About the diffeomorphisms of the 3-sphere and a famous theorem of Cerf ($\Gamma_4 = 0$)

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ABSTRACT. Using a rigidity property of the foliations of $S^2 \times [0, 1]$ that are defined by a nonvanishing closed one-form, we give a rather simple proof of a theorem due J. Cerf, going back to 1968, that the group of direct diffeomorphisms of S^3 is connected.

The famous theorem of Jean Cerf in question in the title states the following.

Every diffeomorphism of the 3-sphere preserving the orientation is isotopic to the identity.

It follows that every diffeomorphism of S^3 extends to the 4-ball. Then, $\Gamma_4 := \text{Diff} S^3 / \rho(\text{Diff} D^4)$ is a trivial group; here, ρ stands for the restriction morphism to the boundary of D^4 . So, $\Gamma_4 = 0$ is a short name for the real theorem of Cerf. Actually in 1992, using recent results at the time in 3-dimensional contact geometry and the theory of holomorphic discs, Y. Eliashberg gave a direct proof of $\Gamma_4 = 0$ [2, section 6], avoiding Cerf's theorem.

In 1979, I solved in [5], with the help of Samuel Blank, a problem raised by J. Moser [7]:

(A) $\begin{cases} On a compact 3-dimensional manifold, two non-vanishing closed one-forms that are tangent to the boundary and cohomologous are isotopic. \end{cases}$

Our proof did not use Cerf's theorem. Applying (A) to $S^2 \times [0, 1]$ immediately implies the original theorem of Cerf, not just $\Gamma_4 = 0.^1$ Nevertheless, I should confess that this isotopy theorem of 1-forms is somehow technical. A simpler proof of (A) is given by N.V. Quê and R. Roussarie [9], but depending on Cerf's theorem.

We name Theorem (A') the particular case of Theorem (A) where the ambient manifold is $S^2 \times [0, 1]$. The present paper aims to give a less technical proof of Theorem (A') than [5], and hence of Cerf's Theorem.

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1. PRELIMINARIES AND PLAN OF PROOF

In the rest of the paper, the ambient manifold will be $S^2 \times [0, 1]$ and z will denote the variable in its second factor.

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¹For completeness, one should apply some classical fibration theorems also due to Cerf [1, Appendice].

1.1. Every non-vanishing closed one-form ω on $S^2 \times [0, 1]$ that is tangent to the boundary reads $\varphi^* dz$ for some diffeomorphism φ that is equal to identity on $S^2 \times \{0\}$. This follows from the fact that every orbit of a vector field X that everywhere fulfills $\omega(X) > 0$ goes from $S^2 \times \{0\}$ to $S^2 \times \{1\}$. Moreover, if $\varphi^* dz = dz$ then φ is isotopic to the identity among the diffeomorphisms of the same type.²

1.2. MOSER'S METHOD. Given two non-vanishing closed 1-forms ω_0 and ω_1 on $S^2 \times [0,1]$, tangent to the boundary, a point $a \in S^2 \times [0, 1]$ is said to be a *negative contact* (resp. a *positive contact*) if there exists $\mu > 0$ (resp. $\mu < 0$) such that $\omega_0(a) \pm \mu \omega_1(a) = 0$. In other words, the kernels of $\omega_0(a)$ and $\omega_1(a)$ coincide but their co-orientations are opposite (resp. equal). The locus of negative (resp. positive) contacts will be denoted by C_{-} (resp. C_{+} .)

Lemma 1.3. If the two non-vanishing closed one-forms are cohomologous and their mutual negative contact locus C_{-} is empty, then the two forms are isotopic.

Indeed, the barycentric combination $\omega_t := (1-t)\omega_0 + t\omega_1$ provides a path $(\omega_t)_{t \in [0,1]}$ of closed 1-forms, from ω_0 to ω_1 , that are cohomologous, tangent to the boundary and nowhere singular. In that case, Moser's *isotopy theorem* applies, as in the case of volume forms [7]. Namely, there is an autonomous flow $(\psi_t)_{t\in[0,1]}$ that conjugates ω_t to ω_0 for every $t\in[0,1]$.³

Therefore, our method to prove the isotopy Theorem (A') on $S^2 \times [0,1]$ will be to cancel the locus C_{-} of negative contacts between $\omega_0 := dz$ and $\omega_1 := \varphi^* dz$.

1.4. REDUCTION TO A MORE SPECIAL CASE. Let f be the function which is equal to the projection onto the factor [0, 1]; we set $\ell := \varphi^* f = f \circ \varphi$. From now on, the two forms ω_0 and ω_1 will be respectively denoted by $\omega_{\mathcal{F}}$ and $\omega_{\mathcal{L}}$; and the foliations they define will be respectively named \mathcal{F} and \mathcal{L} . The leaves of $\omega_{\mathcal{F}}$ will be named the *level sets* (understood of f) while the level sets of ℓ will continue to be named the *leaves* (understood of \mathcal{L} .)

Without loss of generality, to prove Theorem (A') we may assume the following in the rest of the paper:

(1.1) $\begin{cases} \text{The restriction of } f \text{ to every leaf of } \ell, \text{ close enough to the boundary } S^2 \times \{0, 1\}, \\ \text{ has only two critical points, a maximum and a minimum, and these belong to } C_+. \end{cases}$

1.5. PLAN OF THE PROOF OF THEOREM (A'). Generically, the contact points (positive or negative) have a Morse index: 0, 1 or 2. The contact points of index 1 are named saddles. They have also a type: Y, λ , or X. A saddle s on a leaf $L \in \mathcal{L}$ is said to be of type λ (resp. Y) if the connected component of $L \cap [f(s) - \varepsilon, f(s) + \varepsilon]$ which contains s looks like a pair-of-pants (resp. a reversed one.) A saddle is said to be of type X if it is the common limit of two sequences of saddles, one of λ -saddles and the other of Y-saddles, the pair (f, ℓ) being kept fixed (Definition 3.12.) This necessarily involves a pair of saddles both of type X in the same connected component of level curves (Lemma 3.13.) One will speak of λ -, Y-, and X-saddle respectively.

²These facts hold true on $S^n \times [0, 1]$ for every dimension.

³In the case of non-vanishing closed one-forms, a very simple proof is given in [5, Appendice I].

Section 2 is devoted to basics on generic properties. Section 3 reviews elementary isotopies whose effect is to simplify C_{-} . One of them will allow us to kill all the finitely many X-saddles. Typically, this phenomenon occurs in the setting saddle-center-saddle (Definition 3.4.)

The next operation will consist of pushing the saddles of type Y to levels higher than all saddles of type λ . Unfortunately, this operation is obstrued by connecting orbits $Y \to \lambda$ of the \mathcal{L} -gradient of f, a phenomenon that generically appears finitely many times. Here, the \mathcal{L} -gradient of f is the orthogonal projection of $\nabla f = \partial_z$ onto the leaves of ℓ with respect to an understood Riemaniann metric; it will be denoted by $\nabla_{\mathcal{L}} f$.

These obstructions are destroyed thanks to the main operation of the paper, namely the *Dehn modification*. Section 4 will be devoted to this type of 3-dimensional surgery considered in our specific setting. Section 5 makes use of all these tools to prove the main result, namely Theorem (A').

2. Basics on pairs of non-vanishing closed one-forms

2.1. REGULARITY. We set $C = C_{-} \cup C_{+}$. With local coordinates (x, y) on S^{2} , the contact locus is defined by the following system:

(2.1)
$$C = \left\{ \frac{\partial \ell}{\partial x}(x, y, z) = 0, \ \frac{\partial \ell}{\partial y}(x, y, z) = 0 \right\}$$

The subsets C_+ (resp. C_-) are defined by adding the inequation $\frac{\partial \ell}{\partial z}(x, y, z) > 0$ (resp. < 0). By the assumption (1.1), C_- does not approach the boundary and hence is compact.

By the transversality theorem of Thom in jet spaces, some approximation of φ makes maximal the rank of the linearized system associated with system (2.1) at every point of $S^2 \times]0, 1[$. In this case the pair $(\omega_{\mathcal{F}}, \omega_{\mathcal{L}})$ —or (f, ℓ) —is said to be *regular* and C is a smooth curve. The open subset of points in C where

(2.2)
$$\Delta := \frac{\partial^2 \ell}{\partial x^2} \frac{\partial^2 \ell}{\partial y^2} - \left(\frac{\partial^2 \ell}{\partial x \partial y}\right)^2$$

is non-zero is the locus where C is transverse to both foliations \mathcal{F} and \mathcal{L} ; overthere, their contact is quadratic. If a is such a point and L is the leaf of a, this point has a Morse index in $\{0, 1, 2\}$ as a critical point of the function f_L , that is the restriction of f to L. The critical points of f_L are named *minimum*, saddle, maximum, depending on their Morse index.

The remaining points of C are called *inflection points*. Generically, the equation $\Delta(x, y, z) = 0$ is regular and hence, the inflections are isolated in C and do not approach the boundary. So, there are finitely many of them. At an inflection, the tangency of both \mathcal{F} and \mathcal{L} with C is quadratic while the mutual tangency of \mathcal{F} and \mathcal{L} is cubic.

By the versal unfolding theory [6] [8], there are coordinates in a neighborhood of an inflection where the two functions f and ℓ read

(2.3)
$$f(x, y, z) = z \text{ and } \ell(x, y, z) = x^3 \pm y^2 + \lambda(z) x + \mu(z).$$

The sign in the above formula depends on the Morse indices of contact points nearby. The Hessian of f_L at an inflection $I \in L$ has index 0 (resp. 1), and is said to be a *saddle-min* (resp. *saddle-max*) inflection. If the restriction f_C of f to C is locally minimal (resp. maximal) at $p \in C$, p is said to be a *birth* (resp. *cancellation*) inflection.

Since C_{-} is generically a closed—in general non-connex—curve, every of its connected component carries at least two inflections, one maximum and one minimum of the restriction of f to C_{-} .

The Morse index of a quadratic contact is constant on each arc of C ending at an inflection or boundary points. When two arcs in C of quadratic contact have a common inflection in their closure, their indices differ by 1. So, one of these two arcs has a constant index 1; every contact point of this arc is a saddle, and the other arc has the index of an f_L -extremum.

2.2. EXCELLENCE. The pair of functions (f, ℓ) is said to be *excellent* if the following conditions are fufilled.

- (E1) Let a and b be two distinct quadratic contact points at the same level and in the same leaf; let (a(t), b(t)) is a pair of local parametrizations of the respective contact arcs such that a(0) = a, b(0) = b and $\ell(a(t)) = \ell(b(t))$ for every t close to 0. Then we have $\frac{d}{dt}[f(a(t)) f(b(t))] \neq 0$ at t = 0.
- (E2) There are neither three contact points in the same leaf and in the same level set nor two pairs of contact points in the same leaf at two distinct levels.
- (E3) Let σ be an inflection point. Then the leaf of σ (resp. its level set) contains no contact point at the same level (resp. in the same leaf) as σ .

Lemma 2.3. In the space of smooth functions $\ell : S^2 \times [0,1] \rightarrow [0,1]$ such that the pair (f,ℓ) fulfils the assumption (1.1), the subspace such that the pair (f,ℓ) is regular and excellent is an open dense set.

Proof. Such a statement is well known to the topologists when one speaks of a generic path of functions, the variable z being the time. But every path of functions is not the primitive of a non-vanishing one-form $\omega_{\mathcal{L}}$ tangent to the boundary. The main difference comes from the points where $\frac{\partial \ell}{\partial z} = 0$. In our setting, $\frac{\partial \ell}{\partial z}$ is not vanishing at every point where the two other partial derivatives vanish. That means that we work in an open set of the general space of smooth real functions on S^2 . So, Thom's transversality theorems in bi-one-jet⁴ spaces of real functions apply.

3. Elementary isotopies and application.

Here is the list of the isotopies of ℓ that we consider as *elementary*: cancellation—or creation of a *simple loop* in the contact locus C; bypass of the *cusp* singularity x^4 ; exchange (that is another way of bypassing a cusp.) To this list, we add the *isotopies along a satured set*, though they are not related to singularity theory.

As announced, these techniques will allow us to cancel the saddles of type X (in the sense of subsection 1.5.) Before starting with the description of these isotopies, some notation is needed.

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⁴A bi-one-jet is just a pair of two one-jets.

3.1. Given a leaf $L \in \mathcal{L}$ and an extremum m of f_L , the *cone* of m, denoted by $C_{\mathcal{L}}(m)$, is the largest open disc in L—if it exists—that contains m and is bounded by a *singular* simple closed curve $\partial C_{\mathcal{L}}(m)$ in a level set of f such that:

- (i) $\partial C_{\mathcal{L}}(m)$ contains a saddle (or an inflection) and bounds a disc $\Delta(m)$ in its level set;
- (ii) $\Delta(m) \cup C_{\mathcal{L}}(m)$ bounds a (topological) ball $B_{\mathcal{L}}(m)$ in the ambient manifold.

Note that if L is close to the boundary of $S^2 \times [0, 1]$, by (1.1) the cone of the minimum (resp. maximum) is not defined since the leaf contains no saddles.

Definition 3.2. (SIMPLE LOOP.) A simple closed curve Γ in the contact locus C of the pair (f, ℓ) of smooth functions on $S^2 \times [0, 1]$ is said to be a simple loop if the following conditions are fulfilled.

- (1) Γ is made of two open arcs, α and β, of quadratic contact points of the same signs and two inflection points, I₀ and I₁, which make the topological closure of each of the arcs α and β; say α is of Morse index 1; the index of β, 0 or 2, depends on the sign in front of y² in formula (2.3).
- (2) Every leaf L that crosses α at a saddle s_L^5 also crosses β at a unique extremum m_L and conversely. Moreover, s_L belongs to $\partial C_{\mathcal{L}}(m_L)$.
- (3) There is a unique \mathcal{L} -gradient line⁶ γ_L joining s_L to m_L on every leaf L meeting $\alpha \cup \beta$.
- (4) The only arc of definite quadratic contacts located in the ball $B_{\mathcal{L}}(m_L)$ is $\beta \cap B_{\mathcal{L}}(m_L)$.

By formula (2.3), if s_L goes to I_j (j = 0 or 1) then γ_L with its field of unit tangent vectors goes to I_j with $\pm \partial_x$. Note that, by permuting f and ℓ , there are \mathcal{F} -gradient lines connecting pair of points in $\alpha \times \beta$ lying in the same level set of f.

Proposition 3.3. Let Γ be a simple loop in the contact locus of the pair (f, ℓ) . Then, there exists a smooth closed 3-ball N which is an arbitrary small neighbourhood of the union $\bigcup_{m_L \in \beta} B_{\mathcal{L}}(m_L)$ satisfying the following two properties.

- (1) The boundary is foliated by smooth closed curves drawn by $\mathcal{L} \cap \partial N$, with exactly two singularities at $\max \ell_{\partial N}$ and $\min \ell_{\partial N}$.⁷
- (2) The germ of ℓ along ∂N extends to N as a smooth function ℓ' without contact point with f in N. Moreover, ℓ' is isotopic to ℓ by an isotopy supported in the interior of N.

Proof. We may only consider the case where β is an arc of maxima. Let ε be a small positive number. For every $m_L \in \beta$, consider the stable manifold $W^s(m_L, \nabla_{\mathcal{L}} f)$, truncated at the level $f = f(s_L) - \varepsilon$. Then slightly enlarge this surface to a disc $D_{m_L} \subset L$ in order to have a smooth boundary and to contain an ε -neighbourhood of the truncated stable manifold of s_L . If ε is small enough, D_{m_L} avoids all other connected components of the contact locus. If this construction is performed smoothly with respect to m_L with a fixed ε , the desired N is $\bigcup_{m_L \in \beta} D_{m_L}$.

There exists a proper disc $S \subset N$, transverse both to the leaves of ℓ and the level sets of f, that contains all segments γ_L (item (3) in Definition 3.2.) This S can be endowed with coordinates (x, z) where f(x, z) = z and the pair (x, z) allows us to use the normal form

⁵By the Stokes formula this point is unique when it exists.

⁶See the definition in subsection 1.5.

⁷These two points are close to I_0 and I_1 respectively, depending on the distance of ∂N to $\bigcup_{m_L \in \beta} B_{\mathcal{L}}(m_L)$.



FIGURE 1. A simple loop max-saddle.

of inflections. Finally, one finds the y-coordinate by slicing N transversely to S so that the maximum of $f_{|\{x=x_0,\ell(x_0,y,z)=\ell_0\}}$ is quadratic non-degenerate and located in $\{y=0\}$. In Morse theory, such extension of coordinates is named a suspension [3].

We are able to solve the problem of isotopy for a pair of functions like $(f_{|S}, \ell_{|S})$ since the two-dimensional problem is known by the contractibility of Diff $(D^2$ rel. S^1) [10]. An isotopy of S extends to N, relatively to a small neighbourhood of ∂N by suspension. Nothing is changed in $(S^2 \times [0, 1]) \setminus N$. The proof is now complete.⁸

On the opposite, creating a simple loop requires no condition. It can be realized in any open set of the ambient manifold where the two functions f and ℓ have no contact points. The details are left to the reader.

The next elementary isotopy will follow the same idea of suspension. This deals with the configuration of contacts named *saddle-center-saddle*. Center stands for minimum or maximum; here, we only consider the case of a minimum. The case where the center is a maximum is similar.

Definition 3.4. The saddle-center-saddle configuration is the following. The three contact points are in the same leaf L of \mathcal{L} ; the center is a minimum m_L whose cone $C_{\mathcal{L}}(m_L)$ contains two saddles s_L and s'_L in its boundary; there is exactly one \mathcal{L} -gradient line from m_L to s_L , and similarly from m_L to s'_L . So, $f(s_L) = f(s'_L)$.

Let α, β, α' denote the contact arcs containing s_L, m_L, s'_L respectively. Here are the main requirements:

(1) Let L' be a leaf close to L crossing β at $m_{L'}$ with $f(m_{L'}) < f(m_L)$. Then $f(s_{L'}) < f(s'_{L'})$.

(2) Let L" be a leaf close to L crossing β at $m_{L''}$ with $f(m_{L''}) > f(m_L)$. Then $f(s_{L''}) > f(\overline{s'_{L''}})$.

(3) The interior of the ball $B_{\mathcal{L}}(m_L)$ meets no other contact arc than an arc of β .⁹

⁸This proof is inspired by my proof of a theorem of Morse about the cancellation of a pair of critical points of one real function [4].

⁹See subsection 3.1 for notation $B_{\mathcal{L}}(m_L)$.

Here, it is meant that $\{s_{L'}, s_{L"}\} \subset \alpha$ and $\{s'_{L'}, s'_{L"}\} \subset \alpha'$.



FIGURE 2. The leaves are denoted on the left. Vertically are the three contact arcs.

Note that, up permuting α and α' , (1) and (2) above-mentioned follow from the excellence of the pair (f, ℓ) ; moreover, it is easily seen by (3) that s_L , m_L , and s'_L have the same sign as contact point. The planar figure of this configuration is represented in Figure 2. One recognizes—up to sign—the singularity x^4 , named *cusp*, whose versal unfolding—up to conjugation sourcextarget—reads

(3.1)
$$(x,\lambda,\mu) \mapsto -(x^4 + \lambda x^2 + \mu x)$$

where λ and μ are two real parameters. As m_L is a minimum, the suspension consists of adding a *positive square* in the added variable y. Hence, it reads

(3.2)
$$(x, y, \lambda, \mu) \mapsto -(x^4 + \lambda x^2 + \mu x) + y^2$$

The critical points of f restricted to a leaf close to L are given by $y = 0, 4x^3 + 2\lambda x + \mu = 0$. Up to positive coefficients, the equation of this discriminant locus reads

(3.3)
$$4\lambda^3 + 27\mu^2 = 0.$$

The (λ, μ) -space, denoted by $\mathbb{R}_{(\lambda,\mu)}$, is stratified, apart from the origin, by the discriminant locus D of the *fold*, and the half axis $A = \{\lambda < 0\}$, the set of parameters for which the corresponding function has two critical points with the same value. This is exactly the case of f_L . The deformation from L' to L'' corresponds to a path in $\mathbb{R}_{(\lambda,\mu)}$ crossing A transversely (Figure 3.)



FIGURE 3. Discriminant locus of the singular function x^4 or $x^4 - y^2$.

In the setting saddle-center-saddle, we have the 3-ball $B_{\mathcal{L}}(m_L)$ that we can extend up to the level $\{f = f(s_L) + \varepsilon\}$. The closure of this extension contains a piece of the unstable manifolds $W^u(s_L, \nabla_{\mathcal{L}} f)$ and $W^u(s'_L, \nabla_{\mathcal{L}} f)$. Then, we take a neighbourhood N of this closure. And hence, there are coordinates (x, y, z) on N such that the surface $S := \{y = 0\}$ is the locus of the minimum of f on the curve in N defined by $(x, \ell) = (x_0, \ell_0)$ where the pair (x_0, ℓ_0) ranges in a convenient 2-dimensional domain.

Definition 3.5. A bypass of the cusp singularity is any path of functions as in Figure 3.

Figure 4 represents the modification of the contact locus along such a path.

Proposition 3.6. In the above-mentioned setting, there is an isotopy supported in N that, at time 1, carries ℓ to a function ℓ' whose one-parameter family of leaves realizes a bypass of the cusp singularity.



FIGURE 4. Bypass in configuration saddle-center-saddle.

After these first two elementary isotopies, we present more shortly one useful variation of the latter.

Definition 3.7. The configuration min-saddle-min is the following. A quadratic contact m_L of index 0, located in the leaf L, has a cone $C_{\mathcal{L}}(m_L)$ whose boundary has an inflection I_L of the same sign as m_L . Let β be the contact arc containing m_L . It is assumed:

- 1) The ball $B_{\mathcal{L}}(m_L)$ contains no other contact arcs but a sub-arc of β .
- 2) I_L is the cancellation point of a pair saddle-min, that is the common upper bound of an arc α of saddles and an arc $\beta' \neq \beta$ of minima (Figure 6, left-hand side.)

Therefore, there is an \mathcal{L} -gradient line from m_L to I_L . So, if $m_{L'}$ is just below m_L on β , then there is a saddle $s_{L'}$ connected to $m_{L'}$ by an \mathcal{L} -gradient line, and hence, the other branch of the stable manifold $W^s(s_{L'}, \nabla_{\mathcal{L}} f)$ comes from a minimum $m'_{L'}$ distinct from $m_{L'}$.

If one forgets the so-called suspension by $+y^2$, it is clear that the new configuration deals with the cusp singularity x^4 . Instead of crossing the equality of two critical values, one crosses the cancellation of a saddle with a center.

In this setting, bypassing the cusp singularity has the following effect on the contact curves, as shown in Figure 6.

Proposition 3.8. In the configuration min-saddle-min, there is an isotopy supported in a neighbourhood N of the ball $B_{\mathcal{L}}(m_L)$ that, at time 1, carries ℓ to a function whose contact curves in N have the behaviour presented in the right hand side of Figure 6.



FIGURE 5. Another bypassing of the cusp singularity.



FIGURE 6. Min-saddle-min configuration.

We now turn to *isotopies along saturated sets*.

Definition 3.9. Let $x \in S^2 \times [0, 1]$ and $z_0 > f(x)$. The ascending saturated set of x up to level z_0 , denoted by $S_{z_0}(x)$, is the minimal closed subset of $\{f \leq z_0\}$ that contains x and satisfies:

- (i) For every $y \in S_{z_0}(x)$, the positive orbit of y under $\nabla_{\mathcal{L}} f$, truncated at level z_0 , is included in $S_{z_0}(x)$.
- (ii) Let C denote the contact locus of $(\mathcal{F}, \mathcal{L})$. If $y \in (C \cap S_{z_0}(x))$, every arc of C ascending from y and truncated at level z_0 is contained in $S_{z_0}(x)$.

Of course, there is an analogous definition of a descending satured set. Such a subset is stratified in an obvious way. By excellence assumption, $S_{z_0}(x)$ contains finitely many accidents, meaning level sets where the stratified type changes. From one accident to the next one, one proves that $S_{z_0}(x)$ is surrounded by a fundamental system of collapsible domains that, in our setting, are defined as follows.

Definition 3.10. A collapsible domain K is a 3-ball whose boundary ∂K is the angular union of two discs, the lower boundary $\partial_{lo}K$ and the horizontal boundary $\partial_h K$, fulfilling the next conditions:

- (i) $\partial_h K$ is contained in $\{f = z_0\}$.
- (ii) $\partial_{lo} K$ (resp. its interior) is contained in $\{f \leq z_0\}$ (resp. $\{f < z_0\}$).
- (iii) There exists a field of directions on K transverse to the f-level sets and also transverse to $\partial_{lo}K$.

Lemma 3.11. Let K be a collapsible domain which contains $S_{z_0}(x)$ and whose lower boundary is disjoint from $S_{z_0}(x)$. For every $\varepsilon > 0$, there exists an isotopy $(\Phi^t)_{t \in [0,1]}$, supported in the interior of K, such that:

- (1) $\Phi^0 = Id$ and $\Phi^1(S_{z_0}(x))$ is contained in $\{z_0 \varepsilon < f \le z_0\};$
- (2) for every $t \in [0,1]$ and every contact point y of the pair (f, ℓ) , its image $\Phi^t(y)$ is a contact point of the pair $(f, \ell \circ (\Phi^t)^{-1})$, with the same sign and Morse index;
- (3) if γ is an \mathcal{L} -gradient arc, then $\Phi^t(\gamma)$ is transverse to the f-level sets.¹⁰

Such an isotopy is said to be an *isotopy along a saturated set*.

3.12. APPLICATION. Here we give an application of Proposition 3.6, namely, the cancellation of a pair of saddles, that are of the type X. We specify the definition of an X-saddle when the pair (f, ℓ) is excellent.

Definition. A saddle s in a contact arc α is said to be of type X, or an X-saddle, if s is the upper bound on α of an interval of saddles of type λ (resp. Y) and the lower bound of an interval of saddles of type Y (resp. λ).

The following facts are easily checkable when the pair $(\mathcal{F}, \mathcal{L})$ is excellent and the leaves are 2-spheres.

Lemma 3.13. (1) If s is an X-saddle there is another X-saddle s' in some contact arc α' interacting with s, meaning that the two saddles are located in the same connected component \hat{C}_0 of the intersection $L_0 \cap F_0$ of a leaf $L_0 := \{\ell = u_0\}$ and a level set $F_0 := \{f = z_0\}$.

- (2) Both saddles have the same sign and \hat{C}_0 looks like the bold line on Figure 7.
- (3) \hat{C}_0 is made of two closed singular curves C_0 and C'_0 in F_0 meeting in s and s' only.¹¹



FIGURE 7. \hat{C}_0 denotes the connected component of s in the level curve $F_0 \cap L_0$. The dashed (resp. dotted) curves lie in L_0 and in level sets above (resp. below) z_0 .

Proof. It almost clear that a second saddle in \hat{C}_0 is necessary. Indeed, if not, the germ of $\mathcal{L} \cap F_0 \ \hat{C}_0$ is stable in the sense that, up to isotopy, it does not depend on small variations of z_0 . Therefore, s could not be of type X.

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¹⁰See more details in [5, chap. III §4].

¹¹There are two possibilities for the pair (C_0, C'_0) : either it is the limit of the dotted curve from Figure 7 or it is the limit of the dashed curve.



FIGURE 8. The unique example of a chain in S^2 . Uniqueness up to isotopy does not hold in \mathbb{R}^2 .

By excellence, \hat{C}_0 cannot have three saddles (or double points.) Then, it contains exactly two saddles. Up to isotopy of F_0 , there are very few configurations for \hat{C}_0 :

- (a) The one represented in Figure 7.
- (b) A second one that is the image—named *a chain*—of a smooth closed cuve through an immersion with two double points (Figure 8.)

The rule for the co-oriention of \hat{C}_0 in $F_0 = S^2$ near the double points leaves only these two cases. In the same way, one checks that in case (a) the two contact points have the same sign.

In case (b), whatever the signs of contacts the saddles cannot be X-saddles.



FIGURE 9. The gray rectangle represents the domain marked by the ascending saturated set of Γ in the level set $z_0 - \varepsilon$.

Denote by α' the arc of saddles passing through s'. Let s(t) and s'(t) be two local parametrizations of α and α' , with s(0) = s, s'(0) = s', such that $\ell(s(t)) = \ell(s'(t))$ for every t close to 0. Say f(s(t)) and f(s'(t)) are increasing with t. As the pair (f, ℓ) is excellent, the two f-velocities at t = 0 are different. We continue this analysis by specifying that the two contacts are positive.

For ε small enough, let C_- and C'_- be the two smooth closed curves in the level set $\{f = z_0 - \varepsilon\}$ corresponding to C_0 and C'_0 respectively by the \mathcal{L} -gradient flow lines on L_0 . Let T and T' be the lower boundaries of the saturated sets descending from s(0) and s'(0) respectively (Figure 9).

Proposition 3.14. There is an isotopy supported in $\{z_0 - 2\varepsilon \leq f \leq z_0 + \varepsilon\}$ whose effect on ℓ is to decrease by one the number of pairs of contact points with \mathcal{F} that are of type X.

Proof. The idea is to create a configuration saddle-center-saddle which contains s(0) and s'(0). Since the center is missing, one starts by introducing a simple loop Γ of type saddle-min whose



FIGURE 10. The very thick lines are lower boundaries of descending saturated sets of saddles s(0), s'(0) and $\alpha'' \cap \{f = z_0 - \varepsilon\}$.

inflection points I_0 and I_1 satisfy $z_0 - 2\varepsilon < f(I_0) < f(I_1) < z_0 - \varepsilon$ and so that its upper boundary in $\{f = z_0 - \varepsilon\}$ is a position as shown in Figure 9.

The next operation is an isotopy along the satured set $S_{z_0+\varepsilon}(I_1)$. By application of Lemma 3.11, one can collapse this domain above z_0 . After this isotopy, still denoting by ℓ and Γ the carried function and simple loop, one sees a configuration saddle-center-saddle as shown in Figure 10. Then Proposition 3.6 is available that makes the desired cancellation of a pair of X-saddles.

There is a new arc α'' of positive saddles which crosses $\{f = z_0 - \varepsilon\}$ —this is the arc of saddles in the simple loop Γ after the just above mentioned isotopy; the arc of minima is denoted by β . One checks that α'' contains only saddles of type λ . Now we can apply Proposition 3.6. This cancels the pair (s, s') of X-saddles without creating a new one and Proposition 3.14 is proved.

The outcome of Proposition 3.14 is the cancellation of every X-saddle.

4. Dehn modification

After Thom's transversality arguments that generate generic properties, the Dehn modification, also called Dehn surgery, is the main tool to prove Theorem (A), and already regarding its 1-connected version Theorem (A'). We are going to speak of this notion in our setting only: the ambient manifold is $M := S^2 \times [0, 1]$, provided with two foliations, respectively as the level sets of two functions, f and ℓ , that are constant on each component boundary components and whose critical sets are empty.

As an application, we prove that a suitable series of Dehn modifications allows us to move the Y-saddles above the λ -saddles.

Definition 4.1. (DEHN TWIST AND DEHN MODIFICATION.)

1) A left Dehn twist on the standard annulus $\mathbb{A} := S^1 \times [1, 2]$ is a diffeomorphism τ that, in polar coordinates, reads

(4.1)
$$(\theta, r) \xrightarrow{\tau} (\theta + \varphi(r-1), r)$$

where φ is a smooth non-decreasing function from the value $\varphi([0,\varepsilon]) = 0$ to $\varphi([1-\varepsilon,1]) = 2\pi$.

The right Dehn twist is the inverse τ^{-1} of the left one.

2) Let A be an annulus embedded in M and parametrized by A; for $i = 1, 2, \partial_i A$ is parametrized by $\partial_i A := S^1 \times \{i\}$. Let \check{M} be the manifold obtained by cutting M along the interior of A; it is provided with a singular boundary made of two lips A^+ and A^- corresponding to the coorientation of A. The Dehn modification of M along A is the manifold M_{τ} obtained from \check{M} by gluing $x \in A^+$ to $\tau(x) \in A^-$.



FIGURE 11. The two dotted vertical arrows indicate the gluing that makes M_{τ} .

In our setting, A is an annulus, embedded in a level set F_0 of f transversely to \mathcal{L} ; the leaves of $\mathcal{L} \cap A$ are closed curves isotopic to each component of ∂A . The identification with \mathbb{A} is chosen so that these curves are parametrized by the circles of \mathbb{A} . In this setting we shall say that the Dehn modification is *adapted* to the pair (f, ℓ) or $(\mathcal{F}, \mathcal{L})$. By construction, the functions f and ℓ are carried to some uniquely defined functions f_{τ} and ℓ_{τ} .

Important remark 4.2. In the previous setting, a Dehn modification along A keeps the contact locus of the pair (f, ℓ) invariant. Indeed, the support of this modification is away from the contact locus.

Definition 4.3. Let A be an annulus, parametrized by \mathbb{A} , in a level set $F_0 = \{f = z_0\}$. Let $p_A \in S^2$ such that the vertical arc $\{p_A\} \times [0,1] \subset S^2 \times [0,1]$ avoids A and intersects the unique disc in F_0 that contains A and is bounded by $\partial_2 A$.

Let \mathcal{D}_A denote the space of smooth oriented 2-discs embedded in M that contain A as oriented annulus, have algebraic intersection +1 with $\{p_A\} \times [0,1]$ and have $\partial_2 A$ as a boundary.

Proposition 4.4. With this notation, the following holds true.

- 1) Every $D \in \mathcal{D}_A$ defines a unique diffeomorphism $\psi_D : M_\tau \to M$, up to isotopy.
- 2) For every pair (D_0, D_1) of elements in \mathcal{D}_A , then ψ_{D_1} is isotopic to ψ_{D_0} .
- 3) There are two discs D_0 and D_1 , elements of \mathcal{D}_A , such that $(\psi_{D_0})^* df = df_{\tau}$, and $(\psi_{D_1})^* d\ell = d\ell_{\tau}$.
- 4) The functions f and ℓ are isotopic if and only if the functions f_{τ} and ℓ_{τ} are isotopic.

Proof. 1) Since A is an imposed collar of ∂D for every $D \in \mathcal{D}_A$, it is natural to say that, abstractly, D is the disc of radius 2 in the Euclidean plane. This disc may be considered as embedded in M or in M_{τ} as well. Firstly, we consider D in M_{τ} . One endows D with a 3-dimensional collar $N \cong D \times [0, 1]$ on the side of A^+ , that is $A^+ \subset D \times \{0\}$. So, N already existed in \check{M} and M.

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One defines a diffeomorphism $\rho : D \to D$ by $\rho(x) = \tau(x)$ if $||x|| \in [1,2]$ and $\rho(x) = x$ if $||x|| \leq 1$. One chooses an isotopy $(h_t)_{t \in [0,1]}$ among the diffeomorphisms of D from ρ for t = 0 to the identity of D for t = 1, with an extra requirement:

(4.2)
$$||h_t(x)|| = ||x||$$
 for every $t \in [0, 1]$

Then, on defines $\psi_D: M_\tau \to M$ by

(4.3)
$$\begin{cases} \psi_D(y) = y \text{ if } y \in M \smallsetminus N \\ \psi_D(x,t) = (h_t(x),t) \text{ if } (x,t) \in N. \end{cases}$$

Since all choices are made in contractible spaces, ψ_D is uniquely defined up to isotopy.



FIGURE 12. Example of meridional twist. Here, $w_A(D) = -1$.

2) Usual tools of 3-dimensional topology yield the following topological result.¹²

Each 2-disc $D \in \mathcal{D}_A$ has a winding number around A, $w_A(D) \in \mathbb{Z}$, that is the algebraic intersection of D with a vertical arc V in $S^2 \times [0,1]$ starting from the interior of A and ending in $S^2 \times \{1\}$. Moreover, w_A provides a bijection from $\pi_0(\mathcal{D}_A)$ onto \mathbb{Z} .

Here, the linking condition imposed to D for being in \mathcal{D}_A plays an important role. This statement also holds in M_{τ} . Moreover, one can think of A as a flat annulus in a solid torus $T_A \cong S^1 \times D^2$, each ray of A being a diameter of $\{\theta\} \times D^2$. An exterior collar of ∂T_A has coordinates (θ, θ', t) with $(\theta', t) \in \partial D^2 \times [0, 1]$. This allows one to define a *meridional Dehn twist* in the coordinates (θ', t) independent of θ .¹³ Let us denote it by μ . For $D \in \mathcal{D}_A$, one has

(4.4)
$$w_A(\mu(D)) = w_A(D) + 1.$$

This holds true on M and M_{τ} as well because the support of τ and μ are disjoint.

Using that translations on the torus are commuting, one has for every $D \in \mathcal{D}_A$:

(4.5)
$$\psi_{\mu(D)} = \mu \circ \psi_D \circ \mu^{-1}$$

Moreover, μ is isotopic to the identity if A is let free to rotate in T_A around its core. rotating A does not change M_{τ} . Therefore, $\psi_{\mu(D)}$ is isotopic to ψ_D which proves what was desired.

3) The disc D_0 is the unique disc in the level set F_0 fulfilling the requirement $\partial D_0 = \partial_2 A$. One can choose its collar $N \cong D_0 \times [0, 1]$ to be foliated by level sets of f. Then formula (4.3) shows that $(\psi_{D_0})^* f$ is sensitive neither to τ nor to the isotopy (h_t) . So, $(\psi_{D_0})^* f = f$ is clear.

¹²Unfortunately, I have no reference for this exercise. One has to use the classical method of the *innermost* intersection curve for two discs in \mathcal{D}_A with the same winding number w_A .

¹³Use formula (4.1) replacing (θ, r) with (θ', t) .

For D_1 , this is slightly more subtle. There is a disc Δ which is bounded by $\partial_1 A$ in some \mathcal{L} -leaf and intersects $\{p_A\} \times [0, 1]$ positively. The union $A \cup \Delta$ is angular along $\partial_1 A$. But its smoothing is easy: Take a collar N_{Δ} of Δ in the direction outward to A along $\partial_1 A$ and foliated by \mathcal{L} . Then, cap $\partial_1 A$ in N_{Δ} with a disc $\tilde{\Delta}$ tangent to A along $\partial_1 A$ and so that \mathcal{L} induces a foliation by circles with one center. Now, we take $D_1 = A \cup \tilde{\Delta}$.

The last step consists of choosing a collar $N_1 = D_1 \times [0, 1]$ of D_1 in M, that exists in M (on the side of A^+) so that \mathcal{L} induces on each $D_1 \times \{t\}$ a foliation by circles. Therefore, by formula (4.3) the isotopy is compatible with \mathcal{L} . And hence, $\psi_{D_1}(\mathcal{L}_{\tau}) = \mathcal{L}$.

4) This is a formal consequence of items 2) and 3).

We have seen that a Dehn modification adapted to $(\mathcal{F}, \mathcal{L})$ allows us to carry some structures, like f or ℓ , from M to M_{τ} . But an isotopy, for instance along a saturated set, cannot be carried. Fortunately, the last item of Proposition 4.4 allows us to carry the property that the pair $(\mathcal{F}, \mathcal{L})$ is made of isotopic foliations. This is the reason why we introduce the notion of *weak isotopy*.

Definition 4.5. A weak isotopy of \mathcal{L} to \mathcal{F} in M is an alternating finite sequence of Dehn modifications—right or left—and isotopies, $\mathcal{M}_1, (\psi_1^t), \ldots, \mathcal{M}_k, (\psi_k^t), t \in [0, 1]$, and a sequence of pairs of foliations, $(\mathcal{F}, \mathcal{L}), (\mathcal{F}_1, \mathcal{L}_1), \ldots, (\mathcal{F}_k, \mathcal{L}_k)$, fulfilling the following recursive conditions for every integer $j \in [1, k]$.

- (i) \mathcal{M}_j is a Dehn modification adapted to the pair of foliations $(\mathcal{F}_{j-1}, \psi_{j-1}^1(\mathcal{L}_{j-1}));^{14}$
- (ii) \mathcal{M}_j carries \mathcal{F}_{j-1} to \mathcal{F}_j and $\psi_{j-1}^1(\mathcal{L}_{j-1})$ to \mathcal{L}_j ;

(iii)
$$\mathcal{F}_k = \psi_k^1(\mathcal{L}_k).$$

In this language, Theorem (A') reduces to: \mathcal{F} and \mathcal{L} are weakly isotopic.

Now is the announced application of Dehn modification dealing with connecting orbits of saddles. One should first mention that, as for a one-parameter family of functions on a Riemannian manifold, generically on the metric the vector field $\nabla_{\mathcal{L}} f$ has finitely many orbits connecting two saddles; moreover there are connecting orbits neither from a max-saddle inflection to a saddle nor from a saddle to a min-saddle inflection. We now include this property in the definition of *excellence* and assume it in the remainder of this paper. We recall that, by Proposition 3.14, we may assume that no saddles are of type X.

Proposition 4.6. In this settling there exists a weak isotopy of ℓ to a new function ℓ' in M such that all of its Y-saddles are located above its λ -saddles with respect to the order of their f-values.

Proof. Without X-saddles the type of saddles is constant on every saddle arc. The two ends of such an arc are inflections of the same Morse index:¹⁵ if this index is 0 (resp. 1) the saddles are of type λ (resp. Y) as it can be seen near the inflection.

The matter is to destroy the \mathcal{L} -gradient connecting orbits $Y \to \lambda$. Indeed, assume the metric is chosen so that there are no such connections. Consider a level z_0 above all λ -saddles and a contact arc α of Y-saddles. By assumption on the metric, the ascending saturated set $S_{z_0}(\alpha)$

¹⁴Here, $(\mathcal{F}_0, \mathcal{L}_0) = (\mathcal{F}, \mathcal{L}).$

 $^{^{15}}$ For the index of an inflection see Subsection 2.1.

avoids every closure of λ -saddle arcs. Of course, $S_{z_0}(\alpha)$ could meet another Y-saddles not belonging to α or maxima and inflections of Morse index 1. This does not matter, all these contacts have to be incorporated to $S_{z_0}(\alpha)$. An isotopy along $S_{z_0}(\alpha)$ realizes what is wished for α , and so on for the other Y-saddle arcs.

To bypass the connecting orbits we use Dehn modifications. At the beginning, we have finitely many connecting orbits $Y \to \lambda$ lying in distinct leaves L_1, L_2, \ldots, L_n where each $L_i = \ell^{-1}(y_i)$ is a generic leaf. There exists $\varepsilon > 0$ such that $\hat{L}_i := \ell^{-1}([y_i - \varepsilon, y_i + \varepsilon])$ is still made of generic leaves.

The connecting orbit on L_i goes from Y-saddle s_i to λ -saddle s'_i . Denote by α_i (resp. α'_i) the arc of contacts that contains s_i (resp. s'_i). Set $\hat{\alpha}_i := \hat{L}_i \cap \alpha_i$ and $\hat{\alpha}'_i := \hat{L}_i \cap \alpha'_i$; all are named saddle intervals. If ε is small enough there exists η such that every interval in the collection $(f(\hat{\alpha}_i), f(\hat{\alpha}'_j))_{i,j\in[1,n]}$ has a length less than η and every two of them have a mutual distance larger than η . In addition, one may require, for every $j \in [1, n]$ and every pair (a, b) of distinct contact points in a same leaf from \hat{L}_j ,

(4.6)
$$|f(a) - f(b)| > 2\eta.$$



FIGURE 13. T'' is marked by $S_{z_1}(\hat{\alpha}'')$ where $\hat{\alpha}''$ is a saddle interval of type λ in \hat{L}_1 and $f(\hat{\alpha}'') > f(\hat{\alpha}'_1)$.

Say $f(s_1) > f(s_i)$ for every i > 1. Cut \hat{L}_1 at the level $z_1 := f(s'_1) - \eta$. By condition (4.6), this level set of \hat{L}_1 is a finite collection of annuli. Among them, call A_1 the one which meets the connecting orbit $s_1 \to s'_1$. The foliation \mathcal{L} induces on A_1 a foliation by circles; the central circle lies in the leaf L_1 . Knowing that s_1 is of type Y and s'_1 of type λ one sees in A_1 two arcs T and T' joining the two boundary circles: $T = A_1 \cap S_{z_1}(\hat{\alpha}_1)$ and $T' = A_1 \cap S_{z_1}(\hat{\alpha}'_1)$. Here, the first satured set is ascending and the second one is descending. Moreover, $T \cap T'$ is the point where the connecting orbit crosses A_1 —because the connecting orbits are on isolated leaves.

Applying a Dehn twist τ to T'—left or right depending on the sign of intersection $\langle T, T' \rangle$ together with the corresponding Dehn modification to M kills the connecting orbit since $\tau(T') \cap T = \emptyset$.

However, new connecting orbits could have been created in the foliated domain L_1 by this first Dehn modification (this is the case in Figure 13 due to $\tau(T'') \cap T \neq \emptyset$.) One solves this new difficulty as follows.

Look at all saddle intervals of type λ in \hat{L}_1 at level higher than z_1 . Name them $\hat{\beta}_1, ..., \hat{\beta}_k$ such that $f(\hat{\beta}_1) < f(\hat{\beta}_2) < \cdots < f(\hat{\beta}_k)$.¹⁶ The number of them is not affected by forthcoming Dehn modifications, it is kept invariant (Remark 4.2.)

Say that $\hat{\beta}_1 \subset \hat{L}_1$ is the lowest saddle interval of type λ being \mathcal{L}_{τ} -gradient connected to $\hat{\alpha}_1$; if there is no such a connecting orbit, consider $\hat{\beta}_2$, and so on. Take a proper annulus $A'_1 \subset \hat{L}_1$, at the level $z'_1 := \inf f(\hat{\beta}_1) - \eta$, that crosses a connecting orbit from $\hat{\alpha}_1$ to $\hat{\beta}_1$.

The figure is similar to Figure 13, but $S_{z'_1}(\hat{\beta}_1)$ marks a short arc $T'(\hat{\beta}_1)$ joining the two boundary components of A'_1 while $S_{z'_1}(\hat{\alpha}_1)$ marks a long arc like $\tau^{-1}(T)$. These two arcs are transverse to the foliation $\mathcal{L}_{\tau} \cap A'_1$ and, up to isotopy, there is only one intersection point between them. One can repeat the trick of Dehn modification to cancel the connection $\hat{\alpha}_1 \to \hat{\beta}_1$. Of course, new connecting orbits are possibly created, but only from $\hat{\alpha}_1$ to $\hat{\beta}_j$, j > 1. So, the process goes on, without increasing the complexity.

One can destroy all connecting orbits from $\hat{\alpha}_1$ to $\hat{\beta}_1 \cup \cdots \cup \hat{\beta}_k$. As a result, after this long series of Dehn modifications, the ascending saturated set $S_{z_0}(\hat{\alpha}_1)$ avoids every λ -saddle, where z_0 has been chosen at the very beginning of the proof. If the saddles $(s_i)_{i=1}^n$, from which one connecting orbit goes up in the leaf L_i , are ranked in the decreasing order of their *f*-values $(f(s_i))_1^n$, the process continues with the interval $\hat{\alpha}_2$ and so on.

5. Proof of Theorem (A')

5.1. So far, we got that every saddle is of type λ or Y—that is, no X-saddle— and, up to a weak isotopy applied to the function ℓ , every Y-saddle is located at a level higher to all λ -saddles. Recall that an inflection adhering to a contact arc of λ -saddles (resp. Y-saddles) is of type saddle-min (resp. saddle-max). Say that the λ -saddles (resp. le Y-saddles) are located in $\{0 < f < 1/2\}$ (resp. $\{1/2 < f < 1\}$.)

A way to achieve the proof of Theorem (A') is, by Lemma 1.3, to kill every negative contact of the pair (f, ℓ) . This requires some algorithm since the leaves of \mathcal{L} could have a very tricky topology as embedded surface in $S^2 \times [0, 1]$. In [5], this algorithm was based on words in an alphabet with four letters. Here, it is much simpler; it is based on two topological concepts: *basin* and *co-basin* that we are going to define.

Definition 5.2. Let $L = \{\ell = u_0\}$ be a leaf of \mathcal{L} and let $m \in L$ be a negative contact of index 0. The basin of L determined by m is the maximal 3-ball B(m) in $M = S^2 \times [0,1]$ whose boundary is made of two parts: the first part is a horizontal disc $\partial_{\mathcal{F}} B(m)$ whose boundary curve has exactly one negative λ -saddle if L is a generic leaf or one of the following two situations: a negative inflection in $\partial_{\mathcal{F}} B(m)$ or two saddles one of them being negative; the second part is a disc $\partial_{\mathcal{L}} B(m)$ in L with m as unique negative center and no other negative contact with \mathcal{F} in the interior of $\partial_{\mathcal{L}} B(m)$. Its threshold consists of every negative saddle—generically unique—in the boundary curve of $\partial_{\mathcal{L}} B(m)$.

In contrast, $\partial_{\mathcal{L}} B(m)$ may have many positive contacts. If z_0 denotes the level of $\partial_{\mathcal{F}} B(m)$ and u_0 denotes the ℓ -value of the negative saddle(s), we have $B(m) \subset \{f \leq z_0\} \cap \{\ell \leq u_0\}$.

¹⁶Inequalities of intervals mean that their are disjoint and ordered as it is written.

Definition 5.3. A basin B(m) being given as above, a co-basin joint to B(m) is a maximal ball B^* in $\{f \leq z_0\} \cap \{\ell \geq u_0\}$ whose boundary is made of two parts: the first one is a horizontal disc $\partial_{\mathcal{F}} B^*$; the second one is a disc $\partial_{\mathcal{L}} B^*$ in the \mathcal{L} -boundary of B(m). Its threshold is the unique¹⁷ λ -saddle which has only one descending separatrix lying in $\partial_{\mathcal{L}} B^*$ and the other in $\partial_{\mathcal{L}} B(m) \setminus B^*$.

Somehow, a co-basin is like a pocket with respect to a basin. Many co-basins could be glued to a basin. In our setting where the Y-saddles are above all λ -saddles, $\partial_{\mathcal{L}}B^*$ cannot have maxima.



FIGURE 14. I_0 is a birth positive saddle-min inflection; α and β are respectively the saddle and the min locus emanating from I_0 ; γ and γ' form the boundary of the unstable manifold of I_0 . The saddle s is negative.

5.4. LOWER COMPLEXITY. Given a pair (f, ℓ) of functions without critical points, we define its *lower complexity* $\kappa^{-}(f, \ell)$ as the pair (μ, ν) of non-negative integers arranged in lexicographical order and defined in the following way.¹⁸

The entry μ is the number of contact arcs of negative minima.

Let I_0 be the *upper*, unique by excellence, negative inflection which is the *birth* of an arc β of negative minima. Every minimum $m \in \beta$ defines a unique basin. A minimum $m \in \beta$, distinct from a cancellation inflection, is said to be *accidental* if one of the following cases happens:

- the boundary of $\partial_{\mathcal{F}} B(m)$ contains either two saddles or a negative inflection—necessarily a cancellation inflection—by definition of I_0 ,
- $-\partial_{\mathcal{L}}B(m)$ contains a positive inflection.

By definition, the entry ν of the lower complexity is the number of accidental minima in β .

If $\mu = 0$ then there are neither negative minima nor negative λ -saddles anymore and, by convention, $\kappa^{-}(f, \ell)$ is equal to (0, 0); then the set of negative contacts is empty in $\{f < 1/2\}$.

 $^{^{17}}$ The uniqueness of its threshold holds since there are no X-saddles (compare Figure 7.)

¹⁸This pair is meant to measure the topological complexity of \mathcal{L} in the domain $\{0 < f < 1/2\}$.

Similarly, some upper complexity, $\kappa^+(f, \ell)$, is defined that measures the topological complexity of \mathcal{L} in the domain $\{1/2 < f < 1\}$. Its vanishing means that there are no negative contacts in $\{f > 1/2\}$. By symmetry, any result about negative λ -saddles/minima applies to negative Y-saddles/maxima—and conversely. Therefore we are focusing on the first case.

Lemma 5.5. By the choice of I_0 , denoting by β the arc of minima that I_0 generates, the following holds.

- (1) Let B^* be a co-basin joint to a basin B(m) with $m \in \beta$. Every (piece of) leaf in the interior of B^* contains no negative minimum.
- (2) The positive contact arcs descending from contact points in the interior of $\partial_{\mathcal{L}} B^*$ descends to an inflection in the interior of B(m). In particular, m is the absolute minimum of the basin that it defines.

Proof. 1) Let L be a leaf passing through B^* and m' be a negative minimum contact in $L \cap B^*$. The contact arc β' passing through m' cannot cross $\partial_{\mathcal{L}}B^*$ which only contains positive contacts. So, β' descends to an inflection in B^* . But such an inflection should be higher than m, the negative minimum of B(m), that itself is higher than I_0 . This contradicts the choice of I_0 .

2) Suppose such an arc γ crosses $\partial_{\mathcal{L}} B(m)$ twice. The integral of $d\ell$ on γ , oriented in the direction of decreasing z, is positive. This prevents γ from having two distinct points on the same leaf. Hence the lower point of γ is an inflection in the interior of B(m).

The topic of the next lemma is to clean up the basins we are going to deal with in the proof of Theorem (A'), namely to make sure that no co-basins are joint to them. Again, I_0 is the *upper* birth negative saddle-min inflection and β is the contact arc of minima that I_0 generates.

Lemma 5.6. In this setting, let I' be a positive birth saddle-min inflection located in $\partial_{\mathcal{L}} B(m)$ for some $m \in \beta$. Then there exists a contact conjugating¹⁹ isotopy Φ_t , $t \in [0, 1]$, fulfilling the following:

(i) the isotopy is supported in $\{0 < f < 1/2\}$ and keeps the negative contacts fixed;

(ii) $\Phi_1(I')$ is away from every so-called I_0 -basin, that is defined by some minimum in β .

After having iterated such isotopies, every I_0 -basin is free of positive contact.

Proof. For the first part we recall that when the negative threshold of a basin goes down on its own contact arc, meanwhile the positive threshold of the joint co-basin generated by I' goes up on its own contact arc. Hence, there is a level $z_{I'}$ which is the common level of both the threshold of some basin B(m'), $m' \in \beta$, and the threshold of the joint co-basin $B^*_{I'}$ generated by I'.

So, it is natural to consider the ascending saturated set $\Sigma' := S_{z_{I'}+\varepsilon}(I')$ for some small enough ε . Here, we recall that every ascending separatrix of a λ -saddle reaches the level $\{z = 1/2\}$ except if it is bounded from above by a cancellation inflection or another λ -saddle—which will be incorporated to Σ' .

Claim. Σ' does not approach the negative contacts.

Indeed, by Lemma 5.5, there is no negative contact locus in the interior of Σ' —what is equivalent to the interior of a co-basin. What about α , the locus of saddles emanating from I_0 when

¹⁹A diffeomorphism Φ of $S^2 \times [0,1]$ is said to be contact conjugating if Φ maps every contact point of the pair $(\mathcal{F}, \mathcal{L})$ to a contact point of the pair $(\mathcal{F}, \Phi(\mathcal{L}))$ and conversely.

the level is close to $z_{I'}$? Let s' denote the threshold of the basin B(m') and let s'(z) denote that nearby threshold at level z close to $z_{I'}$. Since the crossing can happen, certainly s'(z) has no connecting orbits with points of Σ' if ε is small enough and z ranges in $(z(I') - \varepsilon, z(I') + \varepsilon)$.²⁰ So, for such a z, the negative saddle s'(z) in not in the closure of Σ' .

By Lemma 3.11, there is an isotopy along Σ' that pushes it above the level z(I'). Its support is located in a neighborhood of Σ' . The time one of this isotopy fulfills the demand.

About the last claim, it is sufficient to recall that there are only finitely many such positive birth inflections. Let I'' be one of them.²¹ One can consider its saturated set Σ'' with respect to $\Phi_1(\mathcal{L})$ up to a level which is determined as previously by the crossing argument. An isotopy along Σ'' pushes I'' away from all basins and does not destroy what was gained in the first step. So, cumulating the isotopies related to each positive inflection that is initially contained in a basin makes all basins $B(m), m \in \beta$, free from positive contacts. The order in which the concerned inflections are numbered is irrelevant.

Now we are ready for the decisive part of the proof of Theorem (A'). It will consists of decreasing the lower complexity $\kappa^{-}(f, \ell)$ until it vanishes. By the symmetry $z \to 1 - z$, the same holds true about the upper complexity.

Proposition 5.7. If the set of negative minima contacts of the pair (f, ℓ) is non-empty there exists an isotopy supported in $\{0 < f < 1/2\}$ that carries ℓ to a function ℓ' such that

(5.1)
$$\kappa^{-}(f,\ell') < \kappa^{-}(f,\ell).$$

Proof. There are several cases depending on how the complexity is topologically made.

1) Assume $\mu > 0$ and $\nu = 0$. We continue with the upper negative birth saddle-min inflection I_0 and the arc β of index 0 that it generates. By Lemma 5.6, every basin B(m), $m \in \beta$, is free of positive contact. In this case, for every minimum $m \in \beta$ the cone $C_{\mathcal{L}}(m)$ is standard.²²

If the negative λ -saddle arc α emanating from I_0 and the minimum arc β close up in a common cancellation inflection point then the union $\alpha \cup \beta$ forms a *simple loop* in the sense of Definition 3.2. By Proposition 3.3 there is an isotopy supported in a neighbourhood of $\bigcup_{m \in \beta} C_{\mathcal{L}}(m)$ that eliminates this simple loop and decreases μ by 1.

Let us show that there are no other configurations with $\nu = 0$. Let J_1 be the inflection ending α ; let I_1 , distinct from J_1 , be the inflection ending β and let I_2 be the birth inflection of the λ -arc, named α' , descending from I_1 . We have $f(I_0) > f(I_2)$ by definition of I_0 (Figure 15.) Moreover, $f(J_1) > f(I_1)$; if not, the unique separatrix converging to J_1 starts from a minimum $m_1 \in \beta$, and hence $\nu > 0$.

When $\nu = 0$ and $f(J_1) > f(I_1)$, every $m \in \beta$ ends one separatrix descending from some saddle $s(m) \in \alpha$ and one separatrix descending from some saddle $s'(m) \in \alpha'$. The latter two

²⁰Indeed, for $z = z_{I'}$, the threshold has a separatrix coming from m' for some choice of the Riemannian metric. Hence, the same holds in nearby basins.

²¹Possibly, I" belongs to Σ' . Hence, $\Phi_1(I'')$ is already away from any basin and hence this inflection can be skipped.

 $^{^{22}}$ See Subsection 3.1 for notation.

claims prove similarly: the existence of s(m) is clear near the inflection I_0 and extends to β since no accident happens along this arc; the existence of s'(m) is clear near I_1 and also extends to β for the same reason.

When m is close to I_0 one has f(s'(m)) > f(s(m)) and when s'(m) is close to I_1 one has f(s'(m)) < f(s(m)). Then there is $m_0 \in \beta$ such that

(5.2)
$$f(s(m_0)) = f(s'(m_0)).$$

Hence, $C_{\mathcal{L}}(m_0)$ is in the saddle-center-saddle configuration (Definition 3.4) and ν is positive. Contradiction.



FIGURE 15. The horizontal lines stand for level sets.

2) We assume $\nu > 0$ in the rest of the proof. Consider the first accidental minimum m_0 in β when coming from I_0 and assume that the curve $\partial C_{\mathcal{L}}(m_0)$ contains two saddles s_0 and s'_0 , necessarily negative by the "cleaning" Lemma 5.6. Then we have the saddle-center-saddle configuration; item (3) from Definition 3.4 is fulfilled by the choice of I_0 and the cleaning of the I_0 -basins.



FIGURE 16. Bypass creating a simple loop.

Let α and α' denote the contact arcs containing s_0 and s'_0 respectively; let s_t , s'_t and m_t denote local parametrizations of α , α' and β respectively near t = 0 such that these three contact points are in the same leaf for every t close to 0. If α comes from I_0 and knowing that m_0 is the first accidental point on β , one has $f(m_t) < f(s_t) < f(s'_t)$ for every t < 0. In other words, the f-values of s_t and s'_t are ordered as in Figure 2. Thus every saddle in the sub-arc $(I_0, s_0) \subset \alpha$ is cancellable with the corresponding minimum in β ; for every small t > 0, the pair (s'_t, m_t) can be cancelled. Proposition 3.6 applies and its effect is shown in Figure 4. In particular, this *bypass* creates a simple loop which contains I_0 and a new birth inflection I_1 with the following properties:

- $f(I_1) > f(I_0)$.
- The minimum arc β' emanating from I_1 has less accidental minima than β since it coincides with β in $\{f > f(I_1) + \varepsilon\}$ for ε small enough (Figure 16.)

Once the simple loop has been cancelled the number of birth inflections remains equal to the initial μ but the entry ν of the lower complexity decreases by 1: the arc β' has one less accidental minimum than β .



FIGURE 17. Bypass in configuration min-saddle-min.

3) With the same notation, we assume $\nu > 0$ and the first accidental minimum $m_1 \in \beta$ has a cone with an inflection point J_1 in its boundary. This inflection cannot be a birth since it lies at a higher level than I_0 . So, this is a cancellation inflection, which is a negative contact due to the cleaning Lemma 5.6. Let β denote the index-0 arc starting from I_0 and let I_1 be its upper end. We have $I_1 \neq J_1$ since an arc that is transverse to \mathcal{L} cannot meet the same leaf twice. Here there are two cases.

3-1) Assume that the saddle arc α starting from I_0 ends at J_1 . Here, we are in the configuration min-saddle-min (Definition 3.7). Let β' be the arc of index 0 descending from J_1 . The isotopy from Proposition 3.8 modifies the contact arcs as, including their names, it is shown in Figure 6. The outcome is a simple loop containing I_0 and an arc of minima. After the simple loop has been cancelled, the number of negative birth saddle-min inflections has decreased by one and hence the complexity decreases (Figure 17.)

3-2) Assume that α does not end at J_1 . Let α' and β' respectively denote the saddle arc and the minimum arc ending at J_1 . The arcs (β, α', β') and the leaf L that contains $m_0 \in \beta$ slightly below m_1 also presents a configuration min-saddle-min. The effect of the isotopy from Proposition 3.8 does not produce a simple loop that could be cancelled. But it keeps I_0 as the upper negative birth inflection and the index-0 arc $\check{\beta}$ emanating from I_0 contains no accidental



FIGURE 18. Variant of Figure 17.

minimum. Therefore, the complexity decreases from (μ, ν) to $(\mu, 0)$ (Figure 18.)

When the two complexities vanish, no negative contacts are remaining and the Moser trick (Lemma 1.3) completes the proof of Theorem (A'). \Box

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