gamma \oplus 1_n)(\alpha \oplus 1_n) = (\gamma\alpha \oplus 1_n) \overset{s}{\equiv} \gamma\alpha$. Q.E.D.

We can now restate Theorem (5.1) in the following form.

Theorem (A). *Let K be an unbounded combinatorial manifold. Then for each $\bar{\alpha} \in H_n(K)$ there exists a $\bar{\gamma} \in PL_r^K$ with $\bar{\alpha} \overset{s}{\equiv} \bar{\gamma}$.*

The proof depends on an intermediate construction. For the time being we assume K is an arbitrary complex.

Let α represent an element of $H_n(K)$. Let $\bar{I} = (-1/3, 4/3)$. Let w^α be the identification space of $\bar{I} \times K \times R^n$ under identification of $(1 + t, k, x)$ with $(t, \alpha(k, x))$, $-1/3 < t < 1/3$. And $P_\alpha : \bar{I} \times K \times R^n \to w^\alpha$ the natural projection. Note P_α is a local homeomorphism and

$P_\alpha(\bar{I} \times K \times O) = S' \times K \subseteq w^\alpha$. Further, if α represents an element in PL_n^K, w^α is a microbundle over $S' \times K$.

Define $w^\alpha \overset{i}{\equiv} w^{\alpha'}$ if there is a *p.l.* homeomorphism h of a neighborhood of $S' \times K$ in w^α onto a neighborhood of $S' \times K$ in $w^{\alpha'}$ with $h|S' \times K =$ identity.

Finally it is straightforward to see that $\alpha \overset{i}{\equiv} \alpha' \Rightarrow w^\alpha \overset{i}{\equiv} w^{\alpha'}$.

PROPOSITION (I). *Let α, α' represent elements of $H_n(K)$, γ an element of PL_r^K. If* $w^\alpha \overset{i}{\equiv} w^{\alpha'} \Rightarrow w^{\alpha \oplus \gamma} \overset{i}{\equiv} w^{\alpha' \oplus \gamma}$

Proof. First note commutative diagram

$$\begin{array}{ccccc} \bar{I} \times K \times R^n & \xrightarrow{id \times 0} & \bar{I} \times K \times R^n \times R^r & \xrightarrow{\pi} & \bar{I} \times K \times R^n \\ \downarrow{\scriptstyle P_\alpha} & & \downarrow{\scriptstyle P_\alpha \oplus_\gamma} & & \downarrow{\scriptstyle P_\alpha} \\ w^\alpha & \xrightarrow{\lambda_1} & w^{\alpha \oplus \gamma} & \xrightarrow{\lambda_2} & w^\alpha \end{array}$$

Since P_α, $P_{\alpha \oplus \gamma}$ are local homeomorphisms, $w^{\alpha \oplus \gamma}$ is an r-dimension microbundle over w^α. The restriction of this bundle to $S' \times K$ is clearly w^α. Since $\alpha \sim 1_n$ modulo $K \times O$, it is easily seen that inclusion $S' \times K \to W^\alpha$ is a homotopy equivalence for $W^{1_n} = S' \times K \times R^n$. Thus $W^{\alpha \oplus \gamma}$ considered as a microbundle over W^α is uniquely determined by its restriction to $S' \times K$. Now let $S' \times K \subseteq N^\alpha \subseteq W^\alpha$, N^α a neighborhood of $S' \times K$ in W^α. Then $\lambda_2{}^{-1}(N^\alpha) = W^{\alpha \oplus \gamma}|N^\alpha$ is a neighborhood of $\lambda'(N^\alpha) \subseteq W^{\alpha \oplus \gamma}$. Given an α' representing an element of $H_n(K)$, an $N^{\alpha'}$ where $S' \times K \subseteq N^{\alpha'} \subseteq W^{\alpha'}$, and a homeomorphism $h: N^\alpha \to N^{\alpha'}$ with $h|S' \times K =$ identity; it follows that $W^{\alpha' \oplus \gamma}|N^\alpha$ is equivalent as a microbundle over N^α to $h^*(W^{\alpha' \oplus \gamma}|N^{\alpha'})$. Thus there is a homeomorphism $\bar{h}$ of a neighborhood of $\lambda_1(N^\alpha) \subseteq W^{\alpha \oplus \gamma}|N^\alpha$ onto a neighborhood of $\lambda_1'(N^{\alpha'}) \subseteq W^{\alpha' \oplus \gamma}|N^{\alpha'}$ with $\bar{h}'\lambda_1 = \lambda_1' h$. Since $h|S' \times K =$ identity, $\bar{h}|S' \times K =$ identity and thus $W^{\alpha \oplus \gamma} \overset{i}{\equiv} W^{\alpha' \oplus \gamma}$. Q.E.D.

PROPOSITION (II). *Let α represent an element of $H_n(K)$ and suppose $W^\alpha \overset{i}{\equiv} W^{1_n}$. Then* $\alpha \oplus 1_1 \overset{i}{\equiv} 1_{n+1}$.

Proof. Since we can retract by a homeomorphism $W^{1_n} = S' \times K \times R^n \to$ into any neighborhood of $S' \times K \times O$ leaving a smaller neighborhood fixed we may as well assume the equivalence between W^{1_n} and W^α is given by an embedding $h: S' \times K \times R^n \to W^\alpha$. We further can assume h has the following properties:

1. $hP_{1_n}((-1/4, 1/4) \times K \times R^n) \subseteq P_\alpha((-1/3, 1/3) \times K \times R^n)$
2. $hP_{1_n}((3/4, 5/4) \times K \times R^n) \subseteq P_\alpha((2/3, 4/3) \times K \times R^n)$.

Then one can cover h, by a map $\bar{h}$ so that the following diagram commutes.

$$\begin{array}{ccc} (-1/4, 5/4) \times K \times R^n & \xrightarrow{\bar{h}} & \bar{I} \times K \times R^n \\ \downarrow & & \downarrow{\scriptstyle P_\alpha} \\ S' \times K \times R^n & \xrightarrow{h} & W^\alpha \end{array}$$

$\bar{h}$ is uniquely determined by this and the fact that $\bar{h}(t, k, O) = (t, k, O)$ $t \in (1/4, 5/4)$. Clearly $\bar{h}$ is an imbedding.

Now let $Q_1 = \bar{h}|(-1/4, 1/4) \times K \times R^n$, $Q_2 = \bar{h}|(3/4, 5/4) \times K \times R^n$ $\bar{Q}_2(t, k, y) = Q_2(1 + t, k, y)$, $t \in (-1/4, 1/4)$ $I \times \alpha : (2/3, 4/3) \times K \times R^n \xrightarrow{\mathrm{id} \times \alpha} (2/3, 4/3) \times K \times R^n$. Then we must have $(I \times \alpha)\bar{Q}_2 = Q_1$ on $(-1/4, 1/4) \times K \times R^n$. Now let ρ be a *p.l.* homeomorphism $\rho : R \to (-1/4, 1/4)$ with $\rho|(-1/8, 1/8) = \text{identity}$. Let $\bar{\rho} : K \times R^n \times R \to (-1/4, 1/4) \times K \times R^n$ by $\bar{\rho}(k, x, y) = (\rho(y), k, x)$. Also let

$$\sigma : \tilde{I} \times K \times R^n \xrightarrow{\text{permute}} K \times R^n \times \tilde{I} \xrightarrow{\mathrm{id} \times \text{inclusion}} K \times R^n \times R.$$

Then $\sigma(1 \times \alpha)\bar{Q}_2\rho = \sigma Q_1 \rho$ as maps $K \times R^{n+1} \to K \times R^{n+1}$. Notice that they represent elements of $H_{n+1}(K)$. Consider the map $\bar{H} : I \times K \times R^n \times R \to I \times K \times R^n \times R$ defined as composite of the following

$$I \times K \times R^n \times R \xrightarrow{\mathrm{id} \times \bar{\rho}} I \times (-1/4, 1/4) \times K \times R^n \xrightarrow{\bar{\bar{h}}}$$

$$I \times (-1/3, 4/3) \times K \times R^n \xrightarrow{\text{permute}} I \times K \times R^n \times (-1/3, 4/3) \xrightarrow{\mathrm{id} \times \text{inclusion}} I \times K \times R^n \times R,$$

where $\bar{\bar{h}}(t, t', k, y) = t, \bar{h}(t + t', k, y)$.

Then $\bar{H}$ represents an element of $H_{n+1}(I \times K)$ and $\bar{H}_0 = \sigma Q_1 \bar{\rho}$, $\bar{H}_1 = \sigma Q_2 \bar{\rho}$. Hence $\sigma Q_1 \bar{\rho} \overset{i}{\equiv} \sigma Q_2 \bar{\rho}$ and thus $\sigma(1 \times \alpha)\bar{Q}_2\bar{\rho} \overset{i}{\equiv} \gamma \bar{Q}_2 \bar{\rho}$ and thus $\sigma(1 \times \alpha)\sigma^{-1} \overset{i}{\equiv} 1_{n+1}$ where the left hand side is well defined on $K \times R^n \times (-1/3, 4/3)$ and thus represents an element of $H_{n+1}(K)$. Since $\sigma(1 \times \alpha)\, \sigma^{-1}$ has the same germ as $\alpha \oplus 1_1$, the proof is complete.

COROLLARY TO PROPOSITION (II, I). *Let α represent an element of $H_n(K)$, γ and element of PL_r^K. If $W^{\alpha \oplus 1_s} \overset{i}{\equiv} W^{\gamma \oplus 1_t}$ then $\alpha \overset{s}{\equiv} \gamma$.*

Proof. $W^{\alpha \oplus 1_s} \overset{i}{\equiv} W^{\gamma \oplus 1_t} \Rightarrow W^{\alpha \oplus 1_s \oplus \gamma^{-1}} \overset{i}{\equiv} W^{\gamma \oplus 1_t \oplus \gamma^{-1}} \overset{i}{\equiv} W^{\gamma \oplus \gamma^{-1} \oplus 1_t} = W^{1_{2r} \oplus_t}$. Then by Proposition (II) $\alpha \oplus 1_s \oplus \gamma^{-1} \overset{s}{\equiv} 1_{2r+t}$ or $\alpha \overset{s}{\equiv} \gamma$. Q.E.D.

PROPOSITION (III). *Let α represent an element of $H_n(K)$ where K is an unbounded combinatorial manifold. Then for r sufficiently large there exists a γ representing an element of PL_{n+r}^K with $W^{\alpha \oplus 1_r} \overset{i}{\equiv} W^\gamma$.*

Proof. W^α is also an unbounded combinatorial manifold. Without loss of generality we may assume W^α is orientable [for if not, choose γ representing an element of PL_1^K with the first Stiefel–Whitney class $\neq O$ as microbundle over $S' \times K$. The normal bundle of W^α in $W^{\alpha + \gamma}$ has the first Stiefel–Whitney class $\neq O$ as in proof of Proposition (I). Thus if W^α is non orientable, $W^{\alpha \oplus \gamma}$ is orientable and thus we can work with $\alpha + \gamma$]. Now choose r so large that $S' \times K$ has a normal bundle in $W^{\alpha \oplus 1_r} = W^\alpha \times R^r$. Call bundle ε^{n+r}. Since $W^{\alpha + 1_r}$ is orientable $\varepsilon|S' \times k$, $k \in K$ is trivial (See Lemma (10), Milnor [4]). By construction of $W^{\alpha \oplus 1_r}$, $\varepsilon|p \times K$, $p \in S'$ is trivial.

Thus classifying map $S' \times K \xrightarrow{\varepsilon} BPL_{n+r}$ factors through $S' \times K \to S' \wedge K \to BPL_{n+r}$. The homotopy class of ε' is determined by a characteristic map $\gamma : K \to PL_{n+r}$. And clearly

$W^{\alpha} \equiv \varepsilon_{n+r}$ as a bundle over $S' \times K$. Since ε is a normal bundle for $S' \times K$ in $W^{\alpha \oplus 1_r}$, $W^{\gamma} \overset{i}{\equiv} W^{\alpha \oplus 1_r}$ and proposition follows.

Theorem (A) follows immediately from Corollary and Proposition (III) as does:

THEOREM (B). *$\alpha \overset{i}{\equiv} \alpha'$ if and only if $W^{\alpha \oplus 1_s} \overset{i}{\equiv} W^{\alpha' \oplus 1_r}$. For some s, r, where α represents an element of $H_n(K)$, α' an element of $H_m(K)$, K an unbounded combinatorial manifold.*

Query. Are Theorem (A) or the Weaker Theorem (B) true for arbitrary complexes K? It seems quite likely, yet the proof may be difficult.

THEOREM (5.2). *Let μ, ν be p.l. microbundles over a combinatorial manifold K, and let $f: E(\mu) \to E(\nu)$ be a zero-preserving p.l. homeomorphism. Then for s sufficiently large*

$$F \times 1_s : E(\mu) \times R^s \longrightarrow E(\nu) \times R^s$$

is p.l. isotopic, relative to the zero-section, to a fibrewise map.

Proof. 1. Let λ be any microbundle over K, and λ' any inverse to λ. Let π_μ, π_ν be the bundle projections in μ, ν respectively. Now $E(f^*\pi_\nu^*\lambda) = \{(x, y) \in E(\mu) \times E(\pi_\nu^*\lambda) | f(x) = \rho(y)\}$, ρ the projection $E(\pi_\nu^*\lambda) \to E(\nu)$. Consequently, we have the natural bundle map $\varphi_\lambda : E(f^*\pi_\nu^*\lambda) \to E(\pi_\nu^*\lambda)$, $\varphi_\lambda(x, y) = y$. Similarly, we have $\varphi_{\lambda'} : E(f^*\pi_\nu^*\lambda') \to E(\pi_\nu^*\lambda')$, and finally $\varphi_{\lambda \oplus \lambda'} : E(f^*\pi_\nu^*\lambda \oplus \lambda') \to E(\pi_\nu^*\lambda \oplus \lambda')$. Then

$$\begin{tikzcd}
E(f^*\pi_\nu^*\lambda \oplus f^*\pi_\nu^*\lambda') \arrow[r, "\varphi_\lambda \oplus \varphi_{\lambda'}"] \arrow[d, leftrightarrow] & E(\pi_\nu^*\lambda \oplus \pi_\nu^*\lambda') \arrow[d, leftrightarrow] \\
E(f^*\pi_\nu^*\lambda \oplus \lambda') \arrow[r, "\varphi_{\lambda \oplus \lambda'}"] \arrow[d, leftrightarrow] & E(\pi_\nu^*\lambda \oplus \lambda') \arrow[d, leftrightarrow] \\
E(\mu) \times R^s \arrow[r, "f \times 1_s"] & E(\nu) \times R^s
\end{tikzcd}$$

commute under the natural equivalence. $s = \dim \lambda \times \dim \lambda'$.

2. Since we may assume K is a *p.l.* deformation retract of $E(\mu)$, $f^*\pi^*\lambda \simeq \pi_\mu^*\lambda$. Since $E(\pi^*\lambda) = E(\nu \oplus \lambda)$ and $E(\pi_\mu^*\lambda) = E(\mu \oplus \lambda)$, φ_λ corresponds under identifications to a zero-preserving map $g : E(\mu \oplus \lambda) \to E(\nu \oplus \lambda)$.

Similarly, $\varphi_\lambda + \varphi_{\lambda'}$ corresponds to a bundle map

$$h : E(\pi_{\mu \oplus \lambda}^*\lambda') \to E(\pi_{\nu + \lambda}^*\lambda') \text{ covering } g.$$

Suppose g is isotopic to a fibrewise map, relative to the zero-sections. This isotopy may be covered by a bundle isotopy of h, to a fibrewise map as bundles over K. But then $\varphi_\lambda \oplus \varphi_{\lambda'}$ and hence by (1), $f \times 1^s$, is isotopic to a fibrewise map as bundles over K, relative to zero-sections.

3. Since by uniqueness of stable normal bundles, $\mu \overset{s}{\simeq} \nu$, we can take λ to be an inverse of both μ and ν. Then $g : E(\mu \oplus \lambda) \to E(\nu \oplus \lambda)$ may be considered a zero-preserving map $K \times R^n \to K \times R^n$, $n = \dim \lambda$. Since we may always add a trivial bundle to λ, Theorem (5.1) implies g is isotopic to a fibrewise map for n sufficiently large; and Theorem (5.2) follows from the argument (2) above.

THEOREM (5.3). *Let $f_i : K \to V$, $i = 0, 1$, be p.l. imbeddings, where K and V are combinatorial manifolds without boundary, then*

(a) $f_i(K)$ has a normal microbundle μ_i in $V \times R^n$, n sufficiently large; i.e. there exists *p.l.* homeomorphisms $F_i : E(\mu_i) \to V \times R^n$ such that F_i restricted to the zero-section is $f_i : K \to V \times O$.

(b) If $h : K \times I \to V \times I$ is any *p.l.* isotopy between f_0 and f_1, there exists a *p.l.* isotopy $H : E(\mu_0) \times R^k \times I \to V \times R^{n+k} \times I$, k sufficiently large, covering h, and such that $H_0 = F_0 \times 1_k$, H_1 is a microbundle equivalence of $E(\mu_0) \times R^k$ into $F_1(E(\mu_1)) \times R^k$.

Proof. (a) Milnor [5] proves the existence of a stable normal microbundle.

(b) Extend h to a *p.l.* imbedding $\bar{h} : K \times R \to V \times R$, by setting

$$\begin{aligned}\bar{h}(x, t) &= (f_0(x), t), t < 0\\ &= h(x, t), 0 \leqq t \leqq 1\\ &= (f_1(x), t), t > 1\end{aligned}$$

Then $\bar{h}(K \times R)$ has a normal microbundle in $V \times R \times R^{n+k}$, k sufficiently large. By the covering homotopy Theorem [5], this bundle is of the form $\nu \oplus 1$, where ν is a bundle over K, i.e. there exists a *p.l.* homeomorphism $G : E(\nu) \times R \to V \times R \times R^{n+k}$, $G|K \times R = \bar{h}$ (but not necessarily commuting with projection into R).

Now G may not be an isotopy, but Γ: $E(\nu) \times R \times I \to V \times R \times R^{n+k} \times I$, defined by $\Gamma(x, s, t) = (G(x, s + t), t)$ is a *p.l.* isotopy, for any interval I.

On the other hand, we claim that $F_0 \times 1_{k+1} : E(\mu_0) \times R^{k+1} \to V \times R^{n+k+1}$ is *p.l.* isotopic, relative to the zero-section, to a fibrewise map $\bar{F}_0$ w.r.t. $\Gamma_0 = \Gamma|E(\nu) \times R \times (O)$; i.e.

$$\begin{tikzcd} E(\mu_0) \times R^{k+1} \arrow[rr, "\bar{F}_0"] \arrow[dr, "\phi"'] & & V \times R^{n+k+1} \\ & E(\nu) \times R \arrow[ur, "\Gamma_0"'] & \end{tikzcd}$$

where φ is a microbundle equivalence.

In fact, $F_0 \times 1_{k+1}$ and say Γ_{-1} may both be thought of as normal microbundles of $f_0(K) = \Gamma_{-1}(K \times 0) \subset V \times (-1) \times 0 \subset V \times R \times R^{n+k}$. Hence $F_i \times 1_{k+1}$ is isotopic relative to K (by Theorem (5.2)) to a fibrewise map w.r.t Γ_{-1}. But $\Gamma_{-1}, \Gamma_0 : E(\nu) \times R \to V \times R^{n+k+1}$ are isotopic, relative to K.

Similarly, Γ_1 is isotopic to say Γ_2, which in turn is isotopic to a fibrewise map w.r.t $F_1 \times 1_{k+1}$. Since Γ_0 is also *p.l.* isotopic to Γ_1, we may combine these isotopies to give a *p.l.* isotopy $H : E(\mu_0) \times R^{k+1} \times I \to V \times R^{n+k+1} \times I$ such that $H_0 = F_0 \times 1_{k+1}$ and H_1 is fibrewise w.r.t. $F_1 \times 1_{k+1}$. Hence for k sufficiently large (i.e. writing k for $k + 1$), we get conclusion (b), and the Theorem is proved.

§6. SMOOTHING COMBINATORIAL MANIFOLDS

DEFINITION (6.1). *Let K be a combinatorial n-manifold. A* smoothing *of K is a pair (M, f), where M is a smooth n-manifold and $f: K \to M$ is a non-singular p.d. homeomorphism.*

Two smoothings (M_1, f_1) and (M_2, f_2) are called equivalent if there exists a diffeomorphism $d: M_1 \to M_2$ such that

$$\begin{tikzcd} & M_1 \arrow[dd, "d"] \\ K \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & M_2 \end{tikzcd}$$

commute.

DEFINITION (6.2). *Two smoothings (M_1, f_1), (M_2, f_2) of a combinatorial manifold K, without boundary, are called* concordant, *if there exists a smoothing (W, F) of $K \times I$, $F: K \times I \to W$ such that $\partial W =$ disjoint union of M_1, M_2* and $F|_{k \times O} = f_1$, $F|_{k \times 1} = f_2$.

More explicitly, we are given diffeomorphisms $d_i: M_i \to W$, $i = 1, 2$, $d_1(M_1) \cap d_2(M_2) = 0$, $d_1(M_1) \cup d_2(M_2) = \partial W$ and

$$\begin{tikzcd} K \arrow[r, "f_1"] \arrow[d, "i_0"'] & M_1 \arrow[d, "d_1"] \\ K \times I \arrow[r, "F"] & W \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "f_2"] \arrow[d, "i_1"'] & M_2 \arrow[d, "d_2"] \\ K \times I \arrow[r, "F"] & W \end{tikzcd}$$

commute.

In the remainder of the paper we deal only with combinatorial manifolds of the homotopy type of a finite complex. This will be understood in what follows.

The basic theorem of concordance theory is the Cairns–Hirsch [2] Theorem. We use it in the following form [4].

THEOREM—Cairns–Hirsch. *Let (f, M) be a smoothing of $K \times R^k$. Then there exists a smoothing (g, N) of K, unique up to concordance with the following property.*

There is a piecewise differentiable isotopy $H: K \times R^k \times I \to M \times I$ with $H_0 = f$, and $H_1 = \phi(g \times \mathrm{id})$ where ϕ is an imbedding $\phi: N \times R^k \to M$.

DEFINITION (6.3). *A* trivial triangulation *of a p.d. vector bundle ξ^n over a locally finite complex K is a triangulation $(1^n, \varphi)$, where $E(1^n) = K \times R^n$, $\varphi: K \times R^n \to E(\xi)$. We write (φ, ξ) for a trivially triangulated p.d. bundle.*

Two trivially triangulated *p.d.* bundles (φ_1, ξ_1^n), (φ_2, ξ_2^n) over K are called *equivalent* if there exists a trivially triangulated *p.d.* bundle (φ, ξ) over $K \times I$ such that $\xi|K \times O \simeq \xi_1$, $\xi|K \times 1 \simeq \xi_2$, $\varphi|K \times O \times R^n \simeq \varphi_1$, $\varphi|K \times 1 \times R^n \simeq \varphi_2$.

More explicitly, there exists *p.d.* bundle equivalences

$$\psi: E(\xi_1) \to E(\xi)|_{K \times 0}, \; \psi_2: E(\xi_2) \to E(\xi)|_{K \times 1},$$

such that

$$\begin{array}{ccc} K \times R^n & \xrightarrow{\varphi_1} & E(\xi_1) \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle \psi_1} \\ K \times I \times R^n & \xrightarrow{\varphi} & E(\xi) \end{array} \qquad \begin{array}{ccc} K \times R^n & \xrightarrow{\varphi_2} & E(\xi_2) \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle \psi_2} \\ K \times I \times R^n & \xrightarrow{\varphi} & E(\xi) \end{array}$$

commute.

Alternative definition of equivalence

More simply, if less symmetrically, there exists a *p.d.* bundle equivalence $\psi : E(\xi_1) \to E(\xi_2)$ such that $\psi\varphi_1$ is isotopic to φ_2 through trivial triangulations.

Further, since the Whitney sum of a *p.d.* bundle and a trivial bundle has a naturally defined *p.d.* structure, we may define *stable equivalence* of trivially triangulated *p.d.* bundles over $K: (\varphi_1, \xi_1^k)$, (φ_2, ξ_2^l) are called stably equivalent if there exists a trivially triangulated *p.d.* bundle (φ, ξ^n) over $K \times I$ such that $\xi|_{K\times 0} \overset{s}{\simeq} \xi_1$, $\xi|_{K\times 1} \overset{s}{\simeq} \xi_2$, $\varphi|_{K\times O\times R^n} \overset{s}{\simeq} \varphi_1$, $\varphi|K \times 1 \times R^n \overset{s}{\simeq} \varphi_2$.

We denote the set of stable equivalence classes of trivially triangulated *p.d.* bundles by T. T is a set with unit element, namely the class of $(I, 1)$; $I: K \times R^n \to K \times R^n$ the identity.

Let $C = C(K)$ be the concordance classes of smoothings of a combinatorial manifold K, and let $c_0 \in C$ be a distinguished class. We will denote this set with distinguished class by C_0.

Let $\tilde{K}$ be the s.s. complex corresponding to K. Let $O = \xrightarrow{\text{Lim}} O_n$ be the direct limit of the s.s. complexes defined above. Let $[\tilde{K}, O]$ be the s.s. homotopy classes of s.s. maps of $\tilde{K}$ into O. Define $[\tilde{K}, PD]$, $[\tilde{K}, BO]$, $[\tilde{K}, BPL]$ similarly. Let $\tilde{\mu} : [\tilde{K}, O] \to [\tilde{K}, PD]$ be the map induced by the inclusion $\mu : O \to PD$. Let $\tilde{p} : [\tilde{K}, BO] \to [\tilde{K}, BPL]$ be induced by $p : BO \to BPL$. Define an action of $[\tilde{K}, PD]$ on C_0 as follows: Let $(M, f) \in c \in C_0$, and let $g : K \times R^n \to K \times R^n$ represent a class $(g) \in [\tilde{K}, PD]$. Let $h : K \times R^n \to M \times R^n$ be the composition $K \times R^n \xrightarrow{g} K \times R^n \xrightarrow{f \times 1} M \times R^n$; then h is a non-singular *p.d.* homeomorphism. Let $\{(M_1, f_1)\}$ be the concordance class defined from h by the Cairns–Hirsch theorem. This is clearly independent of the choice of g. Further, it depends only on the concordance class c; as is proved by applying the same construction to the given concordance. We denote this action by λ. Now define a map $\zeta : C_0 \to [KBO]$ by $\zeta\{(M, f)\}$ = homotopy class of the stable classifying map for $\tau \oplus \tau_0^{-1}$, where τ = tangent bundle of M, τ_0 = tangent bundle M_0. It is clear, that this depends only on the concordance class of (M, f), and (M_0, f_0) since concordant manifolds are diffeomorphic [10], and hence have the same stable tangent bundles.

DEFINITION (6.4). *Let G, H, K, L be groups, and C a base-pointed set with H as a group of operators on C. Let $\lambda : C \times H \to C$ be the operation. Let $\mu : G \to K$ a map of base-pointed* *sets. Then we say that*

$$G \xrightarrow{\mu} H \xrightarrow{\lambda} C \xrightarrow{\zeta} K \xrightarrow{\nu} L$$

is exact if the following hold:

(1) $\lambda(h, c) = c$ *if and only if* $h \in \mu(G)$.

(2) $\zeta(c_1) = \zeta(c_2)$ *if and only if* $c_2 = \lambda(c_1, h)$ *for some* $h \in H$.

(3) $\nu(k) = 1$ *if and only if* $k \in \xi(C)$.

Remark. If C is a group, then an exact sequence of groups $G \to H \to C \to K \to L$ is an exact sequence in the above sense, where $\lambda(c, h) = c\bar{\lambda}(h)$.

5—LEMMA. *Let*

$$\begin{tikzcd} G_1 \arrow[r, "\mu_1"] \arrow[d, "\varphi_1"'] & H_1 \arrow[r, "\lambda_1"] \arrow[d, "\varphi_2"'] & C_1 \arrow[r, "\zeta_1"] \arrow[d, "\rho"'] & K_1 \arrow[r, "\nu_1"] \arrow[d, "\psi_1"'] & L_1 \arrow[d, "\psi_2"'] \\ G_2 \arrow[r, "\mu_2"] & H_2 \arrow[r, "\lambda_2"] & C_2 \arrow[r, "\zeta_2"] & K_2 \arrow[r, "\nu_2"] & L_2 \end{tikzcd}$$

be a commutative diagram, where the horizontal rows are exact in the sense of Definition (6.4), $\varphi_1, \varphi_2, \psi_1, \psi_2$ *are group isomorphisms and* ρ *is a map of base-pointed sets with operators. Then* ρ *is an isomorphism of base-pointed sets with operators.*

PROPOSITION (6.1). $[\tilde{K}, O] \xrightarrow{\tilde{\mu}} [\tilde{K}, PD] \xrightarrow{\lambda} C_0 \xrightarrow{\zeta} [\tilde{K}, BO] \xrightarrow{\tilde{p}} [\tilde{K}, BPL]$ *is exact*

Proof. 1. *Order two*

(*a*) $\lambda\tilde{\mu}$

This is just Proposition (3.1) applied to the trivial bundle over M, for using smoothing $f: K \to M$, an element of $[\tilde{K}, O]$ is exactly a *p.d.* vector bundle map $M \times R^n \to M \times R^n$ and the isotopy given by proposition deforms this to a differentiable bundle map $M \times R^n \to M \times R^n$.

(b) $\zeta\lambda$

Let $(g) \in [K, PD]$ and let $h: K \times R^n \to M \times R^n$ be the induced *p.d.* homeomorphism. Then the Cairns–Hirsch theorem gives that h is a *p.d.* isotopic to a *p.d.* homeomorphism $h_1 : K \times R^n \xrightarrow{f_1 \times 1} M_1 \times R^n \xrightarrow{d} M \times R^n$, where d is a diffeomorphism. It follows that M_1 has the same stable tangent bundle as M. Hence $\zeta\lambda((g), c) = K(c)$.

(c) $p\zeta$

Since any two smoothings of K have tangent bundles with the same underlying tangent microbundle, and $\tilde{p}$ is a homomorphism w.r.t. Whitney sum; $\tilde{p}\zeta(c) =$ homotopy class of the classifying map for the trivial microbundle.

2. *Exactness*

(a) $[\tilde{K}, PD]$

Suppose $(g) \in [\tilde{K}, PD]$ is such that $\lambda((g), c) = c$ for some $c \in C_0$. We wish to show that (g) comes from $[\tilde{K}, O]$. Starting with $(M_1, f_1) \in c$, the action of (g) must lead by the construction λ to $(M_2, f_2) \in c$. Let $f: K \times I \to V$ be the concordance between (M_1, f_1) and

(M_2, f_2). The imbedding (which we denote d_2) of $M_2 \times R^n$ in $M_1 \times R^n$ of the λ-construction, can be extended to a smooth imbedding $d: V \times R^n \to M_1 \times I \times R^n$, such that $d|M_2 \times R^n = d_2 : M_2 \times R^n \to M_1 \times O \times R^n$ and $d|M_1 \times O : M_1 \times O \to M_1 \times (1) \times O$ is the identity. In fact, V is diffeomorphic to $M_1 \times I$, and any two imbeddings of $M_1 \times R^n$ are isotopic if n is sufficiently large. Now $d \circ f \times 1$ is *p.d.* isotopic to a *p.l.* imbedding $\bar{f}: K \times I \times R^n \to K \times I \times R^n$, such that $\bar{f}|K \times 1 \times R^n$ is fibrewise *p.d.* isotopic to $d \circ f \times 1|K \times 1 \times R^n \in [\tilde{K}, O]$, and $\bar{f}|K \times O \times R^n : K \times O \times R^n \to K \times O \times R^n$ is g. Now $\bar{f}$ may be extended to a map $h : K \times R \times R^n \to K \times R \times R^n$, by extending the maps over the two boundaries of $K \times I$ in constant fashion. Since $h|K \times R \times O : K \times R \times O \to K \times R \times R^n$ is isotopic to the identity, relative to $K \times (R - \text{Int } I)$, the Tubular Neighborhood Theorem gives a covering isotopy of h to a fibrewise map. But this shows that $h|K \times 1 \times R^n$ and $h|K \times O \times R^n$ are stably isotopic in PD. Hence (g) comes from $[\tilde{K}, O]$.

(b) C_0

Let (M_1, f_1), (M_2, f_2) represent two concordance classes c_1, c_2 respectively, such that $K(c_1) = K(c_2)$; i.e. $\tau_1 \overset{s}{\simeq} \tau_2$. Then there is a smooth imbedding $i : M_2 \times R^n \to M_1 \times R^n$; for n sufficiently large; such that $if_2 : K \to M_2 \times 0 \to M_1 \times R^n$ is homotopic to f_1, and hence *p.d.* isotopic to f_1. By the tubular neighborhood theorem, $i \circ (f_2 \times 1)$ is *p.d.* isotopic to a *p.l.* bundle map $g : K \times R^n \to K \times R^n$. Hence $c_2 = \lambda((g), c_1)$, $(g) \in [\tilde{K}, PD]$.

(c) $[\tilde{K}, BO]$

Let $(h) \in [K, BO]$. Then if $\tilde{p}(h) = 0$, (h) represents a *p.d.* vector bundle ξ which has a trivial triangulation φ. Then $\varphi : K \times R^n \to \tilde{E}(\xi)$, $\tilde{E}(\xi)$ the diff. vector bundle over M_0, defines a concordance class c, with tangent bundle ξ; i.e. $K(c) = (h)$.

PROPOSITION (6.2). $[\tilde{K}, O] \xrightarrow{\tilde{\mu}} [\tilde{K}, PD] \xrightarrow{\lambda} T \xrightarrow{\zeta} [\tilde{K}, BO] \xrightarrow{p} [\tilde{K}, BPL]$ *is exact, where*

(1) $\zeta[(\varphi, \xi)] =$ *the homotopy class of the classifying map for* ξ

(2) $[\tilde{K}, PD]$ *acts on* T *by Whitney sum*; i.e. *given* $(f) \in [\tilde{K}, PD]$, $(\varphi, \xi) \in t \in T$, *let* $\lambda((f), t)$ *be the class of* $\varphi \oplus f : K \times R^n \times R^n \to E(\xi) \times R^n$.

Proof.

1. *Order two*

(a) $\lambda\tilde{\mu}$

Let $(g) \in [\tilde{K}, O]$, $(\varphi, \xi) \in t \in T$. Then $\lambda((g), t)$ is the class of $\varphi \oplus g : K \times R^n \times R^m \to E(\xi) \times R^m$, but $\varphi \oplus g = 1 \oplus g \circ \varphi \oplus 1$, $1 \oplus g : E(\xi) \times R^m \to E(\xi) \times R^m$. Since this last is a *p.d.* bundle equivalence $(\varphi \oplus g, \varphi \oplus 1^m) \in t$.

(b) $\zeta\lambda$

$\zeta(\varphi \oplus f, \varphi \oplus 1^n) = \zeta(\varphi, \xi)$ by definition (see 2) above)

(c) $p\zeta$

If $(\varphi, \xi) \in t \in T$, φ by definition has a trivial triangulation, and $\tilde{p}\zeta(t)$ is the homotopy class of the classifying map for the trivial microbundle.

2. *Exact*

(a) $[\tilde{K}, PD]$

Let $(g) \in [\tilde{K}, PD]$, $(\varphi, \xi) \in t \in T$; and suppose $\lambda((g), t) = t$. Then after stabilizing, we have

$$\begin{tikzcd}
 & E(\xi) \times R^m \arrow[dd, "\psi"] \\
K \times R^m \times R^n \arrow[ur, "\varphi \oplus g"] \arrow[dr, "\varphi \oplus 1"'] & \\
 & E(\xi) \times R^m
\end{tikzcd}$$

commutes up to isotopy through trivial triangulations, where ψ is a *p.d.* bundle equivalence (see alternative definition of equivalent trivial triangulations). Hence $\psi \circ (\varphi \oplus 1)|K \times 0 \times R^m : K \times O \times R^m \to K \times R^m \subset E(\xi) \times R^m$ is isotopic in PD_m to $\varphi \oplus g|K \times O \times R^m : K \times O \times R^m \to K \times R^m \subset E(\xi) \times R^m$. Identifying $K \times O \times R^m$ with $K \times R^m$, $\varphi \oplus g|K \times O \times R^m = g$ and $\psi \circ (\varphi \oplus 1)|K \times O \times R^m = \psi|K \times R^m$. Hence g is isotopic in PD_m to $\psi|K \times R^m$, where $(\psi|K \times R^m) \in [\tilde{K}, O]$.

(b) T

Let $(\varphi_1, \xi_i) \in t_i \in T$, $i = 1, 2$, and suppose that $\zeta(t_1) = \zeta(t_2)$; i.e. $\xi_1 \overset{s}{\simeq} \xi_2$. Then we may suppose $\xi_1 = \xi_2 = \xi$, and $\varphi_i : K \times R^n \to E(\xi)$, $i = 1, 2$, are two triangulations. Then φ_2 is isotopic through triangulations to

$$\begin{tikzcd}
\varphi_2' : K \times R^n \arrow[rr] \arrow[dr, "g"'] & & E(\xi), \quad \varphi_2' = \varphi_1 g, \ g \text{ a } p.l. \text{ microbundle} \\
 & K \times R^n \arrow[ur, "\varphi_1"'] &
\end{tikzcd}$$

homomorphism. I.E. $(g) \in [\tilde{K}, PD]$ and $\lambda((g), t_1) = t_2$.

(c) $[\tilde{K}, BO]$

Let $(h) \in [\tilde{K}, BO]$, and suppose $\tilde{p}(h)$ is trivial. I.E. (h) represents a *p.d.* bundle ξ which has a trivial triangulation φ. Then $\zeta\{(\varphi, \xi)\} = (h)$.

Result from previous section

(6.3) $[\tilde{K}, O] \xrightarrow{\tilde{\mu}} [\tilde{K}, PD] \xrightarrow{\lambda} [\tilde{K}, PD/O] \xrightarrow{\zeta} [\tilde{K}, BO] \xrightarrow{\rho} [\tilde{K}, BPL]$

is an exact sequence of groups.

DEFINITION OF $\rho : T \to C_0$:

Let $f_0 : K \times M_0$ be a given smoothing. Let (φ, ξ) be a triangulated *p.d.* bundle, $\varphi : K \times R^n \to E(\xi)$. Considering ξ as a vector bundle over M_0, $E(\xi)$ may be given a unique differential structure as a diff. vector bundle, up to diff. bundle equivalence. Denote $E(\xi)$ with this diff. structure by $\widetilde{E(\xi)}$. Then there exists a *p.d.* bundle map $\psi : E(\xi) \to \widetilde{E(\xi)}$ over the map $f_0 : K \to M_0$; i.e. ψ is diff. over each simplex of some rectilinear subdivision of K. Then $\psi\varphi : K \times R^n \to \widetilde{E(\xi)}$ is a *p.d.* homeomorphism. By the Hirsch product theorem it defines a concordance class of smoothings of K.

To show this is well-defined, we must show it is independent of the choice of ψ and the choice of representative in the stable equivalence class of (φ, ξ).

By Proposition, if $\psi' : E(\xi) \to \widetilde{E}(\xi)$ is any other *p.d.* bundle map, there exists a diff. bundle equivalence $\lambda : \widetilde{E}(\xi) \to \widetilde{E}(\xi)$ such that

$$\begin{tikzcd} E(\xi) \arrow[r, "\psi"] \arrow[dr, "\psi'"'] & \widetilde{E}(\xi) \arrow[d, "\lambda"] \\ & \widetilde{E}(\xi) \end{tikzcd}$$

commutes up to a *p.d.* bundle isotopy. It follows that $\psi\varphi$ and $\psi'\varphi$ define concordant smoothings.

It is clear that (φ, ξ) and $((\varphi, 1), \varphi + 1)$, $(\varphi, 1) : (K \times R^n) \times R \to E(\xi) \times R$, define concordant smoothings. Now suppose (φ_1, ξ_1) is equivalent to (φ_2, ξ_2); i.e. we are given $\varphi : K \times I \times R^n \to E(\xi)$, φ restricting to (φ_1, ξ_1), (φ_2, ξ_2) resp. at 0 and 1. Again there exists a *p.d.* bundle map $\psi : E(\xi) \to \widetilde{E}(\xi)$, $\widetilde{E}(\xi)$ the diff. bundle over $M \times I$, ψ covering $(f_0, 1) : K \times I \to M_0 \times I$. Then by the Cairns–Hirsch theorem, $\psi\varphi : K \times I \times R^n \to \widetilde{E}(\xi)$ defines a concordance between the smoothings defined by $\psi\varphi_1$ and $\psi\varphi_2$.

Hence the map from stable equivalence classes to concordance classes is well-defined.

Let $\rho : T \to C_0$ be the map defined above.

THEOREM (6.1).

$$\begin{tikzcd} {[\tilde{K}, O]} \arrow[r, "\tilde{\mu}"] & {[\tilde{K}, PD]} \arrow[r, "\lambda"] \arrow[dr, "\lambda"'] & T \arrow[r, "\zeta"] \arrow[d, "\rho"] & {[\tilde{K}, BO]} \arrow[r, "\tilde{p}"] \arrow[dl, "\zeta"] & {[\tilde{K}, BPL]} \\ & & C_0 & & \end{tikzcd}$$

is commutative, and hence ρ is an isomorphism of base pointed sets with operators.

Proof. (a) That $\zeta\rho = \rho$ is immediate from the definition of ρ.

(b) Let $(g) \in [\tilde{K}, PD]$, $(\varphi, \xi) \in t \in T$. Then $\lambda((g), t)$ is represented by $\varphi \oplus g : K \times R^n \times R^m \to E(\xi) \times R^m$ $\rho(\lambda((g), t))$ is determined by the Cairns–Hirsch theorem from $\varphi \oplus g : K \times R^n \times R^m \to E(\xi) \times R^m$. But $\varphi \oplus g$ is *p.d.* isotopic to $\varphi' \oplus g$, where

$$\begin{tikzcd} \phi' : K \times R^n \arrow[r] \arrow[dr, "f_1 \times 1"'] & \widetilde{E}(\xi), \\ & M_1 \times R^n \arrow[u, "d_1"'] \end{tikzcd}$$

where $(M_1, f_1) \in \rho(t)$. Then $f_1 + g : K \times R^m \to M_1 \times R^m$ is *p.d.* isotopic to

$$\begin{tikzcd} h : K \times R^m \arrow[rr] \arrow[dr, "f_2 \times 1"'] & & M_1 \times R^m, \\ & M_2 \times R_m \arrow[ur, "d_2"'] & \end{tikzcd}$$

where $(M_2, f_2) \in \lambda((g), \rho(t))$. But then $(d_1 \times 1) \circ ((f_1 \times 1) \oplus g)$ and hence $\varphi \oplus g$ is *p.d.* isotopic to $(d_1 \times 1) \circ (1 \times d_2) \circ (f_2 \times 1 \times 1)$, i.e. $(M_2, f_2) \in \rho(\lambda((g), t))$. Hence $\rho\lambda = \lambda$.

(c) Since $\rho : T \to C_0$ is a homomorphism of sets with group of operators $[\tilde{K}, PD]$, the usual 5—Lemma argument applies to show that ρ is an isomorphism. Q.E.D.

Define $\sigma : T \to [\tilde{K}, PD/,O]$ as follows:
Let $(\varphi, \xi) \in t \in T$, $\varphi : K \times R^n \to E(\xi)$. Then since φ is *p.d.* homeomorphic over each simplex of K, and $E(\xi)$ is pasted together by elements of O, φ gives a well-defined map $g_\varphi : \tilde{K} \to PD/O$. Further, equivalent pairs give homotopic maps. Consequently, we get a well-defined class $\sigma(t)$.

THEOREM (6.2).

$$\begin{tikzcd} {[\tilde{K}, O]} \arrow[r] & {[\tilde{K}, PD]} \arrow[r, "\lambda"] \arrow[dr, "\lambda"'] & T \arrow[r, "\zeta"] \arrow[d, "\sigma"] & {[\tilde{K}, BO]} \arrow[r, "\tilde{p}"] \arrow[dl, "\zeta"] & {[\tilde{K}, BPL]} \\ & & {[\tilde{K}, PD/O]} & & \end{tikzcd}$$

is commutative, and hence σ is an isomorphism of base-pointed sets with operators.

COROLLARY. *T and hence C_0 may be given a group structure, such that the exact sequences of Proposition* (1, 2) *become exact sequences of groups.*

Proof of Theorem. (a) Let $(g) \in [\tilde{K}, PD]$ and $(\varphi, \xi) \in t \in T$. $\lambda(g)$ is represented by $\varphi \oplus g : K \times R^m \times R^n \to E(\xi) \times R^m$; clearly, $\sigma(\varphi \oplus g, \xi \oplus 1)$ is just the Whitney sum; i.e. $\lambda\sigma = \lambda$.

(b) Let $(\varphi, \xi) \in t \in T$. Let $\tilde{P}$ be the total space of the associated principal s.s. O_n bundle to ξ. Let $f : \Delta_q \to K$ be in $\tilde{K}^{(q)}$. Then $F \in \tilde{P}^{(q)}$ is a *p.d.* bundle map $F : \Delta_q \times R^n \to f^*(\xi)$ over $f \in \tilde{K}^{(q)}$. Further, φ induces a unique *p.d.* homeomorphism φ_f such that

$$\begin{tikzcd} \Delta_q \times R^n \arrow[r, "\varphi_f"] \arrow[d, "f \times 1"'] & f^*(\xi) \arrow[d] \\ K \times R^n \arrow[r, "\varphi"] & E(\xi) \end{tikzcd} \qquad \text{commutes.}$$

Then $F^{-1}\varphi_f : \Delta_q \times R^n \to \Delta_q \times R^n$ is in $PD_n^{(q)}$. Further $(\partial_i F)^{-1}\varphi_{\partial_i f} = \partial_i(F^{-1}\varphi_f)$. Hence φ defines a s.s. map $\tilde{\varphi} : \tilde{P} \to PD_n$. $\tilde{\varphi}$ is a map of principal O_n bundles, since if $R : \Delta_q \times R^n \to \Delta_q \times R^n$ is in $O_n^{(q)}$, $\tilde{\varphi}(FR) = R^{-1}(F^{-1}\varphi_f)$, which is just what the definition of right action of O_n on PD_n calls for.

Now the quotient map $\tilde{K} \to PD/O$ induced by $\tilde{\varphi}$ is just the map g_φ constructed in the definition of σ. Further, $\tilde{P} \to PD_n \subset PD_n \underset{PL}{\times} U$ is an O_n bundle map which induces by passage to quotients

$$\tilde{K} \xrightarrow[g\varphi]{} PD_n/O_n \overset{\zeta_n}{\subset} PD_n \underset{O_n}{} \underset{PL_n}{\times} U = BO_n$$

Hence ζg_φ is a classifying map for ξ, and $\zeta\sigma = \zeta$.

(c) Again the 5—Lemma argument shows that σ is an isomorphism of sets with a group of operators. Q.E.D.

From Theorem (6.1) and (6.2), we have $C_0 \simeq [\tilde{K}, PD/O]$; i.e.

THEOREM (6.3). *Let K be a combinatorial manifold (without boundary), which admits a smoothing M_0, then the concordance classes of smoothings of K are in 1–1 correspondence with the abelian group $[\tilde{K}, PD/O]$ such that the class of M_0 corresponds to the identity element.*

(Theorem (6.3) was announced by Mazur in Seattle)

Remark. Using Hirsch [2] it is easy to generalize Theorem (6.3) to manifolds with boundary).

COROLLARY (1) (Hirsch–Mazur). $\pi_n(PD/O) \simeq \Gamma_n$, *the group of differentiable structures on S^n (under connected sum).*

Proof. $\pi_n(PD/O) \simeq C_0(S^n)$ by the above. But since every n.s. *p.d.* homeomorphism of S^n onto S^n is *p.d.* isotopic to the identity, two smoothings, which are diffeomorphic, are concordant. Hence as sets $C_0(S^n) = \Gamma_n$. If the base point of C_0 is taken to be the standard sphere, then we may show that $C_0(S^n)$ with the group structure induced from $\pi_n(PD/O)$ is group isomorphic to Γ_n.

Note first, that we may define an addition in T by taking Whitney sum of trivial triangulations. Thus $\sigma : T \to [\tilde{K}, PD/O]$ is obviously additive, so that the group structure in T is given by Whitney sum.

Now Whitney sum of vector bundles over a sphere corresponds stably to connected sum of bundles. Defining connected sum of trivializations (stably) in corresponding fashion, we see that under $\rho : T \to C_0$, Whitney sum of trivializations, corresponds to connected sum of spheres.

LEMMA (Milnor). $\pi_i(BO) \overset{p}{\to} \pi_i(BPL)$ *is a monomorphism (and hence $\pi_i(O) \to \pi_i(PD)$ is a monomorphism).*

This follows from Theorems (6.3) and (2.4), and the fact that every homotopy sphere has a stably trivial normal bundle. See [3].

THEOREM (Hirsch–Mazur).

$$O \longrightarrow \pi_n(BO) \xrightarrow{p} \pi_n(BPL) \xrightarrow{q} \Gamma_{n-1} \longrightarrow O$$

is exact, where $q : \pi_n(BPL) \simeq \pi_{n-1}(PL) \simeq \pi_{n-1}(PD) \to \pi_{n-1}(PD/O) \simeq \Gamma_{n-1}$.

Proof. This follows from the above Lemma and Corollary (1).

§7. CONCORDANCE CLASSES OF SMOOTHINGS OF A VECTOR BUNDLE AND SMOOTHINGS OF IMBEDDINGS

Let η be a *p.d.* vector bundle over a unbounded combinatorial manifold K, and let $(E(\mu), \varphi)$ be a triangulation of $E(\eta)$. We wish to show

THEOREM (7.1). *There exists a* 1–1 *correspondence between concordance classes of smoothings of* K *and concordance classes of smoothings of* $E(\mu)$. *Further, this correspondence is obtained by assigning to* (M_α, f_α), *where* $f_\alpha : K \to M_\alpha$ *the concordance class of* $(E(\eta)_\alpha, \varphi_\alpha)$, *where* $\varphi_\alpha : E(\mu) \to E(\eta)_\alpha$ *is the induced triangulation given by* φ, *and* $E(\eta)_\alpha$ *is the total space of differentiable vector bundles defined by* η *over* M_α.

Proof. Let $\rho : T(K) \to C_0(K)$ be as above. Let $c \in C_0$, and $t = \rho^{-1}(c)$. Let $(\psi, \xi) \in t_K$; i.e. ξ is a *p.d.* vector bundle over K and $\psi : K \times R^n \to E(\xi)$ is a trivial triangulation. Let $p : E(\mu) \to K$ be the *p.l.* projection map. Then $(p^*\psi, p^*\xi) \in T(E(\mu))$ where

$$\begin{array}{ccc} E(\mu) \times R^n & \xrightarrow{p^*\psi} & E(p^*\xi) \\ \Big\downarrow{\scriptstyle p\times 1} & & \Big\downarrow{\scriptstyle \tilde{p}} \\ K \times R^n & \xrightarrow{\quad\psi\quad} & E(\xi) \end{array} \qquad \text{commutes.}$$

$E(p^*\xi) = \{(x, y) \in E(\mu) \times E(\xi) \mid p(x) = \pi(y)\}$, $\pi : E(\xi) \to K$ the projection $p^*\psi(x, r) = (x, \psi(p(x), r))$, $\tilde{p}^*(x, y) = y$.

We define $p^* : T(K) \to T(E(\mu))$ by $p^*(t_K) = \{(p^*(\psi), p^*\xi)\}$. Similarly, if $s : K \to E(\mu)$ is the zero-cross-section, then $s^* : T(E(\mu)) \to T(K)$, and $s^*p^* = 1$. This shows that p^* is 1–1 into.

On the other hand, since $s(K)$ is a *p.l.* deformation retract of $E(\mu)$, we have a homotopy $H : E(\mu) \times I \to E(\mu)$ such that $H_0 = 1$, $H_1 = \text{sp}$. Given any trivially triangulated *p.d.* vector bundle $(\psi, \xi) \in t_{E(\mu)}$ over $E(\mu)$, $E(H^*(\xi) \simeq E(\xi) \times I$, and $H^*\psi : E(\mu) \times I \times R^n \to E(H^*(\xi)) \simeq E(\xi) \times I$ gives an equivalence between $(p^*s^*\psi, p^*s^*\xi)$ and (ψ, ξ), showing that $t_{E(\mu)} = p^*s^*t_{E(\mu)}$; i.e. $p^*s^* = 1$, and p^* is 1–1 onto. (The fact that p^* is 1–1 may also be obtained from the fact that $[E(\mu), PD/O] \simeq [\tilde{K}, PD/O]$.

Now let $(M_\alpha, f_\alpha) \in c$, then $t = \rho^{-1}(c)$ is represented by (ψ, ξ), such that $\psi : K \times R^n \to E(\xi)_0$; where $E(\xi)_0$ is the diff. vector bundle over M_0, is *p.d.* isotopic to

$$\psi' : K \times R^n \xrightarrow{f_\alpha \times 1} M_\alpha \times R^n \xrightarrow{d} E(\xi)_0.$$

Then $(p^*\psi, p^*\xi)$ defines a smoothing in $\rho(p^*c)$ on $E(\mu)$ from the Hirsch product theorem using $p^*\psi : E(\mu) \times R^n \to E(p^*\xi)_0$, where $E(p^*\xi)_0$ is the diff. vector bundle over $E(\eta)_0$. We wish to show that this smoothing is $(E(\eta)_\alpha, \varphi_\alpha)$

$E(p^*\xi)_0 = E(\pi^*\eta)_0$ and we have the commutative diagram

$$\begin{array}{ccccc} E(\mu) \times R^n & \xrightarrow{p^*\psi} & E(p^*\xi)_0 & \longrightarrow & E(\pi^*\eta)_0 \\ \Big\downarrow & & & & \Big\downarrow \\ K \times R^n & & \xrightarrow{\qquad\qquad\psi\qquad\qquad} & & E(\xi)_0. \end{array}$$

Consider the *p.d.* vector bundle $E(\psi^*\pi^*\eta) = E(\eta) \times R^n$, we have

$$\begin{tikzcd} & E(\eta) \times R^n \arrow[d, leftrightarrow, "\simeq"] & \\ E(\mu) \times R^n \arrow[r, "\varphi \times 1"] \arrow[d] & E(\psi^*\pi^*\eta) \arrow[r, "\tilde{\psi}"] \arrow[d] & E(\pi^*\eta)_0 \arrow[d] \\ K \times R^n \arrow[r, "\text{identity}"'] & K \times R^n \arrow[r, "\psi"'] & E(\xi)_0 \end{tikzcd} \quad \text{commuting.}$$

The *p.d.* isotopy of ψ to ψ' is covered by a *p.d.* bundle isotopy of $\tilde{\psi}$ to $\tilde{\psi}'$. But then

$$\begin{tikzcd} E(\mu) \times R^n \arrow[r, "\varphi_\alpha \times 1"] \arrow[d] & E(\eta)_\alpha \times R^n \arrow[r, "d"] \arrow[d] & E(\pi^*\eta)_0 \simeq E(p^*\xi)_0 \arrow[d] \\ K \times R^n \arrow[r, "f_\alpha \times 1"'] & M_\alpha \times R^n \arrow[r, "d"'] & E(\xi)_0 \end{tikzcd} \quad \text{commutes.}$$

Since $p^*\psi$ is *p.d.* isotopic to $\tilde{d}(\varphi_\alpha \times 1)$, $(p^*\psi, p^*\xi)$ defines the smoothing $(E(\eta)_\alpha, \varphi_\alpha)$; i.e. $\rho p^* \rho^{-1}\{(M_\alpha, f_\alpha)\} = \{(E(\eta)_\alpha, \varphi_\alpha)\}$. This proves Theorem (7.1).

THEOREM (7.2). *Let μ be an r-dimensional microbundle over a combinatorial manifold K^n, and $i : E(\mu) \to V^{n+r}$ be a p.d. imbedding in a smooth manifold V. Then if μ is a triangulation of some p.d. vector bundle ξ, there exists a smoothing M_α of K such that i is concordant to a smooth imbedding $E(\xi)_\alpha$ in V.*

Explicitly, there exists a *p.d.* imbedding $\lambda : E(\mu) \times I \to V \times I$ such that $\lambda_0 = i : E(\mu) \times 0 \to V \times 0$ and $\lambda_1 : E(\mu) \times 1 \to V \times 1$ may be factored

$$\begin{tikzcd} E(\mu) \arrow[rr, "\lambda_1"] \arrow[dr, "\varphi_\alpha"'] & & V \\ & E(\xi)_\alpha \arrow[ur, "d"'] & \end{tikzcd}$$

where $\varphi : E(\mu) \to E(\xi)$ is the triangulation, and d is a smooth imbedding.

Proof. $i : E(\mu) \to V$ induces a smoothing $(g, E(\mu)\beta)$ of $E(\mu)$, such that

$$\begin{tikzcd} E(\mu) \arrow[rr, "i"] \arrow[dr, "g"'] & & V \\ & E(\mu)_\beta \arrow[ur, "d_0"'] & \end{tikzcd}$$

commutes, where d_0 is smooth

By Theorem (1), $(g, E(\mu)\hat{\beta})$ is concordant to some $(E(\xi)_\alpha, \varphi_\alpha)$. If $\bar{\lambda} : E(\mu) \times I \to W$ is this concordance, then there is a diffeomorphism $e : E(\mu)\hat{\beta} \times I \to W$, such that $e_0 g = \bar{\lambda}_0$. Take $\lambda = (d_0 \times 1) \circ e^{-1} \circ \bar{\lambda}$. Since $\bar{\lambda}_1 = d_1 \varphi_\alpha$, for some diffeomorphism d_1, $\lambda_1 = d\varphi_\alpha$, where $d = d_0 e_1{}^{-1} d_1$ is a diffeomorphism.

THEOREM (7.3). *Let $i: K \to V$ be a p.l. imbedding of a combinatorial manifold in a smooth manifold V, w.r.t $f: L \to V$ a smooth triangulation. Then there exists a non-singular p.d homeomorphism $h: V \to V$ such that $hi(K)$ is a smooth submanifold with normal vector bundle ξ; if and only if $i(K)$ has a normal microbundle μ which triangulates ξ.*

Proof. (a) *Only if*

Let $\varphi: E(\mu) \to E(\xi) \subset V$ be the triangulation. Then φ is *p.d.* isotopic, relative to the zero section, to a *p.l.* imbedding φ_1 w.r.t the triangulation $hf: L \to V$. Then $h^{-1}\varphi_1 : E(\mu) \to V$ is a *p.l.* imbedding which extends the imbedding $i: K \to V$.

(b) *If*

By Theorem (7.2), if $i(K)$ has a normal microbundle μ which triangulates ξ, there exists a *p.d.* imbedding $\lambda: E(\mu) \times I \to V \times I$ such that $\lambda_0 : E(\mu) \to V$ is a *p.l.* imbedding which extends i, and $\lambda_1 = d\varphi_\alpha$; where d is a diffeomorphism, and $E(\xi)_\alpha$ is a differentiable vector bundle over a smoothing M_α of K, which is equivalent as a vector bundle to ξ. Let N be a regular neighborhood of the zero-section in $E(\mu)$. Then $\lambda_0(N)$ is a combinatorial submanifold with boundary of the same dimension as V. Let N_1 be the second regular neighborhood of N in $E(\mu)$; then $N_1 = N \cup \partial N \times I$. It follows by the Cairns–Hirsch theorem, that $\lambda: (N_1 \times I, N_1 \times \partial I) \to (V \times I, V \times \partial I)$ is *p.d.* isotopic, relative to $(N \times I, N \times \partial I)$ to a *p.d.* imbedding $\bar{\lambda}$, such that $\bar{\lambda}(\partial N_1 \times I)$ is a smooth submanifold. Now $\bar{\lambda}(\partial N_1 \times I)$ has a smooth product neighborhood T in $V \times I$. See Fig. 1.
Then $\bar{V}$ is a smooth manifold (after smoothing corners).

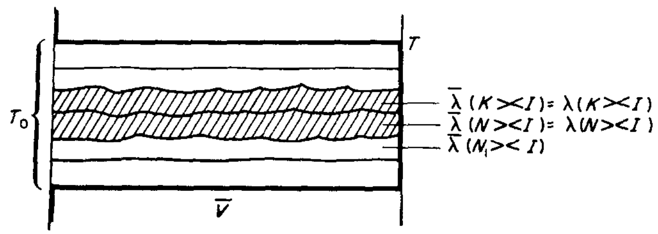

FIG. 1.

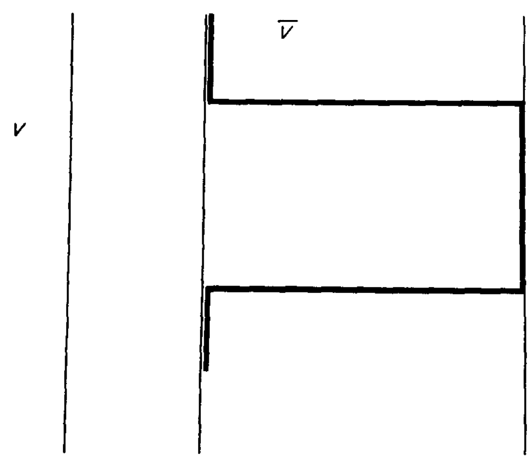

FIG. 2.

Let $\bar{V} = (V - T_0)^{-} \cup \partial T \cup T_1$. This is represented by heavy line in Fig. 1. Since ∂T is diffeomorphic to $\partial T_0 \times I$, and $\bar{\lambda}(N_1 \times 1)$ is homeomorphic to $\bar{\lambda}(N_1 \times O)$, there exists a non-singular *p.d.* homeomorphism $h : V \to \bar{V}$, such that $hi(K)$ is a smooth submanifold of $\bar{V}$ with normal bundle ξ. It remains to show that $\bar{V}$ is diffeomorphic to V.

Extend the left boundary of $V \times I$, then the region between the new boundary and $\bar{V}$ is a smooth manifold which is combinatorially a product (see Fig. 2). Hence V is diffeomorphic to $\bar{V}$, by Munkres [8]–Thom [10].

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