Conference Board of the Mathematical Sciences
REGIONAL CONFERENCE SERIES IN MATHEMATICS

supported by the
National Science Foundation

Number 43

LECTURES ON THREE-MANIFOLD TOPOLOGY

by

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CONTENTS

PREFACE

This manuscript is a detailed presentation of the ten lectures given by the author at the Conference Board of Mathematical Sciences Regional Conference on Three-Manifold Topology, held October 8-12, 1977, at Virginia Polytechnic Institute and State University. The purpose of the conference was to present the current state of affairs in three-manifold topology and to integrate the classical results with the many recent advances and new directions.

The ten principal lectures presented here deal with the study of three-manifolds via incompressible surfaces. At the conference, these lectures were supplemented by eight special lectures: two lectures each by Professor Joan Birman of Columbia University (Heegaard Theory), Professor Sylvain Cappell of the Courant Institute (branched coverings and applications to 4-dimensional topology), Professor Robion Kirby of The University of California/Berkeley (three-dimensional knot theory) and Professor William Thurston of Princeton University (three-dimensional hyperbolic geometry).

I wish to thank the host university, the members of the Department of Mathematics at V.P.I. & S.U., and the Conference Director, Professor Charles Feustel. A special thanks to the Conference Coordinator, Professor Ezra 'Bud' Brown.

I believe that the conference was a success and that much of the success was due to the special lectures given by the previously named participants. I wish to acknowledge my thanks to Professors Birman, Cappell, Kirby and Thurston.

PREFACE

In the preparation of my lectures I had many helpful discussions with my close colleagues, Benny Evans, John Hempel and Peter Shalen. Their help has been extended to the completion of the writing of these lecture notes. In the preparation of the manuscript, I have been assisted by John Rice, the staff at The Institute for Advanced Study, especially Elizabeth Gorman Moyer and Gail Sydow, and at Rice University by Anita Poley.

I dedicate this manuscript to my wife, Linda, who has been understanding and constantly supportive over the long period of time that I have spent in its preparation.

William Jaco

Houston, Texas

July, 1979

INTRODUCTION

This manuscript is intended to present the development of three-manifold topology evolving from the study of incompressible surfaces embedded in three-manifolds. This, of course, is quite a restriction to come under the broad title of the manuscript. But even here, the reader will find many important aspects in the theory of incompressible surfaces missing. I am not trying to make the manuscript all inclusive (I do not believe that a possible task) and I have not tried to make the bibliography complete. The manuscript is exactly what I would do in ten lectures with the above intention.

The reader will find the subject, through the first six chapters, overlappling with the book by John Hempel, [He_1]. I have very high regard for Hempel's book and debated a bit about presuming its contents. However, I decided that this manuscript would better serve if the development began more at the foundations. And in the end, I believe that the reader will find the overlap mostly in spirit and terminology. I have developed the material from my point of view and I give a number of new proofs to the classical results. If I needed material that appears in Hempel's book, and if I felt that the development there was consistent with my development, then I refer the reader to the appropriate result. While this manuscript certainly can be considered independent of Hempel's book, it can also be considered as a sequel to it.

In Chapter I, I wanted to give a unified proof of the Loop Theorem - Dehn's Lemma and the Sphere Theorem, using equivarient surgery. I also had another motive for this approach; namely, to prove that the universal covering space of an orientable, irreducible 3-manifold is itself irreducible. I was only able to carry through equivarient surgery in the limited case of two-sheeted coverings (involutions); hence, I give an equivarient surgery proof of the Loop-Theorem and Dehn's Lemma; but I give no new information on the Sphere Theorem. Since I presented these lectures, W. Meeks and S. T. Yau have given unified proofs of the Loop Theorem - Dehn's Lemma and the Sphere Theorem, using minimal surfaces to accomplish equivarient surgery. Their methods also show that the universal covering space of an orientable, irreducible 3-manifold is irreducible.

In Chapter II the main result is the Prime Decomposition Theorem (II.4) for compact, orientable 3-manifolds. Here, the classical proof of existence is due to H. Kneser [Kn_1] and is a very intriguing proof. However, a more natural approach is an argument based on reasoning by induction. I present such an argument using a theorem of W. Haken [Ha_1], which states that a closed, orientable 3-manifold admitting a connected sum decomposition, admits a connected sum decomposition by closed, orientable 3-manifolds having strictly smaller Heegaard genus (II.7). Even here I give a new proof of Haken's Theorem, which is surprisingly easy and, moreover, is really a 2-dimensional argument.

In Chapter III, I give the definition of an incompressible surface and of a Haken-manifold. I give some sufficient (and some necessary) conditions for the existence of an incompressible surface in a 3-manifold. I have included a number of examples, many of which I use later in the manuscript. Here, however, the main result is the so-called Haken-Finiteness Theorem (III.20). It gives a finiteness condition for collections of pairwise disjoint, incompressible surfaces embedded in a Haken-manifold. It is indespensible to the approach of this author and I have presented it in both detail and generality. I also present a convenient generalization due to Peter Shalen and myself.

I believe the reader will find that Chapter IV is a fun chapter. It is a very important chapter; since the existence of a hierarchy for a Haken-manifold provides an inductive method of proof, which has been a major tool employed in the study of this important class of 3-manifolds. But, in this chapter I introduce the notion of a partial hierarchy and give some fun examples of infinite partial hierarchies for compact 3-manifolds. I also define the (closed) Haken number of a compact 3-manifold and the length of a Haken-manifold; and I discuss different inductive methods of proof (advantages and perils). I end Chapter IV with Theorem IV.19, where I prove that any Haken-manifold has a hierarchy of length no more than four — a result that I have never been able to use.

Chapter V is an abbreviated version of what might have been a revision of my Princeton Lecture Notes on the structure of three-manifold groups. I first indicate the restrictive nature of three-manifold groups by classifying the abelian three-manifold groups. I present the Scott-Shalen Theorem (V.16) that any finitely generated three-manifold group is finitely presented. This is done using the idea of indecomposably covering a group, which I like very much. However, a large part of Chapter V is devoted to open questions about three-manifold groups and properties for three-manifold groups; e.g., the finitely generated intersection property for groups, which is important in later chapters (particularly, Chapter VII).

Chapter VI is in some sense the beginning of the new material. Here I present new results about Seifert fibered manifolds. Namely, I prove that if a finite sheeted covering space of a Haken manifold M is a Seifert fibered manifold, then M itself is a Seifert fibered manifold (VI.29) and I develop the topological study of Seifert fibered manifolds needed in the later chapters to present the recent work of Shalen and myself, Waldhausen and Johannson. Also, I present a proof of the Gordon-Heil prediction that a Haken-manifold, having an infinite cyclic, normal subgroup of its fundamental group, is a Seifert fibered manifold (VI.24) and a description of compact, incompressible surfaces in Seifert fibered manifolds (VI.34). I hope that this chapter on Seifert fibered manifolds will serve as an introduction to this important class of 3-manifolds for the beginners in the subject and provide some enjoyable reading for the more advanced.

I begin Chapter VII by giving general conditions that are sufficient for a noncompact 3-manifold to admit a manifold compactification. The basic result here (VII.1) is after T. Tucker's work [Tu_1]. I give a new proof of J. Simon's Theorem [Si_1] that the covering space of a Haken-manifold corresponding to the conjugacy class of a subgroup of the fundamental group, which has the finitely generated intersection property, admits a manifold-compactification to a Haken-manifold (VII.4). This result is then applied to covering spaces corresponding to finitely generated peripheral subgoups (finitely generated, peripheral subgroups have the finitely generated intersection property (V.20), [J-M] and [J-S_2]) and to covering spaces corresponding to the fundamental groups of well-embedded submanifolds. This latter material is based on work of B. Evans and myself and allows certain isotopes that are used in all the later chapters. While this work on compactifications is itself very important to me, its use here provides the foundation for the existence and uniqueness of the characteristic pair factor of a Haken-manifold pair. This is a new presentation of the main results of the paper [J-S_2]. I use covering space arguments rather than the formal homotopy language used there. I believe that the reader should find the material here much more intuitive. I have not, however, covered many of the surprising results obtained in that manuscript. Time and space would not allow this. I am disappointed in not doing so; but if the reader has interest after completing Chapter VII, I believe revisiting [J-S_2] may be a more pleasant experience.

In Chapter VIII, I present a major part of the joint work of Shalen and myself [J-S_1]. My approach here follows very much the lines of our original approach. While this approach is similar to the presentation in [J-S_1], I do not go into the generalities of that manuscript (thereby, I considerably reduce the notation and length of the presentation) and I have set up a different foundation with the material of Chapter VII. I give proofs of the Essential Homotopy Theorem (VIII.4), the Homotopy Annulus Theorem (VIII.10) and the Homotopy Torus Theorem (VIII.11) and the Annulus-Torus Theorems (VIII.13 and VIII.14).

I believe that the presentation of Chapter IX will introduce the reader to an understanding of the characteristic Seifert pair of a Haken-manifold with a minimum amount of work. In fact, this chapter is really very short. By introducing the idea of a perfectly-embedded Seifert pair in a 3-manifold, I am able to show (under the partial ordering that one perfectly-embedded Seifert pair (Σ',Φ') is less than or equal to another perfectly-embedded Seifert pair (Σ,Φ) if there is an ambient isotopy taking Σ' into Int Σ and Φ' into Int Φ) that a Haken-manifold with incompressible boundary admits a unique (up to ambient isotopy) maximal, perfectly embedded Seifert pair. This unique maximal, perfectly-embedded Seifert pair is the characteristic Seifert pair. The bulk of Chapter IX is in giving a detailed proof of Theorem IX.17, which allows a characterization of the characteristic Seifert pair via homotopy classifications of maps. I conclude Chapter IX with examples of the characteristic Seifert pair of some familiar 3-manifolds and some examples for caution.

I think it is fair to say that I worked the hardest for the results of Chapter X. I guess what is good about this is that I am pleased with the outcome. For here, I present the generalized version of Waldhausen's Theorem on deforming homotopy equivalences (X.7); hence, I have answered conjecture 13.10 of [He_1] affirmatively and given a complete proof of this version of the theorem, which is due to T. Tucker (X.9). Moreover, I have been able to obtain a new proof of the beautiful theorem due originally to K. Johannson [Jo_2] on the deformations of homotopy equivalences between Haken-manifolds with incompressible boundary (X.15 and X.21). This proof is inspired by observations of A. Swarup (VII.22 and VII.24). I also give many examples of "exotic" homotopy equivalences. Most of these examples are well-known.

I have used standard terminology and notation except for the new terms that are introduced; of course, here I give the required definitions and describe the new notation. The reader will find that I have included a lot of detail. However, if there were items that I felt might improve the presentation and I did not want to complete the detail, I set such items off as Exercises. This is in contrast to items labeled as Question; in the case of a question, I simply do not know the answer.

CHAPTER I. LOOP THEOREM-SPHERE THEOREM: The Tower Construction

In this chapter I discuss the Loop Theorem, Dehn's Lemma, the Sphere Theorem and some of their more interesting generalizations. There is a very nice account of these theorems given in the book by John Hempel [He_1]. In particular, one can find there complete proofs that are quite readable.

There are a few reasons why I have chosen to begin these lectures at this point. The fundamental importance of these theorems to the methods of three-manifold topology, and particularly to the approach of this author, cannot be overestimated. I plan to give a modified version of the classical method of proof. The approach that I use here, to prove the Loop Theorem,can be used to give a unified proof of Dehn's Lemma and the Loop Theorem with their generalizations after Shapiro-Whitehead [S-W] and Waldhausen [W_1], respectively. I have here a forum to make a case for a new proof of the Sphere-Theorem using only the Loop-Theorem (or even better, a proof using equivariant surgery in a universal covering, see Chapter VI) and to present some interesting problems arising from the study of these theorems. However, certainly the main pursuasive for starting at this point is that many participants at these lectures are not familiar with the techniques of 3-manifold topology and are here to gain a working knowledge for the study of problems in this area. With this in mind, there is no better place to start.

The results of this chapter can be considered as the first step in the program of studying singular mappings of surfaces into 3-manifolds

For here a singular mapping of a planar surface (a surface is planar if it can be embedded in $\mathbb{R}^2$, the plane) or of a sphere into a 3-manifold is replaced by a non-singular mapping, while preserving certain prescribed conditions. Of first importance, however, is the case when the surface is one of the basic 2-elements, the 2-disk or the 2-sphere.

The version of the Loop Theorem given here is after J. Stallings [St_1

I.1. LOOP THEOREM: Suppose that M is a 3-manifold, S is a connected surface in ∂M and N is a normal subgroup of $\pi_1(S)$. Let $f : \mathbf{D}^2 \to M$ be a map such that $f(\partial\mathbf{D}^2) \subset S \subset \partial M$ and $[f \mid \partial\mathbf{D}^2] \notin N$. Then there exists an embedding $g : \mathbf{D}^2 \to M$ such that $g(\partial\mathbf{D}^2) \subset S \subset \partial M$ and $[g \mid \partial\mathbf{D}^2] \notin N$.

To obtain a possibly better understanding of I.1 consider the following, otherwise not so obvious, consequences.

I.2. COROLLARY: Suppose that M is a 3-manifold and that S is a connected surface in ∂M. Set K equal to the normal subgroup $\ker\{\pi_1(S) \hookrightarrow \pi_1(M)\}$. If $K \neq \{1\}$, then a nontrivial element of K can be represented by a simple closed curve (s.c.c.) in S.

Proof: One simply applies I.1 in the case $N = \{1\}$. ∎

Contrast this to the fact that many normal subgroups of $\pi_1(S)$ have no such element. Indeed, this is already the case for $S = S^1 \times S^1$ and any normal subgroup of $\pi_1(S)$ generated by a nontrivial, non-primitive element. In fact, it is a very interesting question as to when the kernel of a homomorphism that is induced by a mapping between two closed, orientable surfaces needs to contain such an element (see [Ed_1] and [Tu_2]).

I.3. COROLLARY: Suppose that M is a 3-manifold and that S is a *compact surface in* ∂M. *Set* $G = \mathrm{Im}(\pi_1(S) \hookrightarrow \pi_1(M))$. *Then* $G \approx F * G_1 * \ldots * G_k$ *where* F *is a free group and* $G_i \approx \pi_1(S_i)$ *for some closed surface* S_i $(1 \leq i \leq k)$.

Proof: One simply applies I.1 to prove that $K = \ker(\pi_1(S) \hookrightarrow \pi_1(M)$ is normally generated as a subgroup of $\pi_1(S)$ by a finite, pairwise disjoint collection of simple closed curves in S. ■

I.4. EXERCISE: Show I.3 is true in the case where S is not necessarily compact. Of course, the conclusion may need to be modified to admit a possibly infinite number of factors G_i.

If M is a 3-manifold, a subgroup H of $\pi_1(M)$ is *peripheral* if there exists a surface $S \subset \partial M$ such that H is conjugate in $\pi_1(M)$ into a subgroup of $\mathrm{Im}(\pi_1(S) \hookrightarrow \pi_1(M))$.

I.5. EXERCISE: Any finitely generated peripheral subgroup H of a 3-manifold group has the form $H \approx F * H_1 * \ldots * H_n$ where F is a free group and $H_i \approx \pi_1(S_i)$ for some closed surface S_i $(1 \leq i \leq n)$.

The next result is Dehn's Lemma, first formulated by M. Dehn [D_1] in 1910. However, his proof contained a serious gap which was pointed out by H. Kneser [Kn_1] in 1927. A satisfactory solution to Dehn's Lemma was given by C. D. Papakyriakopoulos in 1956 along with his versions of the Loop Theorem and the Sphere Theorem [P_1, P_2].

I.6. DEHN'S LEMMA: *Suppose that* M *is a* 3-*manifold and that* $f : \mathbf{D}^2 \to M$ *is a map such that* $f \mid \partial\mathbf{D}^2$ *is an embedding and* $f^{-1}(f(\partial\mathbf{D}^2)) = \partial\mathbf{D}^2$ (i.e. the singularities of f do not meet $\partial\mathbf{D}^2$).

Then there exists an embedding $g : \mathbf{D}^2 \to M$ *such that* $g \mid \partial\mathbf{D}^2 = f \mid \partial\mathbf{D}^2$.

Sometimes it is helpful to be aware that:

I.7. REMARK: *A contractible simple closed curve is orientation preserving*. Indeed, if two simple closed curves J and K in a 3-manifold M are homologous in M, then J is orientation-preserving iff K is orientation-preserving. This follows from another useful observation. Namely if a s.c.c. in the boundary of a 3-manifold M bounds a surface in M, then it is orientation preserving *in the boundary of* M.

I.8. REMARK: Dehn's Lemma is usually stated by saying that there exists a neighborhood A of $\partial\mathbf{D}^2$ such that $f \mid A$ is an embedding and $f^{-1}(f(A)) = A$ (i.e. that the singularities of f miss a neighborhood of $\partial\mathbf{D}^2$). Such a version clearly follows from the Loop Theorem (I.1). However, it was pointed out to me by John Hempel that the above version of Dehn's Lemma (I.6) also follows from the Loop Theorem (I.1). The proof goes like this: Let U be a small tubular neighborhood of $f(\partial\mathbf{D}^2)$ such that f is in general position with respect to ∂U. Set $M' = M - \overset{\circ}{U}$. Since there are no singularities of f on $\partial\mathbf{D}^2$, there exists a map $f' : \mathbf{D}^2 \to M'$ such that $f'(\partial\mathbf{D}^2) \subset \partial U$ and $[f' \mid \partial\mathbf{D}^2] \neq 1$ in $\pi_1(\partial U)$. Hence, by the Loop Theorem (I.1) there exists an embedding $g : \mathbf{D}^2 \to M'$ such that $g(\partial\mathbf{D}^2) \subset \partial U$ and $[g \mid \partial\mathbf{D}^2] \neq 1$ in $\pi_1(\partial U)$. I want to show that $g \mid \partial\mathbf{D}^2$ is a longitude of U. It follows that U, along with a regular neighborhood of $g(\mathbf{D}^2)$, is a punctured Lens space (allowing $S^2 \times S^1$ and S^3 both as a Lens space). Since $f \mid \partial\mathbf{D}^2$ is the core of U (and as such generates the fundamental group of the Lens space) and is trivial in M, the only possibility is that the Lens space is S^3. Therefore, $g \mid \partial \mathbf{D}^2$ is a longitude of U. The s.c.c. $g(\partial \mathbf{D}^2)$ and $f(\partial \mathbf{D}^2)$ cobound an annulus in U and so the map $f \mid \partial \mathbf{D}^2$ can be extended to an embedding of $\mathbf{D}^2$ into M.

The following theorem is often referred to as the Projective Plane Theorem. The version here is after Epstein [Ep_1] and incorporates essentially all of the other major versions [P_1, Wh_2, St_2].

I.9. SPHERE THEOREM: Suppose that M is a compact 3-manifold and N is a $\pi_1(M)$-invariant subgroup of $\pi_2(M)$. Let $f : S^2 \to M$ be a map such that $[f] \notin N$. Then there exists a covering map $g : S^2 \to g(S^2) \subset M$ such that $g(S^2)$ is two-sided in M and $[g] \notin N$.

I.10. REMARK: The covering map g in the conclusion of I.9 has image $g(S^2)$ either a 2-sphere or a projective plane. In the case of the projective plane it guarantees that $g(S^2)$ is two-sided in M (a 2-sphere is always two-sided in a 3-manifold). If the manifold M is assumed to be orientable (or at least does not admit an embedded, two-sided projective plane), then the conclusion of the Sphere Theorem is that there exists an embedding $g : S^2 \to M$ with $[g] \notin N$. The manifold $M = \mathbf{P}^2 \times S^1$ (which has the property that every embedded 2-sphere in M bounds a 3-cell in M) provides an example where the covering map $g : S^2 \to M$ must be nontrivial and therefore $g(S^2)$ is a two-sided projective plane.

I.11. REMARK: If M is an orientable 3-manifold and $\pi_2(M)$ is not trivial, then it follows from the Sphere Theorem that $\pi_1(M)$ is either infinite cyclic or splits as a nontrivial free product. A partial converse to this (the so-called Kneser Conjecture that if M is a closed, orientable 3-manifold and $\pi_1(M)$ is either infinite cyclic or splits as a nontrivial free product, then there exists an essential 2-sphere embedded in M) can be proved without the aid of the Sphere Theorem (see [He_1], [St_3] and [Wh_2]).

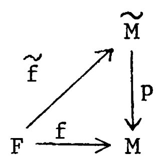
I will give the statements of the two main generalizations of I.1 and I.6.

I.12. GENERALIZED DEHN'S LEMMA [S-W]: *Let M be a 3-manifold and let D be a compact planar surface with boundary components $J_1, \ldots, J_k$. Suppose that $f : D \to M$ is a map such that $f|\partial D$ is an embedding, $f^{-1}(f(\partial D)) = \partial D$ and $f(J_i)$ is orientation preserving in M for each i $(1 \le i \le k)$. Then there exists a compact planar surface D' with boundary components $J'_1, \ldots, J'_{k'}$ and an embedding $f' : D' \to M$ such that for each j $(1 \le j \le k')$ there exists a unique i_j $(1 \le i_j \le k)$ so that $f'(J'_j) = f(J_{i_j})$.*

I.13. GENERALIZED LOOP-THEOREM [W_1]: *Let M be a 3-manifold and let D be a compact, planar surface with boundary components $J_1, \ldots, J_k$. Let $N_1, \ldots, N_k$ be normal subgroups of $\pi_1(M)$. Suppose that $f : D \to M$ is a map such that $f(\partial D) \subset \partial M$, $[f|J_i] \notin N_i$, $f(J_i)$ is orientation preserving in M for each i, and $f(J_i) \cap f(J_j) = \emptyset$ for $i \neq j$. Then given regular neighborhoods U_i of $f(J_i)$ in ∂M there is a compact, planar surface D' with boundary components $J'_1, \ldots, J'_{k'}$ and an embedding $g : D' \to M$ such that for each j $(1 \le j \le k')$ there exists a unique i_j $(1 \le i_j \le k)$ so that $g(J'_j) \subset U_{i_j}$*

and for some j, $[g \mid J'_j] \notin N_{i_j}$.

The "essence" of the proofs of the preceding theorems (I.1, I.6, I.9, I.12 and I.13) is discussed on pages 40-41 of [He_1]. Basically my approach is the same. Given a singular map $f : F \longrightarrow M$, I construct a factorization

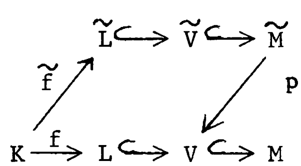


where the corresponding problem for $\tilde{f} : F \longrightarrow \tilde{M}$ has a solution. Then by using techniques of equivariant surgery in covering spaces, I am able to arrive at a solution to the problem for $f : F \longrightarrow M$. This method works well for I.1, I.6 and their generalizations (I.12 and I.13) mentioned above; however, I have had no success using this method to prove the Sphere Theorem (I.9). It is easy enough to find a factorization where $\tilde{f} : F \longrightarrow \tilde{M}$ has a solution (see Exercise I.29). The problem lies in the limited methods of equivariant surgery. This matter is discussed further in Chapter III.

THE TOWER CONSTRUCTION

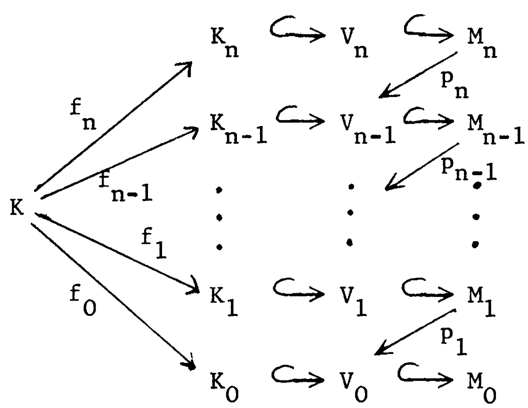
The commutative diagram

I.14



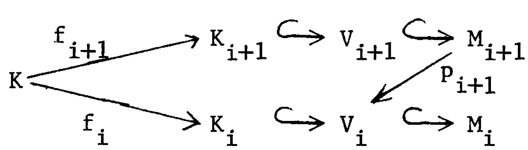
is called *a level over* $f : K \longrightarrow L$ if K, M and $\tilde{M}$ are all simplicial complexes, both of the maps $f : K \to M$, $\tilde{f} : K \to \tilde{M}$ are simplicial maps, $L = f(K)$, $\tilde{L} = \tilde{f}(K)$, $V = U(L)$, $\tilde{V} = U(\tilde{L})$ are relative regular neighborhoods of L in M, $\tilde{L}$ in $\tilde{M}$, respectively, and $\tilde{M}$ is a covering space of V with simplicial covering projection p. The level is said to be <u>trivial</u>, <u>two-sheeted</u>, <u>regular</u> or <u>universal</u> depending on the covering space $(\tilde{M}, p)$ of M being trivial, two-sheeted, regular or universal, respectively. The commutative diagram

I.15



is called a <u>tower</u> (<u>over</u> $f_0 : K \to M_0$) <u>of height</u> n if each subdiagram

I.16



is a level over $f_i : K \to M_i$ for each i, $0 \leq i < n$.

The fundamental lemma for our purposes is:

I.17. LEMMA: <u>A tower of two-sheeted levels having height n can be extended to a tower of two-sheeted levels having height $n + 1$ provided that V_n has a two-sheeted covering M_{n+1} with projection map p_{n+1} such that</u> $(f_n)_*(\pi_1(K)) \subset (p_{n+1})_*\pi_1(M_{n+1})$.

I.18. REMARK: If $\pi_1(K) = 1$, then a tower of two-sheeted levels of height n can be extended to a tower of two-sheeted levels of height $n + 1$ provided that V_n has a two-sheeted covering.

I.19. REMARK: If V is a compact 3-manifold, $\partial V \neq \emptyset$ and some component of ∂V is not a 2-sphere, then V has a two-sheeted covering.

I.20. EXERCISE: If V is a compact, orientable 3-manifold and some component of ∂V is not a 2-sphere, then $\mathrm{Im}(H_1(\partial V; \mathbb{Q}) \hookrightarrow H_1(V; \mathbb{Q}))$ is not trivial.

I.21. EXERCISE: There exists non-orientable 3-manifolds, V_g, with closed, orientable surfaces in ∂V_g having genus g (for arbitrary g); yet, $H_1(V_g)$ is finite.

I.22. LEMMA: *If* K *is a finite simplicial complex,* M_0 *is a triangulated 3-manifold and* $f_0 : K \longrightarrow M_0$ *is simplicial, then there exists a non-negative integer* $\mathcal{C}$ *such that any tower over* $f_0 : K \longrightarrow M_0$ *of height greater than* $\mathcal{C}$ *must have a trivial level.*

Proof: For any simplicial map f of K into a simplicial complex M, define the *complexity of* f, written $\mathcal{C}(f)$, to be the cardinality of the set of pairs of simplicies $\mathcal{C}(f) = \{(\sigma, \tau) \in K \times K : \sigma \neq \tau$ and $f(\mathring{\sigma}) = f(\mathring{\tau})\}$. Since K is finite, $\mathcal{C}(f_0)$ is a nonnegative integer. Set $\mathcal{C} = \mathcal{C}(f_0)$. Then Lemma I.22 follows from the observation that if

$$\begin{array}{ccccccc} & \overset{f_{i+1}}{\nearrow} & K_{i+1} & \hookrightarrow & V_{i+1} & \hookrightarrow & M_{i+1} \\ K & & & & & \swarrow p_i & \\ & \underset{f_i}{\searrow} & K_i & \hookrightarrow & V_i & \hookrightarrow & M_i \end{array}$$

is a level in a tower over $f_0 : K \to M_0$, then $\mathcal{C}(f_{i+1}) \leq \mathcal{C}(f_i)$ and inequality occurs if and only if the level is not trivial. ∎

In the following I shall construct a tower of two-sheeted levels in order to prove the Loop Theorem (and Dehn's Lemma).

The tower for the Loop Theorem is constructed as follows: Set $K = \mathbf{D}^2$ and subdivide both K and M so that f is simplicial. Set $f_0 = f$, $M_0 = M$ and $K_0 = f_0(K)$. Set V_0 equal to a relative regular neighborhood of K_0 in M_0 such that $S_0 = V_0 \cap \partial(M_0)$ is a relative regular neighborhood of $f_0(\partial K)$ in ∂M_0. For $i_0 : S_0 \hookrightarrow \partial M_0$ let $N_0 = (i_0)_*^{-1}(N)$. Having constructed a tower of two-sheeted levels of height k over $f_0 : K \to M_0$, if there exists a two-sheeted covering of V_k, then we can extend this tower to a tower of two-sheeted levels of height $k + 1$ by letting M_{k+1} be a two-sheeted covering of V_k with simplicial covering projection p_{k+1}. The simplicial map $f_k : K \to V_k$ lifts to a simplicial map $f_{k+1} : K \to M_{k+1}$. Set $K_{k+1} = f_{k+1}(K)$. Set V_{k+1} equal to a relative regular neighborhood of K_{k+1} in M_{k+1} such that $S_{k+1} = V_{k+1} \cap \partial M_{k+1}$ is a relative regular neighborhood of $f_{k+1}(\partial K)$ in $\partial(M_{k+1})$. For $i_{k+1} : S_{k+1} \hookrightarrow \partial M_{k+1}$, let $N_{k+1} = (p_{k+1} \circ i_{k+1})_*^{-1}(N_k)$. Notice that if $[f_k \mid \partial \mathbf{D}^2] \notin N_k$, then $[f_{k+1} \mid \partial \mathbf{D}^2] \notin N_{k+1}$.

I.23. LEMMA: <u>Suppose that</u> $f : \mathbf{D}^2 \to M$ <u>satisfies the hypothesis of the Loop theorem. Let</u> I.15 <u>be a tower of two-sheeted levels over</u> $f : \mathbf{D}^2 \to M$ <u>of height</u> n <u>and defined as above. If the tower</u> I.15 <u>cannot be extended to a tower of two-sheeted levels over</u> f <u>of height</u> $n + 1$, <u>then there exists an embedding</u> $g_n : \mathbf{D}^2 \to V_n$ <u>such that</u>

$g_n(\partial \mathbf{D}^2) \subset S_n \subset \partial M_n$ and $[g_n \mid \partial \mathbf{D}^2] \notin N_n$.

Proof: By I.18 and the hypothesis of this lemma, it follows that V_n has no two-sheeted covering spaces. Hence, by I.19 each component of ∂V_n is a 2-sphere. It follows that the 2-manifold S_n is spherical and therefore $\pi_1(S_n)$ is generated by ∂S_n. Since $[f_n \mid \partial \mathbf{D}^2] \notin N_n$, the conjugacy class determined by some component of ∂S_n does not belong to N_n. This class has a representative which is a simple closed curve in S_n and each simple closed curve in S_n bounds a disk embedded in ∂V_n (and therefore bounds a disk embedded in V_n). Let $g_n : \mathbf{D}^2 \to V_n$ be an embedding which realizes such a disk. ■

I.24. REMARK: If we consider the "top" of a tower as the level at which the tower cannot be extended in a desired fashion (in the case of the Loop Theorem a level at which V_n has no two-sheeted coverings), then we can interpret Lemma I.23 as simply stating that there exists a solution to the Loop Theorem at the top of a tower of two-sheeted levels. In the case of the Generalized Loop Theorem (Generalized Dehn's Lemma) again the method of proof is to exhibit a solution at the top of a tower of two-sheeted levels. However, in this case the top occurs at a level in which it may not be true that each component of ∂V_n is a 2-sphere; therefore there is more work in exhibiting a solution. The top of the tower is precisely where this approach falters in the case of the Sphere Theorem. Even though it is necessary that a solution to the Sphere Theorem exists at the top of a tower of two-sheeted levels, I do not know of a direct method for finding it. Notice that it is true, and easy to prove, that for any compact 3-manifold V with $\pi_1(V)$ finite and $\pi_2(V)$ not trivial, then ∂V is a nonempty collection of 2-spheres and $\pi_2(V)$ is generated as a $\pi_1(V)$-module by the components of ∂V.

Next, I consider the method of descending a tower with a solution. For simplicity's sake, as well as continuity of my presentation, I shall give detail only in the case of proving the Loop Theorem (Dehn's Lemma). It is true, however, that particularly subtle points appear in descending a tower with a solution to the Sphere Theorem or with a solution to the Generalized Loop Theorem (Generalized Dehn's Lemma). I will say more about this in the remarks immediately following the proof of Lemma I.25.

I.25. LEMMA: Using the notation established above, suppose that I.15 is a tower of two-sheeted levels for the Loop Theorem. If I.15 has a solution at height k ($k \geq 1$), then I.15 has a solution at height $k - 1$; i.e. if there exists an embedding $g_k : \mathbb{D}^2 \longrightarrow M_k$ such that $g_k(\partial\mathbb{D}^2) \subset S_k \subset \partial M_k$ and $[g_k \mid \partial\mathbb{D}^2] \notin N_k$, then there exists an embedding $g_{k-1} : \mathbb{D}^2 \longrightarrow M_{k-1}$ such that $g_{k-1}(\partial\mathbb{D}^2) \subset S_{k-1} \subset \partial M_{k-1}$ and $[g_{k-1} \mid \partial\mathbb{D}^2] \notin N_{k-1}$.

Proof: Let τ be the nontrivial covering translation. The idea is that a solution $g_k : \mathbb{D}^2 \longrightarrow M_k$ can be found such that $\mathbf{D} \cap \tau(\mathbf{D}) = \emptyset$ where $\mathbf{D} = g_k(\mathbb{D}^2)$. Then it follows that $g_{k-1} = p_k \circ g_k$ is the desired embedding of $\mathbb{D}^2$ into M_{k-1}.

To begin, I take the given embedding $g_k : \mathbb{D}^2 \longrightarrow M_k$ and set $\mathbf{D} = g_k(\mathbb{D}^2)$. I may assume that adjustments have been made so that $\mathbf{D}$

and $\tau(\mathbf{D})$ meet transversally. Then $\mathbf{D} \cap \tau(\mathbf{D})$ is either empty (the desirable situation) or consists of a pairwise disjoint collection of simple closed curves and spanning arcs. I then use the union $\mathbf{D} \cup \tau(\mathbf{D})$ to find a disk $\mathbf{D}'$ such that a map realizing $\mathbf{D}'$ as an embedding of $\mathbf{D}^2$ into M_k is a solution to the Loop theorem, $\mathbf{D}'$ meets $\tau(\mathbf{D}')$ transversely and $\mathbf{D}' \cap \tau(\mathbf{D}')$ has fewer components than $\mathbf{D} \cap \tau(\mathbf{D})$. If I continue to call this new embedding g_k, then it is clear that inductively, I will obtain the desired solution.

Suppose that $\mathbf{D} \cap \tau(\mathbf{D}) \neq \emptyset$.

<u>Case</u> 1. $\mathbf{D} \cap \tau(\mathbf{D})$ has a simple closed curve component (Figure 1.1).

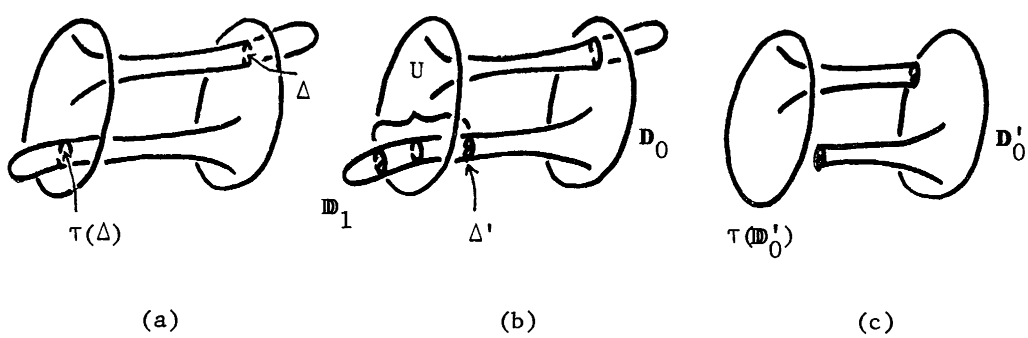
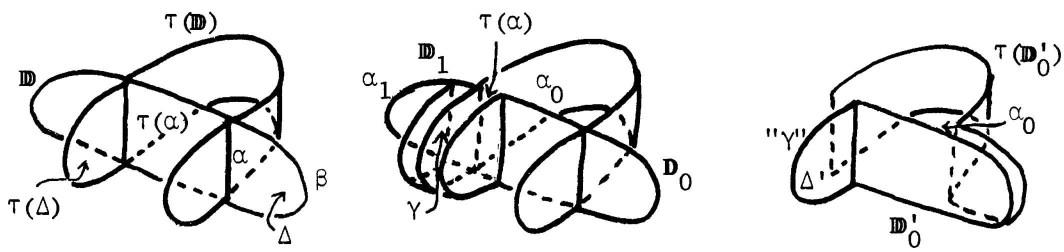


Figure 1.1

In this case let α be a simple closed curve component of $\mathbf{D} \cap \tau(\mathbf{D})$ such that α bounds a disk Δ on $\mathbf{D}$ and $\mathring{\Delta} \cap \tau(\mathbf{D}) = \emptyset$; i.e. Δ is "innermost" on $\mathbf{D}$. The simple closed curve $\tau(\alpha)$ (which may be equal to α if α is invariant) divides $\mathbf{D}$ into two components: an annulus $\mathbf{D}_0$, which contains $\partial\mathbf{D}$, and a disc $\mathbf{D}_1$. Let U be a small product neighborhood of $\tau(\Delta)$ such that U has a parametrization as $I \times I \times I$ with $\tau(\Delta) = I \times I \times \{1/2\}$, $U \cap \mathbf{D} = \partial(I \times I) \times I$, $I \times I \times \{0\} \cap \mathbf{D}_0 \neq 0$ and $I \times I \times \{1\} \cap \mathbf{D}_1 \neq \emptyset$. Set $\mathbf{D}_0' = \overline{(\mathbf{D}_0} - (U \cap \mathbf{D}))$ and set $\Delta' = I \times I \times \{0\}$. Then $\mathbf{D}' = \mathbf{D}_0' \cup \Delta'$ has the property that $\mathbf{D}' \cap \tau(\mathbf{D}')$ has fewer components than $\mathbf{D} \cap \tau(\mathbf{D})$.

Case 2. $\mathbf{D} \cap \tau(\mathbf{D})$ has no s.c.c. components (see Figure 1.2).



(a) (b) (c)

Figure 1.2

Let α be a component of $\mathbf{D} \cap \tau(\mathbf{D})$ (α is a spanning arc of $\mathbf{D}$) so that there exists an arc β in $\partial\mathbf{D}$ having the property that $\partial\alpha = \partial\beta$ and $\alpha \cup \beta$ bounds a disk Δ in $\mathbf{D}$ where $\mathring{\Delta} \cap \tau(\mathbf{D}) = \emptyset$; i.e. α is "outermost" on $\mathbf{D}$. The spanning arc $\tau(\alpha) \subset \mathbb{D} \cap \tau(\mathbf{D})$ (which is disjoint from α) divides $\mathbf{D}$ into two components $\mathbf{D}_0$ and $\mathbf{D}_1$, each of which is a disk, and divides $\partial\mathbf{D}$ into two arcs α_0 and α_1 such that $\alpha_0 \subset \partial\mathbb{D}_0$ and $\alpha_1 \subset \partial\mathbb{D}_1$. Set $\gamma = \tau(\beta)$. By properly choosing orientation we have $[\partial\mathbf{D}] = \alpha_0\alpha_1 = (\alpha_0\gamma)(\gamma^{-1}\alpha_1)$. Therefore, $\alpha_0\gamma \notin N$ or $\gamma^{-1}\alpha_1 \notin N$

(say $\alpha_0\gamma \notin N$). Let U be a small product neighborhood of $\tau(\Delta)$ such that U has a parametrization as $I \times I \times I$ with $\tau(\Delta) = I \times I \times \{1/2\}$, $U \cap \mathbf{D} = \{1\} \times I \times I$, $I \times I \times \{0\} \cap \mathbf{D}_0 \neq \emptyset$ and $I \times I \times \{1\} \cap \mathbf{D}_1 \neq \emptyset$.

Set $\mathbf{D}_0' = \overline{(\mathbf{D}_0 - U \cap \mathbf{D})}$ and set $\Delta' = I \times I \times \{0\}$. Then $\mathbf{D}' = \mathbf{D}_0' \cup \Delta'$ has the property that $\mathbf{D}' \cap \tau(\mathbf{D}')$ has fewer components than $\mathbf{D} \cap \tau(\mathbf{D})$. ∎

In the next two remarks, I am continuing to use the notation of Lemma I.25 and its proof.

I.26. REMARK: In the case of the sphere theorem if one tries to descend a tower of two-sheeted levels with a solution, then the situation is as follows: there exists an embedding $g_k : S^2 \longrightarrow V_k$ with $[g_k] \notin N_k$ and one needs to exhibit an embedding $g_{k-1} : S^2 \longrightarrow V_{k-1}$ with $[g_{k-1}] \notin N_{k-1}$. If $S = g_k(S^2)$, then either $S \cap \tau(S) = \emptyset$ or each component of $S \cap \tau(S)$ is a simple closed curve. If α is such a component of intersection and $\alpha = \tau(\alpha)$, then the induction argument may fail; that is, equivariant surgery, as described in Lemma I.25, may fail to reduce the number of components of intersection between the 2-sphere and its image under the covering translation τ. However, in this case, it follows that V_{k-1} is a nonorientable 3-manifold (in fact, $p_{k-1}(\alpha)$ is an orientation reversing curve in V_{k-1}) and one can find a <u>nontrivial</u> covering map $g_{k-1} : S^2 \longrightarrow g_{k-1}(S^2) \subset V_{k-1}$ with $[g_{k-1}] \notin N_{k-1}$; i.e. $g_{k-1}(S^2)$ is a two-sided projective plane in V_{k-1}.

I.27. REMARK: In the case of the Generalized Loop Theorem (Generalized Dehn's Lemma) if one tries to descend a tower of two-sheeted levels with a solution, then there exists a planar surface F_k with r_k boundary components $b_1^k, \ldots, b_{r_k}^k$ and an embedding $g_k : F_k \longrightarrow V_k$ such that $g_k(b_i^k) \subset S_i^k \subset \partial M_k$ and for some j $[g_k \mid b_j^k] \notin N_j^k$. One needs to exhibit a planar surface F_{k-1} with r_{k-1} boundary components $b_1^{k-1}, \ldots, b_{r_{k-1}}^{k-1}$ and an embedding $g_{k-1} : F_{k-1} \longrightarrow V_{k-1}$ with $g_{k-1}(b_n^{k-1}) \subset S_n^{k-1} \subset \partial M_{k-1}$ and for some m, $[g_{k-1} \mid b_m^{k-1}] \notin N_m^{k-1}$.

Set $F = g_k(F_k)$, then either $F \cap \tau(F) = \emptyset$ or each component of intersection is a spanning arc or a simple closed curve. Now, I have been able to make the method of equivariant surgery presented here successful in this case by using that F is planar and if J, K are components of ∂F, then either $J \cap \tau(K) = \emptyset$ or $J = K$. This latter fact follows from the hypothesis that the singular images of distinct components of the boundary of the planar surface do not meet. Hence, if a component of $F \cap \tau(F)$ is a s.c.c. α, then both α and $\tau(\alpha)$ separate both F and $\tau(F)$ and equivariant surgery can be done. If a component of $F \cap \tau(F)$ is a spanning arc α, then either α has both of its end points in the same component J of ∂F (and, by the above observation (hypothesis), $\tau(\alpha)$ has both of its end points in $\tau(J) \subset \partial\tau(F)$), so both α and $\tau(\alpha)$ separate both F and $\tau(F)$ and equivariant surgery can be done; or both α and $\tau(\alpha)$ have one end point in $J \subset \partial F$ and the other end point in $K \subset \partial F$ $(J \neq K)$ (again, by the above observation (hypothesis)), so together α and $\tau(\alpha)$ separate both F and $\tau(F)$ and equivariant surgery can be done.

I.28. QUESTION: Is the Generalized Loop Theorem (I.13) valid if we eliminate the hypothesis that $f(J_i) \cap f(J_j) = \emptyset$ for $i \neq j$?

I.29. EXERCISE: Suppose that M is a simply connected 3-manifold and

$\pi_2(M) \neq 0$. Using the Loop theorem prove that there exists an essential 2-sphere embedded in M.

I.30. EXERCISE: Prove that Dehn's Lemma, the Loop Theorem, and their generalized versions (I.12 and I.13) follow from the Sphere Theorem.

The next exercises give direct and elementary applications of the theorems discussed in this Chapter.

I.31. EXERCISE: Using Dehn's Lemma prove that the knot $k \subset S^3$ is trivial if and only if $\pi_1(S^3 - k)$ is infinite cyclic.

I.32. EXERCISE: Assume that M is a compact 3-manifold such that every 2-sphere in M bounds a 3-cell in M (M is irreducible). Use the Loop Theorem to prove that M is a cube-with-handles if and only if $\pi_1(M)$ is a free group.

I.33. EXERCISE: Assume that M is a compact, irreducible 3-manifold with connected boundary. Prove that M is a cube-with-handles if and only if $\pi_1(\partial M) \to \pi_1(M)$ is onto.

I.34. EXERCISE: Let k be a knot in S^3. Use the Sphere Theorem to prove that $S^3 - k$ is aspherical.

I.35. EXERCISE: Let M be an irreducible 3-manifold. Let F be a closed component of ∂M and let $F' \neq F$ be a component of ∂M. Suppose that every loop in F is homotopic to a loop in F'. Use the Generalized Loop Theorem to prove that if both $\ker(\pi_1(F) \hookrightarrow \pi_1(M)) = \{1\}$ and $\ker(\pi_1(F') \hookrightarrow \pi_1(M)) = \{1\}$, then M is homeomorphic to $F \times I$ via a homeomorphism taking F onto $F \times \{0\}$.

CHAPTER II. CONNECTED SUMS

Connected sums of 3-manifolds are considered in detail in the book by Hempel [He_1]; and in particular, he gives a proof of the most general version of the Prime Decomposition Theorem. I will present a completely different proof of the existence of prime decompositions for compact, orientable 3-manifolds. My proof is based on a theorem of Haken [H_1], which is of independent interest; moreover, I will give a new proof of Haken's theorem. (The proof that I give is two-dimensional and surprisingly elementary.)

A sphere S in a 3-manifold M is <u>compressible in</u> M if S bounds a 3-cell embedded in M. Otherwise, the 2-sphere S is <u>incompressible in</u> M. For example, a nonseparating 2-sphere S in the 3-manifold M is incompressible in M. A 3-manifold in which every 2-sphere is compressible is called <u>irreducible</u>. A famous theorem of J. W. Alexander [A_1] implies that both $\mathbf{R}^3$ and S^3 are irreducible.

Let M be a 3-manifold and let S be a 2-sphere embedded in M that separates M. Let M_1 and M_2 denote the two 3-manifolds obtained by splitting M along S and capping-off the two resulting 2-sphere boundary components with two 3-cells. Then M is <u>a connected sum</u> of M_1 and M_2, written $M_1 \# M_2$. A 3-manifold M is <u>nontrivial</u> if M is not homeomorphic to S^3. The 3-manifold M is <u>prime</u> (with respect to connected sum) if M is nontrivial and if $M = M_1 \# M_2$ implies that either M_1 or M_2 is trivial; i.e. $M \neq S^3$ and M cannot be written in a nontrivial way as a connected sum.

There is the reverse operation to the one described above in which one starts with two 3-manifolds and builds from them a connected sum. This may result in some ambiguities; and in fact, if care is not taken, the operation is not well-defined. I will discuss this matter at the end of this lecture. This latter problem will not really concern me. I am interested only in splitting a 3-manifold along 2-spheres to arrive at irreducible pieces, along with possible $S^2 \times S^1$'s, and the uniquencess of the seemingly many possible ways of doing this.

II.1. EXERCISE: Let M be a 3-manifold and suppose that S is a non-separating 2-sphere embedded in M. Show that $M = M_1 \# S^2 \times_\varphi S^1$. (If N is a 3-manifold and N is homeomorphic to a surface bundle over S^1 with fiber F and monodromy φ, I shall write $N = F \times_\varphi S^1$. If φ is isotopic to ψ, $\varphi \approx \psi$, then $F \times_\varphi S^1$ is homeomorphic to $F \times_\psi S^1$. Of course, $F \times S^1$ is used for $F \times_\varphi S^1$, $\varphi \approx id_F$.)

II.2. EXERCISE: If some component of ∂M is a 2-sphere, then $M = M_1 \# \mathbf{D}^3$.

II.3. EXERCISE: With the exceptions S^3, $S^2 \times S^1$ and $S^2 \times_\varphi S^1$ with φ orientation reversing on S^2, a manifold is prime if and only if it is irreducible.

I am going to state and prove the Prime Decomposition Theorem for closed, orientable 3-manifolds. The same result is true for compact 3-manifolds in general, orientable or not, with or without boundary (See Exercise II.18). To obtain a proper uniqueness statement in the nonorientable case it is necessary to have a "normal-form" for the decomposition (see either [He_1] or [R_1]). The existence part of the theorem was first proved by

H. Kneser [Kn_1] and the uniqueness part was obtained by J. Milnor [M_1]. Their methods of proof are presented by Hempel [He_1].

II.4. PRIME DECOMPOSITION THEOREM. *Let M be a closed, orientable nontrivial 3-manifold. Then* $M = M_1 \# \ldots \# M_n$ *where each* M_i *is prime. Furthermore, this decomposition for M is unique up to order and homeomorphism.*

Let M be a closed, orientable 3-manifold. A closed, orientable surface F embedded in M *determines a Heegaard splitting* of M if F separates M and the closure of each component of M - F is a cube-with-handles. If the surface F determines a Heegaard splitting of the closed, orientable 3-manifold M, I shall call the pair (M; F) a *Heegaard splitting of* M. The *genus of the Heegaard splitting* (M; F) is by definition the genus of F. The *genus of the* 3-*manifold* M is the minimum of the genera taken over all Heegaard splittings of M. (Lectures on Heegaard theory of 3-manifolds are being given at this Conference by Professor J. Birman and a set of lecture notes has been prepared [B_1].)

II.5. EXERCISE: Every closed, orientable 3-manifold has a Heegaard splitting.

II.6. EXERCISE: If M admits a genus one Heegaard splitting, then either $M \simeq S^3$, $M \simeq S^2 \times S^1$ or M is irreducible. In particular, a genus one 3-manifold is prime.

II.7. THEOREM (Haken) [H_1]: *Let (M; F) be a Heegaard splitting of the closed, orientable 3-manifold M. If M contains an incompressible*

2-sphere, then M contains an incompressible 2-sphere S such that $S \cap F$ is a single simple closed curve.

The proof follows from a lemma about planar surfaces. First, I need some definitions. A properly embedded arc α in a 2-manifold T is *inessential in* T if there exists an arc $\beta \subset \partial T$ such that $\alpha \cup \beta$ bounds a disk in T. Otherwise, α is *essential in* T. A s.c.c. α in T is *inessential in* T if it is contractible in T. Otherwise, α is *essential*. A *hierarchy* for a 2-manifold T is a sequence of pairs $(T_0, \alpha_0), (T_1, \alpha_1), \ldots, (T_n, \alpha_n)$ where $T_0 = T$, α_i is an essential arc or simple closed curve in T_i, T_{i+1} is obtained from T_i by splitting along α_i, and each component of T_{n+1} is a disk.

II.8. LEMMA: *Let T be a planar surface and assume that T has $b > 1$ boundary components* (i.e. $T \neq \mathbf{D}^2$). *Let $(T_0, \alpha_0), \ldots, (T_n, \alpha_n)$ be any hierarchy for T with each α_i an arc. If d is the number of components of T_{n+1}, then $d \leq b - 1$.*

Proof (of Lemma II.8): The proof is via induction on b, $b \geq 2$. If $b = 2$, then T is an annulus and up to isotopy there exists a unique essential arc in T. Hence, $d = 1$ and the conclusion of II.8 is satisfied.

Suppose that the surface T has b boundary components where $b > 2$ and II.8 is true for all planar surfaces having fewer than b boundary components. There are two cases.

Case 1. α_0 does not separate T.

Set b_1 equal to the number of boundary components of T_1.

Since α_0 does not separate T, distinct end points of α_0 are in distinct components of ∂T; and it follows that $b_1 = b - 1$. Hence, by induction, $d = d_1 \leq b_1 - 1 = b - 2$.

Case 2. α_0 separates T.

Let T_1' and T_1'' denote the components of T_1. Set b_1, b_1' and b_1'' equal to the number of boundary components of T_1, T_1' and T_1'', respectively. Since α_0 separates T, distinct end points of α_0 are in the same component of ∂T; and it follows that $b_1 = b + 1$. However, we know that $b_1 = b_1' + b_1''$ and since α_0 is essential, both $b_1' \geq 2$ and $b_1'' \geq 2$. Therefore, $b_1' < b$ and $b_2' < b$. Hence, by induction, $d = d_1 = d_1' + d_1'' \leq (b_1' - 1) + (b_1'' - 1) = b - 1$. ■

II.9. LEMMA: *Let* T *be a planar* 2-*manifold. Assume that* T *has* b *boundary components and* $z \geq 0$ *components which are not disk. Let* $(T_0, \alpha_0), \ldots, (T_n, \alpha_n)$ *be any hierarchy for* T *with each* α_i *an arc. If* d *is the number of components of* T_{n+1}, *then* $d \leq b - z$.

Proof (of Lemma II.9): If $T^{(j)}$ is a component of T which is not a disk $(1 \leq j \leq z)$, then the hierarchy for T determines a hierarchy for $T^{(j)}$. Let $b^{(j)}$ denote the number of boundary components of $T^{(j)}$ $(b^{(j)} \geq 2,\ 1 \leq j \leq z)$, and let $d^{(j)}$ denote the number of components of T_{n+1} coming from $T^{(j)}$. Let k be the number of components of T which are disk. Then $d = k + \sum_{j=1}^{z} d^{(j)}$ and $b = k + \sum_{j=1}^{z} b^{(j)}$. From Lemma II.8 it follows that $\sum_{j=1}^{z} d^{(j)} \leq \sum_{j=1}^{z} (b^{(j)} -$ therefore, $d \leq b - z$. ■

PROOF (OF THEOREM II.7): Let (M; F) be a Heegaard splitting of the closed, orientable 3-manifold M. Denote the closures of the components of $M - F$ by H and H', each of which is a cube-with-handles.

By hypothesis, M contains an incompressible 2-sphere. Therefore, M contains an incompressible 2-sphere which meets F transversely. Among all incompressible 2-spheres in M which meet F transversely, choose one, S, such that the number of components of $S \cap F$ is a minimum.

I shall generalize the notions of compressible and incompressible for 2-spheres to all properly embedded 2-manifolds. This will be simply more convenient language to use. I will discuss these notions in great detail in Chapter III. A 2-manifold T, properly embedded in a 3-manifold N, is <u>compressible</u> in N if either $T = S^2$ and T bounds a 3-cell or $T \neq S^2$ and there exists a disk $D \subset N$ such that $D \cap T = \partial D \cap T$ is a noncontractible curve in T. Otherwise, T is <u>incompressible</u> in N.

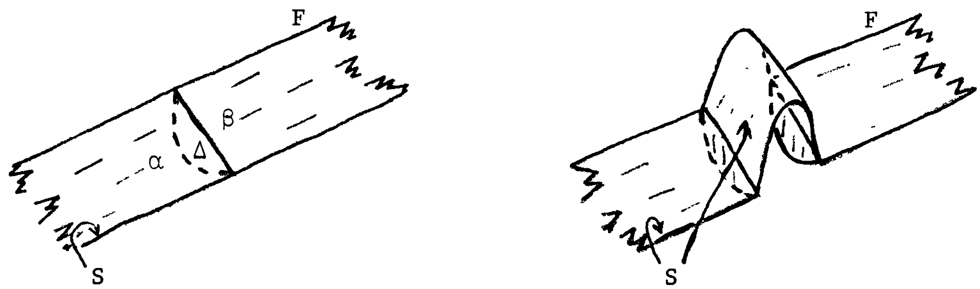
<u>CLAIM</u> 1: Both $S \cap H$ and $S \cap H'$ are incompressible.

This follows from the choice of S such that the number of components of $S \cap F$ is a minimum.

<u>CLAIM</u> 2: $S \cap F$ <u>consists of a single simple closed curve</u>.

Since S is incompressible and a cube-with-handles is irreducible, it follows that $S \cap F \neq \emptyset$. In order to establish Claim 2, I need to describe a certain isotopy of S in M.

Suppose that Δ is a disk in M such that $\Delta \cap S = \alpha$ is an arc in $\partial\Delta$, $\Delta \cap F = \beta$ is an arc in $\partial\Delta$, $\partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial\Delta$ (see Figure 2.1a).



(a) (b)

Figure 2.1

An <u>isotopy</u> <u>of</u> <u>Type</u> A <u>at</u> α is performed by sliding α across Δ and past β (see Figure 2.1). Such an isotopy shoves a part of S that is on one side of F in M through F and into the other side of F in M. The result of this isotopy to that part of S which was on the same side of F as α before the isotopy is equivalent to the result of cutting along α.

Now, suppose that some component of $S \cap H$ is not a disk. Set $T = S \cap H$. By Claim 1, T is incompressible in H. In this situation it is not difficult to prove that there exists a hierarchy $(T_0, \alpha_0), \ldots, (T_n, \alpha_n)$ for T which gives rise to a sequence of isotopies of S in M where the first isotopy is of Type A at α_0, the second isotopy is of Type A at $\alpha_1, \ldots,$ and the $(n+1)$-st isotopy is of Type A at α_n. (For example, look at a complete system of disks for H and intersect them transversely with T.)

Set S' equal to the image of S after this sequence of isotopies Each component of $S' \cap H$ is a disk. The number of components of $S \cap F$ is b, the number of components of ∂T. The number of components of

$S' \cap F$ is d, the number of components of T_{n+1}. However, by Lemma II.9, $d \leq b - z$, where $z \geq 1$ is the number of components of T which are not disk. So, S' would be an incompressible 2-sphere in M meeting F transversely and the number of components of $S' \cap F$ would be less than the number of components of $S \cap F$.

Similarly, if some component of $S \cap H'$ is not a disk, then we could find an incompressible 2-sphere S' in M meeting F transversally and the number of components of $S' \cap F$ would be less than the number of components of $S \cap F$.

Therefore, by the earlier choice of S, each component of $S \cap H$ is a disk and each component of $S \cap H'$ is a disk. This establishes Claim 2 and proves the theorem. ∎

PROOF OF II.4: I will only prove existence here. (The original proof by Milnor [M_1] of uniqueness is very easy to follow.) The proof is via induction on the Heegaard genus of the 3-manifold M. For manifolds having Heegaard genus one, this is just Exercise II.6. Now, assume that each closed, orientable 3-manifold having Heegaard genus $< n$ $(n \geq 2)$ admits a prime decomposition. Let M be a closed, orientable 3-manifold having Heegaard genus n.

Since M has Heegaard genus n, there exists a Heegaard splitting $(M; F)$ of M having genus n. Also, M is nontrivial (Heegaard genus of S^3 is zero). If M is prime, then there is nothing to prove. So, assume that M is not prime. Hence, there exists an incompressible 2-sphere in M. By Theorem II.7, there exists an incompressible 2-sphere S in M such that $S \cap F$ consists of a single simple closed curve. By a slight modification, if necessary, I can find such an S such that S

separates M. The 2-sphere S allows M to be written as $M = M_1 \# M_2$ for some 3-manifolds M_1 and M_2. Moreover, since $S \cap F$ is connected, there are induced Heegaard splittings of M_1 and M_2. Therefore, the Heegaard genus of M_i is less than n for $i = 1,2$. The induction hypothesis now gives the desired conclusion. ■

II.10. COROLLARY: <u>The Heegaard genus of the sum is the sum of the Heegaard genera</u>; i.e., if $M = M_1 \# \ldots \# M_n$, then genus $M = \sum_1^n$ genus M_i.

Now, let's consider the problem of recapturing a 3-manifold by knowing its prime factors.

Given two 3-manifolds M_1 and M_2 I can choose a point in the interior of each manifold, remove the interior of small 3-cells chosen about these points from each manifold and attach the resulting 2-sphere boundary components via some homeomorphism. The resulting 3-manifold is <u>a connected sum</u> of M_1 and M_2 written $M_1 \# M_2$. It turns out that this operation does not, in general, lead to a unique (up to homeomorphism) 3-manifold. However, by the Newman and Gugenheim homogeneity theorem [Gu_1] the operation does not depend on the points or the 3-cells chosen. The problem is in the two distinct isotopy classes of homeomorphisms on the 2-sphere.

II.11. EXERCISE: Show that isotopic attaching maps on the 2-spheres result in homeomorphic 3-manifolds.

There is a way around this problem. Namely, work in the category of oriented manifolds and orientation preserving homeomorphisms. Then the orientations on M_1 and M_2 induce orientations on the 2-spheres (as the boundaries of oriented 3-cells). Now, require that the attaching map be orientation reversing as a map between oriented 2-spheres.

II.12. EXERCISE: In the category of oriented 3-manifolds and orientation preserving homeomorphisms, the operation of connected sum is well-defined. Furthermore, up to orientation preserving homeomorphisms, it is an associative and commutative operation with S^3 acting as an identity element. (Corollary II.10 can be used to show that there are no inverses.)

If M is an oriented manifold, let $-M$ denote the oriented manifold determined by M and its opposite orientation.

II.13. EXERCISE: Form $M \# M$ and $M \# -M$ by requiring the attaching map on 2-spheres to be orientation reversing. Then $M \# M$ is topologically equivalent to $M \# -M$ if and only if M admits an orientation reversing self-homeomorphism.

II.14. REMARK: I do not know if there is any general information available on orientation reversing self-homeomorphisms of 3-manifolds. (Some study has been done on orientation reversing involutions [Kw_1], and [B_2].) However, there are 3-manifolds that _do_ _not_ admit orientation reversing self-homeomorphisms.

II.15. LEMMA: _The lens space_ $L(p, q)$ _admits an orientation reversing self-homeomorphism if and only if_ $q^2 \equiv -1 \pmod{p}$.

PROOF: Set $L = L(p, q)$, $0 < q < p$. Then L has a genus one Heegaard splitting $(L; F)$ such that if H and H' are the closures of $L - F$, then F has a framing α, β such that β is contractible in H and $p\alpha + q\beta$ is contractible in H'.

I shall prove the theorem under the additional assumption that the supposed homeomorphism $h : L \to L$ is isotopic to a homeomorphism which is invariant on F. (I do not know if this assumption introduces any limitations. I suspect this is always the case. See Question II.16.) A proof not using this assumption may be obtained by modeling the proof of Lemma 3.23 in [He_1].

I may assume that $h(F) = F$. There are two possible cases.

Case 1. $h(H) = H$.

The matrix of the homeomorphism $h|F$ must have the form $\begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ n & 1 \end{pmatrix}$, since $h(\beta) = \pm\beta$. However, it must also be true that $h(p\alpha + q\beta) = \pm(p\alpha + q\beta)$. Hence, the only possibility is $\begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix}$ with $n = 1$ and therefore $p = 2$, $q = 1$; i.e. $L = L(2, 1)$ is real projective 3-space.

Case 2. $h(H) = H'$.

The matrix of the homeomorphism $h|F$ must have the form $\begin{pmatrix} r & p \\ s & q \end{pmatrix}$ or $\begin{pmatrix} -r & -p \\ -s & -q \end{pmatrix}$ where $rq - ps = +1$, since $h(\beta) = \pm(p\alpha + q\beta)$. In this case it must also be true that $h(p\alpha + q\beta) = \pm\beta$. For $\begin{pmatrix} r & p \\ s & q \end{pmatrix}$ one calculates that $h(p\alpha + q\beta) = -\beta$ is the only possibility and then $q^2 \equiv -1 \pmod p$. For $\begin{pmatrix} -r & -p \\ -s & -q \end{pmatrix}$ one calculates that $h(p\alpha + q\beta) = \beta$ is the only possibility and then $q^2 \equiv -1 \pmod p$.

In Case 1 and Case 2 we arrive at $q^2 \equiv -1 \pmod p$. This proves that the condition is necessary. However, the proof also shows how to construct such homeomorphisms if $q^2 \equiv -1 \pmod p$; therefore showing that this condition is also sufficient. ■

II.16. QUESTION: Let L be a lens space and let $(L; F)$ be a genus one Heegaard splitting of L. Suppose that $h : L \longrightarrow L$ is a homeomorphism. Then is h isotopic to $h' : L \longrightarrow L$ such that $h'(F) = F$?

II.17. REMARK: The problem of decomposing 3-manifolds via "disk-sums" has been studied [G_1, G_2], [Sw_1]. The results in this case are similar to the results presented here on connected sums.

II.18. EXERCISE: Let M be a compact, orientable 3-manifold distinct from S^3. Then $M = M_1 \# \ldots \# M_n$ where each M_i is prime. Furthermore, this decomposition for M is unique up to order and homeomorphism. [Hint: This is the generalization of Theorem II.4 to manifolds with possibly non-empty boundary. First, assume that $\partial M \neq \emptyset$ and each component of ∂M is incompressible. Let $2M$ be the manifold obtained by doubling M along ∂M. By II.4, $2M$ has a prime decomposition. Set F equal to the incompressible 2-manifold in $2M$ corresponding to ∂M. Among all collections of incompressible 2-spheres in $2M$ giving a prime decomposition of M, choose S such that the number of components of $S \cap F$ is minimal. Argue that $S \cap F = \emptyset$ and the collection $S \cap M$ gives a prime decomposition of M. It is now easy to extend the result to manifolds M with possibly compressible boundary. Milnor's proof [M_1] for the closed case works equally well here.]

CHAPTER III. 2-MANIFOLDS EMBEDDED IN 3-MANIFOLDS

There have been two very successful approaches to the study of 3-manifolds by studying embeddings of 2-manifolds in them. One approach uses the Heegaard surface. Here the underlying philosophy is to embed a surface into a 3-manifold so that the components of its complement are as "simple" as possible. The other approach uses the incompressible surface. Here the underlying philosophy is to embed a surface into a 3-manifold so that the surface is as "simple" as possible and carries both geometric and algebraic information. (Recent success using differential geometry in 3-dimensional manifold theory exploits minimal surfaces, totally geodesic surfaces, etc.)

A surface F embedded in a compact 3-manifold M is a <u>Heegaard surface</u> in M provided that F bounds a cube-with-handles U in M and M is obtained from U by adding a finite number of two and three handles.

III.1. REMARK: Using this definition we allow a compact 3-manifold with boundary to have a Heegaard surface. Note that if M is closed, then F is a Heegaard surface in M iff the closure of each component of M - F is a cube-with-handles of genus equal to the genus of F.

I plan to direct my attention to the study of incompressible surfaces. As mentioned earlier, J. Birman has written a set of notes on Heegaard surface theory to supplement her lectures here at this conference [B_1].

A surface F properly embedded in the 3-manifold M (or embedded in ∂M) is compressible in M if either

(i) $F = S^2$ and F bounds a 3-cell in M, or

(ii) there exists a disk $D \subset M$ such that $D \cap F = \partial D$ and $[\partial D]$ is not trivial in F.

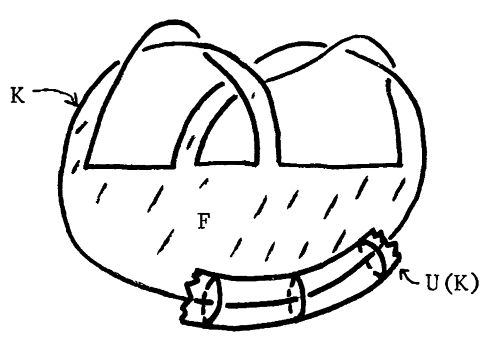
Otherwise, F is incompressible in M.

Notice that any properly embedded disk in M is incompressible in M. This is very convenient for later definitions and it creates no extra considerations.

III.2. EXAMPLES:

(a) Let H_n be a cube-with-handles of genus n, then ∂H_n is compressible in H_n.

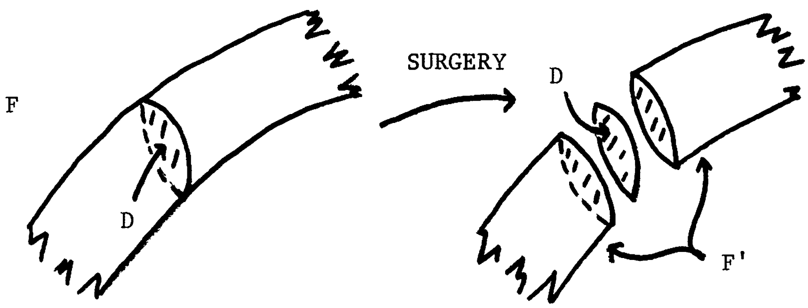
(b) Let K be a knot in S^3 and let $U(K)$ be a tubular neighborhood of K. Set $M(K) = S^3 - \overset{\circ}{U}(K)$. Let F be the intersection of a minimal genus, orientable Seifert surface for K with $M(K)$. Then F is incompressible in $M(K)$ (Figure 3.1).



K = TREFOIL KNOT

Figure 3.1

III.3. REMARK: If F is compressible in M, then we may do "surgery" (see Figure 3.2) on F to obtain a 2-manifold F' such that the Euler Characteristic of each component of F' is greater than the Euler Characteristic of F, $\chi(F)$.



$\chi(F') = \chi(F) + 2$

Figure 3.2

(c) Let K be a knot in S^3 and let $U(K)$ be a tubular neighborhood of K. Set $M(K) = S^3 - \overset{\circ}{U}(K)$. Then $\partial M(K) \approx S^1 \times S^1$ is incompressible in $M(K)$ iff K is non-trivial.

(d) Let K_1 and K_2 be non-trivial knots in S^3; let $M(K_i)$ be the knot-space corresponding to K_i, $i = 1, 2$. Let $h : \partial M(K_1) \to \partial M(K$ be a homeomorphism. If $M = M(K_1) \underset{h}{\cup} M(K_2)$ is obtained from the disjoint union of $M(K_1)$ and $M(K_2)$ via the identification $x \sim h(x)$ for each $x \in \partial M(K_1)$, then $T = \partial M(K_1) = \partial M(K_2)$ is incompressible in M.

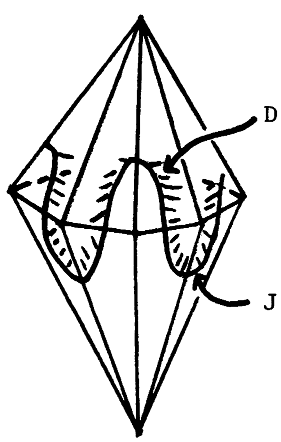
III.4. REMARK: If framings of $\partial M(K_i)$ (Example (d)), say μ_i, λ_i, are chosen so that μ_i is a "meridian" and λ_i is a "longitude" $(i = 1, 2)$ and $h : \partial M(K_1) \to \partial M(K_2)$ has the property that $h(\mu_1) = \lambda_2$ and

$h(\lambda_1) = \mu_2$, then $M = M(K_1) \underset{h}{\cup} M(K_2)$ is a homology 3-sphere containing an incompressible surface.

(e) $\mathbf{P}^2 \subset \mathbf{P}^3$ is incompressible (one-sided).

III.5. REMARK: A useful observation is that if F is a surface in M and $D \subset M$ is a disk such that $D \cap F = \partial D$, then ∂D is an orientation preserving curve on F.

(f) Let $M(n)$ be the Lens space $L(2n, 1)$. Then there exists an incompressible non-orientable surface with n-cross caps embedded in $M(n)$ (see Figure 3.3).



$P(n) \equiv$ suspension of $2n$-edge polygon

$D \equiv \mathbf{D}^2$

$J \equiv \partial D$

$P(n)$ identifies to $M(n)$ and

D identifies to F.

Figure 3.3

III.6. EXERCISE: Show that the Lens space $L(p, q)$ contains a closed, incompressible surface iff p is even. [Hint: Wait until Lemma III.9.]

III.7. REMARK: The surface $F \subset M$ is <u>injective</u> if $\ker(\pi_1(F) \hookrightarrow \pi_1(M))$ is trivial. An injective surface in M is incompressible in M; however, the converse is not true. For example, there is an incompressible Klein bottle embedded in $L(4, 1)$ (see Example III.2 (f)). For two-sided surfaces there is the following lemma.

III.8. LEMMA: A two-sided surface distinct from S^2 in a 3-manifold is incompressible in M iff it is injective in M.

The next lemma, while quite elementary, is very important to the study of 3-manifolds via incompressible surfaces. It is implicit in the work of Stallings [St_4]; and this version comes from Waldhausen [W_2].

III.9. LEMMA: Let M be a 3-manifold, N an n-manifold and $L \subset N$ a compact, properly embedded $(n-1)$-submanifold with $\ker(\pi_1(L) \hookrightarrow \pi_1(N)) = \{1\}$, $\pi_2(N) = \pi_2(N - L) = \{0\}$, and $\pi_3(N) = 0$. Suppose that $f : M \to N$ is a compact map. Then there exists a homotopy f_t $(0 \leq t \leq 1)$ such that

(i) $f_0 = f$

(ii) f_1 is transverse to L

(iii) Each component of $f_1^{-1}(L)$ is a properly embedded, incompressible surface in M.

Furthermore, if $f \mid \partial M$ is transverse to L we may take $f_t \mid \partial M = f \mid \partial M$ $(0 \leq t \leq 1)$.

Proof: First deform f via a proper homotopy so that the resulting map (also called f) is transverse to L. Set n_k = number of components of $f^{-1}(L)$ having Euler Characteristic equal to k. (Each component of $f^{-1}(L)$ is a compact, properly embedded surface in M.) Using surgery on the map f, induct on the set of finitely nonzero tuples $(\ldots, n_k, \ldots, n_{-1}, n_0, n_1, n_2)$ lexicographically ordered from the left (see [He_1], Lemma 6.5). ∎

The next two theorems give the most commonly used results for existence of incompressible surfaces in 3-manifolds.

III.10. THEOREM: Let M be a compact 3-manifold. Then M contains a two-sided incompressible surface if at least one of the following is true:

(1) $H_1(M, \mathbb{Q}) \neq 0$,

(2) $\pi_1(M) \approx A *_C B$ or $A *_C$, or

(3) $\partial M \neq \emptyset$.

Proof: In case (1) there exists a map $f : M \longrightarrow S^1$ which has the property that f cannot be homotoped off of any point of S^1. Set $N = S^1$ and L = a point in S^1 and apply Lemma III.9.

In case (2) there are classical constructions using either mapping cylinders or a mapping torus to construct a manifold N and a submanifold L so that there exists a map $f : M \longrightarrow N$ which has the property that f cannot be homotoped off of L. Again apply Lemma III.9.

In case (3) any properly embedded disk will work. This, however, is a rather noninstructuve surface; it is more instructive to note that if some component of ∂M is not S^2 and M is orientable, then (1) is also satisfied (see Exercise I.20). ∎

III.11. THEOREM: Let M be a noncompact 3-manifold. Then M contains a compact, two-sided, incompressible surface if at least one of the following is true:

(1) M has at least two ends, or

(2) $\pi_1(M)$ is not locally free.

Proof: In case (1) there exists a compact map of M onto $\mathbb{R}^1$.

In case (2) if $\pi_1(M)$ is not locally free, then $\pi_1(M)$ contains a finitely presented subgroup G which is neither infinite cyclic nor a nontrivial free product. Let X be a compact 2-complex with $\pi_1(X) \approx G$. Then there exists a map $g : X \longrightarrow M$ such that g_* induces an embedding of $\pi_1(X)$ onto G. Using surgery techniques (see the proof of the Corollary on page 343 of [J_1] for details), some neighborhood, U, of $g(X)$ has a boundary component which is incompressible. (If each component of ∂U is a 2-sphere, we invoke the hypothesis that M is not compact; hence, getting an incompressible 2-sphere. This is the only place, in case (2), that the hypothesis that M is noncompact is used.) ■

III.12. REMARK: Useful information may be gained in the case that the 3-manifold M contains a one-sided incompressible surface. In particular if M contains a properly embedded, one-sided surface, then M contains an incompressible, properly embedded, one-sided surface. Such a condition is satisfied if M is orientable, M does not contain an incompressible two-sided surface and $H_1(M; \mathbb{Z}_2) \neq 0$. I considered some of these problems in 1971. Most of the information that I gained was later published by Hempel ([He_2] and [He_1]). To date the most extensive work on one-sided surfaces is that by Rubenstein [R_1].

A surface F in a 3-manifold M is ∂-compressible (boundary-compressible) if either

(i) F is a disk and F is parallel to a disk in ∂M or

(ii) F is not a disk and there exists a disk $D \subset M$ such that $D \cap F = \alpha$ is an arc in ∂D, $D \cap \partial M = \beta$ is an arc in ∂D, with

$\alpha \cap \beta = \partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial D$, and either α does not separate F or α separates F into two components and the closure of neither is a disk (see Figure 3.4). Otherwise, F is ∂-incompressible.

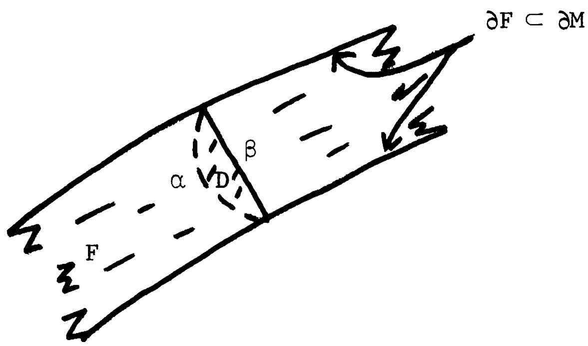


Figure 3.4

III.13. EXAMPLES:

(a) If F is an incompressible, two-sided surface in a solid torus, then F is homeomorphic to either a disk or an annulus. If F is an incompressible and ∂-incompressible surface in a solid torus, then F is homeomorphic to a disk.

(b) The only incompressible and ∂-incompressible surface in a cube-with-handles is a disk.

I want to give a construction for embedding incompressible surfaces in the product of a 2-manifold with an interval. I will give the construction in two special cases; the reader should then be able to generalize to the many possibilities that come out of this method. I first gave such a construction in [J_2].

III.14. EXAMPLES:

(a) Let S be a disk with two holes. Let α and β be essential spanning arcs in S as shown in Figure 3.5 and let $U(\alpha)$, $U(\beta)$ be regular neighborhoods of α, β respectively, having the property that $U(\alpha) \cap U(\beta) = \emptyset$. Denote the arcs in $\mathrm{Fr}\, U(\gamma)$ by γ^+, γ^- where $\gamma = \alpha, \beta$. Set $S' = S - (\mathring{U}(\alpha) \cup \mathring{U}(\beta))$.

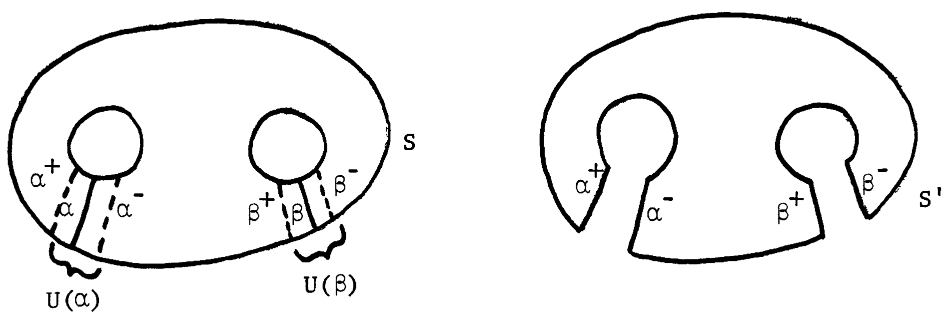


Figure 3.5

Given an integer $n \geq 1$, let $S_i' = S' \times \{\frac{i}{n+1}\}$, $1 \leq i \leq n$. Then $S_i' \subset S \times I$. Set $\gamma_i^+ = \gamma^+ \times \{\frac{i}{n+1}\}$, $\gamma_i^- = \gamma^- \times \{\frac{i}{n+1}\}$, $\gamma_0 = \gamma \times \{0\}$ and $\gamma_{n+1} = \gamma \times \{1\}$, $\gamma = \alpha, \beta$, $1 \leq i \leq n$. Let S_n be the surface obtained from the union $\bigcup_{i=1}^{i=n} S_i'$ by attaching γ_1^+ to γ_0, γ_i^+ to γ_{i-1}^- $(1 < i \leq n)$ γ_{n+1} to γ_n^- $(\gamma = \alpha, \beta)$ with "nearly vertical" disk (see Figure 3.6 for $n = 3$).

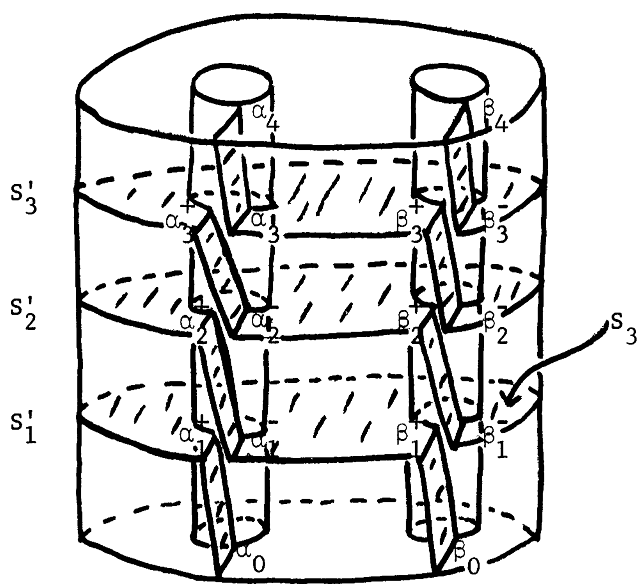


Figure 3.6

Using an induction argument, it is easily seen that (i) $\chi(S_n) = 2 - n$, (ii) for $n = 2k - 1$, an odd integer, the number of components of ∂S_n is one and the genus of S_n is $k - 1$, and (iii) for $n = 2k$, an even integer, the number of components of ∂S_n is two and the genus of S_n is $k - 1$.

There are different ways to see that S_n is incompressible. One method (geometric) is given in [J_2]. However, an algebraic method is available as follows: A set of free generators $\{z_1, \ldots, z_{n-1}\}$, for $\pi_1(S_n)$, a free group of rank $n - 1$, can be chosen so that inclusion sends $z_i \to x^i y^{-i}$ where $\{x, y\}$ are the obvious free generators of $\pi_1(S \times I)$, a free group of rank two. The subgroup generated by $\{xy^{-1}, \ldots, x^{n-1}y^{1-n}\}$ in the free group of rank two generated by $\{x, y\}$

is freely generated by these symbols.

(b) Let S be a closed, orientable surface having genus two. Let J be a nonseparating simple closed curve in S and let $U(J)$ be a regular neighborhood of J in S. Denote the s.c.c. components of $\mathrm{Fr}\, U(J)$ by J^+ and J^-. Set $S' = S - \overset{\circ}{U}(J)$. (See Figure 3.7.)

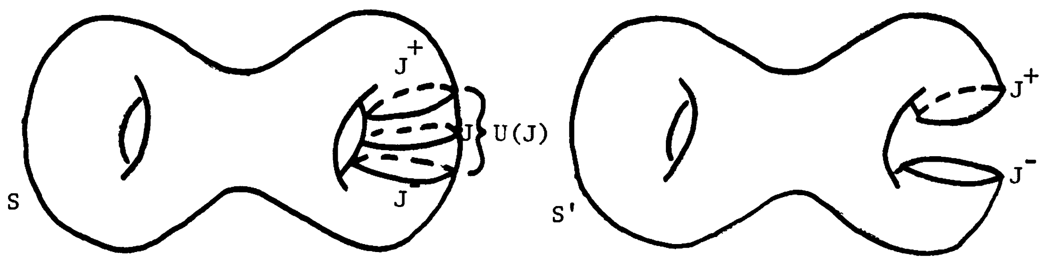


Figure 3.7

Given an integer $n \geq 1$, let $S_i' = S' \times \{\frac{i}{n+1}\}$, $1 \leq i \leq n$. Then $S_i' \subset S \times I$. Set $J_i^+ = J^+ \times \{\frac{i}{n+1}\}$, $J_i^- = J^- \times \{\frac{i}{n+1}\}$, $J_0 = J \times \{0\}$ and $J_{n+1} = J \times \{1\}$. Let S_n be the surface obtained from the union $\bigcup_{i=1}^{n} S_i'$ by attaching J_1^+ to J_0, J_i^+ to J_{i-1}^- $(1 < i \leq n)$ and J_{n+1} to J_n^- with "nearly vertical" annuli.

As before it is easily seen that

(i) $\chi(S_n) = -2n$

(ii) The number of components of ∂S_n is two and the genus of S_n is n.

In this case free generators $\{x_1, y_1, \ldots, x_n, y_n, z\}$ for $\pi_1(S_n)$, a free group of rank $2n + 1$, can be chosen so that inclusion sends $x_i \longrightarrow d^i a d^{-i}$, $y_i \longrightarrow d^i b d^{-i}$, $z \longrightarrow d^n c d^{-n}$ where $\{a, b, c, d\}$ are the obvious generators of $\pi_1(S)$. Again we have this subgroup a free subgroup which is freely generated by the selected generators.

III.15. CONCLUSION: Let H_2 be the cube-with-handles of genus two. Then for every genus g, H_2 contains a properly embedded, incompressible surface of genus g (with one boundary component). Analogous conclusions may be drawn for arbitrary cubes-with-handles. Notice that our construction gives nonseparating surfaces.

III.16. QUESTION: Does H_2 (respectively, H_n), the cube-with-handles of genus 2 (respectively, n), contain a separating incompressible surface of arbitrarily high genus?

Another interesting conclusion may be drawn from the above examples.

III.17. CONCLUSION: Using the notation of Example III.14, let $M = S \times S^1$. For each n, the surface S_n constructed in Examples III.14 gives rise to an incompressible surface $\bar{S}_n$ in M. Also, it can be shown, [J_2], that M fibers over S^1 with fiber $\bar{S}_n$. In particular, it follows that if S is the closed, orientable surface of genus 2, then for any integer n, $M = S \times S^1$ fibers over S^1 with fiber the closed, orientable surface of genus n. (In [Nw_1], D. Neumann shows that this phenomenon is quite common for $M = F \times_\varphi S^1$. Also, see [Th_1] for many examples of incompressible surfaces in bundles which are not fibers in any fibration.)

These examples show that a 3-manifold may admit many distinct embeddings of surfaces which are incompressible. On the other hand, there exists closed, orientable, irreducible 3-manifolds with infinite fundamental group which admit no embedded incompressible surfaces [W_2, Th_2, J-R]. I shall discuss these matters in later chapters.

I now want to consider the very important Kneser-Haken Finiteness Theorem [Kn_1, H_1, H_2]. I shall prove the theorem in the most general case for surfaces, with or without boundary, and for compact 3-manifold irreducible or not. Also, I shall present a special case of the new result of P. Shalen and myself, which eliminates the condition of ∂-incompressibility on the surfaces, which is necessary in the work of Haken ([H_1, H_2]).

A compact, orientable, irreducible 3-manifold is called a *Haken-manifold* if it contains a two-sided incompressible surface.

III.18. EXAMPLES:

(a) A 3-cell is a Haken-manifold.

(b) A knot space in S^3 is a Haken-manifold.

(c) The manifolds S^3, $S^2 \times I$, $S^2 \times S^1$ and $L(p, q)$ are not Haken-manifolds.

III.19. REMARK: A Haken-manifold is a $K(\pi, 1)$.

III.20. THEOREM: *Let M be a compact, orientable 3-manifold. There is a nonnegative integer $n_0(M)$ such that if $\{F_1, \ldots, F_n\}$ is any collection of pairwise disjoint, incompressible, and ∂-incompressible surfaces in M, then either $n < n_0(M)$ or for some $i \neq j$, F_i is parallel to F_j in M.*

Proof: I shall prove this theorem in a number of steps. I begin with M a special type of Haken-manifold; and then, by progressively relaxing conditions, I arrive at the situation of the theorem.

STEP 1: Suppose that M is a ∂-irreducible Haken-manifold; i.e. each component of ∂M is incompressible.

Proof in Step 1: The reader can easily fashion a proof after the proofs of Lemma 3.14 and Lemma 13.2 of [He_1], both of which are very well done. However, I do mention a significant difference. If T is a fixed triangulation of M, then the collection $\{F_1, \ldots, F_n\}$ can be put in "normal form" with respect to T; however, now there are two types of "disk replacements", both of which may be accomplished by an ambient isotopy of M. The first is precisely as the "D-modification" used in Lemma 3.14 of [He_1] and can be accomplished by an isotopy since each F_i $(1 \leq i \leq n)$ is incompressible and M is irreducible. The second is not considered in [He_1], since there each F_i $(1 \leq i \leq n)$ was assumed to be closed. For this second "disk-replacement" let $\overline{D}$ be a 2-cell in M with $\overline{D} \cap \cup F_i = \alpha$, an arc in $\partial\overline{D}$, $\overline{D} \cap \partial M = \beta$, an arc in $\partial\overline{D}$ with $\alpha \cap \beta = \partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial\overline{D}$. Then $\alpha \subset F_i$ for some i. Since each F_i is ∂-incompressible, there is a disk E in F_i such that $E \cap \partial M = \gamma$ is an arc in ∂E, $\alpha \cap \gamma = \partial\alpha = \partial\gamma$ and $\alpha \cup \gamma = \partial E$. Put $\overline{F}_i = (F_i - E) \cup \overline{D}$. This is called a $\overline{D}$-modification of $\{F_1, \ldots, F_n\}$. A $\overline{D}$-modification can be accomplished by an ambient isotopy of M, since M is ∂-irreducible and irreducible.

In Step 1, for a fixed triangulation T of M, the number $2\beta_1(M; \mathbf{Z}_2) + 6|T^3|$, where $|T^3|$ is the number of 3-simplexes of T, satisfies the conclusion of Theorem III.20.

STEP 2: _Suppose that_ M _is a Haken-manifold._

Now, there is the possibility that some component of ∂M is compressible. The argument that I give is fashioned after the arguments presented in §VI of [Wr_1]; however, I point out that I do not arrive at the same number given in the proof of Lemma B of [Wr_1]. I need two elementary results which I state as the next two lemmas.

III.21. LEMMA: _Let_ M _be a compact Haken-manifold. Then there is a finite collection of pairwise disjoint disks_ $D_1, \ldots, D_r$ _properly embedded in_ M _such that each component of_ M _split open at_ $\bigcup_1^r D_i$ _is a_ ∂_-irreducible Haken-manifold._

Proof: Induct on the complexity of M using $\chi(\partial M)$.

If M is a compact 3-manifold a collection $\{D_1, \ldots, D_r\}$ of pairwise disjoint, properly embedded disks in M is called a _complete system of disks for_ M if each component of M split at $\bigcup_1^r D_i$ is ∂-irreducible. (Sometimes it may be the case that for such a collection to be a complete system it also satisfies the condition that each member D_i is ∂-incompressible.)

III.22. LEMMA: _Let_ M _be a_ ∂_-irreducible Haken-manifold. Let_ $\{D_1, \ldots, D_r\}$ _be any collection of pairwise disjoint disks in_ ∂M. _If_ $\{E_1, \ldots, E_n\}$ _is any collection of pairwise disjoint, properly embedded disks in_ $M - \bigcup_1^r D_i$ _and_ $n \geq 2r$, _then either some_ E_i _is parallel to a disk in_ $\partial M - \bigcup_1^r D_i$ _or for some_ $i \neq j$, E_i _is parallel to_ E_j _in_ $M - \bigcup_1^r D_i$.

Proof: The proof is via induction on r. For $r = 1$, $n \geq 2$.

Since M is ∂-irreducible and irreducible, both E_1 and E_2 are parallel into ∂M. Hence either E_1 is parallel to E_2 in $M - D_1$, or at least one of E_1 or E_2 is parallel into $\partial M - D_1$.

Now assume that the lemma is true for all collections $\{D_1, \ldots, D_r\}$ where $r \leq k$, $k \geq 1$. Let $\{D_1, \ldots, D_{k+1}\}$ be a collection of $(k+1)$ pairwise disjoint disks in ∂M. Let $\{E_1, \ldots, E_n\}$ be a collection of pairwise disjoint, properly embedded disks in M, with $n \geq 2(k + 1)$. It follows from induction that either for some i, E_i is parallel to a disk in $\partial M - \bigcup_1^k D_i$ or for some $i \neq j$, E_i is parallel to E_j in $M - \bigcup_1^k D_i$. In the first situation either E_i is parallel to a disk in $\partial M - \bigcup_1^{k+1} D_i$ or E_i is parallel in $\partial M - \bigcup_1^k D_i$ to D_{k+1}. If for some i, E_i is parallel to a disk in $\partial M - \bigcup_1^{k+1} D_i$, then this is one of the possible conclusions and the induction step follows. Therefore assume that no E_i is parallel to a disk in $M - \bigcup_1^{k+1} D_i$. If E_i is parallel in $M - \bigcup_1^k D_i$ to D_{k+1}, then consider the collection $\{E_1, \ldots, E_n\} - \{E_i\}$. Since $n \geq 2(k + 1)$, $(n - 1) \geq 2k + 1$ and it follows from induction and the above assumption that either for some i', $E_{i'}$ is parallel in $M - \bigcup_1^k D_i$ to D_{k+1} or for some $i' \neq j$, $E_{i'}$ is parallel to E_j in $M - \bigcup_1^k D_i$. If $E_{i'}$ is parallel in $M - \bigcup_1^k D_i$ to D_{k+1}, then $E_{i'}$ is parallel to E_i in $M - \bigcup_1^{k+1} D_i$. This is one of the possible conclusions and the induction step follows. Therefore in addition to the previous assumption, assume further that only E_i is parallel in $M - \bigcup_1^k D_i$ to D_{k+1}. So, $E_{i'}$ is parallel to E_j in $M - \bigcup_1^k D_i$. The collection $\{E_1, \ldots, E_n\} - \{E_i, E_{i'}\}$ has $n - 2 \geq 2k$ disks and therefore by the induction step and all of the previous assumptions, it follows that for some $i'' \neq j'$, $E_{i''}$ is parallel in $M - \bigcup_1^k D_i$ to $E_{j'}$ (one of i'' on j' may be j). Therefore one of these pairs $E_{i'}, E_j$ or $E_{i''}, E_{j'}$ is parallel in $M - \bigcup_1^{k+1} D_i$. This is one of the possible conclusions and the induction step follows.

To complete the induction step, I may assume that for some $i \neq j$, E_i is parallel to E_j in $M - \bigcup_1^k D_i$ and that no E_i is parallel to a disk in $\partial M - \bigcup_1^k D_i$. This is one of the special situations considered above.

The induction step and, hence, the lemma follow. ∎

I shall now prove the theorem under the assumptions of Step 2.

Let Δ be any complete system of disks for M. Let $|\Delta|$ denote the cardinality of Δ and let $\{M_1, \ldots, M_k\}$ be the components of M split open at Δ. Define $n_0(M, \Delta) = \Sigma\, n_0(M_i) + 6|\Delta|(|\Delta| + 1)$, where $n_0(M_i)$ is the number for M_i determined in Step 1. Now, by Lemma III.21 M has a complete system of disk, say Δ_0. Set $n_0(M) = n_0(M, \Delta_0)$.

Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be any collection of incompressible and ∂-incompressible surfaces in M. If Δ is a complete system of disk in M and Δ is transverse to $\mathcal{F}$, define the <u>complexity of</u> Δ <u>and the collection</u> $\mathcal{F}$, written $\mathcal{C}(\Delta, \mathcal{F})$, to be the total number of components of the intersection of Δ and $\mathcal{F}$ (each component being either a spanning arc or a simple closed curve in some member of Δ and some F_i).

Now, among all collections Δ of complete systems of disks for M having the property that $n_0(M, \Delta) = n_0(M, \Delta_0)$, choose one having minimal complexity with the collection $\mathcal{F}$. Also denote this collection Δ

First, I shall prove that for this Δ, $\mathcal{C}(\Delta, \mathcal{F}) = 0$. Then I shall prove that if $\mathcal{C}(\Delta, \mathcal{F}) = 0$, then either $n < n_0(M)$ or for some $i \neq j$

the surface F_i is parallel to F_j.

Since the collection $\mathcal{F}$ is incompressible and M is irreducible, I can assume that no component of the intersection of Δ and $\mathcal{F}$ is a s.c.c. So, if $\mathcal{C}(\Delta, \mathcal{F}) \neq 0$, there exists a component of intersection, say α, which is a spanning arc in some member of Δ and in some $F_i \subset \mathcal{F}$. I may assume that α is "outermost" in the member of Δ. Since F_i is ∂-incompressible, α must separate F_i and one component of $F_i - \alpha$ must have closure which is a disk, say E. Consider $E \cap \Delta$. This is a collection (possibly empty) of spanning arcs in E along with α. Choose one of these, say β, which is "outermost away from α" or in the absence of any other components of intersection, set $\beta = \alpha$. Then $\beta \neq \alpha$ separates E into two disks and one, say E', has the property that $E' \cap \Delta = \beta$; or $\beta = \alpha$ and $E' = E$ has the property that $E' \cap \Delta = \beta$. Now, $E' \subset M_j$ for some j and since Δ is a complete system of disk, M_j is irreducible and ∂-irreducible. Hence E' splits M_j into M_j' and a 3-cell B; and $E' \cap \Delta = E' \cap D_r = \beta$ where β splits D_r into two disks D' and D" with D", say, in ∂B. Let Δ' be the new collection of disks obtained from Δ' by setting $D_i' = D_i$, $i \neq r$, and $D_r' = D' \cup E'$. Then Δ' is a complete system of disks for M and $\mathcal{C}(\Delta', \mathcal{F}) < \mathcal{C}(\Delta, \mathcal{F})$; yet, $|\Delta'| = |\Delta|$ and each component of M split at Δ' is homeomorphic to one and only one component of M split at Δ. It follows, that $n_0(M, \Delta') = n_0(M, \Delta) = n_0(M, \Delta_0)$. This contradicts our choice of Δ having the property that $\mathcal{C}(\Delta, \mathcal{F})$ is minimal for all complete systems of disk satisfying such an equality. So, we have established that $\mathcal{C}(\Delta, \mathcal{F}) = 0$.

Now, with $\mathcal{C}(\Delta, \mathcal{F}) = 0$, suppose that $n \geq n_0(M) = n_0(M, \Delta)$

$= \Sigma\, n_0(M_i) + 6|\Delta|(|\Delta| + 1)$. Since, $\mathcal{C}(\Delta, \mathcal{F}) = 0$, each F_i is contained in some component M_j of M split at Δ. Furthermore, a component M_j of M split at Δ has at most $2|\Delta|$ disk in its boundary corresponding to the splitting of M at Δ. If n_j denotes the number of members of $\mathcal{F}$ in M_j, then $\Sigma\, n_j = n$. On the other hand, there are at most $|\Delta| + 1$ components of M split at Δ and by assumption $n \geq n_0(M)$; hence, for some component M_j of M split at Δ, it must be true that $n_j \geq n_0(M_j) + 6|\Delta|$. Since each component of $\mathcal{F}$ is incompressible and ∂-incompressible in M; only those members of $\mathcal{F}$ in M_j which are disk are not incompressible and ∂-incompressible in M_j. However, since at most $2|\Delta|$ disk in ∂M_j corresponding to the splitting of M at Δ, it follows from Lemma III.22 that the number of members of $\mathcal{F}$ in M_j which are disk is either less than $4|\Delta|$ or two of them are parallel in a fashion which makes them parallel in M. So, I suppose that there are less than $4|\Delta|$ such disks from $\mathcal{F}$ in M_j. Hence, there must be at least $n_0(M_j) + 2|\Delta| + 1$ members of $\mathcal{F}$ in M_j which are incompressible and ∂-incompressible in M_j. Therefore, there are at least $2|\Delta| + 1$ distinct pairs which are parallel in M_j. Since at most $2|\Delta|$ disk in ∂M_j correspond to the splitting of M at Δ, it follows that at least one of these pairs is parallel in M. This completes this step

STEP 3. M _is an arbitrary compact_ 3-_manifold_.

If the reader survived Step 2, then this step is straightforward; simply replace Lemma III.21 by the prime decomposition theorem using 2-spheres now rather than disk. As a guide the reader may follow

Section VI of [Wr_1]. However, as I remarked earlier, beware of Lemma B on page 493 of [Wr_1]. In the proof of that lemma, n should be set equal to $n(\hat{C}^3) + 3k$. For one gets a guarentee of at most $n - 2k + 1$ of these surfaces incompressible in $\hat{C}^3$ (notation as in [Wr_1]). This completes the proof of Theorem III.20. ■

Let M be a compact 3-manifold. For M, the collection of nonnegative integers $n_0(M)$ satisfying the conclusion of Theorem III.20 is not empty. Set $h(M)$ equal to the minimal such number. The number $h(M)$ is called the _Haken number of_ M.

III.23. REMARK: If M is a compact 3-manifold there exists a nonnegative integer defined for M as above where only collections of _closed_ incompressible surfaces are considered. I shall denote this number by $\bar{h}(M)$ and call it the _closed Haken number of_ M when I have occasion to refer to it.

The next theorem is a new theorem due to P. Shalen and myself. I shall state it in general but prove it in a very simple special case.

III.24. THEOREM: _Let_ M _be a compact_ 3-_manifold_. _Then there is an integer_ $N_0(M)$ _such that if_ $\{F_1, \dots, F_n\}$ _is any collection of pairwise disjoint, incompressible surfaces in_ M, _then either_ $n < N_0(M)$, _some_ F_i _is an annulus or a disk parallel into_ ∂M, _or for some_ $i \neq j$, F_i _is parallel to_ F_j _in_ M.

SPECIAL CASE: _Assume that each_ F_i _in the collection is an annulus_.

Proof: Set $N_0(M) = 3h(M)$. I may assume that no F_i is parallel to an annulus in ∂M and at most $h(M) - 1$ of the annuli are also ∂-incompressible. Hence, there must be at least $2h(M)$ annuli in the collection none of which are parallel into ∂M and each of which is ∂-compressible. By successively performing ∂-compressions, I arrive at a collection of at least $2h(M)$ incompressible and ∂-incompressible disks in M. Now, by Theorem III.20 at least 3 of these disks are pairwise parallel. It is now straightforward to reconstruct the original annuli and conclude that at least two of them are parallel in M. ■

CHAPTER IV. HIERARCHIES

The results of this chapter are basic to the study of Haken manifolds; and they exhibit one of the major roles of incompressible surfaces. In Chapter II, I used the notion of a hierarchy for a 2-manifold. After reading this chapter, I believe that it would be instructive for the reader to think about 2-dimensional results based on inductive methods using hierarchies for 2-manifolds; and then try to find the analogous results for 3-manifolds and extend the methods. An example of this parallel is carried out between Theorem 13.1 and Theorem 13.6 of [He_1].

Let M be a compact 3-manifold. A <u>partial hierarchy</u> for M is a finite or infinite sequence of pairs

$$(M_1, F_1), \ldots, (M_n, F_n), \ldots$$

where $M_1 = M$, F_n is a two-sided, incompressible surface in M_n, which is not boundary-parallel, and M_{n+1} is the manifold obtained from M_n by splitting M_n at F_n (i.e. for some regular neighborhood $U(F_n)$ of F_n in M_n, then $M_{n+1} = M_n - \overset{o}{U}(F_n)$).

A partial hierarchy for M is called a <u>hierarchy for</u> M if for some n, each component of M_{n+1} is a 3-cell.

IV.1. REMARK: Necessarily, a <u>hierarchy for</u> M is a <u>finite</u> sequence of pairs $(M_1, F_1), \ldots, (M_n, F_n)$. For such a hierarchy, the integer n is called the <u>length of the hierarchy</u>.

IV.2. EXAMPLE: Partial hierarchies for a cube-with-handles of genus 2

(a) A hierarchy of length two (see Figure 4.1).

Let M be the cube-with-handles of genus 2. Let $M_1 = M$ and let F_1 be a nonseparating disk properly embedded in M_1. Let M_2 be M_1 split at F_1, then M_2 is a solid torus. Let F_2 be a non-separating disk properly embedded in M_2. The hierarchy for M is $(M_1, F_1), (M_2, F_2)$.

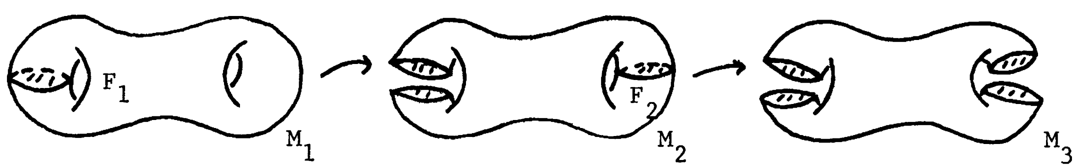


Figure 4.1

(b) An infinite partial hierarchy where each M_n is a cube-with-handles of genus 2 and each F_n is an incompressible annulus (see Figure 4.2).

Let M be the cube-with-handles of genus 2. Then M is homeomorphic to $T \times I$ where T is a once-punctured torus (i.e. T is homeomorphic to $S^1 \times S^1$ minus an open disk). Let J be a nonseparating simple closed curve in T. Set $M_1 = M$. There is a homeomorphism $h_1 : T \times I \longrightarrow M_1$. Set $F_1 = h_1(J \times I)$. For $n \geq 1$, define M_{n+1} to be M_n split at F_n. Then there is a homeomorphism $h_{n+1} : T \times I \longrightarrow M_{n+1}$. Set $F_{n+1} = h_{n+1}(J \times I)$. The sequence of pairs $(M_1, F_1), \ldots, (M_n, F_n), \ldots$ is the proposed partial hierarchy.

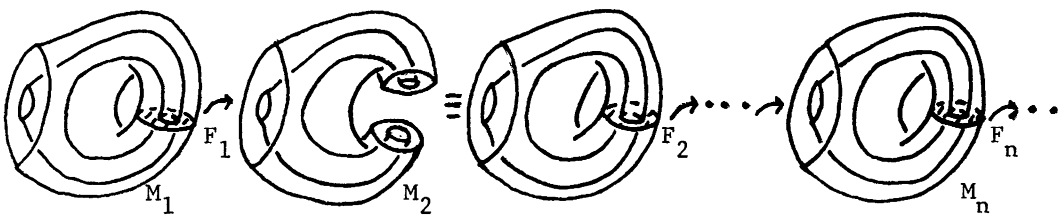


Figure 4.2

(c) An infinite partial hierarchy where each M_n $(n \geq 2)$ is a cube-with-handles of genus $(2^{n-1} + 1)$ and each F_n $(n \geq 2)$ is an incompressible, two-sided, nonseparating surface having $\chi(F_n) = -2^{n-1}$ (see Figure 4.3b).

Let M be the cube-with-handles of genus two. Then M is homeomorphic to $T \times I$ where T is the once-punctured torus. Set $M_1 = M$ and set $T_1 = T$. Using the construction of Example III.14 (using a single nonseparating s.c.c. J_1 in T_1 in this case (see Figure 4.3a)), there is an incompressible, two-sided, nonseparating surface F_1 in $M_1 = T_1 \times I$ having $\chi(F_1) = \chi(T_1) = -1$. Let M_2 be M_1 split at F_1, then M_2 is a cube-with-handles (Exercise I.32) and $\chi(M_2) = \frac{1}{2}\chi(\partial M_2) = \frac{1}{2}[\chi(\partial M_1) + 2\chi(F_1)] = -2$. So, M_2 has genus 3. Since M_2 is a cube-with-handles of genus 3, M_2 is homeomorphic to the product $T_2 \times I$ where T_2 is a twice-punctured torus (i.e. T_2 is homeomorphic to $S^1 \times S^1$ minus two disjoint open disks). Using the construction of Example III.14 (using a single nonseparating s.c.c. J_2 in T_2 in this case (see Figure 4.3a)), there is an incompressible, two-sided, nonseparating surface F_2 in $M_2 = T_2 \times I$ and $\chi(F_2) = \chi(T_2)$

Now, for $n > 2$ and each i, $2 \leq i < n$, suppose that M_i and F_i have been constructed satisfying the requirements of (c). Let M_n be M_{n-1} split at F_{n-1}. Then M_n is a cube-with-handles (Exercise I.32) and since $\chi(M_n) = \frac{1}{2}\chi(\partial M_n) = \chi(\partial M_{n-1}) = 2\chi(M_{n-1})$, it follows that genus $M_n = 1 - \chi(M_n) = 2^{n-1} + 1$ (by assumption, genus $M_{n-1} = 2^{n-2} + 1$ and $\chi(F_{n-1}) = -2^{n-2}$).

Since M_n is a cube-with-handles of genus $(2^{n-1} + 1)$, M_n is homeomorphic to the product $T_n \times I$ where T_n is a (2^{n-1})-punctured torus (i.e. T_n is homeomorphic to $S^1 \times S^1$ minus 2^{n-1} pairwise disjoint open disks). Using the construction of Example III.14 (again using a single nonseparating s.c.c. J_n in T_n in this case (see Figure 4.3a)), there is an incompressible, two-sided, nonseparating surface F_n in $M_n = T_n \times I$ and $\chi(F_n) = \chi(T_n) = -2^{n-1}$

The sequence $(M_1, F_1), (M_2, F_2), \ldots, (M_n, F_n), \ldots$ is the proposed partial hierarchy (see Figure 4.3b).

Notice that the sequence of genera for the cubes-with-handles M_n obtained in this example is $\{2, 3, 5, 9, 17, 33, \ldots, 2^{n-1}+1, \ldots$ $n \geq 2$. Furthermore, each F_n is planar.

IV.3. REMARK: Even though there is an analogy between the splitting of 3-manifolds at incompressible surfaces and the splitting of 2-manifolds at essential simple closed curves and spanning arcs, Example IV.2 should show that some caution is required in the 3-dimensional case. In (a) the manifold was simplified, in (b) the manifold was unchanged and in (c) the manifold was made more complex. It will be seen that the key to simplifying is to use incompressible and ∂-incompressible surfaces.

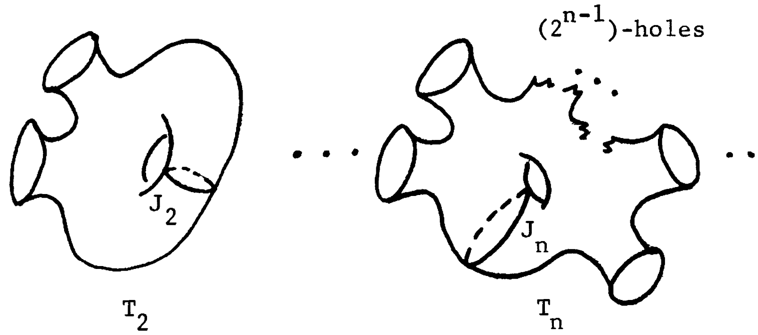


Figure 4.3a

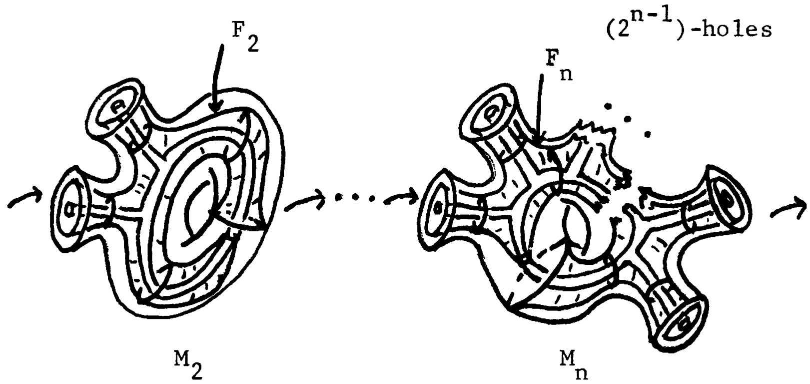


Figure 4.3b

IV.4. EXERCISE: Let M be a compact 3-manifold. Suppose that $(M_1, F_1), \ldots, (M_n, F_n), \ldots$ is a partial hierarchy for M. If for all but a finite number of n's the surface F_n is a disk, then the partial hierarchy is finite.

IV.5. EXERCISE: Let M be a compact 3-manifold. Suppose that $(M_1, F_1), \ldots, (M_n, F_n), \ldots$ is a partial hierarchy for M. If S is an incompressible, closed surface in M_n, then S is also incompressible in M.

The next lemma shows that given any partial hierarchy for M, the order of surfaces can be arranged in such a fashion to obtain a new partial hierarchy for M which postpones cuts along disks as long as possible.

IV.6. LEMMA: *Let M be a compact 3-manifold. Suppose that $(M_1, F_1), \ldots, (M_n, F_n), \ldots$ is a partial hierarchy for M and for some integer k at least k of the surfaces F_n are not disk. Then M has a partial hierarchy $(M_1', F_1'), (M_2', F_2'), \ldots, (M_k', F_k'), \ldots$ such that for $1 \leq j \leq k$, the surface F_j' is not a disk.*

Proof: This follows easily from induction and the observation that if M has a partial hierarchy $(M_1, F_1), \ldots, (M_n, F_n), \ldots$ where F_n is a disk, then M has a partial hierarchy $(M_1', F_1'), \ldots, (M_n', F_n'), \ldots$ where $F_i' \cong F_i$, $1 \leq i < n$, $F_n' \cong F_{n+1}$, $F_{n+1}' \cong F_n$ and for $i > n+1$, $F_i' \cong F_i$. ■

Recall that if M is a compact 3-manifold, then $\bar{h}(M)$ is the *closed* Haken-number for M; hence, if $\{S_1, \ldots, S_k\}$ is a collection of pairwise disjoint, closed, incompressible surfaces in M and $k > \bar{h}(M)$, then for some $i \neq j$, S_i is parallel to S_j in M (see Remark III.23).

IV.7. THEOREM: *Let M be a compact 3-manifold. Suppose that $(M_1, F_1), \ldots, (M_n, F_n), \ldots$ is a partial hierarchy for M. If for each n, the surface F_n is both incompressible and ∂-incompressible in M_n, then there exists at most $3\bar{h}(M)$ integers n for which F_n is not a disk.*

Proof: Suppose that for some integer k at least k of the surfaces F_n are *not* disk. It follows from Lemma IV.6 that I may assume that for $1 \leq j \leq k$, the surface F_j is not a disk. I must show that $k \leq 3\bar{h}(M)$.

To prove that $k \leq 3\bar{h}(M)$ I need the following lemma. This lemma is a simple generalization of Lemma III.21 and its proof follows from the same methods as those used in Step 2 of the proof of Theorem III.20. Note that the hypothesis of ∂-incompressibility is used here (it is also used at a crucial step later in the argument).

IV.8. LEMMA: *Let M be a compact 3-manifold and let F be a two-sided, incompressible and ∂-incompressible surface in M. Then there exists a complete system of disks $\{D_1, \ldots, D_r\}$ for M such that $F \cap \bigcup_1^r D_i = \emptyset$.* ■

I shall construct a collection $\{S_1, \ldots, S_k\}$ of pairwise disjoint, incompressible, closed surfaces in M. This is done as follows:

If F_j is closed and $1 \leq j \leq k$ set $S_j = F_j$. Then S_j is an incompressible, closed surface in M_j and by Exercise IV.5, S_j is an incompressible, closed surface in M. If F_j is not closed and $1 \leq j \leq k$, then F_j is a two-sided, incompressible and ∂-incompressible surface in M_j; hence, by Lemma IV.8 there exists a complete system of disks $\{D_1^j, \ldots, D_{r_j}^j\}$ for M_j such that $F_j \cap \bigcup_1^{r_j} D_i^j = \emptyset$. Let M_j' denote the component of M_j split at $\cup D_i^j$ that contains the surface F_j. Each component of $\partial M_j'$ is an incompressible, closed surface in M_j; let S_j' be a component of $\partial M_j'$ such that $S_j' \cap F_j \neq \emptyset$. Set S_j equal to the surface obtained by a small isotopy of S_j' into $\overset{\circ}{M}{}_j'$. Since S_j is an incompressible, closed surface in M_j, it follows from Exercise IV.5 that S_j is an incompressible, closed surface in M.

If $k > 3\bar{h}(M)$, then for some set of four distinct integers $\{p, q, r, s\}$, the surfaces S_p, S_q, S_r, and S_s are pairwise parallel in M. Hence, there are two possibilities:

(1) For at least two of the integers $\{p, q, r, s\}$, say p and q, $S_p = F_p$ and $S_q = F_q$ where both F_p and F_q are closed; or

(2) For at most one of the integers $\{p, q, r, s\}$, say s, $S_s = F_s$ where F_s is closed.

In case of (1), there is no loss to assume that $p < q$. However, it then follows that F_q is parallel into ∂M_q. This is a contradiction.

In case of (2), the three surfaces S_p, S_q, and S_r are pairwise parallel; so, one of them is "between" the other two, say S_p is "between S_q and S_r. Since S_p has only two-sides, one of S_q or S_r must be contained in M_p' (notation as above), say S_q. Therefore, $p < q$ and it follows that F_p is parallel in M_p' to a surface in $\partial M_p'$.

I claim that F_p is not parallel in M_p' to a surface in $\partial M_p'$. It is at this point that I use that F_j $(1 \leq j \leq k)$ is not a disk and is both incompressible and ∂-incompressible. For, if F_p is parallel to a surface F'_p in $\partial M'_p$, then F'_p is not in ∂M_p; but F'_p minus a finite union of disks is in ∂M_p. Now, since F_p is not a disk, F'_p is not a disk and there exists an essential spanning arc β in $F'_p \cap \partial M_p$. Using that F_p is parallel to F'_p in M'_p, there is an essential spanning arc α in F_p and a disk D in M_p such that $D \cap F_p = \alpha$, $\alpha \cup \beta = \partial D$ and $\alpha \cap \beta = \partial\alpha = \partial\beta$. This contradicts the hypothesis that F_p is ∂-incompressible in M_p.

For both (1) and (2) I arrive at a contradiction. Therefore $k \leq 3\bar{h}(M)$ is the only possibility. ■

IV.9. REMARK: By Exercise IV.4 and Theorem IV.7, any partial hierarchy for M in which each surface is both incompressible and ∂-incompressible is finite.

Now, I want to restrict my attention to partial hierarchies for Haken-manifolds. Here I shall state and prove, as a Corollary to Theorem IV.7, one of the most useful tools for the study of Haken-manifolds (it gives us the means for an inductive method of proof).

IV.10. EXERCISE: If $(M_1, F_1), \ldots, (M_n, F_n), \ldots$ is a partial hierarchy for the Haken-manifold M, then each M_n is a Haken-manifold.

IV.11. EXERCISE: If the Haken-manifold M is not a cube-with-handles, then there exists a two-sided, incompressible and ∂-incompressible surface F in M, which is not a disk (and is not parallel into ∂M). [Hint: Consider M split open at a complete system of disks for M.]

IV.12. COROLLARY: _A Haken-manifold has a hierarchy._

Proof: Let M be a Haken-manifold. Define a partial hierarchy for M as follows. Set $M_1 = M$. If M_1 is a cube-with-handles, then M_1 has a hierarchy and thus so does M. If M_1 is not a cube-with-handles, then by Exercise IV.11, M_1 contains a two-sided, incompressible and ∂-incompressible surface which is not a disk and is not parallel into ∂M_1. Let F_1 be such a surface in M_1.

Now, suppose that M is not a cube-with-handles and for some $n \geq 1$ the finite partial hierarchy $(M_1, F_1), \ldots, (M_n, F_n)$ for M has been defined such that each F_i is both incompressible and ∂-incompressible and is not a disk.

Let M_{n+1} be the manifold obtained by splitting M_n open at F_n. By Exercise IV.9 both M_n and M_{n+1} are Haken-manifolds. If each component of M_{n+1} is a cube-with-handles, then M_{n+1} has a hierarchy and thus so does M. If some component of M_{n+1} is not a cube-with-handles, then by Exercise IV.11, M_{n+1} contains a two-sided, incompressible and ∂-incompressible surface which is not a disk and is not parallel into ∂M_{n+1}. Let F_{n+1} be such a surface in M_{n+1}.

Then $(M_1, F_1), \ldots, (M_n, F_n), (M_{n+1}, F_{n+1})$ is a partial hierarchy for M such that each F_i is both incompressible and ∂-incompressible and is not a disk. However, by Theorem IV.7 the number of pairs in such a partial hierarchy for M is bounded above by $3\bar{h}(M)$. It follows from Exercise IV.11 that for some $n \leq 3\bar{h}(M)$, Each component of M_{n+1} is a cube-with-handles. Hence, M_{n+1} has a hierarchy; and therefore M has a hierarchy. ∎

IV.13. REMARK: Let M be a Haken-manifold. Let $(M_1, F_1), \ldots, (M_n, F_n)$

be a partial hierarchy for M such that each F_n is incompressible, ∂-incompressible and not a disk (necessarily a finite partial hierarchy). Under these conditions, the number $3\bar{h}(M)$ places a bound on the integers n; i.e. the number of pairs is always $\leq 3\bar{h}(M)$. Define the integer $\nu(M)$ to be the maximal number of pairs occuring in any such partial hierarchy for M. Then $\nu(M)$ is called the <u>length of</u> M. I was first introduced to this idea in a conversation with B. Evans. He tells me that W. Haken also used such a notion.

Clearly, whenever M is a Haken-manifold and F is an incompressible and ∂-incompressible surface in M, which is not a disk and is not parallel into ∂M, then the manifold, M', obtained by splitting M open at F, is a Haken-manifold and $\nu(M') < \nu(M)$. This allows us to use $\nu(M)$ for an inductive method of proof. (Notice that $\bar{h}(M)$ does not have this same property.)

There are some important differences between the use of the <u>length of</u> M and the use of the <u>length of a hierarchy for</u> M in an inductive method of proof. The former allows freedom of choice in finding the surface (of course, the surface must be incompressible, ∂-incompressible, not a disk and not parallel into ∂M); however, induction starts at an arbitrary cube-with-handles. The latter gives no freedom of choice in finding the surface (the surface comes as a member of a fixed hierarchy having a certain length (see Example IV.14)); however, in this case induction starts with the 3-cell.

There is another important invariant for an inductive method of proof. This is the <u>handle complexity</u> used by Waldhausen in [W_3]. Here, one also uses incompressible <u>and</u> ∂-incompressible surfaces (necessary to normalize). However, an advantage to handle complexity is that it allows the use of cuts along ∂-incompressible disks.

IV.14. EXAMPLE: There exists a Haken-manifold M such that for any integer n_0, M has a hierarchy of length greater than n_0.

Let $M = F \times S^1$ where F is a compact surface with $\partial F \neq \emptyset$ and $\chi(F) < 0$. For any integer n_0, there is a compact surface F' with $-\chi(F') \geq n_0$ and $M = F' \times_\varphi S^1$ (see III.17). Set $M_1 = M$, $F_1 = F'$ and set M_2 equal to M_1 split at F_1. Then $M_2 \cong F' \times I$ is a cube-with-handles and genus $M_2 = 1 - \chi(F')$. Hence, any hierarchy for M_2 has length greater than or equal to genus $M_2 = 1 - \chi(F') > n_0$ thereby giving M a hierarchy of length greater than n_0. Notice that such a hierarchy for M also has the property that each surface is incompressible and ∂-incompressible.

IV.15. EXERCISE: M a Haken-manifold. Show that $\bar{h}(M) = 0$ iff M is a cube-with-handles.

IV.16. EXERCISE: Give an example of a Haken-manifold M such that $\nu(M) = 3\bar{h}(M)$ (and M is not a cube-with-handles); such that $\nu(M) < 3\bar{h}(M)$. Notice that an example with $\nu(M) = 3\bar{h}(M)$ shows that $3\bar{h}(M)$ is, in general, the best possible bound for Theorem IV.7.

IV.17. EXERCISE: Give an example of a Haken-manifold M having the property that for any integer n_0 there exists a partial hierarchy $(M_1, F_1), \ldots, (M_{n_0}, F_{n_0}), \ldots$ for M such that each F_i is incompressible and ∂-incompressible, yet, M_{n_0+1} is <u>not</u> a cube-with-handles. (How does such an example co-exist with Theorem IV.7?)

IV.18. REMARK: Given a Haken-manifold M, it is possible to determine the length of a hierarchy for M.

IV.19. THEOREM: _Let_ M _be a Haken-manifold. If the_ 2-_manifolds in a hierarchy are not required to be connected, then_ M _has a hierarchy_

$$(M_1, F_1), (M_2, F_2), (M_3, F_3), (M_4, F_4)$$

of length four.

Proof: Set $M_1 = M$. Let F_1 be a maximal (in number) collection of pairwise disjoint, incompressible closed surfaces in M_1, none of which is ∂-parallel in M_1 and no two of which are parallel in M_1 (such a collection exists, although possibly empty). Let M_2 be M_1 split open at F_1 (if $F_1 = \emptyset$, then $M_2 = M_1$ and the hierarchy starts at (M_2, F_2)).

Each component of M_2 is a Haken-manifold and therefore contains an incompressible surface which is not ∂-parallel (otherwise, some component of M_2 is a 3-cell and hence M was a 3-cell). Let F_2 be a collection of pairwise disjoint, incompressible surfaces in M_2, no component of which is ∂-parallel and F_2 meets each component of M_2. Each component of F_2 has nonempty boundary. Let M_3 be M_2 split open at F_2.

Each component of M_3 is a "disk-sum" of cubes-with-handles and the product of a number of closed surfaces with I. If any of the latter appear, let F_3 be the union of essential annuli, one from each such product chosen to be properly embedded in M_3. Let M_4 be M_3 split open at F_3.

Each component of M_4 is a cube-with-handles. ■

IV.20. REMARK: Notice that in the preceeding theorem, each component of F_1 is closed, each component of F_3 is an annulus and each component of F_4 is a disk.

CHAPTER V. THREE-MANIFOLD GROUPS

There is a vast amount of information on three-manifold groups. In 1971, I wrote a set of notes on the subject [J_3]; since that time many questions posed in those notes have been answered and much new information has been gained. If such a set of notes were written today, it could easily be double in size. I shall be able to barely introduce the reader to the subject in this chapter. However, I shall present The Scott-Shalen Theorem, which says that a finitely-generated three-manifold group is finitely-presented; and I shall give some of the most outstanding problems associated with the study of three-manifold groups.

Suppose that $\langle X:R\rangle$ is a finite presentation of the group G. The <u>deficiency of</u> $\langle X:R\rangle$, written def $\langle X:R\rangle$, is Card(X) - Card(R); i.e. the number of generators of the presentation minus the number of relations of the presentation. If C is any group, I shall use the notation $\rho(C)$ to denote the minimum number of generators of C. The next result, due to D. Epstein [Ep_2], allows a definition for the deficiency of a finitely presented group.

V.1. LEMMA: <u>Let</u> G <u>be a finitely presented group</u>. <u>For any finite presentation</u> $\langle X:R\rangle$ <u>of</u> G, <u>the inequality</u> def $\langle X:R\rangle \leq \text{Rank } H_1(G; \mathbb{Z}) - \rho H_2(G; \mathbb{Z})$ <u>holds</u>. ■

Now, if G is a finitely presented group, the <u>deficiency of</u> G, written def G, is the supremum of the set $\{\text{def}\langle X:R\rangle : G \equiv \langle X:R\rangle$ is a finite presentation of $G\}$.

V.2. EXAMPLES: (a) If A is a finitely generated abelian group,

$$A = \bigoplus_1^r \mathbb{Z} + \bigoplus_1^s \mathbb{Z}_{p_i} \qquad (p_i \mid p_{i+1}, \quad 1 \leq i < s) \quad ,$$

then $\operatorname{def} A = r - \binom{r+s}{2}$ where $\binom{r+s}{2}$ is the binomial coefficient. In particular, $\operatorname{def}(\mathbb{Z} + \mathbb{Z} + \mathbb{Z}) = 0$ while $\operatorname{def}(\mathbb{Z} + \mathbb{Z} + \mathbb{Z} + \mathbb{Z}) = -2$.

(b) If F is a finitely generated free group, then $\operatorname{def} F = \operatorname{Rank} F$.

(c) If G has a one-relator presentation $\langle X:R \rangle$; i.e., $\operatorname{Card} R = 1$, and G is not free, then $\operatorname{def} G = \operatorname{def} \langle X:R \rangle$ (see [Ep_2]).

V.3. LEMMA: <u>Let</u> M <u>be any compact</u> 3-<u>manifold</u>. <u>Then</u>

$$\operatorname{def} \pi_1(M) \geq \begin{cases} 1 - \chi(M), & \partial M \neq \emptyset, \\ 0, & \partial M = \emptyset \end{cases} .$$

Proof: The 3-manifold M has a cell decomposition consisting of <u>one</u> 3-cell attached along a 2-complex C. The 2-complex C gives a natural presentation for $\pi_1(M) \equiv \langle X_C : R_C \rangle$ where X_C is a set of free generators for the 1-skeleton, $C^{(1)}$, of C and R_C is the set of words coming from the boundaries of the 2-skeleton, $C^{(2)}$, of C.

Let p_i denote the number of i-cells of C and let $C^{(i)}$ denote the i-skeleton of C, $0 \leq i \leq 2$. Then $\operatorname{Card}(X_C) = 1 - \chi(C^{(1)}) = 1 + p_1 - p_0$; and $\operatorname{Card}(R_C) = p_2$. By definition

$$\operatorname{def} \langle X_C : R_C \rangle = 1 + p_1 - p_0 - p_2 = 1 - \chi(C) \quad .$$

If $\partial M \neq \emptyset$, then $\chi(M) = \chi(C)$ and $\operatorname{def} \pi_1(M) \geq \operatorname{def} \langle X_C : R_C \rangle = 1 - \chi(M)$.

If $\partial M = \emptyset$, then $\chi(M) = \chi(C) - 1$ and $\operatorname{def} \pi_1(M) \geq \operatorname{def} \langle X_C : R_C \rangle$

$= -\chi(M) = 0.$ ■

V.4. EXERCISE: Suppose that M is an irreducible 3-manifold. Show that $H_2(M)$ is isomorphic to $H_2(\pi_1(M))$. [Note: $\pi_2(M)$ may not be trivial.]

V.5. EXERCISE: Let M be a compact, irreducible 3-manifold. Show that

$$\text{def } \pi_1(M) = \text{def } \langle X_C : R_C \rangle$$

where C is any cell-decomposition of M.

The next theorem first appeared in [J_1] (for an outline of the proof, see Theorem 8.1 of [He_1]).

V.6. THEOREM: <u>Let M be a 3-manifold and let G be any finitely presented subgroup of $\pi_1(M)$. Then there is a compact 3-manifold N and an immersion $f : N \to M$ such that $f_* : \pi_1(N) \to \pi_1(M)$ is an isomorphism onto G.</u> ■

V.7. REMARK: In [Sc_2], Scott improves Theorem V.6 to show that if the subgroup G is all of $\pi_1(M)$ $(G = \pi_1(M))$, then f can be taken to be an embedding. In [J_1], I could only do this in the case that $G = \pi_1(M)$ was indecomposable (with respect to free product).

I will now give an example of the use of the rather elementary information developed thus far in this chapter.

V.8. EXAMPLE: <u>The following is a complete list of all finitely generated abelian groups which are 3-manifold groups</u> (along with a representative manifold).

GROUP	1	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}_2$	$\mathbb{Z}_p$, $p \geq 2$
REALIZATION	$\mathbf{D}^2 \times I$	$\mathbf{D}^2 \times S^1$	$\mathbf{T}^2 \times I$	$\mathbf{T}^2 \times S^1$	$\mathbf{P}^2 \times S^1$	Lens Space $L(p,1)$

Proof: Let A be a finitely generated abelian group. Then A is a finitely presented group. If A is a 3-manifold group, then by Theorem V.6, there is a compact 3-manifold N with $\pi_1(N) \approx A$.

For the present I shall assume that N does not contain a two-sided $\mathbf{P}^2$. If any component of ∂N is a 2-sphere, it can be "capped-off" with a 3-cell without any alteration to $\pi_1(N)$. I shall assume that no component of ∂N is a 2-sphere. According to Remark I.11, either $\pi_2(N) = 0$ or $A \approx \pi_1(N)$ is infinite cyclic. I shall assume that $\pi_2(N) = 0$; hence, I can even go further and assume that N is irreducible (Theorem II.4).

Under the preceeding assumptions, $\chi(\partial N) \leq 0$; and therefore, $\chi(N) \leq 0$. So, by Lemma V.3, it follows that $\operatorname{def} A = \operatorname{def} \pi_1(N) \geq 0$. Now, I use Example V.2(a) to conclude that the only possibility for the group A is one of the groups 1, $\mathbb{Z}$, $\mathbb{Z} + \mathbb{Z}$, $\mathbb{Z} + \mathbb{Z} + \mathbb{Z}$, $\mathbb{Z} + \mathbb{Z}_p$ $(p \geq 2)$, or $\mathbb{Z}_p$ $(p \geq 2)$.

There is an easy realization for each of these groups with the exceptions of the groups $\mathbb{Z} + \mathbb{Z}_p$ $(p \geq 3)$; and no member of the groups $\mathbb{Z} + \mathbb{Z}_p$ $(p \geq 3)$ is a 3-manifold group. To see this, recall that $H_2(N)$ can only have torsion when N is closed and nonorientable; and then $H_2(N)$ has only one factor of 2-torsion. By Exercise V.4, $H_2(A) \approx H_2(N)$. However, $H_2(\mathbb{Z} + \mathbb{Z}_p)$ has p-torsion; and so, $p = 2$.

Now, suppose that I allow the possibility that N contains a two-sided $\mathbb{P}^2$. Then $A \approx \pi_1(N)$ is a split $\mathbb{Z}_2$ extension of (and hence a direct sum of $\mathbb{Z}_2$ and) one of the groups 1, $\mathbb{Z}$, $\mathbb{Z} + \mathbb{Z}$, $\mathbb{Z} + \mathbb{Z} + \mathbb{Z}$, $\mathbb{Z} + \mathbb{Z}_2$, or $\mathbb{Z}_p$ $(p \geq 2)$. Again, by applying Exercise V.4, the homology excludes any additional groups being added to the collection already obtained; and $\mathbb{P}^2 \times S^1$ gives a realization of the group $\mathbb{Z} + \mathbb{Z}_2$. ∎

V.9. REMARK: A nonfinitely generated abelian group is a 3-manifold group iff it is a subgroup of the additive group of rational numbers [E-M]. Realizations of such groups come from the fundamental group of the compliment of a solenoid in $\mathbb{R}^3$. Hence, all abelian groups that are 3-manifold groups are known.

Let $\tilde{G}$, G be groups and suppose that $\varphi : \tilde{G} \twoheadrightarrow G$ is a group epimorphism. Then φ is <u>decomposable</u> if there is a consistent diagram of group epimorphisms

$$\begin{array}{ccc} \tilde{G} & \overset{\varphi}{\twoheadrightarrow} & G \\ {\scriptstyle\alpha}\searrow & & \nearrow{\scriptstyle\beta} \\ & X * Y & \end{array}$$

where $X * Y$ is a non-trivial free product. Otherwise, φ is <u>indecomposable</u> If $\varphi : \tilde{G} \twoheadrightarrow G$ is indecomposable, then $\tilde{G}$ <u>indecomposably covers</u> G and φ is a <u>covering epimorphism</u>.

V.10. REMARK: If $\tilde{G}$ indecomposably covers G, then both $\tilde{G}$ and G are indecomposable groups.

V.11. EXERCISE: A group homomorphism $\tilde{\alpha} : \tilde{G} \longrightarrow \tilde{X} * \tilde{Y}$ is <u>inessential</u> if there is an inner automorphism γ of $\tilde{X} * \tilde{Y}$ such that $\gamma \circ \tilde{\alpha}(\tilde{G})$ is contained in either $\tilde{X}$ or $\tilde{Y}$. Otherwise, $\tilde{\alpha}$ is <u>essential</u>. Show that

$\varphi : \widetilde{G} \twoheadrightarrow G$ is indecomposable iff there is a consistent diagram of group homomorphisms

$$\begin{array}{ccc} \widetilde{G} & \overset{\varphi}{\twoheadrightarrow} & G \\ {\scriptstyle\widetilde{\alpha}}\searrow & & \nearrow{\scriptstyle\widetilde{\beta}} \\ & \widetilde{X} * \widetilde{Y} & \end{array}$$

where $\widetilde{X} * \widetilde{Y}$ is a non-trivial free product and $\widetilde{\alpha}$ is essential.

V.12. EXAMPLE: Suppose that G is a two generator group (i.e. $\rho(G) = 2$). Then either

(i) $G \approx A * B$ where both A and B are nontrivial cyclic groups; or

(ii) G can be indecomposably covered by a finitely presented group $\widetilde{G}$ with $\rho(\widetilde{G}) = 2$.

This is rather straightforward to establish; however, it does use that a free product of cyclic groups is Hopfian (a group is Hopfian iff it is not a proper quotient of itself). The result appeared in [J-M]. (I always liked this particular manuscript, and I think it has a lot of interesting material; however, we decided not to publish it after it was rejected by Topology.)

It is given that $\rho(G) = 2$. Suppose that (i) does not occur. Let F be the free group of rank 2 and let $\theta : F \twoheadrightarrow G$ be an epimorphism of F onto G. Among all consistent diagrams of epimorphisms

$$\begin{array}{ccc} F & \overset{\theta}{\twoheadrightarrow} & G \\ {\scriptstyle\alpha}\searrow & & \nearrow{\scriptstyle\beta} \\ & A * B & \end{array}$$

with $A \neq 1 \neq B$ (so both A and B are nontrivial and cyclic), choose

A, B, α and β so that A and B have the smallest possible orders, where first the order of A is minimized and then the order of B is minimized. Since (i) does not occur, $\ker(\beta) \neq 1$. Let $1 \neq \omega \in \ker(\beta)$. Set $\tilde{G} = A * B/\langle\omega\rangle$; let $\varphi : \tilde{G} \twoheadrightarrow G$ be the natural epimorphism.

I shall show that $\tilde{G}$ indecomposably covers G with covering epimorphism φ. Since $A * B$ is finitely presented, $\tilde{G}$ is finitely presented.

Suppose that there is a consistent diagram of epimorphisms

$$\begin{array}{ccccccc} F & \twoheadrightarrow & A * B & \twoheadrightarrow & \tilde{G} & \xrightarrow{\varphi} & G \\ & & & & \searrow & & \nearrow \\ & & & & & C * D & \end{array}$$

with $C \neq 1 \neq D$ (so both C and D are nontrivial and cyclic). At least one of C or D must have finite order. For otherwise, $C * D$ is isomorphic to F and the sequence of epimorphisms $F \twoheadrightarrow A * B \twoheadrightarrow \tilde{G} \twoheadrightarrow C * D$ would contradict that F is Hopfian. Choose the notation so that $|C| \leq |D|$ where $|\ |$ denotes order. By the choice of A and B, it follows that $|A| \leq |C|$ and $|B| \leq |D|$. Now, pass to the first homology and use that the epimorphism $A * B \twoheadrightarrow \tilde{G} \twoheadrightarrow C * D$ above induces an epimorphism of $A \times B$ onto $C \times D$ to conclude that strict inequality cannot hold (i.e. $|A| = |C|$ and $|B| = |D|$). So, $A * B$ is isomorphic to $C * D$; however, the epimorphism $A * B \twoheadrightarrow \tilde{G} \twoheadrightarrow C * D$ has $\omega \neq 1$ in its kernal. This contradicts the fact that $A * B$ is Hopfian. ∎

The importance of a 3-manifold group being indecomposably covered is captured by the next theorem (see [G-H-M], Theorem 2.1).

V.13. THEOREM: *Let* M *be a* 3-*manifold. Suppose that* $\pi_1(M)$ *can be indecomposably covered by a finitely presented group* $\tilde{G}$. *Then there exists a compact manifold* N *embedded in* M *such that* $\pi_1(N) \hookrightarrow \pi_1(M)$ *is an isomorphism. In particular,* $\pi_1(M)$ *is finitely presented.*

Proof: I shall give the proof from [J_2] (see Remark V.7) where I obtained the same conclusion with the hypothesis that $\pi_1(M)$ was finitely presented and indecomposable.

By hypothesis there is a finitely presented group $\tilde{G}$ indecomposably covering $\pi_1(M)$ with covering epimorphism φ. Let K be a finite 2-complex with $\pi_1(K) \approx \tilde{G}$. Then there is a map $f : K \longrightarrow \mathring{M}$ with $f_* = \varphi$. Let $U = U(f(K))$ be a regular neighborhood of $f(K)$ in $\mathring{M}$. Then there is a consistent diagram of maps

$$\begin{array}{ccc} \tilde{G} & \overset{\varphi}{\twoheadrightarrow} & \pi_1(M) \\ & {\scriptstyle \alpha_U} \searrow \quad \nearrow {\scriptstyle \beta_U} & \\ & \pi_1(U) & \end{array}$$

where U is a compact submanifold of $\mathring{M}$ and β_U is induced by inclusion.

For any compact submanifold U of M let n_k denote the number of boundary components of U with Euler Characteristics equal to k. The complexity of U is defined to be the tuple (finitely nonzero) $(\ldots, n_k, \ldots, n_{-1}, n_0, n_1, n_2)$. The set of all such tuples is lexicographically ordered from the left (see the proof of Lemma III.9).

Now, consider the entire set of compact submanifolds U of $\mathring{M}$ such that there exists homomorphisms $\alpha_U : \tilde{G} \longrightarrow \pi_1(U)$ and $\beta_U : \pi_1(U) \longrightarrow \pi_1(M)$, induced via inclusion, with $\beta_U \circ \alpha_U$ conjugate to φ. Choose one, say N, such that the complexity of N is a minimum.

I claim that each component of ∂N is incompressible in M or $\pi_1(M)$ is cyclic. If some component of ∂N is compressible in M, then there is a component B of ∂N and a disk $D \subset M$ such that $D \cap \partial N = D \cap B = \partial D \cap B = \partial D$ is not contractible in B.

If D is not contained in N, then simply add a two-handle to N at D to obtain N'. The inclusion map $\pi_1(N) \hookrightarrow \pi_1(M)$ factors through the inclusion map of $\pi_1(N') \hookrightarrow \pi_1(M)$. However, the complexity of N' is strictly less than the complexity of N. This contradicts our choice of N.

If D is contained in N, then $\pi_1(N)$ splits as a nontrivial free product $A * B$ or $\pi_1(N)$ is infinite cyclic. If $\pi_1(N)$ is infinite cyclic, then $\pi_1(M)$ is cyclic. If $\pi_1(N)$ splits as a nontrivial free product $A * B$, then there is a consistent diagram of maps

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\varphi_N} & \pi_1(M) \\ & {}_{\alpha_N}\searrow & \uparrow \beta_N \\ & & \pi_1(N) \approx A * B \end{array}$$

where φ_N is conjugate to φ (and therefore is a covering epimorphism) It follows from Exercise V.11 that α_N is inessential; hence, I may choose notation so that the image $\alpha_N(\tilde{G})$ is contained in A (after possibly a conjugation in $\pi_1(N)$). Let N' be the component of N obtained by surgery along D so that $\pi_1(N') = A$. The complexity of N' is strictly less than the complexity of N. This contradicts our choice of N. So, I have established the claim.

If $\pi_1(M)$ is infinite cyclic, then there is a solid torus $N \subset \mathring{M}$ such that $\pi_1(N) \hookrightarrow \pi_1(M)$ is an isomorphism. If $\pi_1(M)$ is finite cyclic, then the results of [Ep_2] find a compact submanifold N of M (each component of ∂N is a 2-sphere or projective plane) such that $\pi_1(N) \hookrightarrow \pi_1(M)$ is an isomorphism. If $\pi_1(M)$ is not cyclic, then the N found above has incompressible boundary. Since $\pi_1(N) \hookrightarrow \pi_1(M)$ is onto, it is an isomorphism. ■

To prove that a finitely generated 3-manifold group is finitely presented, it is sufficient to prove that a finitely generated, indecomposable 3-manifold group can be indecomposably covered by a finitely presented group. In light of Example V.12 this leads to the following question:

V.14. QUESTION: Can each finitely generated, indecomposable group be indecomposably covered by a finitely presented group?

I do not know the answer to this question; however, Peter Scott and Peter Shalen independently answered it for 3-manifold groups.

V.15. THEOREM: *Let G be a finitely generated, indecomposable group. If G is isomorphic to the fundamental group of some 3-manifold, then G can be indecomposably covered by a finitely presented group.*

Proof: The proof of this theorem can be found in [Sc_1] or extracted from the proofs of Lemma 8.5 and Theorem 8.2 of [He_1] by inducting on $\rho(G)$. ■

V.16. COROLLARY (SCOTT-SHALEN): *Let G be a finitely generated group. If G is isomorphic to the fundamental group of some 3-manifold, then G is finitely presented.*

Before leaving the concept of a group indecomposably covering a group, I would like to give another example.

V.17. EXAMPLE: Given two finitely presented, indecomposable groups $\tilde{G}$ and G, one can ask if $\tilde{G}$ indecomposably covers G; i.e., does there exist a covering epimorphism $\varphi : \tilde{G} \twoheadrightarrow G$. In particular, suppose that $\tilde{G}$ is the group

$$\tilde{G} \equiv \langle x_1, y_1, \ldots, x_n, y_n : \prod_1^n [x_i, y_i] \rangle \quad .$$

Let F_n be the free group of rank n. Now, there are many different epimorphisms $\varphi : \tilde{G} \twoheadrightarrow F_n \times F_n$.

V.18. QUESTION: Does $\tilde{G}$ indecomposably cover $F_n \times F_n$?

If the answer to Question V.18 is negative, then the 3-dimensional Poincaré Conjecture is true [St_5]. If the answer to Question V.18 is positive, then the 3-dimensional Poincaré Conjecture is false [J_4].

There are a couple of other aspects of the study of 3-manifold groups that really intrigue me. One of these is the <u>finitely generated intersection property</u> (hereafter always referred to as f.g.i.p.). If G is a group, then a finitely generated subgroup H of G has f.g.i.p. if for every finitely generated subgroup K of G, the group $H \cap K$ is finitely generated. If every finitely generated subgroup of G has f.g.i.p., then G is said to have f.g.i.p.

V.19. EXAMPLES:

(a) A free group has f.g.i.p. (see [J-M] for a geometric proof).

(b) The fundamental group of a surface has f.g.i.p. (see [J-M] for a geometric proof).

(c) Let F_n be the free group of rank n. The group $F_n \times \mathbf{Z}$, $n \geq 2$, does <u>not</u> have f.g.i.p.

To see this suppose that F_n is free on the set $\{x_1, \ldots, x_n\}$. Let t denote the generator of $\mathbf{Z}$. Let H be the subgroup generated by $\{x_1, \ldots, x_{n-1}, tx_n\}$. Then if $K = F_n$, the intersection $H \cap K$ is the infinitely generated free group generated by $\{x_n^i x_j x_n^{-1} : 1 \leq j < n, -\infty < i < \infty\}$.

This example can be studied geometrically by considering the examples given in Example III.14. Also, see Conclusion III.17. In this way a 3-manifold M can be constructed having two-sided incompressible surfaces S_1 and S_2 such that $S_1 \cap S_2$ is a single arc; yet $\pi_1(S_1) \cap \pi_1(S_2)$ is <u>not</u> a finitely generated group.

The group $F_n \times \mathbf{Z}$ of Example V.19 (c) is a 3-manifold group. Hence, in general, a 3-manifold group does not have f.g.i.p. However, I believe that a slight generalization of example (c) may essentially describe the extend to which a 3-manifold group does not have f.g.i.p. I give this as the next example. It depends strongly on the results of Chapter VII and Chapter VIII, as well as some knowledge from Chapter VI; however, I believe it fits in best with the present discussion.

(d) (with B. Evans) Let $M = F \times_\varphi S^1$ be a bundle over S^1 with fiber F. If $\chi(F) < 0$, then there is a two-generator subgroup H of $\pi_1(M)$ such that $H \cap \pi_1(F)$ is a nonfinitely generated free group.

I need to make two definitions. A 3-manifold M is <u>algebraically simple</u> if each $\mathbf{Z} + \mathbf{Z}$ subgroup of $\pi_1(M)$ is peripheral (for the definition of peripheral, see Exercise I.5); M is (geometrically) <u>simple</u> if each incompressible torus embedded in M is parallel to a torus in ∂M.

A Haken-manifold which is not simple is not algebraically simple. Conversely, as a result of the version of the torus theorem (Chapter VIII) due to Shalen and myself, a Haken-manifold which is not algebraically simple, is either a special Seifert fibered manifold or is not simple. I need another result due to Shalen and myself, [J-S_1]: If the Haken-manifold M is algebraically simple and H is a two-generator subgroup of $\pi_1(M)$, then either

(i) H is free,

(ii) H is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and is peripheral, or

(iii) The index of H in $\pi_1(M)$ is finite.

I shall now establish the claims of the example. I do this first in a special case.

<u>CASE</u> 1. Let $M = F \times_\varphi S^1$ be a bundle over S^1 with fiber F. If $\chi(F) < 0$ <u>and</u> M is algebraically simple, then there is a two-generator subgroup H of $\pi_1(M)$ such that $H \cap \pi_1(F)$ is a nonfinitely generated free group.

Proof in Case 1: Since $\chi(F) < 0$, there exists an $x \in [\pi_1(F), \pi_1(F)]$, which is not peripheral. Let $t \in \pi_1(M)$ map onto the generator of the fundamental group of the circle under the given fibration of M. Set $H = gp(x, t)$, the subgroup of $\pi_1(M)$ generated by x and t.

If $y \in H$, then $y = t^{\delta_1} x^{\varepsilon_1} \cdots t^{\delta_k} x^{\varepsilon_k}$. If y is also in $\pi_1(F)$, then $\Sigma\, \delta_i = 0$. Hence, if $y \in H \cap \pi_1(F)$, $y \in gp(t^{-i} x t^i : -\infty < i < \infty)$. Conversely, since $\pi_1(F)$ is normal in $\pi_1(M)$, $gp(t^{-i} x t^i : -\infty < i < \infty) \subset \pi_1(F)$ and so, it is contained in $H \cap \pi_1(F)$. Furthermore, since $x \in [\pi_1(F), \pi_1(F)]$ and $[\pi_1(F), \pi_1(F)]$ is a characteristic subgroup of $\pi_1(F)$,

$H \cap \pi_1(F) \subset [\pi_1(F), \pi_1(F)]$.

By choosing x not peripheral, H is not peripheral. Also, the index of H in $\pi_1(M)$ is infinite; in fact if f_1 and f_2 are in distinct cosets of $[\pi_1(F), \pi_1(F)]$ in $\pi_1(F)$, then f_1 and f_2 are in distinct cosets of H in $\pi_1(M)$. (If $f_1, f_2 \in \pi_1(F)$ and $f_1f_2^{-1} \in H$, then $f_1f_2^{-1} \in H \cap \pi_1(F) \subset [\pi_1(F), \pi_1(F)]$.)

By the result quoted above, the only possibility is that H is free. Clearly, the rank of H is two and so H is freely generated by x and t. The subgroup $H \cap \pi_1(F)$ is therefore a nonfinitely generated free group and is freely generated by the set $\{t^{-i}xt^{i} : -\infty < i < \infty\}$.

<u>CASE</u> 2. The general case.

The proof is by induction on $-\chi(F) \geq 1$.

Suppose that $\chi(F) = -1$. Then F is either a disk-with-two-holes or a once-punctured torus.

If F is a disk-with-two-holes, then φ is isotopic to a periodic homeomorphism and M has a finite sheeted covering $(\widetilde{M}, p)$ with $\widetilde{M}$ homeomorphic to $F \times S^1$. Since $\pi_1(\widetilde{M}) = \pi_1(F) \times \mathbf{Z}$ and $\pi_1(F)$ is free of rank two, the subgroup H of example (c) has the property that H is a two-generator group and $H \cap \pi_1(F)$ is not finitely generated. The monomorphism p_* carries these properties into $\pi_1(M)$.

If F is a once-punctured torus, then either M is algebraically simple, M is not simple or M is a special Seifert fibered manifold. If M is algebraically simple, then by Case 1, $\pi_1(M)$ contains the desired subgroup. If M is not simple, then there is an incompressible torus T embedded in M which is not parallel into ∂M. By considering the possibilities for the components of $T \cap F$, it is easy to conclude that there is a nonboundary parallel simple closed curve J in F and either $\varphi(J) = J$ or $\varphi^2(J) = J$. Hence, there is a finite sheeted covering $(\tilde{M}, p)$ of M (having one or two sheets), $\tilde{M} = F \times_{\tilde{\varphi}} S^1$ where $\tilde{\varphi}(J) = J$ $(\tilde{\varphi} = \varphi$ or $\tilde{\varphi} = \varphi^2)$. The annulus $J \times I$ in $F \times I$ identifies to a torus $\tilde{T}$ in $\tilde{M}$ and $\tilde{M}$ split at $\tilde{T}$ fibers over S^1 with fiber F' which is F split at J; hence, F' is a disk-with-two-holes. The previous considerations give a two-generator subgroup H of $\pi_1(M')$ which has the property that $H \cap \pi_1(F')$ is not finitely generated. The inclusion induced monomorphisms of $\pi_1(M') \hookrightarrow \pi_1(M)$ and $\pi_1(F') \hookrightarrow \pi_1(F)$ and the monomorphism p_* carry these properties into $\pi_1(M)$. If M is a special Seifert fibered manifold, then φ is periodic (Chapter VI) and M has a finite sheeted covering $(\tilde{M}, p)$ with $\tilde{M}$ homeomorphic to $F \times S^1$. Now, there is a simple closed curve J in F which is not parallel into ∂F and a torus $\tilde{T} = J \times S^1$ in $\tilde{M}$ such that $\tilde{M}$ split at $\tilde{T}$ is homeomorphic to $F' \times S^1$ where F' is F split at J; hence, F' is a disk-with-two-holes. Again, the previous considerations give the conclusion.

The induction argument follows the above lines. Namely, suppose that for any bundle $M' = F' \times_{\varphi'} S^1$ where $-\chi(F') < n$ $(n \geq 2)$ there is a two-generator subgroup H of $\pi_1(M')$ and $H \cap \pi_1(F')$ is not finitely generated.

Let $M = F \times_{\varphi} S^1$ where $-\chi(F) = n$ $(n \geq 2)$. Then either M is algebraically simple, M is not simple or M is a special Seifert fibered manifold. If M is algebraically simple, then by Case 1, $\pi_1(M)$ contains the desired subgroup. If M is not simple, then as before there is an incompressible torus T embedded in M which is not parallel into ∂M. In this case there is a nonboundary parallel simple closed curve J in F and for some integer k, $\varphi^k(J) = J$. The argument is exactly as before only $(\tilde{M}, p)$ is a k-sheeted covering of M, $\tilde{M} = F \times_{\tilde{\varphi}} S^1$ where $\tilde{\varphi}(J) = J$ $(\tilde{\varphi} = \varphi^k)$, and after splitting $\tilde{M}$ at the torus $\tilde{T} = J \times I/\tilde{\varphi}$, the argument is completed by the use of induction. If M is a special Seifert fibered manifold, then φ is periodic and M has a finite sheeted covering $(\tilde{M}, p)$ with M homeomorphic to $F \times S^1$. There is a simple closed curve J in F which is not parallel into ∂F and if $\tilde{M}$ is split open at the torus $\tilde{T} = J \times S^1$, then induction gives the desired conclusions. ∎

I shall finish my discussion of f.g.i.p. with a very useful result. It first appeared in [J-M]. An updated proof is being published in [J-S_2].

V.20. THEOREM: Let M be a 3-manifold. Then any finitely generated, peripheral subgroup of $\pi_1(M)$ has f.g.i.p.

As I mentioned at the beginning of this Chapter, there are many aspects to the study of 3-manifold groups. However, I shall conclude this chapter with just a few of the more intriguing questions.

V.21. QUESTION: Suppose that M is a closed, irreducible 3-manifold with $\pi_1(M)$ infinite (or even say, M has a hyperbolic structure). Does $\pi_1(M)$ contain a subgroup isomorphic to the fundamental group of a closed surface?

V.22. QUESTION: If G is a group, then a subgroup H of G is subgroup separable if H is an intersection of subgroups of finite index (equivalently, for each $g \in G - H$ there is a homomorphism φ mapping G onto a finite group with $\varphi(g) \notin \varphi(H)$).

Is each finitely generated subgroup of the fundamental group of a Haken-manifold subgroup separable?

Is each finitely generated subgroup of the fundamental group of a Haken-manifold M, which is isomorphic to $\pi_1(F)$ for some two-sided incompressible surface $F \subset M$, subgroup separable?

Is each finitely generated, peripheral subgroup of the fundamental group of a Haken-manifold subgroup separable?

W. Thurston claims that the answer to this last question is positive if one also assumes that the peripheral subgroup is $\mathbb{Z} + \mathbb{Z}$. Notice that $\{1\} \subset G$ being subgroup separable is the familiar property of G being residually finite.

V.23. QUESTION: A subgroup H of the fundamental group of a manifold M is geometric in $\pi_1(M)$ if there is a codimension zero submanifold $N \subset M$ such that $\pi_1(N) \hookrightarrow \pi_1(M)$ is an isomorphism onto a conjugate of H in $\pi_1(M)$. The subgroup H is almost geometric if there is a finite sheeted covering $(\tilde{M}, p)$ of M such that $p_*^{-1}(H \cap p_*\pi_1(\tilde{M}))$ is geometric in $\pi_1(\tilde{M})$.

Is every finitely generated subgroup of the fundamental group of a Haken-manifold almost geometric? (See [Sc_3].)

Is every finitely generated, indecomposable subgroup of the fundamental group of a Haken-manifold almost geometric?

Is every finitely generated, peripheral subgroup of the fundamental group of a Haken-manifold almost geometric?

CHAPTER VI. SEIFERT FIBERED MANIFOLDS

This is a chapter that has been very difficult for me to be successful at trying not to cover too much. There is a considerable amount of literature on Seifert fibered manifolds. To start, there is Seifert's paper [Se_1] (or an excellent English translation by W. Heil); there is the paper by Orlik, Vogt and Zieschang [O-V-Z]; and there is the book [O_1] by Orlik. Much of the topology of Seifert fibered manifolds, of interest to me, was introduced in the papers by Waldhausen [W_2, W_4]. This approach was taken by Hempel in Chapter 12 of his book [He_1]; and it is this approach that is closest in spirit to the study by Shalen and myself in [J-S_1] and to the study presented here. There will, naturally, be a considerable overlap of the material of this Chapter with these and other manuscripts on Seifert fibered manifolds. In fact, there will be no new results in the sense of formerly being completely unknown in the existing literature. This presentation is designed primarily for support of the material of Chapter VIII-Chapter X. In the later chapters, I restrict any considerations to orientable 3-manifolds and therefore only need to draw from the study of orientable Seifert fibered manifolds. However, it was my original plan to present Seifert fibered manifolds, orientable or nonorientable. Having progressed quite a way in this generality, I was finally convinced that an adequate presentation of the nonorientable case must allow a generalization of the classical concept of Seifert fibered manifold to include fibered neighborhoods which are homeomorphic to a fibered solid Klein bottle. It was then very apparent that to present Seifert fibered manifolds in this generality was beyond the scope of these lectures. So, I have, quite reluctantly, settled on presenting a view of orientable Seifert fibered manifolds. I hope that it will serve as an introduction to this important class of 3-manifolds for the beginners in the subject and also provide some enjoyable reading for the more advanced.

Let (μ, ν) be a pair of relatively prime integers. Let $\mathbf{D}^2$ be the unit disk in $\mathbf{R}^2$ defined in polar coordinates as $\mathbf{D}^2 = \{(r, \theta) : 0 \leq r \leq 1\}$. A fibered solid torus of type (μ, ν) is the quotient of the cylinder $\mathbf{D}^2 \times I$ via the identification $((r, \theta), 1) = ((r, \theta + \frac{2\pi\nu}{\mu}), 0)$. The fibers are the image of the arcs $\{x\} \times I$. Thus the core (or former cylinder axis) arises from the identification of $\{0\} \times I$ and meets the meridinal disk $\mathbf{D}^2 \times \{0\}$ once. Every other fiber meets $\mathbf{D}^2 \times \{0\}$ exactly μ times. Up to fiber-preserving homeomorphism, I may assume that $\mu > 0$ and $0 \leq \nu < \mu/2$. The integer μ is called the index. If $\mu > 1$, the fibered solid torus is said to be exceptionally fibered and the core is an exceptional fiber; otherwise, the fibered solid torus is regularly fibered and each fiber is a regular fiber.

VI.1. REMARK: Suppose that T is a fibered solid torus of type (μ, ν) where $0 \leq \nu < \mu/2$ and T' is a fibered solid torus of type (μ', ν') where $0 \leq \nu' < \mu'/2$. There is a fiber-preserving homeomorphism between T and T' iff $\mu' = \mu$ and $\nu' = \nu$.

An orientable 3-manifold M is said to be a Seifert fibered manifold if M is a union of pairwise disjoint simple closed curves, called fibers, such that each fiber has a closed neighborhood, consisting of a union of fibers, which is homeomorphic to a fibered solid torus via a fiber-preserving homeomorphism. Such a neighborhood of a fiber is called a fibered-neighborhood. Notice that if T_1 and T_2 are fibered neighborhoods of the fiber τ, then τ is exceptional in T_1 iff τ is exceptional in T_2. Hence, it makes sense to call a fiber τ of the Seifert fibered manifold M an exceptional fiber, if τ has a fibered neighborhood which is homeomorphic to a fibered solid torus via a fiber-preserving homeomorphism taking τ to an exceptional fiber. Otherwise, a fiber τ of the Seifert fibered manifold M is a regular fiber.

VI.2. REMARK: If M is a Seifert fibered manifold, then it is implicit that M has a fixed Seifert fibration; i.e. M has a partition into simple closed curves satisfying the above conditions. Two Seifert fibered manifolds are equivalent if there is a fiber-preserving homeomorphism between them. (See Example VI.5 and Theorem VI.17.)

VI.3. REMARKS: If T is a fibered solid torus, then T is a Seifert fibered manifold and every fiber of T, with the possible exception of the core, is a regular fiber.

If τ is any non-contractible simple closed curve in the boundary of a solid torus T, then T has a representation as a fibered solid torus with τ a regular fiber.

The quotient space obtained from a Seifert fibered manifold M by identifying each fiber to a point is a 2-manifold which is connected if M is connected and disconnected otherwise. This quotient space is called the orbit-manifold (of the Seifert fibered manifold M). There is a natural projection map from M to its orbit-manifold called the projection map. The projection of an exceptional fiber in M is an exceptional point.

VI.4. REMARKS:

(a) If M is a compact Seifert fibered manifold, then since the fibered neighborhoods give a covering of M, it follows that there are at most a finite number of exceptional fibers in the Seifert fibration of M.

(b) By convention, if M is a Seifert fibered manifold, then M is orientable; however, the orbit-manifold may or may not be orientable (see Example VI.5).

(c) If M is a Seifert fibered manifold with orbit-manifold B, then $\partial M \neq \emptyset$ iff $\partial B \neq \emptyset$. Furthermore, if M is compact (equivalently, if B is compact), then each component of ∂M is a fibered torus and each fiber in ∂M is a regular fiber in the fibration of M.

(d) A useful way to think of a Seifert fibered manifold M is to think of M as being obtained from an orientable S^1-bundle over a 2-manifold by removing a finite number of regular fibers and replacing them with exceptional fibers.

VI.5. EXAMPLES:

(a) Let B be a compact, orientable 2-manifold. Set $M = B \times S^1$. Then M is a Seifert fibered manifold with orbit-manifold B and projection map $M \to B$ projection onto the first factor. Hence, M is fibered over B with no exceptional fibers.

(b) Let M be an S^1-bundle over S^2 (necessarily, M is orientable). Then M is a Seifert fibered manifold with orbit-manifold S^2 and projection map $M \longrightarrow B$ bundle projection. Hence, M is fibered over S^2 with no exceptional fibers. Of course, one such M is just $S^2 \times S^1$. The Lens spaces $L(n, 1)$, $n \geq 0$, are the manifolds obtained in this fashion ($S^2 \times S^1$ is $L(0, 1)$). Notice that for $n = 1$, this example gives the Hopf fibration of S^3 over S^2; and for $n = 2$ this example gives $\mathbf{P}^3$ as an S^1-bundle over S^2.

(c) The Lens space $L(n, m)$ is a Seifert fibered manifold with orbit-manifold. S^2 and one exceptional fiber of index $\mu \equiv \pm m^{-1}$ (mod n). Here is an example showing that the index of an exceptional fiber is not determined by the manifold. This example can also be used to show that the number of exceptional fibers is not determined by the manifold; for $L(n, m)$ is a Seifert fibered manifold with orbit-manifold S^2 and two exceptional fibers. This lack of uniqueness can be described and is not the rule for Seifert fibered manifolds in general (see Theorem VI.17).

(d) The twisted I-bundle over the Klein bottle is a Seifert fibered manifold in two distinct ways. It is a Seifert fibered manifold with orbit-manifold the disk and two exceptional fibers, each of index two. And, it is a Seifert fibered manifold with orbit-manifold the Möbius band and no exceptional fibers.

VI.6. EXERCISE: Show that the knot space determined by a torus knot in S^3 is a Seifert fibered manifold. (Lemma VI.3.4 of [J-S_1] characterizes all compact Seifert fibered manifolds embedded in $\mathbf{R}^3$.)

Let X and Y be spaces. Suppose that $f : X \longrightarrow Y$ is a map. A subset Z of X is <u>saturated</u> (with respect to f) if $Z = f^{-1}(f(Z))$.

In particular, if M is a Seifert fibered manifold with orbit-manifold B and projection map $p : M \longrightarrow B$, then a subset $Z = p^{-1}(p(Z))$ is saturated iff Z is a union of fibers of M. Notice, if α is a spanning arc in B and α does not meet an exceptional point, then $p^{-1}(\alpha)$ is a saturated annulus in M ($p^{-1}(\alpha)$ is two-sided and incompressible in M); and if α is a simple closed curve in B and α does not meet an exceptional point, then $p^{-1}(\alpha)$ is a saturated torus or Klein bottle in M ($p^{-1}(\alpha)$ is a torus iff α is two-sided; and $p^{-1}(\alpha)$ is incompressible if either α does not bound a disk $D \subset B$ or whenever α bounds a disk $D \subset B$, then D contains at least two exceptional points). In the case that $p^{-1}(\alpha)$ is a Klein bottle, the fibering of the Klein bottle is by orientation preserving circles. However, there are examples of saturated Klein bottles in Seifert fibered manifolds where the fibering is not by orientation preserving curves alone (of course such Klein bottles are not saturated over simple closed curves; see Example VI.5 (d)).

VI.7. LEMMA: <u>With the exceptions of the manifolds homeomorphic to either $S^2 \times S^1$ or $\mathbf{P}^3 \# \mathbf{P}^3$, an orientable Seifert fibered manifold is irreducible</u>.

Proof: Let M be a Seifert fibered manifold with orbit-manifold B and projection map $p : M \longrightarrow B$.

Suppose that $\partial M \neq \emptyset$ ($\partial B \neq \emptyset$). Then it is easy to show that M is irreducible by using an induction argument which splits B along spanning arcs and splits M along the corresponding saturated annuli.

Suppose that $\partial M = \emptyset$ ($\partial B = \emptyset$). Here the idea is to split M

along a torus which is saturated over a two-sided simple closed curve α in B. However, for this method to be effective, the torus must also be incompressible in M; i.e. the torus is saturated over a simple closed curve that either does not bound a disk $D \subset B$ or if it bounds a disk $D \subset B$, then D contains at least two exceptional points. Such a simple closed curve can be found in B with the exceptions that $B = S^2$ and the Seifert fibration has no more than three exceptional fibers or $B = P^2$ and the Seifert fibration has no more than one exceptional fiber. These cases must be considered individually. The arguments are straightforward. I shall only list the conclusions.

Case 1. M is Seifert fibered with orbit-manifold S^2 and no exceptional fibers (i.e. M is an S^1-bundle over S^2).

In this case, with the exception $S^2 \times S^1$, M is irreducible.

Case 2. M is Seifert fibered with orbit-manifold S^2 and one exceptional fiber (i.e. M is a Lens space including all examples of Case 1).

In this case, with the exception $S^2 \times S^1$, M is irreducible.

Case 3. M is Seifert fibered with orbit-manifold S^2 and two exceptional fibers (i.e. M is a Lens space and is already among the examples of Case 2)

In this case, with the exception $S^2 \times S^1$, M is irreducible.

Case 4. M is Seifert fibered with orbit-manifold S^2 and three exceptional fibers.

In this case, M is irreducible (see Example VI.13).

Case 5. M is Seifert fibered with orbit-manifold P^2 and no exceptional fibers (i.e. M is an S^1-bundle over P^2).

In this case M can be viewed as the orientable S^1-bundle over the Möbius band (the twisted I-bundle over the Klein bottle) with a solid torus attached. The twisted I-bundle over the Klein bottle admits a Seifert fibration in precisely two distinct ways (see Example VI.5 (d)); however, in this case the fibers are the circles as viewed in the structure of an S^1-bundle over the Möbius band; and therefore, they are orientation preserving curves on the Klein bottle sitting over the "center" curve of the Möbius band. Hence, there is only one way to attach the solid torus and not get an irreducible manifold. This exception gives $M = \mathbf{P}^3 \# \mathbf{P}^3$.

Case 6. M is Seifert fibered with orbit-manifold $\mathbf{P}^2$ and one exceptional fiber.

In this case, there are no additions to the previous list. In fact, the exception $\mathbf{P}^3 \# \mathbf{P}^3$ does not appear and each such M is irreducible. ∎

VI.8. EXERCISE: Let M be a Seifert fibered manifold with $\partial M \neq \emptyset$. With the exception of $\mathbf{D}^2 \times S^1$, M is irreducible and ∂-irreducible.

The fundamental group of a Seifert fibered manifold has some very interesting structure. In fact, it is conjectured that the structure is so characteristic that it distinguishes Seifert fibered manifolds within the class of 3-manifolds (see Theorem VI.24 and Remark VI.30 (b)).

Let M be a compact, connected, orientable Seifert fibered manifold with orbit-manifold B.

VI.9. B is orientable, has genus g and p boundary components and M has q exceptional fibers.

$$\pi_1(M) \equiv \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_q, d_1, \ldots, d_p, h : \\ a_i h a_i^{-1} = h,\ b_i h b_i^{-1} = h,\ c_j h c_j^{-1} = h,\ d_k h d_k^{-1} = h, \\ c_j^{\alpha_j} = h^{\beta_j},\ h^b = \prod [a_i, b_i] c_1 \cdots c_{q-1} c_q d_1 \cdots d_p \rangle$$

where each α_j is the index of the jth exceptional fiber $0 < \beta_j < \alpha_j$ $(1 \leq j \leq q)$ and b is an integer.

VI.10. B is nonorientable, has g cross caps and p boundary components and M has q exceptional fibers.

$$\pi_1(M) \equiv \langle a_1, \ldots, a_g, c_1, \ldots, c_q, d_1, \ldots, d_p, h : \\ a_i h a_i^{-1} = h^{-1},\ c_i h c_i^{-1} = h,\ d_k h d_k^{-1} = h,\ c_j^{\alpha_j} = h^{\beta_j}, \\ h^b = a_1^2 \cdots a_g^2 c_1 \cdots c_q d_1 \cdots d_p \rangle$$

where each α_j is the index of the jth exceptional fiber $0 < \beta_j < \alpha_j$ $(1 \leq j \leq q)$ and b is an integer.

VI.11. OBSERVATIONS: (notation as above)

(a) The element h generates a cyclic, normal subgroup of $\pi_1(M)$ and h may be represented by any regular fiber. The element h (and, therefore, the subgroup generated by h) depends on the fibration. However, if $\pi_1(M)$ is infinite, then $\langle h \rangle$ is infinite cyclic.

(b) The quotient group $\pi_1(M)/\langle h \rangle$ is a member of a well-studied family of groups, the Fuchsian groups (see §II.3 of [J-S_1] for a brief "topological" study of Fuchsian groups, which is basically adequate for the purposes of studying Seifert fibered manifolds).

(c) The group $\pi_1(M)$ is either $\mathbf{Z}$, $\mathbf{Z}_2 * \mathbf{Z}_2$ or finite iff $\pi_1(M)/\langle h\rangle$ is finite. Hence, $\pi_1(M)$ is finite iff M is not homeomorphic to $S^2 \times S^1$ or $\mathbf{P}^3 \# \mathbf{P}^3$ and one of the following holds:

i) B is orientable, $g = 0$, $p = 0$ and either $q \leq 2$ or $q = 3$ and $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1$. If $q \leq 2$, then M is a Lens space, and if $q = 3$, there are four possible sets of triples $(\alpha_1, \alpha_2, \alpha_3)$ satisfying the condition that $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1$, the "platonic triples": $(2, 2, \alpha_3)$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$. Or

ii) B is nonorientable, $g = 1$, $p = 0$ and $q \leq 1$. Notice that these manifolds already appear among those listed in (i). They are the Lens spaces $L(4n, 2n-1)$ or the "prism-manifolds" corresponding to the triples $(2, 2, \alpha_3)$.

(d) If g is an element of a group G, then $\zeta(g) = \{x \in G : x^{-1}gx = g\}$ is the centralizer of g in G. The centralizer, $\zeta(h)$, of h in $\pi_1(M)$ has index ≤ 2. If $\pi_1(M)$ is infinite, then for $x \in \langle h\rangle$, $\zeta(x) = \zeta(h)$; and for $x \notin \langle h\rangle$, $\zeta(x)$ contains an abelian subgroup of index ≤ 2 in $\zeta(x)$ (in fact, $\zeta(x) \cap \zeta(h)$ is abelian). If $\pi_1(M)$ is infinite and $x \notin \zeta(h)$, then $\zeta(x)$ is cyclic.

(e) An immediate consequence of (d) is the useful fact that if N is a Seifert fibered manifold and C is an infinite, cyclic, normal subgroup of $\pi_1(N)$, then there exists a Seifert fibered manifold M and a homeomorphism $f : N \longrightarrow M$ such that $f_*(C) \subset \langle h\rangle$. This can be stated simply that any infinite, cyclic, normal subgroup of the fundamental group of a compact Seifert fibered manifold is generated by a power of a regular fiber in some Seifert fibration. I only need this result in the case that $\partial N \neq \emptyset$ (see VI.25 and Remark VI.21). The proof in the case $\partial N \neq \emptyset$ (Lemma II.4.8 of [J-S_1]) uses the fact that any compact, orientable, irreducible, 3-manifold with nonempty boundary having an abelian group as a subgroup of finite index in its fundamental group must be homeomorphic to $\mathbf{D}^2 \times S^1$, $S^1 \times S^1 \times I$ or a twisted I-bundle over the Klein bottle. If $\partial N = \emptyset$, then it is necessary to analyze certain manifolds covered by $S^1 \times S^1 \times S^1$. The situation is not too bad; by using some algebraic manipulation and the fact that the manifold N is orientable, it can be shown that either the statement holds or N is double covered by $S^1 \times S^1 \times S^1$, and therefore is the torus bundle over S^1 given as Example 12.3 (2) in [He_1]. This manifold is the double of the twisted I-bundle over the Klein bottle. The result can be proved directly in this case.

(f) If $x \in \pi_1(M)$ and $x^n \in \langle h \rangle$, then x is represented by a power of a fiber of M (possibly an exceptional fiber - see Lemma II.4.2 of [J-S_1]).

VI.12. EXERCISE: Prove the statements VI.11 (a) through (f).

VI.13. EXAMPLE: [W_2] Let M be a Seifert fibered manifold with orbit-manifold S^2 and with three exceptional fibers. Then M is irreducible and M contains a two-sided incompressible surface (M is a Haken-manifold) iff $H_1(M)$ is infinite (see [E-J] for a detailed discussion of these manifolds).

The claims that such a Seifert fibered manifold M is irreducible in general, and that such a Seifert-fibered manifold M contains a two-sided incompressible surface iff $H_1(M)$ is infinite can be proved simultaneously. Notice that $\pi_1(M)$ has a presentation (VI.9)

$$\pi_1(M) \cong \langle c_1, c_2, h : c_1hc_1^{-1} = h, \quad c_2hc_2^{-1} = h, \quad c_1^{\alpha_1} = h^{\beta_1}, \quad c_2^{\alpha_2} = h^{\beta_2}, \quad (c_1c_2)^{\alpha_3} = h^{\beta_3}\rangle \quad ;$$

and the Fuchsian quotient $\pi_1(M)/\langle h\rangle$ has a presentation (VI.11 (b))

$$\pi_1(M)/\langle h\rangle \cong \langle c_1, c_2 : c_1^{\alpha_1} = c_2^{\alpha_2} = (c_1c_2)^{\alpha_3} = 1\rangle$$

where $|\alpha_i| > 1$, $1 \leq i \leq 3$. Hence, $\pi_1(M)$ is neither infinite cyclic nor a nontrivial free product. It follows that any incompressible 2-sphere S in M must separate M; and, the closure of one component of $M - S$ is a homotopy cell.

If $H_1(M)$ is infinite, then by Theorem III.10, M contains an incompressible two-sided surface. So, I shall assume that $H_1(M)$ is finite. Thus any incompressible surface S in M must separate M.

From the above, I shall have established my claim if I can show that M does not contain a separating incompressible surface. Let f_1, f_2, f_3 denote the exceptional fibers in the given Seifert fibration of M; let T_i be a fibered neighborhood of f_i $(1 \leq i \leq 3)$. Then each T_i is a fibered solid torus.

Among all separating incompressible surfaces in M choose one, say S, such that each component of $S \cap T_i$ is a disk $(1 \leq i \leq 3)$ and the number of components is minimal. Let $M' = M - \bigcup_1^3 \overset{\circ}{T}_i$ and let $S' = S \cap M'$. Then M' is homeomorphic to the product of a disk-with-two-holes and S^1; and S' is both incompressible and ∂-incompressible in M'. Furthermore, S' must separate M'. Since $\pi_1(M')$ has nontrivial center, S' is either a 2-sphere, a torus or an annulus. A 2-sphere in M' is compressible (bounds a 3-cell) in M' and an incompressible torus in M' is parallel to one of the tori in $\partial M'$ and hence could not be incompressible in M. The only possibility is that S' is an annulus, both components of $\partial S'$ are on the same torus T_1, say, and h corresponds to the generator of $\pi_1(S')$. This is impossible; for if this were the case, then h would be trivial in T_1 and the given Seifert fibration of M would not have three exceptional fibers. ■

VI.14. REMARK: Except for those Seifert fibered manifolds where the triple $(\alpha_1, \alpha_2, \alpha_3)$ corresponds to a "platonic-triple", each manifold of Example VI.13 has infinite fundamental group. These were the first known examples of irreducible 3-manifolds with infinite fundamental group which are not Haken-manifolds [W_2]. Other examples of such manifolds have since been discovered (Thurston [Th_2] by doing surgery on the compliment of the "figure-eight" knot; and, more generally, [J-R] by doing surgery on a once-punctured torus bundle over S^1). All known examples of compact, irreducible 3-manifolds with infinite fundamental group have the property that they can be covered in a finite sheeted fashion by a Haken-manifold (i.e. almost sufficiently-large or almost Haken). However, this question is not answered in general. If M is a closed, irreducible 3-manifold, it is unknown if $\pi_1(M)$ contains a subgroup isomorphic to a closed surface group (see Question V.21); it is unknown if $(\tilde{M}, p)$ is a covering of M, if $\tilde{M}$ is itself irreducible (see Chapter I --the only success on this problem has been in the case of two-sheeted coverings by using equivariant surgery) and it is unknown, even if the answer to both of these previous questions is affirmative, if

M must be almost Haken. These are good problems. P. Shalen has made some success toward their answer in the case where M is assumed to have a hyperbolic structure. This not only avoids the bothersome irreducible problem, it also gives a large amount of literature to draw from.

The arguments are now complete as to which Seifert-fibered-manifolds are irreducible and which have finite fundamental group.

VI.15. THEOREM: With the exceptions Lens spaces (including $S^2 \times S^1$ and S^3) and $\mathbf{P}^3 \# \mathbf{P}^3$, a compact Seifert-fibered-manifold is either a Haken-manifold or is Seifert fibered with orbit-manifold S^2 and with precisely three exceptional fibers. In the latter case M is a Haken-manifold iff $H_1(M)$ is infinite. ∎

In Theorem VI.24, I give a characterization of those orientable Haken-manifolds which are Seifert-fibered-manifolds.

In VI.5 I gave examples of Seifert fibered manifolds which admitted distinct (no fiber-preserving homeomorphism) Seifert fiberings. In the closed case these manifolds are included among the "small" Seifer fibered-manifolds. (See [O_1], §5.4.) This term is a bit misleading since some "small" manifolds are "sufficiently large" in the sense of [W_3]. In Chapter VIII, I shall have occasion to single out some of these manifolds (in the bounded case); however, my purpose will be other than the fact that they admit distinct Seifert fiberings.

VI.16. NON-UNIQUE SEIFERT FIBERINGS:

(a) Lens Spaces (including $S^2 \times S^1$ and S^3).

(b) *Prism-manifolds*. These, by definition, are Seifert fibered with orbit-manifold S^2 and with three exceptional fibers of index corresponding to the triple $(2, 2, \alpha_3)$. Since $\alpha_3 > 1$, each Prism-manifold admits a Seifert fibering with orbit-manifold $\mathbf{P}^2$ and with no more than one exceptional fiber. Note that there are orientable Seifert fibered manifolds with orbit-manifold $\mathbf{P}^2$ and with no more than one exceptional fiber which are *not* Prism-manifolds (Lens spaces $L(4n, 2n-1)$ and the manifold $\mathbf{P}^3 \# \mathbf{P}^3$).

(c) *Double of twisted* I-*bundle over Klein bottle*. This manifold is Seifert fibered with orbit-manifold S^3 and with four exceptional fibers of index $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 2$. It is also Seifert fibered with orbit-manifold the Klein bottle and with no exceptional fibers (an S^1-bundle over the Klein bottle). Notice that it also has the structure of a torus bundle over S^1 (see Example 12.3 (2) of [He_1]).

(d) *The solid torus*: (see Remark VI.1).

(e) *A twisted* I-*bundle over the Klein bottle*: (see Example VI.5 (d)).

VI.17. THEOREM: *Let* M *and* N *be connected Seifert-fibered-manifolds and let* $f : M \to N$ *be a homeomorphism. Either* M (and therefore N) *appears on the list* VI.16 *or there is a fiber-preserving homeomorphism from* M *to* N.

The idea of proving such a theorem is to actually prove a stronger conclusion, with more Seifert-fibered-manifolds as exceptions than those listed in VI.16, and then, by hand, and case by case arrive at the desired conclusion. I shall do precisely this in the case that the manifolds in question have nonempty boundary. The version of the theorem needed in this case is given as Theorem VI.18. The general case of this theorem was proved by Waldhausen in [W_4].

VI.18. THEOREM: Let M and N be connected Seifert-fibered-manifolds and let $f : M \to N$ be a homeomorphism. Suppose that M (and therefore N) has the property that $\partial M \neq \emptyset$ and M is not homeomorphic to $D^2 \times S^1$, $S^1 \times S^1 \times I$ or a twisted I-bundle over the Klein bottle. Then f is isotopic to a fiber-preserving homeomorphism.

Proof: I shall prove this by establishing two lemmas. The first lemma can be stated simply that a fiber in the boundary of a Seifert-fibered-manifold completely determines the Seifert fibering:

VI.19. LEMMA: Let M and N be connected Seifert-fibered-manifolds and let $f : M \to N$ be a homeomorphism. Suppose that for some fiber τ in ∂M that $f(\tau)$ is a fiber in ∂N. Then f is isotopic (rel τ) to a fiber-preserving homeomorphism.

Proof: Let B_N be the orbit-manifold of the Seifert fibered manifold N with projection map $p_N : N \to B_N$. The proof is via induction on the "complexity" of B_N.

If B_N is a disk and there is at most one exceptional fiber in N, then N (and hence M) is a fibered solid torus; and since f is a homeomorphism, τ is a fiber in M and $f(\tau)$ is a fiber in N, it follows that both M and N are fibered solid tori of the same type and the desired isotopy can be constructed.

Now, suppose that either B_N is not a disk or if B_N is a disk then there are at least two exceptional fibers in N. There is an arc α_N in B_N from the point $p_N(f(\tau))$ in ∂B_N to a distinct point in ∂B_N such that α_N is not homotopic (rel $\partial\alpha_N$) into ∂B_N missing an exceptional point. Hence, $p_N^{-1}(\alpha_N) = A_N$ is an incompressible, saturated annulus in N; one component of $\partial A_N = f(\tau)$ and A_N is not homotopic (rel ∂A_N) into ∂N.

Let $A_M = f^{-1}(A_N)$. Then A_M is an incompressible annulus in M; one component of ∂A_M is τ and A_M is not homotopic (rel ∂A_M) into ∂M.

It is not hard to prove that A_M is isotopic (rel τ) to a saturated annulus (see VI.25 for a generalization). That is, after an isotopy of f (rel τ), I may assume that $A_M = f^{-1}(A_N)$ is saturated. I can now split N at A_N and M at A_M and apply the induction argument to complete the proof of Lemma VI.19. ∎

The second lemma can be stated simply that with certain exceptions there exists only one isotopy class of s.c.c. in each boundary component which can be a fiber in any Seifert fibration of M.

VI.20. LEMMA: Let M be homeomorphic to a compact, connected, Seifert-fibered-manifold with nonempty boundary. Suppose that M is not homeomorphic to $\mathbf{D}^2 \times S^1$, $S^1 \times S^1 \times I$ or a twisted I-bundle over a Klein bottle. Let T be a component of ∂M. Then up to ambient isotopy of M there exists a unique simple closed curve in T which is mapped to a fiber.

Proof: Suppose that there exists compact, connected Seifert fibered manifolds N_1 and N_2 and homeomorphisms $f_i : M \longrightarrow N_i$ $(i = 1, 2)$. For $i = 1, 2$, let τ_i be a simple closed curve in T such that $f_i(\tau_i)$ is a fiber in N_i.

Now, either N_i (and hence M) is a solid torus, which is not the case, or there exists saturated annuli $A_i' \subset N_i$ with one component of their boundary equal to $f_i(\tau_i)$, the other component of their boundary a fiber in ∂N_i distinct from $f_i(\tau_i)$ and A_i' is not homotopic (rel ∂A_i) in N_i into ∂N_i $(i = 1, 2)$. Set $A_i = f_i^{-1}(A_i')$.

Since M is not homeomorphic to $\mathbf{D}^2 \times S^1$ and M is homeomorphic to a Seifert fibered manifold, each component of ∂M is incompressible (Exercise VI.8). Hence, the covering space of M corresponding to $\mathrm{Im}(\pi_1(T) \hookrightarrow \pi_1(M))$ compactifies [Si$_1$] to the product $T \times I$. Both A_1 and A_2 lift and each has distinct boundary components in distinct components of the boundary of the covering. It follows from Theorem 1 of [J$_1$] that either there is an ambient isotopy of M taking τ_1 to τ_2 or M is an I-bundle over a torus or an I-bundle over the Klein bottle. By hypothesis, neither of the latter two is the case. ∎

It is now easy to prove Theorem VI.18. By hypothesis the manifold M is not homeomorphic to $\mathbf{D}^2 \times S^1$, $S^1 \times S^1 \times I$ on a twisted I-bundle over the Klein bottle. Setting one homeomorphism equal to the identity on M and the other one equal to the given homeomorphism $f : M \to N$; it follows from Lemma VI.20 that up to isotopy of M there is a fiber τ in ∂M such that $f(\tau)$ is a fiber in ∂N. Hence, by Lemma VI.19, the homeomorphism f is isotopic to a fiber-preserving homeomorphism. ∎

VI.21. REMARK: In order to prove Theorem VI.18, it was not necessary to have Lemma VI.20. I stated and proved VI.20 to establish a strong geometric form of the uniqueness of a fiber. This corresponds to the algebraic result given in VI.11 (e) for infinite, cyclic normal subgroups of the fundamental group. I could have used VI.11 (e) in place of VI.20 to prove Theorem VI.18.

VI.22. EXERCISE: Suppose that M and N are Seifert fibered manifolds and each is homeomorphic to the trivial I-bundle over the torus. Prove that there is a fiber-preserving homeomorphism from M to N.

VI.23. EXERCISE: Prove Theorem VI.17 in the general case.

There is a very nice characterization of those orientable Haken-manifolds which are homeomorphic to Seifert fibered manifolds. The version presented here first appeared in [J-S_1]; however, major contributions to this theorem are due to [B-Z], [W_2], [G-H] and [To_1]. Also see [He-J]. Some of the steps in proving this theorem are important in their own right and will be used later. For these reasons I assign them a reference number.

VI.24. THEOREM: Let M be a compact, orientable Haken-manifold. Then $\pi_1(M)$ has an infinite, cyclic, normal subgroup iff M is homeomorphic to a Seifert fibered manifold.

Proof: A compact, orientable Haken-manifold M is either a 3-cell or has infinite fundamental group. So, if M is homeomorphic to a Seifert fibered manifold, then it follows from VI.11 (a) that $\pi_1(M)$ has an infinite cyclic, normal subgroup.

Conversely, suppose that M is a compact, orientable Haken-manifold and that C is an infinite, cyclic, normal subgroup of $\pi_1(M)$. To prove that M is homeomorphic to a Seifert fibered manifold, I shall induct on the length of M (see Remark IV.13).

The induction starts when M is a cube-with-handles. The only possibility is for M to be a solid torus; and hence, M is homeomorphic to a Seifert fibered manifold (Remark VI.3).

Suppose that the manifold M satisfies the hypothesis of Theorem VI.24, that M has length n $(n \geq 2)$ and that any manifold of length less than n satisfying the hypothesis of VI.24 is homeomorphic to a Seifert fibered manifold.

Let F denote a two-sided, incompressible and ∂-incompressible surface in M (such a surface exists by Exercise IV.11); and if possible, choose such an F that does not separate M. This condition is possible iff $H_1(M)$ is infinite, which is always the case if $\partial M \neq \emptyset$.

I shall consider separately three possibilities: the infinite, cyclic, normal subgroup $C < \pi_1(F)$; the infinite, cyclic, normal subgroup $C \not< \pi_1(F)$ and F does not separate M; the infinite, cyclic, normal subgroup $C \not< \pi_1(F)$ and F separates M.

Only the first case uses the induction step.

VI.25. _If_ $C < \pi_1(F)$, _then_ F _is an annulus or a torus and_ M _is homeomorphic to a Seifert fibered manifold via a homeomorphism taking_ F _to a union of fibers_ (i.e. F is saturated in some Seifert fibration of M)

Proof of VI.25: Since $\pi_1(F)$ contains C, F must be an annulus or a torus. Let M' denote the manifold obtained by splitting M at F. Then each component of M' is a compact, orientable Haken-manifold having length strictly less than the length of M and having an infinite, cyclic, normal subgroup of its fundamental group. It follows by induction that each component of M' is homeomorphic to a Seifert fibered manifold and therefore by VI.11 (e), each component of M' admits a Seifert fibration with C generated by a power of a fiber. It follows that each component of M' admits a Seifert fibration with the surfaces corresponding to F saturated. Clearly, if F is an annulus, the Seifert fibration(s) on the component(s) of M' can be extended to a Seifert fibration of M with F saturated. If F is a torus, then since C is generated by a power of a fiber and each element of $\pi_1(F)$ has a unique primitive represented by a simple closed curve on F, the Seifert fibration(s) on the component(s) of M' can be extended to a Seifert fibration of M with F saturated. ■

VI.26. <u>If</u> $C \not< \pi_1(F)$ <u>and</u> F <u>does not separate</u> M, <u>then</u> C <u>is central in</u> $\pi_1(M)$, $M = F \times_\varphi S^1$ <u>is fibered over</u> S^1 <u>with fiber</u> F <u>and sewing map</u> φ <u>and</u> φ <u>is periodic</u>.

Proof of VI.26: This result is well known and appears frequently in the existing literature. I shall only outline the ideas.

The first step is to go to the infinite cyclic covering of M determined by F (F is two-sided and does not separate M) and prove that it is the covering of M corresponding to $\pi_1(F)$. This method of proof was introduced in [B-Z] and exploited by Waldhausen in [W_2]. The proof is carried out in detail as Case 1 in the proof of Theorem 12.7 of [He_1] where it is also shown that C must be central in $\pi_1(M)$. It follows that $\pi_1(M)$ can be written as an extension $\pi_1(F) \longrightarrow \pi_1(M) \longrightarrow \mathbf{Z}$ where $F \neq \mathbf{P}^2$. Hence, by [St_4] the manifold $M = F \times_\psi S^1$ fibers over

S^1 with fiber F and sewing map ψ. Now, by applying the original work of [B-Z], which is done in the proof of Lemma 12.6 of [He_1], it can be shown that some power of ψ_* is an inner automorphism of $\pi_1(F)$. Therefore, ψ is homotopic (hence isotopic) to a periodic homeomorphism, φ [N_1]. ∎

It remains to prove that M is homeomorphic to a Seifert fibered manifold. The next result is well known. It is implicit in the work of Waldhausen [W_2]. Also, it is implicit in the statement of Lemma 12.6 of [He_1] and a proof may be found within the proof of that lemma.

VI.27. *Let* $M = F \times_\varphi S^1$ *be a surface bundle over* S^1 *with fiber* F *and sewing map* φ. *If* φ *is homotopic to a periodic homeomorphism, then* M *is homeomorphic to a Seifert fibered manifold.*

Now, I shall establish that M is homeomorphic to a Seifert fibered manifold in the remaining case.

VI.28. *If* $C \not< \pi_1(F)$ *and* F *separates* M, *then* M *is closed and* $M = M_1 \cup M_2$ *with* $M_1 \cap M_2 = \partial M_1 = \partial M_2 = F$ *and each* M_i $(i = 1, 2)$ *is a twisted* I-*bundle over a closed, nonorientable surface*.

Proof of VI.28: Since F separates M, (by the choice of F) $H_1(M)$ is finite; so, M is closed; and $M = M_1 \cup M_2$ with $M_1 \cap M_2 = \partial M_1 = \partial M_2 = F$. At the fundamental group level, $\pi_1(M)$ splits as a free product of the groups $\pi_1(M_1)$ and $\pi_1(M_2)$ with amalgamation over $\pi_1(F)$. It is straightforward, from the algebra, that either $C < \pi_1(F)$ or $C \cap \pi_1(F) = \{1\}$ and $\pi_1(F)$ has index precisely two in both $\pi_1(M_1)$ and $\pi_1(M_2)$. By hypothesis, the former is not the case. The latter implies that each M_i $(i = 1, 2)$ is a twisted I-bundle over a closed, nonorientable surface. ■

It remains, in this case, to prove that M is homeomorphic to a Seifert fibered manifold. This step was the only remaining step prior to [J-S_1]. I again want to emphasize that <u>only</u> VI.25 uses the induction hypothesis.

Notice that if VI.28 is satisfied, then M has a canonical two-sheeted covering $(\tilde{M}, p)$, F lifts to $\tilde{M}$ and if $\tilde{F}$ is one of the components of $p^{-1}(F)$, then $\tilde{M} = \tilde{F} \times_{\varphi} S^1$ is a surface bundle over S^1 with fiber $\tilde{F}$ and sewing map φ. Furthermore, if $\tilde{C} = p_*^{-1}(C \cap p_*\pi_1(\tilde{M}))$, $\tilde{C}$ is an infinite, cyclic, normal subgroup of $\pi_1(\tilde{M})$; and since $C \cap \pi_1(F) = \{1\}$, $\tilde{C} \not< \pi_1(\tilde{F})$. As I have pointed out above, VI.26 does not depend on the induction hypothesis; hence, $\tilde{M}$ is homeomorphic to a Seifert fibered manifold. The results of [To_1] can be applied to show that either M is homeomorphic to a Seifert fibered manifold or $\tilde{M}$ is closed and Seifert fibered with orbit-manifold the 2-sphere with precisely three exceptional fibers. Lemma II.5.2 of [J-S_1] eliminates the second possibility. I shall state in VI.29 what seems to be a more general statement than is needed here; however, it can be argued that VI.29 is equivalent to proving that if M and $\tilde{M}$ satisfy the above situation, then M is homeomorphic to a Seifert fibered manifold.

VI.29. <u>Let</u> M <u>be a connected, orientable Haken-manifold. Suppose that</u> $(\tilde{M}, p)$ <u>is a finite-sheeted covering space of</u> M. <u>Then</u> $\tilde{M}$ <u>is homeomorphic to a Seifert fibered manifold iff</u> M <u>is homeomorphic to a Seifert fibered manifold</u>.

Lemma VI.29 is proved as Lemma II.5.1 and Lemma II.5.3 of [J-S_1].

This completes the proof of VI.28 and the proof of Theorem VI.24. ∎

VI.30. REMARKS: (a) Theorem VI.24 is not true in the nonorientable case. However, it appears to be valid if the class of manifolds considered as Seifert fibered manifolds is enlarged to admit orientation reversing curves as fibers, as suggested in the opening remarks of this Chapter.

(b) It is conjectured that any closed, connected 3-manifold M having a finite-sheeted covering $(\tilde{M}, p)$ where $\tilde{M}$ is homeomorphic to a Seifert fibered manifold implies that M is homeomorphic to a Seifert fibered manifold. Such an M must have an infinite, cyclic, normal subgroup of its fundamental group. However, it is not known if the center of $\pi_1(M)$ must be finitely-generated.

I shall finish this chapter with two consequences of VI.24 and its sublemmas VI-25 through VI.29.

In VI.27 I observed that a surface bundle over S^1 admits a Seifert fibration if the sewing map is periodic. The next lemma is a partial converse to VI.27.

VI.31. LEMMA: Let M be homeomorphic to a Seifert fibered manifold. If $M = F \times_\varphi S^1$ is a surface bundle over S^1 with fiber F and sewing map φ, either φ is homotopic to a periodic homeomorphism or F is a torus and M is an S^1-bundle over the torus or Klein bottle.

Proof: By hypothesis $\pi_1(M)$ contains an infinite, cyclic, normal subgroup C. If $C \not< \pi_1(F)$, then by VI.26 C is central in $\pi_1(M)$ and φ is homotopic to a periodic homeomorphism. If $C < \pi_1(F)$, then F is an annulus or a torus. If F is an annulus, then φ is homotopic to a periodic homeomorphism. If F is a torus, then it can be shown that the monodromy of φ_* is represented by a matrix $A = \begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix}$. If $\mathrm{TR}\,A = 2$, then M is an S^1-bundle over a torus and if $\mathrm{TR}\,A = -2$, then M is an S^1-bundle over a Klein bottle. ■

VI.32. THEOREM: Let M be an orientable Seifert fibered manifold. Then $M = F \times_\varphi S^1$ is a surface bundle over S^1 with fiber F and sewing map φ iff either h has infinite order in $H_1(M)$ or M is an S^1-bundle over a torus, a Klein bottle, an annulus or a Möbius band. ■

VI.33. REMARK: If M is a Seifert fibered manifold with orbit-manifold B and B is nonorientable, then h has finite order in $H_1(M)$. The order of h in $H_1(M)$ can be computed (e.g. see page 122 [O_1]).

The following theorem, which concludes this chapter, characterizes the two-sided incompressible surfaces embedded in a Seifert fibered manifold.

VI.34. THEOREM: Let M be a compact, orientable Seifert fibered manifold. If F is a two-sided, incompressible surface in M then one of the following alternatives holds:

(i) F is a disk or an annulus and F is parallel into ∂M.

(ii) F does not separate M and F is a fiber in a fibration of M as a surface bundle over S^1.

(iii) F does separate M and $M = M_1 \cup M_2$ where $M_1 \cap M_2 = \partial M_1 = \partial M_2 = F$ and M_i $(i = 1, 2)$ is a twisted I-bundle over a compact surface (possibly with boundary).

(iv) F is an annulus or a torus and F is saturated in some Seifert fibration of M.

Proof: This is just a matter of collecting the results proved earlier.

Fix a Seifert fibration of M and let $h \in \pi_1(M)$ be represented by a regular fiber (by VI.11 (a), $\langle h \rangle$ is a cyclic, normal subgroup of $\pi_1(M)$).

If $\langle h \rangle < \pi_1(F)$, then by VI.25 alternative (iv) holds.

If $\langle h \rangle \not< \pi_1(F)$ and F does not separate M, then by VI.26 alternative (ii) holds. If $\langle h \rangle \not< \pi_1(F)$ and F does separate M, then either alternative (i), (iii), or (iv) holds. If the splitting of $\pi_1(M)$ determined by F is not a nontrivial free product with amalgamation over $\pi_1(F)$, then F is a disk, an annulus or a torus parallel into ∂M. If F is a torus, then alternative (iv) holds. If the splitting of $\pi_1(M)$ determined by F is a nontrivial free product with amalgamation over $\pi_1(F)$, then an argument analogous to that for VI.28 proves that alternative (iii) holds. ∎

CHAPTER VII. PERIPHERAL STRUCTURE

If M is a 3-manifold with boundary, any subgroup of the fundamental group of a component of ∂M defines a conjugacy class of subgroups of $\pi_1(M)$, namely the conjugacy class of subgroups induced by inclusion of ∂M into M. Such a class of "peripheral" subgroups is particularly important, in the study of the structure of $\pi_1(M)$, in the case the class corresponds to the fundamental group of an incompressible surface in ∂M. For instance, if M is a Haken-manifold, then M admits a hierarchy (Chapter IV); so $\pi_1(M)$ may be described, via an inductive procedure, in terms of peripheral subgroups defined on the fundamental groups of certain "simpler" 3-manifolds and these peripheral subgroups correspond to incompressible surfaces in the boundaries of the simpler 3-manifolds.

The techniques developed in this chapter for studying the peripheral structure will be fundamental to the work of the remaining chapters. It is used in the analysis and proof of the homotopy versions of the Annulus-Torus Theorems (Chapter VIII), in the proof of the existence and uniqueness of the Characteristic Seifert Pair (Chapter IX) and in the study of the extent to which the fundamental group of a Haken-manifold determines its topological type (Chapter X).

Recall (Exercise I.5) that if M is a 3-manifold, a subgroup H of $\pi_1(M)$ is <u>peripheral</u> if there exists a surface $S \subset \partial M$ such that H is conjugate in $\pi_1(M)$ into a subgroup of $\operatorname{Im}(\pi_1(S) \hookrightarrow \pi_1(M))$. Since a peripheral subgroup is only determined up to conjugacy, it is natural to pass to the covering space uniquely determined by such a conjugacy class of subgroups of the fundamental group. The study of these covering spaces turns out to be particularly rewarding due to two beautiful theorems, one theorem of T. Tucker [Tu_1] and one theorem of J. Simon [Si_1]. These results apply to the compactification of 3-manifolds and particularly, to the compactification of certain coverings of Haken-manifolds.

A manifold M admits a manifold-compactification if there exists a compact manifold Q and an embedding $\varphi: M \to Q$ such that $\text{Int}\, Q \subset \varphi(\text{Int}\, M)$. Notice that a manifold M admits a manifold compactification iff there exists a compact manifold Q such that M is homeomorphic to Q minus a closed subset of ∂Q. There are many interesting examples of 3-manifolds which do or do not admit manifold compactifications in [Tu_1], [Tu_2] and [J_6]. For my purposes, the main theorem, as to when a 3-manifold admits a manifold compactification, is due to T. Tucker [Tu_1].

VII.1. THEOREM: _Let M be an irreducible, orientable 3-manifold. Then M admits a manifold compactification if and only if for each compact polyhedron C in M, each component of $M - C$ has finitely generated fundamental group._

I wish to apply this theorem to certain covering spaces of Haken-manifolds. This is the point that the work of Simon [Si_1] is used. In [Si_1], Simon gives a sufficient condition for a covering space of a Haken manifold to admit a manifold compactification. I shall give a modified version of his theorem. In the proof I give, it should be noticed that it is not necessary to construct a manifold compactification as Simon does indeed, the theorem of Simon follows easily from Theorem VII.1. This has been independently observed by Tucker [Tu_2].

VII.2. THEOREM: *Let* M *be a Haken-manifold and suppose that* H *is a finitely generated subgroup of* $\pi_1(M)$. *Let* $\widetilde{M}(H)$ *be the covering space of* M *corresponding to the conjugacy class of* H *in* $\pi_1(M)$ *and let* $p : \widetilde{M}(H) \longrightarrow M$ *denote the covering projection. If there exists a two-sided, incompressible 2-manifold* F *in* M *such that each component of* $p^{-1}(M-\overset{\circ}{U}(F))$ *admits a manifold compactification to a Haken-manifold, where* $U(F)$ *is a product neighborhood of* F *in* M, *then* $\widetilde{M}(H)$ *admits a manifold compactification to a Haken-manifold.*

Proof: To prove that $\widetilde{M}(H)$ admits a manifold compactification it is sufficient to prove that if C is a compact polyhedron in $\widetilde{M}(H)$, then each component of $\widetilde{M}(H) - C$ has finitely generated fundamental group.

First, observe that if C and C' are compact polyhedra in $\widetilde{M}(H)$, $C' \subset C$ and each component of $\widetilde{M}(H) - C$ has finitely generated fundamental group, then each component of $\widetilde{M}(H) - C'$ has finitely generated fundamental group. Using this observation and the hypothesis that $\pi_1(\widetilde{M}(H))$ is finitely generated (being isomorphic to H), I only need to consider compact, connected polyhedra C in $\widetilde{M}(H)$ which have the property that the inclusion induced homormorphism of $\pi_1(C)$ into $\pi_1(\widetilde{M}(H))$ is an epimorphism.

So, let C be a compact, connected polyhedron in $\widetilde{M}(H)$ which has the property that the inclusion $\pi_1(C)$ into $\pi_1(\widetilde{M}(H))$ is an epimorphism. Since C is compact, there are only finitely many components of $p^{-1}(F)$ which meet C nontrivially and only finitely many components of $\widetilde{M}(H) - p^{-1}(F)$ whose closure meets C nontrivially. Let $\widetilde{N}$ be the union of the closures of those components of $\widetilde{M}(H) - p^{-1}(F)$ whose closure meets C nontrivially. Then $\widetilde{N}$ involves only finitely many components of $\widetilde{M}(H) - p^{-1}(F)$, the inclusion of $\pi_1(\widetilde{N})$ into $\pi_1(\widetilde{M}(H))$ is an epimorphism and, what is more, the fundamental group of any component of $\widetilde{M}(H) - C$ is isomorphic to the fundamental group of some component of $\widetilde{N} - C$.

Each component of $\widetilde{N} - C$ is a union of a finite number of components of $\widetilde{M}(H) - p^{-1}(\overset{o}{U}(F)) - C$ connected by a finite number of components of $p^{-1}(U(F)) - C$. Since each component of $\widetilde{M}(H) - p^{-1}(\overset{o}{U}(F))$ admits a manifold compactification, each component of $\widetilde{M}(H) - p^{-1}(\overset{o}{U}(F)) - C$ has finitely generated fundamental group. Hence it follows from the Seifert-Van Kampen Theorem that the fundamental group of a component of $\widetilde{N} - C$ is a quotient of a finite free product of finitely generated groups and is, therefore, a finitely generated group. As observed above, this implies that each component of $\widetilde{M}(H) - C$ has a finitely generated fundamental group.

It remains to show that the compactification of $\widetilde{M}(H)$ is a Haken-manifold. This is really straightforward. The manifold $\widetilde{M}(H)$ is orientable and since each component of $p^{-1}(M-\overset{o}{U}(F))$ compactifies to a Haken-manifold, the closure of each component of $p^{-1}(F)$ is irreducible, which implies that $\widetilde{M}(H)$ is irreducible. It follows that the compactification of $\widetilde{M}(H)$ is orientable and irreducible. The compactification either has nontrivial boundary (and so is a Haken-manifold) or is closed, in which case the compactification is $\widetilde{M}(H)$. If this latter case holds, then M is closed, $\widetilde{M}(H)$ is a finite sheeted covering of M and $p^{-1}(F)$ is an incompressible 2-manifold in $\widetilde{M}(H)$. I have already shown that $\widetilde{M}(H)$ is orientable and irreducible. So, in this case the compactification is a Haken-manifold. ■

VII.3. REMARK: J. Simon conjectured that the covering space corresponding to a finitely generated subgroup of the fundamental group of a Haken-manifold admits a manifold compactification to a Haken-manifold. If M is a Haken-manifold and H is a finitely generated subgroup of $\pi_1(M)$, then it is very tempting to try to prove the existence of a two-sided surface F in M so that M, H and F all satisfy the conditions of the hypothesis of Theorem VII.2; thereby, establishing Simon's Conjecture in the affirmative. However, the conclusions of Example V.19(d) provide numerous examples where such an attempt will fail. Recall that in Example V.19(d) it is shown that if $M = F \times_\varphi S^1$ is a bundle over S^1 with fiber F where $\chi(F) < 0$, then there is a two-generation subgroup H of $\pi_1(M)$ such that $H \cap \pi_1(F)$ is <u>not</u> finitely generated. Furthermore, there are examples of bundles $M = F \times_\varphi S^1$ where $\chi(F) < 0$ and, up to isotopy, the fiber F is the only two-sided incompressible surface in M (e.g, if M is obtained by zero framed surgery on a (p,q)-torus knot manifold where $|(p-1)(q-1)/_2| > 1$). In this situation if $\widetilde{M}(H)$ is the covering space of M corresponding to the conjugacy class of H in $\pi_1(M)$ and $p : \widetilde{M}(H) \to M$ is the covering projection, then there exists components of $p^{-1}(M-\overset{o}{U}(F))$ which do not have finitely generated fundamental groups; and so, such components of $p^{-1}(M-\overset{o}{U}(F))$ could not admit manifold compactifications. The status of Simon's Conjecture is unknown at this time.

However, there are reasonable conditions which can be placed on a subgroup H of the fundamental group of a Haken-manifold M which guarantee that $\widetilde{M}(H)$ admits a manifold compactification. The next theorem is due to J. Simon.

VII.4. THEOREM: <u>Let</u> M <u>be a Haken-manifold,</u> H <u>a finitely generated subgroup of</u> $\pi_1(M)$ <u>and</u> $\widetilde{M}(H)$ <u>the covering space of</u> M <u>corresponding to the conjugacy class of</u> H <u>in</u> $\pi_1(M)$. <u>If</u> H <u>has the finitely generated</u>

intersection property (f.g.i.p.), *then* $\widetilde{M}(H)$ *admits a manifold compactification to a Haken-manifold.*

Proof: The proof is via induction on the length of a hierarchy for M.

If M is a 3-cell, then there is nothing to prove. So, suppose that M has a hierarchy of length n $(n \geq 1)$ and for any Haken-manifold M' having a hierarchy of length $< n$ and any finitely generated subgroup H' of $\pi_1(M')$, which has f.g.i.p., the covering space of M' corresponding to the conjugacy class of H' in $\pi_1(M')$ admits a manifold compactification to a Haken-manifold.

Let F be the first surface in a hierarchy of length n for M. Let $p : \widetilde{M}(H) \longrightarrow M$ denote the covering projection. If $\widetilde{N}$ is a component of $p^{-1}(M-\overset{o}{U}(F))$, then for some component M' of $M - \overset{o}{U}(F)$, $\widetilde{N}$ is the covering space of M' corresponding to the conjugacy class in $\pi_1(M')$ of the subgroup $\pi_1(M') \cap H^g$ for some $g \in \pi_1(M)$. (H^g is the conjugate (in $\pi_1(M)$) of H by the element $g \in \pi_1(M)$.) Set $H' = \pi_1(M') \cap H^g$. I claim that H' has f.g.i.p. (in $\pi_1(M')$). Let K be a finitely generated subgroup of $\pi_1(M')$. Then $K \cap H' = K \cap \pi_1(M') \cap H^g$ is isomorphic to $K^g \cap H$; and so, $K \cap H'$ is finitely generated, since H has f.g.i.p.

Since M' has a hierarchy of length $< n$, $\widetilde{N}$ admits a manifold compactification to a Haken-manifold. However, $\widetilde{N}$ was an arbitary component of $p^{-1}(M-\overset{o}{U}(F))$. It follows from Theorem VII.2 that $\widetilde{M}(H)$ admits a manifold compactification to a Haken-manifold. ■

VII.5. COROLLARY: *Let* M *be a Haken-manifold,* H *an abelian subgroup of* $\pi_1(M)$ *and* $\widetilde{M}(H)$ *the covering space of* M *corresponding to the* _conjugacy class of_ H _in_ $\pi_1(M)$. _Then_ $\tilde{M}(H)$ _admits a manifold compactification to a Haken-manifold._

Proof: An abelian subgroup of the fundamental group of a Haken-manifold must be finitely generated and so has f.g.i.p. □

VII.6. COROLLARY: _Let_ M _be a Haken-manifold. Then the universal cover of_ M _admits a manifold compactification to a_ 3-_cell._ □

VII.7. QUESTION: _Let_ M _be a closed, aspherical_ ($\pi_n(M)=0$, $n \geq 2$) 3-_manifold. Does the universal cover of_ M _admit a manifold compactification to a_ 3-_cell?_

VII.8. COROLLARY: _Let_ M _be a Haken-manifold,_ H _a finitely generated peripheral subgroup of_ $\pi_1(M)$ _and_ $\tilde{M}(H)$ _the covering space of_ M _corresponding to the conjugacy class of_ H _in_ $\pi_1(M)$. _Then_ $\tilde{M}(H)$ _admits a manifold compactification to a Haken-manifold._

Proof: By Theorem V.20 a finitely generated peripheral subgroup H of $\pi_1(M)$ has f.g.i.p. ■

The preceding corollary is the major result in my approach to the study of the peripheral structure of 3-manifolds. However, before leaving the results on compactifications and investigating the consequences of Corollary VII.8, there is one more compactification theorem, due to B. Evans and myself, that is needed in Chapters VIII and IX. I need to introduce some notation prior to stating this theorem. This notation will be used through the next three chapters, since the major results of these chapters can be stated best in relative terms using this notation.

A pair (X,Y) is called a polyhedral pair if X is a polyhedron and Y is a sub-polyhedron of X. A polyhedral pair (X,Y) is connected if X is connected. A component of a polyhedral pair (X,Y) is a polyhedral pair (X',Y') where X' is a component of X and $Y' = Y \cap X'$. A polyhedral pair (X,Y) is compact if both X and Y are compact. The polyhedral pair (X',Y') is contained in the polyhedral pair (X,Y), written $(X',Y') \subset (X,Y)$, if X' is a subpolyhedron of X and Y' is a subpolyhedron of Y.

An n-manifold pair is a polyhedral pair (M,T) where M is an n-manifold and T is an $(n-1)$-manifold contained in ∂M. Let M be a 3-manifold. A 3-manifold pair $(M',T') \subset (M,\partial M)$ is well-embedded in M if (i) $M' \cap \partial M = T'$ and (ii) $Fr_M(M')$ is incompressible in M.

VII.9. THEOREM: *Let M be a Haken-manifold. Suppose that the connected 3-manifold pair $(M',T') \subset (M,\partial M)$ is well-embedded in M. Then the covering space of M corresponding to the conjugacy class of $\pi_1(M')$ in $\pi_1(M)$ admits a manifold compactification to a Haken-manifold.*

VII.10. REMARK: Theorem VII.9 yields the same conclusion as the earlier results, yet for quite different reasons. In this case the subgroup $\pi_1(M')$ may not have f.g.i.p. (e.g., see V.19(c)).

Proof of Theorem VII.9: Set $H = \pi_1(M')$ and set $F = Fr_M(M')$. Let $\widetilde{M}(H)$ denote the covering space of M corresponding to the conjugacy class of $H = \pi_1(M')$ in $\pi_1(M)$ and let $p : \widetilde{M}(H) \longrightarrow M$ denote the covering projection. Set $U(F)$ equal to a product neighborhood of F in M. I shall show that the hypotheses of Theorem VII.2 are satisfied; that is, each component of $p^{-1}(M-\overset{\circ}{U}(F))$ admits a manifold compactification to a

Haken-manifold.

There is a lifting of $M' - \mathring{U}(F)$ to a component $\widetilde{M}'$ of $p^{-1}(M-\mathring{U}(F))$ such that $p|\widetilde{M}' : \widetilde{M}' \longrightarrow M' - \mathring{U}(F)$ is a homeomorphism; furthermore, the inclusion induced homomorphism of $\pi_1(\widetilde{M}') \hookrightarrow \pi_1(\widetilde{M}(H))$ is an isomorphism. It follows that each component of $p^{-1}(F)$ separates $\widetilde{M}(H)$. Now, for $\widetilde{f}$ a component of $p^{-1}(F)$, choose notation so that if M_1 and M_2 are the components of $\widetilde{M}(H) - \widetilde{f}$, then $\widetilde{M}' \subset M_1$. Since the inclusion $\pi_1(\widetilde{M}') \hookrightarrow \pi_1(\widetilde{M}(H))$ is an isomorphism and $p^{-1}(F)$ is incompressible, the inclusion $\pi_1(\widetilde{M}') \hookrightarrow \pi_1(\overline{M}_1)$ is an isomorphism and the inclusion $\pi_1(\widetilde{f}) \hookrightarrow \pi_1(\overline{M}_2)$ is an isomorphism. I claim that each component of $p^{-1}(F)$ has finitely generated fundamental group. This follows from a lemma, which is of interest in its own right.

VII.11. LEMMA: *Let* R *be a* 3-*manifold and let* S *be a surface in* ∂R. *If* $\pi_1(R)$ *is finitely generated, then* $\mathrm{Im}(\pi_1(S) \hookrightarrow \pi_1(R))$ *is finitely generated*.

Proof: Let R' be a homeomorphic copy of R which is disjoint from R. Choose a homeomorphism $h : R \longrightarrow R'$. Let $S' = h(S)$. Let $2R$ be the 3-manifold obtained from the disjoint union $R \cup R'$ by identifying $s \in S$ with $s' = h(s) \in S'$. Set $G = \mathrm{Im}(\pi_1(S) \hookrightarrow \pi_1(R))$ and $G' = I_M(\pi_1(S') \hookrightarrow \pi_1(R'))$. Let $\varphi : G \longrightarrow G'$ be the isomorphism of groups induced by h. Then $\pi_1(2R)$ can be represented as a free product with amalgamation $\pi_1(2R) \approx (\pi_1(R), \pi_1(R'), G, G'; \varphi)$. The group $\pi_1(2R)$ is a finitely generated 3-manifold group. If G, and hence G', are not finitely generated, then $\pi_1(2R)$ cannot be finitely presented [Ne_1]. However, this contradicts Corollary V.16. ∎

So, if $\tilde{f}$ is a component of $p^{-1}(F)$, then, with the above notation, $\tilde{f}$ is a surface in $\partial\overline{M}_1$. Since $\pi_1(\overline{M}_1)$ is isomorphic to $\pi_1(\tilde{M}')$, which in turn is isomorphic to H, a finitely generated group, the group $\pi_1(\overline{M}_1)$, is finitely generated. By Lemma VII.11, $\mathrm{Im}\,(\pi_1(\tilde{f}) \hookrightarrow \pi_1(\overline{M}_1))$ is finitely generated. However, $\tilde{f}$ is incompressible, so $\mathrm{Im}(\pi_1(\tilde{f}) \hookrightarrow \pi_1(\overline{M}_1))$ is isomorphic to $\pi_1(\tilde{f})$. This establishes my claim.

Now, let $\tilde{N}$ be a component of $p^{-1}(M-\mathring{U}(F))$. Then $\tilde{N}$ is a covering space of a component N of $M-\mathring{U}(F)$ with covering projection $p|\tilde{N}$. Furthermore, by the preceding, either $\tilde{N} = \tilde{M}'$ or $\tilde{N}$ is the covering space of N corresponding to the conjugacy class in $\pi_1(N)$ of a finitely generated peripheral subgroup of $\pi_1(N)$. If $\tilde{N} = \tilde{M}'$, then $\tilde{N}$ is itself a compact Haken-manifold; otherwise, by Corollary VII.8 $\tilde{N}$ admits a manifold compactification to a Haken-manifold. It now follows from Theorem VII.2 that $\tilde{M}(H)$ admits a manifold compactification to a Haken-manifold. ■

VII.12. COROLLARY: *Let M be a Haken-manifold. Suppose that F is a two-sided, incompressible surface in M or in ∂M. Then the covering space of M corresponding to the conjugacy class of $\pi_1(F)$ in $\pi_1(M)$ admits a manifold compactification to a Haken-manifold.*

Proof: Let M' be a regular neighborhood of F in M which meets ∂M in a regular neighborhood T' of $F \cap \partial M$ in ∂M. The pair (M',T') is a well-embedded manifold pair in (M,∂M) and $\pi_1(M') = \pi_1(F)$. ■

As I mentioned earlier (Remark VII.3) it is unknown if the covering space of a Haken-manifold M corresponding to the conjugacy class of a finitely generated subgroup of $\pi_1(M)$ admits a manifold compactification.

There is a perplexing set of examples to study. Namely, I have the following question.

VII.13. QUESTION: Let $M = F \times_{\varphi} S^1$ be a bundle over S^1 with fiber the compact surface F. Does the covering space of M corresponding to any finitely generated subgroup of $\pi_1(M)$ admit a manifold compactification?

A 3-manifold pair (M,T) is called a Haken-manifold pair if M is a Haken-manifold and T is an incompressible 2-manifold in ∂M.

Let (M,T) be a Haken-manifold pair. I shall describe a method, based on Corollary VII.8, for analyzing the peripheral structure of M at the components of T. Let T_r be a component of T. Let $\widetilde{M}_r$ denote the covering space of M corresponding to the conjugacy class in $\pi_1(M)$ of $\pi_1(T_r)$. Let $p_r\colon \widetilde{M}_r \longrightarrow M$ denote the covering projection. There is a component $\widetilde{T}_r$ of $p_r^{-1}(T_r)$ such that $p_r|\widetilde{T}_r\colon \widetilde{T}_r \longrightarrow T_r$ is a homeomorphism. By Corollary VII.8 $\widetilde{M}_r$ admits a manifold compactification to a Haken-manifold. It follows from $[Br_1]$ that the manifold compactification of $\widetilde{M}_r$ is homeomorphic to $T_r \times I$ via a homeomorphism with $\widetilde{T}_r$ corresponding to $T_r \times \{0\}$. Therefore $\widetilde{M}_r$ may be viewed as the product $T_r \times I$ with a closed subset of $T_r \times \{1\}$ missing and $\widetilde{T}_r$ corresponding to $T_r \times \{0\}$. With this identification let $\Pi_r\colon \widetilde{M}_r \longrightarrow \widetilde{T}_r$ denote the corresponding product projection.

Each component of $p_r^{-1}(T)$ is an incompressible surface in $\partial\widetilde{M}_r$; and by the preceding, the product structure of $\widetilde{M}_r$ has been chosen so that each component of $p_r^{-1}(T)$, except $\widetilde{T}_r$, is a submanifold of $T_r \times \{1\}$ in the compactification of $\widetilde{M}_r$ to $T_r \times I$. Let $\widetilde{\mathcal{J}}_r$ denote the incompressible 2-manifold in $\widetilde{T}_r$ which is the image under the product projection Π_r of the 2-manifold $p_r^{-1}(T) - \widetilde{T}_r \subset T_r \times \{1\}$. Set $\mathcal{F}_r = p_r(\widetilde{\mathcal{F}}_r)$.

Carry out this same procedure for each component T_r of T. Then for each component T_r of T there is the collection of pairwise disjoint, incompressible surfaces $\mathcal{F}_r$ in T_r, which can be thought of as a recording of the "shadows" on T_r of the components above T in the covering $\widetilde{M}_r$ of M which corresponds to $\pi_1(T_r)$ (of course, all of this is after an appropriate product structure has been chosen for $\widetilde{M}_r$). Set $\mathcal{F} = \bigcup_r \mathcal{F}_r$. Then $\mathcal{F}$ is an incompressible 2-manifold in T.

Now, some more notation and definitions. Suppose that (P,Q) and (X,Y) are polyhedral pairs. A PL-map $f: P \longrightarrow X$ is a <u>map of pairs</u> if $f(Q) \subset Y$. This is written $f : (P,Q) \longrightarrow (X,Y)$. The map $g : (P,Q) \longrightarrow (X,Y)$ is <u>homotopic to</u> f (<u>as a map of pairs</u>) if there exists a map of pairs

$$h : (P \times I,\ Q \times I) \longrightarrow (X,Y)$$

with $h(p,0) = g(p)$ and $h(p,1) = f(p)$ for $p \in P$.

Let (M,T) be an n-manifold pair and let (X,Y) be a connected polyhedral pair. A map of pairs $f : (X,Y) \longrightarrow (M,T)$ is <u>essential</u> if f is not homotopic, as a map of pairs, to a map $g : (X,Y) \longrightarrow (M,T)$ such that $g(X) \subset T$. A map of a general polyhedral pair into (M,T) is <u>essential</u> iff it is essential on each component. Otherwise, such a map is <u>inessential</u>.

This preliminary discussion makes a very nice setting in which to describe essential homotopies into M having their initial and terminal ends in T. These homotopies are quite restricted and in the nicest possible case such an essential homotopy can be thought of as being deformable in M, while keeping the initial and terminal ends in T, so that its image is "level-by-level" in an embedded I-bundle in M.

Let P be a polyhedron and suppose that $f : (P \times I, P \times \partial I) \longrightarrow (M, T)$ is a map of pairs. The map $f_t : P \longrightarrow M$ where $f_t(p) = f(p,t)$ is defined for each t; and in particular, f_0 and f_1 map P into T. The map f_0 is called the <u>initial end</u> of f and the map f_1 is the <u>terminal end</u> of f. The map f is a <u>homotopy from</u> f_0 <u>to</u> f_1. If P is connected and $f : (P \times I, P \times \partial I) \longrightarrow (M, T)$ has $f_0(P) \subset T_r$, a component of T, then f lifts to $\tilde{f} : (P \times I, P \times \partial I) \longrightarrow (\tilde{M}_r, p^{-1}(T))$ such that $\tilde{f}_0(P) \subset \tilde{T}_r$. Furthermore, if f is essential, then $\tilde{f}_1(P) \not\subset \tilde{T}_r$; however, $\tilde{f}_1(P) \subset p^{-1}(T)$.

VII.14. OBSERVATIONS: Let (M, T) be a Haken-manifold pair, P a polyhedron and $f: (P \times I, P \times \partial I) \longrightarrow (M, T)$ an essential map of pairs. All other notation as above:

(a) The map of pairs $\bar{f} : (P \times I, P \times \partial I) \longrightarrow (M, T)$ defined as $\bar{f}(p,t) = f(p, 1-t)$ is an essential map of pairs with initial end $\bar{f}_0 = f_1$ and terminal end $\bar{f}_1 = f_0$.

(b) There exists $g : (P \times I, P \times \partial I) \longrightarrow (M, T)$ homotopic to f as a map of pairs such that both $g_0(P)$ and $g_1(P)$ are contained in $\mathcal{F}$.

Proof: Suppose that P is connected and $f_0(P) \subset T_{r_0}$, $f_1(P) \subset T_{r_1}$. The map f_i (i=0,1) is homotopic in T into $\mathcal{F}$ via the homotopies $p_{k_0} \circ \Pi_{k_0} \circ \tilde{f}$ and $p_{k_1} \circ \Pi_{k_1} \circ \tilde{\tilde{f}}$, respectively. In general, the homotopy is defined by restricting to each component of P. ∎

(c) If the map $f_1 : P \longrightarrow T$ is an embedding of P into T, then

f_0 is *homotopic in* T to an embedding of P into $\mathcal{T}$; i.e., if a map of a polyhedron into T is homotopic *in* M to an embedding in T (possibly different components of T), then the map is homotopic *in* T to an embedding in T. ∎

Let M be an n-manifold and let S be a subset of ∂M (S is not necessarily connected). An essential map of pairs $d : (S \times I, S \times \partial I) \to (M, \partial M)$ is a *spatial deformation* if

(i) d_0 is the inclusion $S \to M$ (i.e., $d_0(s) = s$ for every $s \in S$) and

(ii) d_1 embeds S into ∂M.

If d is a spatial deformation and S' is the homeomorphic image of S under d_1, then d is a *spatial deformation* from S to S'.

VII.15. LEMMA: *Let* (M,T) *be a Haken-manifold pair. Using the preceding notation, if* F *is a component of* $\mathcal{F}$, *then there exists a component* F' *of* $\mathcal{F}$ *and a spatial deformation from* Int F *to* Int F' (possibly F' = F).

Proof: Let F be a component of $\mathcal{F}$. I may assume that F is not simply connected; for otherwise the conclusion is immediate. By the definition of $\mathcal{F}$ and Observation VII.14(c), there exists a spatial deformation d from F into T. Furthermore, by Observation VII.14(b) I may assume that $d_1(F)$ is contained in a component of $\mathcal{F}$, say F'. Now, consider $\bar{d}$. It follows that $\bar{d}$ is a spatial deformation from F' into F. Since both F and F' are incompressible and the embeddings d_1 and $\bar{d}_1$ induce isomorphisms, it follows that Int F and Int F' are homeomorphic. The desired spatial deformation is now easy to construct. ∎

Let (M,T) be a Haken-manifold pair. A compact, incompressible 2-manifold $\Phi \subset T$ is a <u>characteristic pair factor</u> for (M,T) if

(i) for any component φ of Φ there is a spatial deformation from φ to a component φ' of Φ (possibly $\varphi' = \varphi$),

(ii) for any connected polyhedron P and any essential map $f : (P \times I, P \times \partial I) \longrightarrow (M,T)$ such that the image under f_* of $\pi_1(P \times I)$ is not trivial, the map f can be factored through a spatial deformation from some component φ of Φ to a component φ' of Φ; i.e., there exists a component φ of Φ, a spatial deformation d from φ to a component φ' of Φ and a map $g : P \longrightarrow \varphi$ such that $d \circ (g \times \mathrm{id})$ is homotopic to f as a map of pairs, and

(iii) for no proper subcollection of components of Φ are (i) and (ii) satisfied.

VII.16. THEOREM: <u>Let</u> (M,T) <u>be a Haken-manifold pair</u>. <u>A characteristic pair factor for</u> (M,T) <u>exists and is unique</u> (up to ambient isotopy of M fixed on $M - T$).

Proof: Given the Haken-manifold pair (M,T) let $\mathfrak{F}$ be an incompressible 2-manifold embedded in T and defined as above. Since T is compact and each component of $\mathfrak{F}$ is incompressible, each component of $\mathfrak{F}$ has finitely generated fundamental group. Therefore, if F is a component of $\mathfrak{F}$, there is a compact, incompressible surface $C_F \subset \mathrm{Int}\, F$ so that the inclusion $C_F \longrightarrow F$ is a homotopy equivalence. The surface

C_F is called a *core for* F and is unique up to ambient isotopy of F, fixed off of some compact subset of Int F. An easy Euler characteristic argument shows that $\mathfrak{F}$ has at most a finite number of components having a core with negative Euler characteristic and those cores of components of $\mathfrak{F}$, which have Euler characteristic zero, represent at most a finite number of ambient isotopy classes in T. I am not concerned with the simply connected components of $\mathfrak{F}$.

Define Φ to be the compact, incompressible 2-manifold in T consisting of cores from components of $\mathfrak{F}$, one to represent each component of $\mathfrak{F}$ having a core with negative Euler characteristic and one to represent each isotopy class in T of cores of components of $\mathfrak{F}$ with Euler characteristic zero. I claim that Φ is a characteristic pair factor for (M, T).

If C is a core of any component of $\mathfrak{F}$ and C is not simply connected, then there exists a component φ of Φ and an ambient isotopy of T, fixed on $\partial M - T$, taking C onto φ. So, by Lemma VII.15, part (i) is satisfied.

If P is a connected polyhedron and $f : (P \times I, P \times \partial I) \longrightarrow (M, T)$ is essential, then by Observation VII.14(b) I may assume that both f_0 and f_1 have image in $\mathfrak{F}$. Now, the image under f_* of $\pi_1(P \times I)$ is not trivial; so, I may further assume, by the definition of Φ, that f_0 has image in some component φ_0 of Φ and f_1 has image in some component φ_1 of Φ. If both φ_0 and φ_1 have negative Euler characteristic, then the spatial deformation of part (i) beginning in φ_0 must take φ_0 to φ_1, and f may be factored through it with $g = f_0$. If either of φ_0 or φ_1 has zero Euler characteristic, then the situation is a bit more subtle. Suppose this is the case. Choose notation so that φ_1 has zero Euler characteristic (this is possible, use $\bar{f}$ if necessary) Let T_0 be the component of T containing φ_0 and let $\widetilde{M}_0$ denote the covering space of M corresponding to $\pi_1(T_0)$ with covering projection $p_0: \widetilde{M}_0 \longrightarrow M$. I shall use notation consistent with the above in studying this situation. The map f lifts to $\widetilde{f} : (P \times I, P \times \partial I) \longrightarrow (\widetilde{M}_0, p_0^{-1}(T))$ with $\widetilde{f}_0$ having image in $\widetilde{\varphi}_0 \subset \widetilde{T}_0$. There is a unique component $\widetilde{F}_0$ of $p_0^{-1}(T)$, distinct from $\widetilde{T}_0$, such that $\widetilde{\varphi}_0$ is a core of $\Pi_0(\widetilde{F}_0)$ and there is a unique component $\widetilde{F}_1$ of $p_0^{-1}(T)$, distinct from $\widetilde{T}_0$ such that $\widetilde{f}_1$ has image in $\widetilde{F}_1$. If $\widetilde{F}_1 = \widetilde{F}_0$, then there is a component φ_1' of Φ (possibly $\varphi_1' = \varphi_1$; and in fact, $\varphi_1' = \varphi_1$ if φ_0 is an annulus; otherwise, φ_1 is an annulus "parallel" to a component of $\partial\varphi_1'$) and a spatial deformation from φ_0 to φ_1' through which f can be factored with $g = f_0$. If $\widetilde{F}_1 \neq \widetilde{F}_0$, then there is a component φ_o' of Φ (possibly $\varphi_0' = \varphi_0$; and in fact, $\varphi_0' = \varphi_0$ if φ_0 is an annulus; otherwise, φ_0' is an annulus "parallel" to a component of $\partial\varphi_0$) and a spatial deformation from φ_0' to φ_1 through which f can be factored with $g = f_0$. So, part (ii) is satisfied. Clearly, part (iii) is satisfied.

It remains to prove that a characteristic pair factor for (M,T) is unique. To this end, suppose that both Φ and Φ' satisfy conditions (i), (ii), and (iii) above. Then from conditions (i) and (ii) it follows that there is an ambient isotopy of M, fixed on $M - T$, taking Φ into Φ' and an ambient isotopy of M, fixed on $M - T$, taking Φ' into Φ. By condition (iii) it follows that there is an ambient isotopy of M, fixed on $M - T$, taking Φ to Φ' iff up to ambient isotopy of M, fixed on $M - T$, Φ and Φ' have the same classes of annuli. This can be easily arrived at by modifying the above argument. All details are given in Lemma 4.3 of [J-S_2]. ∎

VII.17. REMARK: Theorem VII.16 is a major observation leading to the results of Chapters VIII and IX. Indeed, the use of the term characteristic pair factor anticipates this work. Many surprising and very useful finiteness properties involving peripheral information are immediate corollaries of Theorem VII.16; e.g., Corollarly 3.6, Theorem 4.5 and Corollaries 4.6, 4.7, 4.8, and 4.9 of [J-S_2]. I would like very much to present these results here; however, space is limited and these latter results will not be needed in the remaining chapters of these lectures. I shall conclude this chapter with an "algebraic characterization" of a peripheral subgroup. It makes the work of Chapter X much easier and quite civilized.

Let $\widetilde{X}$ be a polyhedron. If C is a compact subpolyhedron of $\widetilde{X}$, then $\widetilde{X} - C$ has at most a finite number of components; some of which are unbounded (i.e., have noncompact closure) if $\widetilde{X}$ is not itself compact. Let $e(\widetilde{X} - C)$ denote the number of unbounded components of $\widetilde{X} - C$. The <u>number of ends of</u> $\widetilde{X}$, written $e(\widetilde{X})$, is defined to be $e(\widetilde{X}) \equiv \sup \{e(\widetilde{X} - C)$: C is a compact subpolyhedron of $\widetilde{X}\}$ if this number exists; otherwise, $e(\widetilde{X})$ is infinite.

VII.18. EXAMPLES:

(a) Let $\widetilde{X}$ be compact. Then $e(\widetilde{X}) = 0$.

(b) Let $\mathbb{R}^1$ denote the real line, then $e(\mathbb{R}^1) = 2$.

(c) Let $\mathbb{R}^n$ denote n-dimensional Cartesian space $(n \geq 2)$. Then $e(\mathbb{R}^n) = 1$.

(d) Let X denote the wedge of two circles $(X = S^1 \vee S^1)$. Let $\widetilde{X}$ denote the universal covering space of X. Then $e(\widetilde{X})$ is infinite.

VII.19. EXERCISE: Let X be the closed, orientable surface of genus two. For each $n \geq 0$, show that X has a covering space $\widetilde{X}_n$ such that $e(\widetilde{X}_n) = n$.

Now, I want to use the above ideas to define an invariant of a pair (G,H) where G is a finitely presented group and H is a subgroup of G. Recall that if G is a finitely presented group, then there exists a compact polyhedron X such that $\pi_1(X) \approx G$. Hence, suppose that G is a finitely presented group and H is a subgroup of G. Let X be a compact polyhedron with $\pi_1(X) \approx G$ and let $\widetilde{X}$ denote the covering space of X corresponding to the conjugacy class of H in $G \approx \pi_1(X)$. Define $e(G,H) = e(\widetilde{X})$ to be the <u>number of ends of the pair</u> (G,H). Then $e(G,H)$ depends only on G and the cosets of H in G [Ep_3]. In particular, $e(G,\{1\})$ is the <u>number of ends of</u> G; and if H is normal in G, then $e(G,H)$ is the number of ends of G/H.

VII.20. EXERCISE: If H is normal in G, show that $e(G,H)$ is equal to 0, 1, 2 or is infinite.

VII.21. EXERCISE: Give an example of a finitely presented group G which has the property that for each $n \geq 0$ there is a finitely presented subgroup H_n of G such that $e(G,H_n) = n$.

The next two propositions were originally observed by G. A. Swarup and reported to me by Peter Scott. If the characteristic pair factor of a Haken-manifold pair $(M,\partial M)$ is empty, then they give a pleasant algebraic characterization of a peripheral subgroup of the fundamental group of M. In Chapter X I shall give a generalization of these results.

VII.22. PROPOSITION: Let $(M,\partial M)$ be a Haken-manifold pair. Set $G = \pi_1(M)$ and let H be a subgroup of G isomorphic to the fundamental group of a closed, orientable surface. If $e(G,H) = 1$, then H is peripheral.

Proof: Let $\widetilde{M}(H)$ denote the covering space of M corresponding to the conjugacy class of H in $G = \pi_1(M)$. Let $p : \widetilde{M}(H) \longrightarrow M$ denote the covering projection.

Let F be a closed, orientable surface with $\pi_1(F) \approx H$. There exists a map $f : F \longrightarrow M$ such that the induced homomorphism $f_*: \pi_1(F) \longrightarrow \pi_1(M)$ is an isomorphism onto H. Hence, f lifts to $\tilde{f} : F \longrightarrow \widetilde{M}(H)$. Since $H \approx \pi_1(F)$ is neither infinite cyclic nor a nontrivial free product, it follows by Theorem V.13 that there exists a compact submanifold N in $\widetilde{M}(H)$ such that the inclusion induced homomorphism of $\pi_1(N)$ to $\pi_1(\widetilde{M}(H))$ is an isomorphism. Now, by hypothesis $e(\widetilde{M}(H)) = 1$; so $\widetilde{M}(H) - N$ has precisely one unbounded component. I may assume that $\widetilde{M}(H)-N$ is connected (and unbounded). However, $\pi_1(N)$ being isomorphic to the fundamental group of a closed surface implies that N is an I-bundle; and the orientability implies that N is a product I-bundle. Hence, one component $\widetilde{B}$ of ∂N is contained in $\partial\widetilde{M}(H)$ and the inclusion $\pi_1(\widetilde{B}) \hookrightarrow \pi_1(\widetilde{M}(H))$ is a homotopy equivalence. Set $B = p(\widetilde{B})$. Then B is a component of ∂M and H is conjugate in $\pi_1(M)$ into $\mathrm{Im}(\pi_1(B) \hookrightarrow \pi_1(M))$; i.e., H is peripheral. ■

VII.23. REMARK: Notice that if M is a compact 3-manifold, $G = \pi_1(M)$ and

H is a finitely-generated subgroup of G, which has infinite index in G, then $e(G,H)$ finite implies that ∂M is incompressible. In particular, in the preceding proposition I did not need to assume that ∂M is incompressible; it is necessarily incompressible by the assumptions on H.

VII.24. PROPOSITION: *Let $(M,\partial M)$ be a Haken-manifold pair. Set $G = \pi_1(M)$ and let H be a subgroup of G isomorphic to the fundamental group of a closed, orientable surface. If the characteristic pair factor of $(M,\partial M)$ is empty and H is peripheral, then* $e(G,H) = 1$.

Proof: Since H is peripheral, there is a component B of ∂M such that H is conjugate into $\mathrm{Im}(\pi_1(B) \hookrightarrow \pi_1(M))$. Furthermore, since H is a closed surface group, the conjugacy class of H in $\mathrm{Im}(\pi_1(B) \hookrightarrow \pi_1(M))$ has finite index. So, it is sufficient to show that if $\widetilde{M}(B)$ is the covering space of M corresponding to the conjugacy class of $\mathrm{Im}(\pi_1(B) \hookrightarrow \pi_1(M))$, then $\widetilde{M}(B)$ has only one end. By Corollary VII.8, $\widetilde{M}(B)$ admits a manifold compactification to the Haken-manifold $B \times I$ in such a way that $\widetilde{M}(B)$ is homeomorphic to $B \times I$ minus a closed subset Z of $B \times \{1\}$. Since the characteristic pair factor for $(M,\partial M)$ is empty, it follows that Z is connected; so, $\widetilde{M}(B)$ has only one end. This completes the proof. ■

VII.25. REMARK: Let M be a Haken-manifold and suppose that ∂M is incompressible. Then there is an algebraic characterization of when the pair $(M,\partial M)$ has a nonempty characteristic pair factor; namely, the pair $(M,\partial M)$ has a nonempty characteristic pair factor iff $\pi_1(M)$ splits as a nontrivial free product with amalgamation along the infinite cyclic group.

CHAPTER VIII. ESSENTIAL HOMOTOPIES (THE ANNULUS-TORUS THEOREMS)

In this chapter I shall give a method for the study of certain maps into Haken-manifolds. In particular, I shall prove the homotopy versions of the Annulus-Torus Theorems. Immediate corollaries of these theorems are the Annulus-Torus Theorems announced by Waldhausen [W_5] and the Characteristic Seifert Pair Theorem, Chapter IX.

This work was done originally in collaboration with P. Shalen [J-S_1], [J-S_2], [J-S_3] and [J-S_4]. The presentation given here follows the lines of our original approach. It is similar to the presentation in [J-S_1]; however, I do not go into the generalities of the manuscript [J-S_1], I have changed the emphasis somewhat and I am much less formal in this presentation.

A different approach is outlined in [J-S_3] and is carried out in detail in two unpublished manuscripts. An independent and different approach from either of the above mentioned methods is given by K. Johannson [Jo_1]. I understand that still another approach has been obtained by P. Scott.

A Haken-manifold pair (S,F) is an I-<u>pair</u> if there exists a homemorphism h of S onto the total space of an I-bundle over a compact 2-manifold, not necessarily orientable, such that $h(F)$ is the total space of the corresponding ∂I-bundle. For example, if B is a compact, orientable surface, then the pair $(B \times I, B \times \partial I)$ is an I-pair. If S is homeomorphic to a product I-bundle, then the I-pair (S,F) is a product I-pair; otherwise, the I-pair (S,F) is a twisted I-pair. In the latter case, F is connected iff S is connected.

A Haken-manifold pair (S,F) is an S^1-pair if there exists a homeomorphism h of S onto the total space of a Seifert fibered 3-manifold such that h(F) is a saturated subset in some Seifert fibration.

A Haken-manifold pair (S,F) is a Seifert pair if each component is either an I-pair or an S^1-pair.

VIII.1. REMARKS:

(a) Some Haken-manifold pairs are both I-pairs and S^1-pairs; e.g., $(S^1 \times S^1 \times I,\ S^1 \times S^1 \times \partial I)$. In fact, any I-pair over a surface with Euler characteristic zero is an S^1-pair.

(b) If (S,F) is an S^1-pair, then S is a Seifert fibered manifold with orbit manifold B and projection $p : S \longrightarrow B$ such that F is saturated (with respect to p) over some 1-manifold $\bar{F} \subset \partial B$; i.e., $F = p^{-1}(\bar{F})$.

Before getting into the results of this chapter, I need a few more definitions. If (M,T) is a 3-manifold pair, a map $f : (S^1 \times I,\ S^1 \times \partial I) \longrightarrow (M,T)$ is nondegenerate if f is essential (as a map of pairs; see Chapter VII) and $f_*: \pi_1(S^1 \times I) \longrightarrow \pi_1(M)$ is injective. A map $f : (S^1 \times S^1, \emptyset) \longrightarrow (M,T)$ is nondegenerate if f is essential (as a map of pairs) and $f_*: \pi_1(S^1 \times S^1) \longrightarrow \pi_1(M)$ is injective.

The next definitions are for technical reasons and later help in keeping track of homotopies and piecing manifolds together. Let L be a polyhedron and let T be a 2-manifold. A submanifold T' of T carries f if f is homotopic to a map $f': L \to T$ such that $f'(L) \subset T'$ and each component of T' meets $f'(L)$. The map $f': L \to T'$ fills T' if every incompressible 2-manifold in T' which carries f is a deformation retract of T'. The following observations will be helpful through the remainder of this chapter.

VIII.2. OBSERVATIONS:

(a) If L is a disconnected polyhedron and T' is an annulus, then no map of L into T' can fill T'. This gives examples of maps of a polyhedron into a 2-manifold which do not fill even if their restrictions to some subpolyhedron do fill. This causes a technical pain; however, the phenomenon can be analyzed.

(b) Let L be a polyhedron, T a 2-manifold and $f : L \to T$ a map. Then there exists a submanifold T' of T and a map $f': L \to T'$, homotopic to f, such that f' fills T'.

I shall now state the main technical results of this chapter.

VIII.3. PROPOSITION A: Let (M,T) be a Haken-manifold pair and let $f : (S^1 \times I, S^1 \times \partial I) \to (M,T)$ be a map of pairs such that $f|S^1 \times \partial I : S^1 \times \partial I \to T$ is an embedding. If f is nondegenerate, then there exists a well-embedded S^1-pair $(\Sigma, \Phi) \subset (M,T)$ and a map $g : (S^1 \times I, S^1 \times \partial I) \to (M,T)$ homotopic to f (rel $S^1 \times \partial I$) such that $g(S^1 \times I) \subset \Sigma$ and $g(S^1 \times \partial I) \subset \Phi$.

VIII.4. ESSENTIAL HOMOTOPY THEOREM: Let (M,T) be a Haken-manifold pair and let K be a compact polyhedron. Suppose that $f : (K \times I, K \times \partial I) \to (M,T)$ is a map of pairs such that K has no component k <u>for which the subgroup</u> $(f|k \times I)_* (\pi_1(k \times I)) \subset \pi_1(M)$ <u>is trivial</u>. <u>If</u> f <u>is essential, then there exists a well-embedded Seifert pair</u> $(\Sigma, \Phi) \subset (M, T)$ <u>and a map</u> $g : (K \times I, K \times \partial I) \longrightarrow (M, T)$ <u>homotopic to</u> f (as a map of pairs) <u>such that</u> $g(K \times I) \subset \Sigma$, $g(K \times \partial I) \subset \Phi$ <u>and</u> $g|K \times \partial I : K \times \partial I \longrightarrow \Phi$ <u>fills</u> Φ.

VIII.5. DISCUSSION: Notice that Proposition A is a special case of the Essential Homotopy Theorem. The hypothesis in Proposition A, that the map $f|S^1 \times \partial I$ is an embedding, allows the conclusion that the Seifert pair is actually an S^1-pair. This type of distinction is not possible in the Essential Homotopy Theorem. Clearly, in general, such a distinction could not be possible; but even if $K = S^1$ in the Essential Homotopy Theorem, the Seifert pair may be an I-pair and not an S^1-pair (e.g., see [J_5]).

Proposition A, while of independent interest, basically has the role of a technical result needed in the proof of the Essential Homotopy Theorem. The two results are proved together via induction on the length of the Haken-manifold M (see Chapter IV). The proof of Proposition A is really straightforward in idea; however, the fact that $f|S^1 \times \partial I$ maps $S^1 \times \partial I$ into T requires everything to be done in a relative fashion. The resulting technical and notational problems cost insight into what is really going on. So, I only plan to outline the idea of the proof of Proposition A. The analogous "absolute" version of proof is given in detail later (the proof of the Homotopy Torus Theorem, Theorem VIII.11). If the reader is interested in the details of a proof of Proposition A, they can be found in [J-S_1]. In this Chapter, I shall prove that Proposition A implies the Essential Homotopy Theorem.

Before I outline the proof of Proposition A, I think it might help the reader if Theorem VII.16, the discussion just preceding that theorem, and the next paragraph are read with the Essential Homotopy Theorem in mind.

For convenience, in the statement of the Essential Homotopy Theorem suppose that T is connected and K is connected. Let $\widetilde{M}$ be the covering space of M corresponding to the conjugacy class of $\pi_1(T)$ in $\pi_1(M)$. Let $p : \widetilde{M} \longrightarrow M$ denote the covering projection. Then $\widetilde{M}$ is homeomorphic to $T \times I$ minus a closed subset of $T \times \{1\}$; and if $\widetilde{T}$ corresponds to $T \times \{0\}$, the map $p|\widetilde{T} : \widetilde{T} \longrightarrow T$ is a homeomorphism. Now, $f : (K \times I, K \times \partial I) \longrightarrow (M,T)$; so, f lifts to $\widetilde{f} : (K \times I, K \times \partial I) \longrightarrow (\widetilde{M}, p^{-1}(T))$ with $\widetilde{f}|K \times \{0\} : K \times \{0\} \longrightarrow \widetilde{T}$ and $\widetilde{f}|K \times \{1\}:K \times \{1\} \longrightarrow p^{-1}(T)$. Furthermore, by f being essential it follows that $\widetilde{f}|K \times \{1\}$ maps $K \times \{1\}$ into a component of $p^{-1}(T)$ distinct from $\widetilde{T}$. The product structure on $\widetilde{M}$ guarantees a well-embedded Seifert-pair (actually, always a product I-pair) $(\widetilde{\Sigma}, \widetilde{\Phi})$ in $(\widetilde{M}, p^{-1}(T))$ and a map $\widetilde{g} : (K \times I, K \times \partial I) \longrightarrow (\widetilde{M}, p^{-1}(T))$ homotopic to $\widetilde{f}$ (as a map of pairs) such that $\widetilde{g}(K \times I) \subset \widetilde{\Sigma}$ and $\widetilde{g}(K \times \partial I) \subset \widetilde{\Phi}$. So, in the covering space $\widetilde{M}$ there exists a well-embedded Seifert pair $(\widetilde{\Sigma}, \widetilde{\Phi})$ and a map $\widetilde{g}$ satisfying the conclusions of the Essential Homotopy Theorem with respect to the map $\widetilde{f}$ and the 3-manifold pair $(\widetilde{M}, p^{-1}(T))$. The idea is to consider the map $p|\widetilde{\Sigma} : \widetilde{\Sigma} \longrightarrow M$ and try to make it an embedding. Then set $g = p \circ \widetilde{g}$. From the work of Chapter VII, I have that $p|\widetilde{\Phi}$ embeds each component of $\widetilde{\Phi}$. However, $p|Fr\widetilde{\Sigma}$, each component of $Fr\,\widetilde{\Sigma}$ is an incompressible annulus, may be singular. In the case that $(\widetilde{\Sigma}, \widetilde{\Phi})$

is not the I-pair $(S^1 \times I \times I,\ S^1 \times I \times \partial I)$, I can apply Proposition A to the map $p|\mathrm{Fr}\,\tilde{\Sigma}$ and actually arrange the pair $(\tilde{\Sigma}, \tilde{\Phi})$ so that $p|\tilde{\Sigma} : \tilde{\Sigma} \longrightarrow M$ is a covering map onto its image. The pair (Σ, Φ) with $\Sigma = p(\tilde{\Sigma})$ and $\Phi = p(\tilde{\Phi})$ is a well-embedded I-pair in M and together with the map $g = p \circ \tilde{g}$ satisfies the conclusions of the Essential Homotopy Theorem. In the case that $(\tilde{\Sigma}, \tilde{\Phi})$ is the I-pair $(S^1 \times I \times I, S^1 \times I \times \partial I)$ I cannot, in general, alter the pair $(\tilde{\Sigma}, \tilde{\Phi})$ so that $p|\tilde{\Sigma} : \tilde{\Sigma} \longrightarrow M$ is a covering map onto its image. However, I can still apply Proposition A in this case and in a more direct fashion. Namely the pair $(\tilde{\Sigma}, \tilde{\Phi}) = (S^1 \times I \times I,\ S^1 \times I \times \partial I)$ naturally collapses onto $(S^1 \times I,\ S^1 \times \partial I)$. The map $p|S^1 \times I : (S^1 \times I,\ S^1 \times \partial I) \longrightarrow (M, T)$ can be arranged so that $p|S^1 \times \partial I$ is an embedding; so, by Proposition A, there is an S^1-pair $(\Sigma, \Phi) \subset (M, T)$ and the map $\tilde{g} : (K \times I,\ K \times \partial I) \longrightarrow (\tilde{M}, p^{-1}(T))$ may be chosen so that $g = p \circ \tilde{g}$ is homotopic to f (as a map of pairs) and g maps $(K \times I,\ K \times \partial I)$ into (M, T) with $g(K \times I) \subset \Sigma$ and $g(K \times \partial I) \subset \Phi$. The fact that $g|K \times \partial I : K \times \partial I \longrightarrow \Phi$ fills Φ comes out naturally. The work of Chapter VII enables me to present the proof of the Essential Homotopy Theorem in a more formal and slick fashion than the preceding discussion.

There is a technical lemma which is needed in the proof of Proposition A; in fact, this lemma will be used several times later in this chapter and in the next chapter.

VIII.6. LEMMA: _Let_ (M, T) _be a Haken-manifold pair. Suppose that_ $(\Sigma_1, \Phi_1), \ldots, (\Sigma_n, \Phi_n)$ _are well-embedded Seifert pairs in_ (M, T) _such that_ $\partial\Phi_i \cap \partial\Phi_j = \emptyset$, $i \neq j$. _Then there exists a well-embedded Seifert_ pair $(\Sigma, \Phi) \subset (M, T)$ and homeomorphisms $J_i : M \to M$ each isotopic to the identity on M with $J_i(\Sigma_i) \subset \Sigma$ and $J_i(\Phi_i) \subset \Phi$.

Proof: This is one of those proofs that has close to a million cases which need to be considered. I would like to just skip over it; however, this lemma plays an important role and I feel that I should put in something of its proof.

It is sufficient to consider the case that M is connected. Also, since I do not require that a Seifert pair be connected, if I reason inductively on the number, n, of Seifert pairs, it is sufficient to prove the lemma for two Seifert pairs (Σ_1, Φ_1) and (Σ_2, Φ_2).

With an ambient isotopy of M (fixed on ∂M) make $\mathrm{Fr}\,\Sigma_1$ and $\mathrm{Fr}\,\Sigma_2$ meet transversely. I shall induct on the number of components of $\mathrm{Fr}\,\Sigma_1 \cap \mathrm{Fr}\,\Sigma_2$, each of which is a simple closed curve $(\partial\Phi_1 \cap \partial\Phi_2 = \emptyset)$.

I have to first consider the case that $\mathrm{Fr}\,\Sigma_1 \cap \mathrm{Fr}\,\Sigma_2 = \emptyset$. If $\Sigma_1 \cap \Sigma_2 = \emptyset$, then set $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\Phi = \Phi_1 \cup \Phi_2$. If $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, then to find the desired pair (Σ, Φ) I induct on the number of components of (Σ_1, Φ_1) which meet (Σ_2, Φ_2). Suppose that (σ^*, φ^*) is a component of (Σ_1, Φ_1) and $\sigma^* \cap \Sigma_2 \neq \emptyset$. If (σ^*, φ^*) is contained in a component of (Σ_2, Φ_2), then let (Σ_1', Φ_1') be obtained from (Σ_1, Φ_1) by discarding the component (σ^*, φ^*). The lemma follows from induction using the pairs (Σ_1', Φ_1') and (Σ_2, Φ_2). So, suppose that (σ^*, φ^*) is not contained in a component of (Σ_2, Φ_2) and if (σ_2, φ_2) is a component of (Σ_2, Φ_2) meeting (σ^*, φ^*), then (σ_2, φ_2) is not contained in (σ^*, φ^*) (any components of (Σ_2, Φ_2) entirely contained in (σ^*, φ^*) can be completely forgotten).

If (σ^*, φ^*) is an I-pair and <u>not</u> an S^1-pair, then via an ambient isotopy of $\underline{M}$ moving components of (Σ_2, Φ_2) only in a neighborhood of (σ^*, φ^*), I can assume that either (σ^*, φ^*) no longer meets (Σ_2, Φ_2) or (σ^*, φ^*) only meets components of (Σ_2, Φ_2) that are I-pairs and not S^1-pairs. The union of (σ^*, φ^*) and all components of (Σ_2, Φ_2) which meet (σ^*, φ^*) is an I-pair, say $(\sigma^{**}, \varphi^{**})$. Let (Σ_1', Φ_1') be obtained from (Σ_1, Φ_1) by discarding the component (σ^*, φ^*). Let (Σ_2', Φ_2') be obtained from (Σ_2, Φ_2) by discarding the components of (Σ_2, Φ_2) which meet (σ^*, φ^*) and adding the I-pair $(\sigma^{**}, \varphi^{**})$. The lemma follows from induction using the pairs (Σ_1', Φ_1') and (Σ_2', Φ_2').

If (σ^*, φ^*) is an (S^1-pair), then consider any component of intersection of (σ^*, φ^*) with a component of (Σ_2, Φ_2). First, since (σ^*, φ^*) is an S^1-pair and not contained in any component of (Σ_2, Φ_2) I can assume that after an isotopy, either (σ^*, φ^*) does not meet (Σ_2, Φ_2) or (σ^*, φ^*) meets (Σ_2, Φ_2) only in components which are S^1-pairs. So, the particular component of intersection of (σ^*, φ^*) with an S^1-pair, (σ_2, φ_2), of (Σ_2, Φ_2) has frontier, which consists of saturated annuli and tori in the Seifert fibration of (σ^*, φ^*) and the Seifert fibration of (σ_2, φ_2). (Notice that any annulus in the frontier is clearly saturated in both fibrations; however, to conclude that the tori are also saturated in both fibrations, I am using Theorem VI.34 from which I can conclude that either the tori are saturated in both fibrations or one of σ^* or σ_2 is closed, and therefore either $(\sigma^*, \varphi^*) \subset (\sigma_2, \varphi_2)$ or $(\sigma_2, \varphi_2) \subset (\sigma^*, \varphi^*)$, both of which I ruled out earlier by convention.) I can now conclude that the component of intersection of (σ^*, φ^*) and (σ_2, φ_2)

admits a Seifert fibration which is compatible with the Seifert fibration of (σ^*, φ^*) and the Seifert fibration of (σ_2, φ_2). I can adjust the Seifert fibrations to agree unless the component of intersection admits distinct Seifert fibrations. It follows from VI.18 that the component of intersection may admit distinct Seifert fibrations only if it is a solid torus, a twisted I-bundle over the Klein bottle or the product $S^1 \times S^1 \times I$. Under the present circumstances the only possibility is $S^1 \times S^1 \times I$, in which case I can make a small isotopy and eliminate the component of intersection between (σ^*, φ^*) and (σ_2, φ_2). Eventually, I obtain an S^1-pair $(\sigma^{**}, \varphi^{**})$ that contains (σ^*, φ^*) and possibly some components of (Σ_2, Φ_2). Let (Σ_1', Φ_1') be obtained from (Σ_1, Φ_1) by discarding the component (σ^*, φ^*) and let (Σ_2', Φ_2') be obtained from (Σ_2, Φ_2) by discarding the components of (Σ_2, Φ_2) which meet (σ^*, φ^*) and adding the S^1-pair $(\sigma^{**}, \varphi^{**})$. The lemma follows from induction using the pairs (Σ_1', Φ_1') and (Σ_2', Φ_2').

We must go on. The case now is when $\mathrm{Fr}\ \Sigma_1 \cap \mathrm{Fr}\ \Sigma_2 \neq \emptyset$. As observed earlier, each component of $\mathrm{Fr}\ \Sigma_1 \cap \mathrm{Fr}\ \Sigma_2$ is a simple closed curve. I shall induct on the number of components of $\mathrm{Fr}\ \Sigma_1 \cap \mathrm{Fr}\ \Sigma_2$.

If some component J of $\mathrm{Fr}\ \Sigma_1 \cap \mathrm{Fr}\ \Sigma_2$ is contractible in M, it is contractible in both $\mathrm{Fr}\ \Sigma_1$ and $\mathrm{Fr}\ \Sigma_2$. Hence, via an isotopy of M (fixed on ∂M), J may be eliminated from the intersection while no components of intersection are added. So, for the remainder of the argument, I shall assume that no component of the intersection, $\mathrm{Fr}\ \Sigma_1 \cap \mathrm{Fr}\ \Sigma_2$, is contractible in M.

Let F be a component of $\mathrm{Fr}\ \Sigma_1$ which meets $\mathrm{Fr}\ \Sigma_2$. Now, F is either an annulus or a torus; and since no component of $\mathrm{Fr}\ \Sigma_1 \cap \mathrm{Fr}\ \Sigma_2$

is contractible, $F \cap \mathrm{Fr}\ \Sigma_2$ is a collection of parallel simple closed curves on F dividing F up into a collection of (more than one) annuli. Let A be one of these annuli such that $A \cap \Sigma_2$ is either one or both components of ∂A ; i.e., A is not contained in Σ_2.

If A has only one component of ∂A in Σ_2, then F is an annulus and $\partial A = J_0 \cup J_1$, where J_0 is a component of ∂F contained in T and $J_1 = A \cap \Sigma_2$. Let A_1 denote the component of $F \cap \Sigma_2$ where $A_1 \cap A = J_1$. Then A_1 is an annulus properly embedded in a component (σ_2, φ_2) of (Σ_2, Φ_2). If (σ_2, φ_2) is an I-pair and not an S^1-pair, then A_1 is parallel into $\mathrm{Fr}\ \sigma_2$ and via an isotopy of M (fixed on ∂M), the number of components of $\mathrm{Fr}\ \Sigma_1 \cap \mathrm{Fr}\ \Sigma_2$ can be reduced. So, I may assume that (σ_2, φ_2) is an S^1-pair and A_1 is not parallel into $\mathrm{Fr}\ \sigma_2$. It follows from VI.20 that J_1 is a fiber in some Seifert fibration of σ_2. I have to check that such a Seifert fibration of σ_2 can be chosen so that φ_2 is saturated. This is the case (Theorem VI.18) unless σ_2 is homeomorphic to $\mathbb{D}^2 \times S^1$, the twisted I-bundle over the Klein bottle or $S^1 \times S^1 \times I$. Since $J_1 \cap \varphi_2 = \emptyset$, it follows in each of these cases that the Seifert fibration of σ_2 may be made compatible with both J_1 and φ_2. Let $(\sigma_2^*, \varphi_2^*)$ be the S^1-pair obtained from (σ_2, φ_2) by adding a regular neighborhood of A to σ_2, and a regular neighborhood of J_0 in ∂M to φ_2. Let (Σ_2^*, Φ_2^*) be obtained from (Σ_2, Φ_2) by discarding (σ_2, φ_2) and adding $(\sigma_2^*, \varphi_2^*)$. Then $\mathrm{Fr}\ \Sigma_1 \cap \mathrm{Fr}\ \Sigma_2^*$ contains fewer components than $\mathrm{Fr}\ \Sigma_1 \cap \mathrm{Fr}\ \Sigma_2$ and the lemma follows by induction on the pairs (Σ_1, Φ_1) and (Σ_1^*, Φ_1^*).

If A has both components of ∂A in Σ_2, then F may be either an annulus or a torus. Set $\partial A = J_0 \cup J_1$. Let A_i denote the component of $F \cap \Sigma_2$ where $A_i \cap A = J_i$, $i = 0,1$. Then A_i $(i = 0,1)$ is an annulus properly embedded in a component $(\sigma_2^i, \varphi_2^i)$ of (Σ_2, Φ_2). From the above argument, I only need to consider the case that both $(\sigma_2^0, \varphi_2^0)$ and $(\sigma_2^1, \varphi_2^1)$ are S^1-pairs and both A_0 and A_1 are not parallel into $\mathrm{Fr}\ \sigma_2^0$ and $\mathrm{Fr}\ \sigma_2^1$, respectively. Again from VI.20, I have that J_0 is a fiber in a Seifert fibration of σ_2^0 and J_1 is a fiber in a Seifert fibration of σ_2^1. These fiberings can be chosen to be compatible with φ_2^0 and φ_2^1 and compatible with each other in the case that $\sigma_2^0 = \sigma_2^1$. Let $(\sigma_2^*, \varphi_2^*)$ be the S^1-pair obtained by adding a regular neighborhood of A to $\sigma_2^0 \cup \sigma_2^1$. Now, $(\sigma_2^*, \varphi_2^*)$ is an S^1-pair; however, $(\sigma_2^*, \varphi_2^*)$ may not be well-embedded; i.e., it may have a component of its frontier that is a compressible torus in M. A compressible torus in the irreducible manifold M either bounds a solid torus in M or is contained in a 3-cell in M. It can be argued that in the present situation a compressible torus in the frontier of $(\sigma_2^*, \varphi_2^*)$ actually bounds a solid torus in M; and in fact, it bounds a solid torus in the closure of $M - \sigma_2^*$. Since (Σ_1, Φ_1) is well-embedded in M and both J_0 and J_1 are fibers in σ_2^*, the Seifert fibering on $(\sigma_2^*, \varphi_2^*)$ extends over such a solid torus. I shall continue to call the resulting S^1-pair $(\sigma_2^*, \varphi_2^*)$. In this way if I let (Σ_2^*, Φ_2^*) be obtained from (Σ_2, Φ_2) by discarding $(\sigma_2^0, \varphi_2^0)$ and $(\sigma_2^1, \varphi_2^1)$ and adding $(\sigma_2^*, \varphi_2^*)$, then $\mathrm{Fr}\ \Sigma_1 \cap \mathrm{Fr}\ \Sigma_2^*$ contains fewer components than $\mathrm{Fr}\ \Sigma_1 \cap \mathrm{Fr}\ \Sigma_2$. By induction on the pairs (Σ_1, Φ_1) and (Σ_2^*, Φ_2^*), the lemma follows.

And, that is what the proof of Lemma VIII.6 is all about. ∎

VIII.7. REMARK: For technical reasons it is sometimes helpful to notice that Φ, in the conclusion of Lemma VIII.6, may be taken as $\Phi = \bigcup_{i=1}^{n} \Phi_i$.

VIII.8. OUTLINE OF PROOF FOR PROPOSITION A. Proposition A is proved via induction on the length of the manifold M (defined in Chapter IV). The proof of Proposition A given in [J-S_1] is not via induction but requires that M admit a certain type of hierarchy; and such a hierarchy is shown to exist for any bounded Haken-manifold.)

Remember that this method of inductive proof begins with the validity of the theorem for handlebodies. The proof of Proposition A in the case that M is a handlebody is via induction on the genus of M. Both inductive steps can be proved at once.

First, Proposition A is vacuously true if M is a 3-cell. So, suppose that (M, T) is a Haken-manifold pair and M is not a 3-cell (M may or may not be a handlebody). If M is a handlebody, let F be a properly embedded disk in M such that (i) F is not parallel into ∂M, (ii) the map $f : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M, T)$ is transverse to F and (iii) among all disks in M satisfying (i) and (ii) the number of components of $f^{-1}(F)$ is a minimum. If M is not a handlebody, let F be a two-sided incompressible and ∂-incompressible surface in M such that (i) F is not a disk and is not parallel into ∂M (see Exercise IV.11), (ii) the map $f : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M, T)$ is transverse to F and (iii) among all incompressible and ∂-incompressible surfaces F in M satisfying (i) and (ii) the number of components of $f^{-1}(F)$ is a minimum.

Since in both situations F is incompressible and the number of components of $f^{-1}(F)$ is a minimum, it can be shown by using standard arguments that no component of $f^{-1}(F)$ is a simple closed curve which is contractible in $S^1 \times I$. Since in both situations F is incompressible and ∂-incompressible and the number of components of $f^{-1}(F)$ is a minimum, it can be shown by using the loop theorem in the manifold obtained by splitting M at F that no component of $f^{-1}(F)$ is a spanning arc having both of its end points in the same component of $S^1 \times \partial I$. So, there are three cases to consider.

The first case is when $f^{-1}(F) = \emptyset$. In this case the image of the map f is in a component of the manifold obtained by splitting M at F. If M is a handlebody, then such a component is a handlebody having genus strictly less than the genus of M; so, by induction, the conclusion of Proposition A follows. If M is not a handlebody, then such a component is a Haken-manifold having length strictly less than the length of M (Chapter IV); so, by induction, the conclusion of Proposition A follows.

The second case is when $f^{-1}(F) \neq \emptyset$ and each component of $f^{-1}(F)$ is a simple closed curve. (This case is not possible if M is a handlebody.) If the number of components of $f^{-1}(F)$ is n $(n \geq 1)$, then the product $S^1 \times I$ can be reparametrized so that the components of $f^{-1}(F)$ correspond to the simple closed curves $S^1 \times \{i/n+1\}$, $1 \leq i \leq n$. Furthermore, it follows from VII.14(c), transversality and an induction argument, that by modifying f, by at most a homotopy (rel $S^1 \times \partial I$), I can assume that the entire set of simple closed curves $S^1 \times \{i/n+1\}$, $1 \leq i \leq n$, is embedded in F.

Let M' be the manifold obtained by splitting M at F and let $T' \subset \partial M'$ be the incompressible 2-manifold (each component of which is an annulus) consisting of a regular neighborhood in $\partial M'$ of the collection of pairwise disjoint simple closed curves $f(S^1 \times \{i/n+1\})$, $0 \leq i \leq n+1$. Now, with a slight abuse of notation, the map $f_i = f|S^1 \times [i/n+1, i+1/n+1]$ maps the pair $(S^1 \times [i/n+1, i+1/n+1], S^1 \times \{i/n+1, i+1/n+1\})$ into (M',T'). Observe that f_i is nondegenerate, since f is nondegenerate and $f^{-1}(F)$ is minimal, and f_i embeds $S^1 \times \{i/n+1, i+1/n+1\}$ into T'. Each component of the Haken-manifold M' is a Haken-manifold having length strictly less than the length of M (I already observed that this case does not happen if M is a handlebody); so by induction there exists a well-embedded S^1-pair $(\Sigma_i, \Phi_i) \subset (M',T')$ and maps g_i homotopic to f_i (rel $S^1 \times \{i/n+1, i+1/n+1\}$) such that $g_i(S^1 \times [i/n+1, i+1/n+1]) \subset \Sigma_i$ and $g_i(S^1 \times \{i/n+1, i+1/n+1\}) \subset \Phi_i$. This can be done for each i, $0 \leq i \leq n$.

So, I get the collection of well-embedded S^1-pairs $(\Sigma_0, \Phi_0), \ldots, (\Sigma_n, \Phi_n)$ where $\Phi_i \cap \Phi_j = \emptyset$, $i \neq j$. I now use Lemma VIII.6 to find an S^1-pair $(\Sigma', \Phi') \subset (M',T')$ such that each (Σ_i, Φ_i) is isotopic (as a pair) into (Σ', Φ'). It is easy to argue that the components of (Σ', Φ') fit together to give the desired S^1-pair $(\Sigma, \Phi) \subset (M,T)$ and that g may be defined from the g_i's.

The third case is when $f^{-1}(F) \neq \emptyset$ and each component of $f^{-1}(F)$ is an essential spanning arc; i.e., has its end points in distinct components of $S^1 \times \partial I$. This is the case that requires a messy relativization. The idea (carried out in the absolute case in Theorem VIII.11) is this: let M' be the manifold obtained from M by splitting at F. Now, forget T and let T' be the two copies of F in $\partial M'$ obtained from the splitting. The components of $f^{-1}(F)$ split $S^1 \times I$ into disks (at least one) which may be parametrized as $I \times I$ in such a way that

f restricted to one of these disks may be thought of as a map of pairs taking $(I \times I, I \times \partial I)$ into (M', T') such that the arcs $\partial I \times I$ embed in $(M' - T')$. Now, if M is a handlebody, then each component of M' is a handlebody having genus strictly less than the genus of M, so, by induction Proposition A is valid in each component of (M', T'). If M is not a handlebody, then each component of M' is a Haken-manifold having length strictly less than the length of M; so, again by induction, Proposition A is valid in each component of (M', T'). Using Proposition A it is possible to prove a relative version of the Essential Homotopy Theorem which applies to the maps of the pair $(I \times I, I \times \partial I)$ into (M', T') (see Proposition B in [J-S_1]). As in the second case above, a Seifert pair (Σ', Φ') is obtained in (M', T') in such a way that (Σ', Φ') identifies to a Haken-manifold pair $(\Sigma, \Phi) \subset (M, T)$; furthermore, there is a map $g : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M, T)$ homotopic to $f(\mathrm{rel}\ S^1 \times \partial I)$ with $g(S^1 \times I) \subset \Sigma$ and $g(S^1 \times \partial I) \subset \Phi$. The burden of argument now switches to proving that (Σ, Φ) is a Seifert-pair. I shall carry out this argument in the proof of Theorem VIII.11. This completes the outline of the proof of Proposition A.

VIII.9. LEMMA: Proposition A implies the Essential Homotopy Theorem.

Proof: I believe that the statement of this lemma should be clear. Namely, I am assuming that (M, T) is a Haken-manifold pair and if f is any non-degenerate map of $(S^1 \times I, S^1 \times \partial I)$ into (M, T) with $f | S^1 \times \partial I$ an embedding, then f can be homotoped (rel $S^1 \times \partial I$) to a map of $(S^1 \times I, S^1 \times \partial I)$ into (M, T) having its image in a well-embedded S^1-pair in (M, T); and I claim that the Essential Homotopy Theorem (VIII.4) is valid for the pair (M,T).

So, let the notation be as in the statement of VIII.4. In view of Lemma VIII.6 it is sufficient to prove the result in the case that K is connected. Now, by Theorem VII.16 the Haken-manifold pair (M,T) has a characteristic pair factor Φ^*; so, the map $f : (K \times I, K \times \partial I) \longrightarrow (M,T)$ can be factored through a spatial deformation, d, from some component φ of Φ^* to a component φ' of Φ^*, possibly $\varphi = \varphi'$. I need to consider two cases.

The first case is when the Euler characteristic of φ is zero (φ is an annulus). Now, the map $f : (K \times I, K \times \partial I) \longrightarrow (M,T)$ factors through the spatial deformation $d : (\varphi \times I, \varphi \times \partial I) \longrightarrow (M,T)$. Since φ is an annulus, it follows from the definition of a spatial deformation that f factors through an essential map $f' : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M,T)$ with $f'|S^1 \times \partial I$ an embedding. However, by the hypothesis that the subgroup $f_*(\pi_1(K \times I)) \subset \pi_1(M)$ is not trivial, it follows that f' is actually nondegenerate. Hence, by Proposition A (applied to the map f') there exists a Seifert pair $(\Sigma, \Phi) \subset (M,T)$ (in this case an S^1-pair) and a map $g' : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M,T)$ homotopic to f'(rel $S^1 \times \partial I$) such that $f'(S^1 \times I) \subset \Sigma$ and $f'(S^1 \times \partial I) \subset \Phi$. The map f factors through the map g'; hence, I can conclude that there exists a map $g : (K \times I, K \times \partial I) \longrightarrow (M,T)$ homotopic to f (as a map of pairs) with $g(K \times I) \subset \Sigma$ and $g(K \times \partial I) \subset \Phi$.

The second case is when the Euler characteristic of φ is negative (φ is not an annulus). In this case I will show that the spatial deformation d can be chosen so that $d : (\varphi \times I, \varphi \times \partial I) \longrightarrow (M,T)$ is a covering map onto the image of d, $d(\varphi \times I)$; so, if I set $\Sigma = d(\varphi \times I)$

and $\Phi = d(\varphi \times \partial I)$, the pair (Σ, Φ) will be the desired Seifert pair (in this case an I-pair; e.g., see Lemma 4.1 of $[W_3]$, reported as Theorem 10.3 in $[He_1]$).

If $\partial\varphi = \emptyset$, then $d|\partial(\varphi \times I)$ is a covering map and by the work of $[W_3]$, reported as Theorem 13.6 in $[He_1]$, either d is homotopic (rel $\varphi \times \partial I$) to a covering map or d is homotopic (as a map of pairs) to a map taking $\varphi \times I$ into T. The latter situation is not possible since d is a spatial deformation and is, in particular, essential (as a map of pairs). It follows, in the case that $\partial\varphi = \emptyset$, that I can choose d as above.

If $\partial\varphi \neq \emptyset$, then $d|\partial(\varphi \times I)$ is not necessarily a covering map; however, it misses only on $d|(\partial\varphi \times I)$. Now, d restricted to a component of $\partial\varphi \times I$ is a nondegenerate map of $(S^1 \times I, S^1 \times \partial I) \longrightarrow (M, T)$. It follows from Proposition A and Lemma VIII.6 (with Remark VIII.7) that there exists a well-embedded S^1-pair $(\Sigma', \Phi') \subset (M, T)$ and the spatial deformation d can be chosen so that $d(\partial\varphi \times I) \subset \Sigma'$ and $d(\partial\varphi \times \partial I) \subset \Phi'$; i.e., I can assume that there is a well-embedded S^1-pair $(\Sigma', \Phi') \subset (M, T)$ and I can encase the singular annuli $d|\partial\varphi \times I$ in (Σ', Φ').

Now, consider the collection $\mathfrak{a} = d^{-1}(\mathrm{Fr}\ \Sigma')$ of two-sided surfaces in $\varphi \times I$. First, I may choose Φ' to be a regular neighborhood of $\partial\varphi \cup \partial\varphi'$ in ∂M (Remark VIII.7). Since d is a spatial deformation, it follows that $\partial\mathfrak{a} \neq \emptyset$; in fact, $\partial\mathfrak{a}$ can be very precisely described: for each component J of $\partial\mathfrak{a}$, there is one and only one component of $\partial\mathfrak{a}$ in $\varphi \times \{0\}$ parallel to $J \times \{0\}$ and one and only one component of $\partial\mathfrak{a}$ in $\varphi \times \{1\}$ parallel to $J \times \{1\}$. By Lemma III.9 I can also choose d so that each component of $\mathfrak{a}$ is incompressible in $\varphi \times I$. Now, here is where I use that the Euler characteristic of φ is negative. I know from the fact that each component of $\mathfrak{a}$ is incompressible and that each component of Fr Σ' is either an annulus or a torus, that each component of $\mathfrak{a}$ is either a disk, an annulus or a torus. Since I know that each component of $\partial\mathfrak{a}$ is parallel in either $\varphi \times \{0\}$ or $\varphi \times \{1\}$ to a component of $\partial\varphi \times \{0\}$ or $\partial\varphi \times \{1\}$, no component of $\mathfrak{a}$ is a disk. From [W_2] or section 8 of [H_1], if any component of $\mathfrak{a}$ is a torus then φ would be a torus; but I am in the situation that $\partial\varphi \neq \emptyset$. So, I conclude that each component of $\mathfrak{a}$ is an annulus. Now, again by [W_2] or section 8 of [H_1], each annulus component of $\mathfrak{a}$ is either "vertical" in the product structure of $\varphi \times I$ or is parallel into $\varphi \times \{0\}$ or $\varphi \times \{1\}$; since I know exactly what $\partial\mathfrak{a}$ is, the only way that a component of $\mathfrak{a}$ could be parallel into $\varphi \times \{0\}$ or $\varphi \times \{1\}$ is for φ to be itself an annulus (Euler characteristic of φ be zero). Since the Euler characteristic of φ is negative, I can conclude that each component of $\mathfrak{a}$ is a "vertical" annulus in the product structure of $\varphi \times I$; and so, for each component J of $\partial\varphi$ there is one and only one component of $\mathfrak{a}$ parallel in $\varphi \times I$ to $J \times I$. It follows from d being essential that d can be chosen so that d embeds each component of $\partial\varphi \times I$ onto a two-sided annulus in M (in Fr Σ') in such a way that $d|\partial(\varphi \times I)$ is a covering map. Now, by using [W_2], it follows that I can choose d so that d is itself a covering map onto its image. This completes the proof of Lemma VIII.9. ■

I shall now state and prove the strong homotopy versions of the

Annulus-Torus Theorems.

VIII.10. HOMOTOPY ANNULUS THEOREM: Let (M, T) be a Haken-manifold pair. If $f : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M, T)$ is a nondegenerate map, then there exists a well-embedded Seifert pair $(\Sigma, \Phi) \subset (M, T)$ and a map $g :$ $(S^1 \times I, S^1 \times \partial I) \longrightarrow (M, T)$ homotopic to f (as a map of pairs) such that $g(S^1 \times I) \subset \Sigma$ and $g(S^1 \times \partial I) \subset \Phi$. Furthermore, if $f | S^1 \times \partial I$ is an embedding, then the map g may be chosen so that $g | S^1 \times \partial I = f | S^1 \times \partial I$.

Proof: This is just a special case of the Essential Homotopy Theorem (VIII.4) in which $K = S^1$. The refinement in the case that $f | S^1 \times \partial I$ is an embedding follows from Proposition A. ■

VIII.11. HOMOTOPY TORUS THEOREM: Let (M, T) be a Haken-manifold pair. If $f : (S^1 \times S^1, \emptyset) \longrightarrow (M, T)$ is a nondegenerate map, then there exists a well-embedded Seifert pair $(\Sigma, \emptyset) \subset (M, T)$ and a map $g : (S^1 \times S^1, \emptyset)$ $\longrightarrow (M, T)$ homotopic to f such that $g(S^1 \times S^1) \subset \Sigma$.

Proof: The proof of this Theorem can be considered as the "aboslute case of the proof of Proposition A (VIII.3). Here I prove the theorem via induction on the length of a hierarchy for the Haken-manifold M (see Chapter IV).

The induction starts with M a 3-cell, in which case the theorem is vacuously true. So, suppose that (M, T) is a Haken-manifold pair and that there exists a two-sided incompressible surface $W \subset M$ such that for each component M' of M split open at W and each nondegenerate map $f' : (S^1 \times S^1, \emptyset) \longrightarrow (M', \emptyset)$ there exists a well-embedded Seifert pair $(\Sigma', \emptyset) \subset (M', \emptyset)$ and a map $g' : (S^1 \times S^1, \emptyset) \longrightarrow (M', \emptyset)$ homotopic to f' such that $g'(S^1 \times S^1) \subset \Sigma'$.

Considering those maps in the homotopy class of f which are transverse to W, I may assume that the number of components of $f^{-1}(W)$ is a minimum. Then either $f^{-1}(W) = \emptyset$ or each component of $f^{-1}(W)$ is a noncontractible simple closed curve in the torus $S^1 \times S^1$. If $f^{-1}(W) = \emptyset$, then for some component M' of M split open at W, the map f factors through a nondegenerate map $f': (S^1 \times S^1, \emptyset) \longrightarrow (M', \emptyset)$. The desired conclusions follow from induction in this case. (This is the only place in the argument that induction is used.) So, I only need to consider the case that $f^{-1}(W) \neq \emptyset$.

Suppose that $f^{-1}(W)$ consists of n components $(n \geq 1)$. Then I may parametrize the torus $S^1 \times S^1$ as the quotient space obtained from $S^1 \times I$ by identifying $(x,1)$ with $(x,0)$ in such a way that $f^{-1}(W)$ corresponds to the set of simple closed curves $S^1 \times \{i/n\}$, $0 \leq i \leq n$. Let (M',T') be the Haken-manifold pair where M' is the Haken-manifold obtained by splitting M at W and T' is the incompressible 2-manifold in $\partial M'$ corresponding to the union of the two copies of W in $\partial M'$. Then with a slight abuse of notation the map f splits as a union of maps of pairs $f_i: (S^1 \times [i/n, i+1/n], S^1 \times \{i/n, i+1/n\}) \longrightarrow (M',T')$, $0 \leq i < n$; and furthermore, since I chose f so that $f^{-1}(W)$ is a minimum in the class of maps homotopic to f and transverse to W, each f_i is non-degenerated. I now want to apply the Essential Homotopy Theorem.

Let K be n disjoint copies of S^1 and identify $(K \times I, K \times \partial I)$ with $\cup(S^1 \times [i/n, i+1/n], S^1 \times \{i/n, i+1/n\})$. Let $f': (K \times I, K \times \partial I) \longrightarrow (M',T')$ be the map where $f'|S^1 \times [i/n, i+1/n] = f_i$, $0 \leq i \leq n$. Then since f is nondegenerate, K has no component k for which the subgroup $(f'|k \times I)_*(\pi_1(k \times I)) \subset \pi_1(M')$ is trivial; and (as noted earlier) since each f_i is essential, f' is essential. Hence, by the Essential Homotopy Theorem (VIII.4) there exists a Seifert pair (Σ', Φ') $\subset (M',T')$ and a map $g': (K \times I, K \times \partial I) \longrightarrow (M',T')$ homotopic to f' (as a map of pairs) such that $g'(K \times I) \subset \Sigma'$, $g'(K \times \partial I) \subset \Phi'$ and $g'|K \times \partial I : K \times \partial I \longrightarrow \Phi'$ fills Φ'. This last technical statement now is very important. By transversality of the original map f, it follows that the components of Φ' may be matched in pairs so that when I recover M by identifying the components of T' to get W, the components of Σ' are attached along the corresponding components of Φ' to give me a well-embedded Haken-manifold pair $(\Sigma,\emptyset)$ with the components of Φ' identifying to a two-sided incompressible surface $F \subset \Sigma$. (There is the standard "cut and paste" argument to show that Fr Σ is incompressible in M and it uses that Fr Σ' is incompressible in M'.) Furthermore, I can consider the map g' as being made up of n maps g_i' $(0 \leq i < n)$ where $g_i': (S^1 \times [i/n, i+1/n], S^1 \times \{i/n, i+1/n\})$ $\longrightarrow (M',T')$ with $g_i'(S^1 \times [i/n, i+1/n]) \subset \Sigma'$ and $g_i'(S^1 \times \{i/n, i+1/n\})$ $\subset \Phi'$; and so, g_i' is homotopic to f_i as a map of pairs, $0 \leq i < n$. Since "$f_i|S^1 \times \{\frac{i+1}{n}\} = f_{i+1}|S^1 \times \{\frac{i+1}{n}\}$" (reduction modulo n), it follows that after identification of the components of T' to W in order to recover M, the map "$g_i'|S^1 \times \{\frac{i+1}{n}\}$ is homotopic in W to $g_{i+1}'|S^1 \times \{\frac{i+1}{n}\}$" (again reduction modulo n and, of course, a bit of abuse of notation). Therefore, "$g_i'|S^1 \times \{\frac{i+1}{n}\}$ is homotopic in F to $g_{i+1}'|S^1 \times \{\frac{i+1}{n}\}$." This enables me to define a map $g:(S^1 \times S^1, \emptyset) \longrightarrow (M,\emptyset)$ homotopic to f such that $g(S^1 \times S^1) \subset \Sigma$. I simply set g equal to g_i' on the appropriate part of $S^1 \times S^1$ in the chosen parametrization.

I now need to show that $(\Sigma, \emptyset)$ is a Seifert pair. Of course, I will need to use that (Σ', Φ') is a Seifert pair; and again I will need to use that the map $g'|K \times \partial I$ fills Φ'.

It may help the reader if I organize the situation at this point into a statement. Namely, $(\Sigma, \emptyset)$ is a well-embedded Haken-manifold pair in (M, T). The 2-manifold F is two-sided and incompressible in Σ. If I split Σ at F, I obtain Σ'; and if I set Φ' equal to the two copies of F in $\partial\Sigma'$, the pair (Σ', Φ') is a Seifert pair; i.e., each component of (Σ', Φ') is either an I-pair or an S^1-pair. In general, the pair $(\Sigma, \emptyset)$ would not have to be a Seifert pair; however, in this situation I also have the nondegenerate map $g : (S^1 \times S^1, \emptyset) \longrightarrow (\Sigma, \emptyset)$ having the property that $g|g^{-1}(F) : g^{-1}(F) \longrightarrow F$ fills F and the number of components of $g^{-1}(F)$ is a minimum.

I will reason by induction on the number of components of F. I need to consider two cases.

Case 1. <u>Each component of</u> (Σ', Φ') <u>is an I-pair</u>.

I will break this case into two subcases. The first subcase is when each component of (Σ', Φ') is a <u>product</u> I-pair. This situation can easily be reduced to the consideration in which F is connected. It follows that Σ is a surface bundle over the circle, $\Sigma = F \times_\gamma S^1$, with fiber F and sewing map γ. Since $g|g^{-1}(F) : g^{-1}(F) \longrightarrow F$ fills F and the domain of g is $S^1 \times S^1$, it follows that γ is homotopic to a periodic homeomorphism (see [J-S_5]). So, by VI.27, Σ admits a

Seifert fibration and the pair $(\Sigma,\emptyset)$ is a Seifert pair.

The second subcase is when some component of (Σ',Φ') is a twisted I-pair. As before, this situation can be reduced to the consideration in which F is connected. It follows that Σ is the union of two twisted I-bundles (the two components of Σ') attached along their associated ∂I-bundles (the two components of Φ') which identify to the surface F. Now, Σ has a canonical two-sheeted covering $\tilde{\Sigma}$ with covering projection p, F lifts to $\tilde{\Sigma}$ and if $\tilde{F}$ is one of the components of $p^{-1}(F)$, then $\tilde{\Sigma} = \tilde{F} \times_{\tilde{\gamma}} S^1$ is a surface bundle over the circle with fiber $\tilde{F}$ and sewing map $\tilde{\gamma}$. If I choose the covering space of $S^1 \times S^1$ corresponding to the subgroup $g_*^{-1}(p_*\pi_1(\tilde{\Sigma}))$ of $\pi_1(S^1 \times S^1)$, the map g "lifts" to a map $\tilde{g} : S^1 \times S^1 \longrightarrow \tilde{\Sigma}$. The map $\tilde{g}|\tilde{g}^{-1}(\tilde{F}) : \tilde{g}^{-1}(\tilde{F}) \longrightarrow \tilde{F}$ fills $\tilde{F}$ and as in the first subcase, I can argue that $\tilde{\gamma}$ is homotopic to a periodic homeomorphism. So, by VI.27, $\tilde{\Sigma}$ admits a Seifert fibration It follows from VI.29 that Σ is itself a Seifert fibered manifold; and so, the pair $(\Sigma,\emptyset)$ is a Seifert pair.

Case 2. Some component of (Σ',Φ') is an S^1-pair and not an I-pair.

As before, this situation can be reduced to the consideration in which F is connected. It follows that F is either an annulus or a torus and each component of (Σ',Φ') is an S^1-pair. The situation is that Σ' has either one or two components (Seifert fibered manifolds) and Φ' has two components which are saturated annuli in the Seifert fibration of Σ' or tori boundary components. Clearly if F is an annulus the Seifert fibration of Σ' extends to a Seifert fibration of Σ. So, I only need to consider the situation when F is a torus.

I am going to prove that in Case 2, F cannot be a torus, by proving that if F is a torus, then each component of (Σ', Φ') must be an I-pair. First, recall that $g|g^{-1}(F) : g^{-1}(F) \longrightarrow F$ fills F. Each component of $g^{-1}(F)$ is a simple closed curve. If F is a torus, then any loop in F is homotopic in F to a power of a simple closed curve. It follows that if $g|g^{-1}(F)$ fills F, then $g^{-1}(F)$ has at least two components J and J' such that $g|J : J \longrightarrow F$ is not homotopic in F to a power of the same simple closed curve as $g|J' : J' \longrightarrow F$. Now, if I split Σ at F I obtain Σ' with the two copies of F in $\partial\Sigma'$, which I am denoting by Φ'. Suppose that $g|J : J \longrightarrow F$ is a power of the simple closed curve α and $g|J' : J' \longrightarrow F$ is a power of the simple closed curve α'. Let φ_1' and φ_2' denote the components of Φ' (φ_1' and φ_2' are tori which identify to the torus F). Let α_i and α_i' denote the simple closed curves in φ_i', $i = 1,2$, which identify to α and α' respectively. By transversality and the choice of g so that the number of components of $g^{-1}(F)$ is minimal, both α_i and α_i' are the initial ends of essential homotopies in the component of Σ' containing φ_i', $i = 1,2$. It follows from Chapter VI that there exists a Seifert fibration of Σ' having α_1 a fiber, one having α_1' a fiber, one having α_2 a fiber and one having α_2' a fiber. However, for $i = 1,2$, α_i and α_i' are in the same component φ_i' of Φ' and are not homotopic in φ_i'. It follows that for $i = 1,2$ the component of Σ' containing φ_i' admits distinct Seifert fibrations. By Theorem VI.18 and the fact that F (and Φ') are incompressible, it follows that each component of (Σ', Φ') is an I-pair. So, as I wanted to show, F cannot be a torus in Case 2.

This completes the proof of Theorem VIII.11. ■

VIII.12. REMARK: Notice that the Seifert pair $(\Sigma, \emptyset)$ in the statement of the Homotopy Torus Theorem (VIII.11) is an S^1-pair. This is implicit since the second term in an I-pair can never be empty.

I shall finish this chapter with the so-called Annulus Theorem and Torus Theorem.

VIII.13. ANNULUS THEOREM: _Let_ (M, T) _be a Haken-manifold pair. Suppose that_ $f : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M, T)$ _is a map of pairs. If_ f _is nondegenerate, then there exists an embedding_ $g : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M, T)$ _that is nondegenerate. Furthermore, if_ $f | S^1 \times \partial I$ _is an embedding, then_ g _may be chosen so that_ $g | S^1 \times \partial I = f | S^1 \times \partial I$.

Proof: It follows from VIII.10 that there exists a well-embedded Seifert pair $(\Sigma, \Phi) \subset (M, T)$ and a map $f' : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M, T)$ homotopic to f (as a map of pairs) such that $f'(S^1 \times I) \subset \Sigma$ and $f'(S^1 \times \partial I) \subset \Phi$. Furthermore, in the case that $f | S^1 \times \partial I$ is an embedding, then $f' | S^1 \times \partial I = f | S^1 \times \partial I$ and the homotopy may be taken (rel $S^1 \times \partial I$).

I have to alter the pair (Σ, Φ) for technical reasons which I shall point out later in the argument. Now, each component of $\mathrm{Fr}\, \Sigma$ is either an incompressible annulus or an incompressible torus. Furthermore, if some component A of $\mathrm{Fr}\, \Sigma$ is an incompressible annulus and is parallel into an annulus $A' \subset T$, then $A \cup A'$ cobounds a solid torus K_A. Observe that since f' is nondegenerate as a mapping into (M, T) and $f'(S^1 \times I) \subset \Sigma$, the Seifert pair (Σ, Φ) is not contained in K_A. Let K be the collection of all such solid tori K_A taken over the annular components of Fr Σ that are parallel into an annulus in T. Set $\Sigma' = \Sigma \cup K$ and set $\Phi' = \Sigma' \cap T$. Then (Σ', Φ') is a well-embedded Seifert pair in (M, T) and if A is an annular component of Fr Σ', then A is not parallel to an annulus in T.

Since $f'(S^1 \times I) \subset \Sigma'$ and $f'(S^1 \times \partial I) \subset \Phi'$ (with a slight abuse of notation), I can consider the map f' as a map from $(S^1 \times I, S^1 \times \partial I)$ into (Σ', Φ'), a Seifert pair. Furthermore, as a map $f': (S^1 \times I, S^1 \times \partial I) \longrightarrow (\Sigma', \Phi')$, f' is nondegenerate. Now, I can use that the Annulus Theorem is true in the case that the Haken-manifold pair is a Seifert pair (see Exercise VIII.15). Hence, there exists an embedding $g : (S^1 \times I, S^1 \times \partial I) \longrightarrow (\Sigma', \Phi')$ which is nondegenerate; and if $f'|S^1 \times \partial I$ is an embedding (in particular the case if $f|S^1 \times \partial I$ is an embedding), then g may be chosen so that $g|S^1 \times \partial I = f'|S^1 \times \partial I$. By the way in which I constructed the pair (Σ', Φ') above, it follows that g must be nondegenerate considered as a map of $(S^1 \times I, S^1 \times \partial I)$ into (M, T). This completes the proof of the theorem. ■

A Seifert fibered manifold S is <u>special</u> if S is homeomorphic to

(a) a Seifert fibered manifold with orbit-manifold a disk and at most two exceptional fibers.

(b) a Seifert fibered manifold with orbit-manifold an annulus and at most one exceptional fiber.

(c) a Seifert fibered manifold with orbit-manifold a disk-with-two-holes and no exceptional fibers.

(d) a Seifert fibered manifold with orbit-manifold a Möbius band and no exceptional fibers.

(e) a Seifert fibered manifold with orbit-manifold a 2-sphere and at most three exceptional fibers.

(f) a Seifert fibered manifold with orbit-manifold a projective plane and at most one exceptional fiber.

VIII.14. TORUS THEOREM: *Let* (M, T) *be a Haken-manifold pair. Suppose that* $f : (S^1 \times S^1, \emptyset) \longrightarrow (M, T)$ *is a map of pairs. If* f *is nondegenerate, then either there exists an embedding* $g : (S^1 \times S^1, \emptyset) \longrightarrow (M, T)$ *that is nondegenerate or* $T = \partial M$ *and* M *is homeomorphic to a special Seifert fibered manifold.*

Proof: It follows from VIII.11 that there exists a well-embedded Seifert pair $(\Sigma, \emptyset) \subset (M, T)$ and a map $f' : (S^1 \times S^1, \emptyset) \longrightarrow (M, T)$ homotopic to f such that $f'(S^1 \times S^1) \subset \Sigma$.

As in the proof of Theorem VIII.13, I have to possibly alter the pair $(\Sigma, \emptyset)$. Each component of $\mathrm{Fr}\ \Sigma$ is an incompressible torus in M. If some component C of $\mathrm{Fr}\ \Sigma$ is parallel into a torus C' in T, then $C \cup C'$ cobounds a product K_C homeomorphic to $S^1 \times S^1 \times I$. Observe that since f' is nondegenerate as a mapping into (M, T) and $f'(S^1 \times S^1) \subset \Sigma$, the Seifert pair $(\Sigma, \emptyset)$ is not contained in K_C. Let K denote the collection of all such products K_C taken over the components C of $\mathrm{Fr}\ \Sigma$ that are parallel into a torus in T. Set $\Sigma' = \Sigma \cup K$ and set $\Phi' = \Sigma' \cap T$. Then (Σ', Φ') is a well-embedded Seifert pair in (M, T) and if C is a component of $\mathrm{Fr}\ \Sigma'$, then C is an incompressible torus in M and C is not parallel to a torus in T.

Since $f'(S^1 \times S^1) \subset \Sigma'$, I can consider f' as a map from $(S^1 \times S^1, \emptyset)$ into (Σ', Φ'), a Seifert pair. Furthermore, as a map f': $(S^1 \times S^1, \emptyset) \longrightarrow (\Sigma', \Phi')$, f' is nondegenerate. This time I can use the Torus Theorem for the case that the Haken-manifold pair is a Seifert pair (Exercise VIII.16). Hence, there exists an embedding $g : (S^1 \times S^1, \emptyset) \longrightarrow (\Sigma', \Phi')$ that is nondegenerate or $\Phi' = \partial\Sigma'$ and Σ' is a special Seifert fibered manifold.

If there exists an embedding $g : (S^1 \times S^1, \emptyset) \longrightarrow (\Sigma', \Phi')$ that is nondegenerate, then by the way in which I constructed the pair (Σ', Φ') above, it follows that g must be nondegenerate considered as a map of $(S^1 \times S^1, \emptyset)$ into (M, T). On the other hand, if $\Phi' = \partial\Sigma'$ and Σ' is a special Seifert fibered manifold, then $\partial\Sigma' \subset T \subset \partial M$. It follows that $T = \partial M$ and $M = \Sigma'$ is a special Seifert fibered manifold. This completes the proof of Theorem VIII.14. ■

VIII.15. EXERCISE: Let (Σ, Φ) be a Seifert pair. Suppose that $f : (S^1 \times I, S^1 \times \partial I) \longrightarrow (\Sigma, \Phi)$ is a map of pairs. If f is nondegenerate, then there exists an embedding $g : (S^1 \times I, S^1 \times \partial I) \longrightarrow (\Sigma, \Phi)$ that is nondegenerate. Furthermore, if $f|S^1 \times \partial I$ is an embedding, then g may be chosen so that $g|S^1 \times \partial I = f|S^1 \times \partial I$.

VIII.16. EXERCISE: Let (Σ, Φ) be a Seifert pair. Suppose that $f : (S^1 \times S^1, \emptyset) \longrightarrow (\Sigma, \Phi)$ is a map of pairs. If f is nondegenerate, then either there exists an embedding $g : (S^1 \times S^1, \emptyset) \longrightarrow (\Sigma, \Phi)$ that is nondegenerate or $\Phi = \partial\Sigma$ and Σ is a special Seifert fibered manifold.

VIII.17. EXERCISE: [W_5] Let M be a Haken-manifold. Suppose that $f : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M, \partial M)$ is a map such that both $f_*: \pi_1(S^1 \times I) \longrightarrow \pi_1(M)$ and $f_*: \pi_1(S^1 \times I, S^1 \times \partial I) \longrightarrow \pi_1(M, \partial M)$ are injections. Then there exists an embedding $g : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M, \partial M)$ such that both $g_*: \pi_1(S^1 \times I) \longrightarrow \pi_1(M)$ and $g_*: \pi_1(S^1 \times I, S^1 \times \partial I) \longrightarrow \pi_1(M, \partial M)$ are injections.

VIII.18. EXERCISE: Let M be a Haken-manifold. Suppose that $f : S^1 \times S^1 \longrightarrow M$ is a map such that $f_*: \pi_1(S^1 \times S^1) \longrightarrow \pi_1(M)$ is an injection. Then either there exists an embedding $g : S^1 \times S^1 \longrightarrow M$ such that $g_*: \pi_1(S^1 \times S^1) \longrightarrow \pi_1(M)$ is an injection or $\partial M = \emptyset$ and M is homeomorphic to a Seifert fibered manifold with orbit-manifold S^2 with three exceptional fibers.

For a more extensive treatment of the Annulus-Torus Theorems see Chapter IV of [J-S_1].

CHAPTER IX. CHARACTERISTIC SEIFERT PAIRS

If a Haken-manifold, M, is closed or has incompressible boundary, the results of the last chapter show that for a certain type of mapping f into M, there is a choice of a well-embedded Seifert pair (Σ, Φ) in $(M, \partial M)$ and a mapping g, homotopic to f, such that the image of g is contained in Σ. In this chapter I shall prove that the pair (Σ, Φ) can be chosen independently of the mapping f and of its domain; indeed, there is a canonical choice for (Σ, Φ), given the Haken-manifold M. This canonical pair is the characteristic Seifert pair. The characteristic Seifert pair can be used to show that a Haken-manifold that is closed or has incompressible boundary admits a canonical decomposition into submanifolds that are either Seifert fibered manifolds, I-bundles, or simple Haken-manifolds (defined later in this chapter) by splitting it along annuli and tori.

As a corollary to the main results of this chapter, I extend the Homotopy Annulus Theorem (VIII.10) and the Homotopy Torus Theorem (VIII.11) to give a homotopy description of nondegenerate mappings from general Seifert pairs into Haken-manifolds. In fact, it is this result (Theorem IX.17) that represents the bulk of this chapter.

Recall that if M is a 3-manifold, then the 3-manifold pair $(\Sigma, \Phi) \subset (M, \partial M)$ is well-embedded (in M) if $\Sigma \cap \partial M = \Phi$ and $Fr_M \Sigma$ is incompressible. In order to later obtain certain uniqueness statements about 3-manifold pairs $(\Sigma, \Phi) \subset (M, \partial M)$, I need to refine this notion of being well-embedded. I shall say that the 3-manifold pair $(\Sigma, \Phi) \subset (M, \partial M)$ is perfectly-embedded (in M) if the following are satisfied:

(i) (Σ, Φ) is well-embedded in M .

(ii) Each component of $Fr_M\Sigma$ is essential (rel ∂M), i.e., if F is a component of $Fr_M\Sigma$, then the inclusion map of the pair $(F,\partial F)$ into $(M,\partial M)$ is essential (as a map of pairs). This is equivalent to no component of $Fr_M\Sigma$ being parallel into a surface in ∂M .

(iii) No component (σ,φ) of (Σ, Φ) can be homotoped (as a pair) into $(\Sigma-\sigma, \Phi-\varphi)$.

Most of the work needed to establish the results of this chapter has already been done. I shall organize the proof of the existence and uniqueness of a characteristic Seifert pair by stating and proving a number of lemmas; many of these lemmas are of independent interest.

I state but shall not prove the first lemma. I have used variations of this lemma before in these lectures; however, since it is repeatedly used in this chapter and there is no explicit statement in these lectures concerning it, I believe that it will be helpful to have, at least, a statement readily available. It first appeared as a lemma in section 8 of [H_1] and reappeared in [W_3].

IX. 1. LEMMA: *Let* S *be a compact 2-manifold. Let* F *be a two-sided, incompressible surface in* $S \times I$ *such that* $\partial F \subset \partial S \times I$. *Then either* F *is a disk or an annulus and is parallel in* $S \times I$ *to a disk or an annulus, respectively, in* $\partial S \times I$, *or* F *is homeomorphic to* S *and is parallel in* $S \times I$ *to both* $S \times \{0\}$ *and* $S \times \{1\}$.

IX. 2. LEMMA: *Let* M *be a Haken-manifold. Let* $(\Sigma_0, \Phi_0), \ldots, (\Sigma_n, \Phi_n), \ldots$ *be a sequence of perfectly-embedded 3-manifold pairs contained in* $(M,\partial M)$ *such that* $\Sigma_n \subset \mathring{\Sigma}_{n+1}$ *for each* $n \geq 0$. *Then either the sequence is finite or there exists an integer* $n_0 \geq 0$ *such that for each* $n > n_0$, Σ_{n+1} *is a* regular neighborhood of Σ_n in M and Φ_{n+1} is a regular neighborhood of Φ_n in ∂M.

Proof: The only effort in this proof is in setting it up in a fashion to avoid a lot of cases. Of course, there needs to be some subtlety, since the deletion of any one of the criteria for the pairs (Σ_n, Φ_n) to be perfectly embedded leads to a counterexample.

First I need to make a definition. If F and F' are disjoint, two-sided surfaces in M, then Q is a co-product from F to F' if there exists an embedding $h : F \times I \longrightarrow M$ with $h(F \times I) = Q$, $h(F \times \{0\}) = F$, $h(F \times \{1\}) = F'$ and $h(\partial F \times I) \subset \partial M$. Notice that if Q is a co-product from F to F', then Q is a co-product from F' to F. Also, the disjoint surfaces F and F' are parallel in M iff there is a co-product from F to F' in M.

Assume that the sequence of pairs $(\Sigma_0, \Phi_0), \ldots, (\Sigma_n, \Phi_n), \ldots$ is not finite and for no n is the pair $(\Sigma_n, \Phi_n) = (M, \partial M)$.

It follows from Theorem III.20 that there exists an integer $n_1 \geq 0$ such that if F is a component of $\mathrm{Fr}\Sigma_i$ and $i > n_1$ then there exists a $k \leq n_1$ (k depends on F) and a component F' of $\mathrm{Fr}\Sigma_k$ with F parallel to F' in M.

Now, in general, if $i > j$ and C_i is a component of $M - \mathring{\Sigma}_i$, then there exists a unique component C_j of $M - \mathring{\Sigma}_j$ with $C_i \subset \mathring{C}_j$. Since $C_i \subset \mathring{C}_j$, a component F_i of $\mathrm{Fr}C_i$ is a two-sided, incompressible surface in $\mathring{C}_j$; so, if C_j is a co-product from the component F_j of $\mathrm{Fr}\Sigma_j$ to the component F_j' of $\mathrm{Fr}\Sigma_j$, it follows from Lemma IX.1 that F_i is either a disk or an annulus and is parallel into ∂M or F_i is parallel in M to both F_j and F_j'. Since (Σ_i, Φ_i) is perfectly-embedded, the component F_i

of $\mathrm{Fr}\Sigma_i$ is not parallel into ∂M. It follows that if C_i is not a co-product from one component of $\mathrm{Fr}\Sigma_i$ to another component of $\mathrm{Fr}\Sigma_i$, then C_j is not a co-product from one component of $\mathrm{Fr}\Sigma_j$ to another component of $\mathrm{Fr}\Sigma_j$.

In the case that $i > j > n_1$, there is an even more intimate relationship between C_i and C_j (as above). Namely,

CLAIM 1. Suppose that $i > j > n_1$ and C_i is a component of $M - \overset{o}{\Sigma}_i$ that is not a co-product from some component of $\mathrm{Fr}\Sigma_i$ to another component of $\mathrm{Fr}\Sigma_i$. If C_j is the unique component of $M - \overset{o}{\Sigma}_j$ with $C_i \subset C_j$, then C_j is a regular neighborhood of C_i in M and $C_j \cap \partial M$ is a regular neighborhood of $C_i \cap \partial M$ in ∂M.

Proof of Claim 1: Let C_i be as in the hypothesis of the claim. If F_i is a component of $\mathrm{Fr}C_i$, then F_i is a component of $\mathrm{Fr}\Sigma_i$; so, there exists $k \leq n_1$ and a component F_k of $\mathrm{Fr}\Sigma_k$ such that F_i is parallel to F_k in M. Let Q_{F_i} be the co-product from F_i to F_k in M. Since $\Sigma_k \subset \overset{o}{\Sigma}_i$ and C_i is not a co-product from some component of $\mathrm{Fr}\Sigma_i$ to another component of $\mathrm{Fr}\Sigma_i$, it follows, as above, from Lemma IX.1, that $Q_{F_i} \cap C_i = F_i$.

Now, since $\Sigma_k \subset \overset{o}{\Sigma}_j$, there exist components of $\mathrm{Fr}\Sigma_j$ in Q_{F_i}; in particular, there exists a component F_j of $\mathrm{Fr}C_j$ ($\subset \mathrm{Fr}\Sigma_j$) in Q_{F_i}. Again, as above, it follows from Lemma IX.1 that there exists a co-product from F_i to F_j contained in C_j. This is true for each component F_i of $\mathrm{Fr}\Sigma_i$. Hence, the claim follows.

The relationship set up in Claim 1 between the components of $M - \overset{o}{\Sigma}_i$ that are not co-products and the components of $M - \overset{o}{\Sigma}_j$ that are not co-products, $i > j > n_1$, allows me to choose an integer $n_0 \geq n_1$ such that if $i > j > n_0$, then the components of $M - \mathring{\Sigma}_i$ that are not co-products and the components of $M - \mathring{\Sigma}_j$ that are not co-products are in one-one correspondence. More precisely, I have

<u>CLAIM</u> 2: There exists an integer $n_0 \geq n_1$ such that if $i > j > n_0$ and C_i is a component of $M - \mathring{\Sigma}_i$, then either C_i is not a co-product from some component of $\mathrm{Fr}\Sigma_i$ to another component of $\mathrm{Fr}\Sigma_i$ and the unique component C_j of $M - \mathring{\Sigma}_j$ containing C_i is a regular neighborhood of C_i in M while $C_j \cap \partial M$ is a regular neighborhood of $C_i \cap \partial M$ in ∂M or C_i is a co-product from some component of $\mathrm{Fr}\,\Sigma_i$ to another component of $\mathrm{Fr}\,\Sigma_i$ and the unique component C_j of $M - \mathring{\Sigma}_j$ containing C_i is a co-product from some component of $\mathrm{Fr}\,\Sigma_j$ to another component of $\mathrm{Fr}\,\Sigma_j$.

I shall now show that the integer n_0 of Claim 2 is the integer desired in the statement of Lemma IX.2.

To do this I shall show that if $i > j > n_0$ and F_i is a component of $\mathrm{Fr}\,\Sigma_i$, then there exists a unique component F_j of $\mathrm{Fr}\,\Sigma_j$ and a co-product Q from F_i to F with $Q \subset \Sigma_i - \mathring{\Sigma}_j$. To this end, let $i > j > n_0$ be given and let F_i be any component of $\mathrm{Fr}\,\Sigma_i$. Let C_i be the unique component of $M - \Sigma_i$ such that F_i is a component of $\mathrm{Fr}\,C_i$.

If C_i is not a co-product from some component of $\mathrm{Fr}\,\Sigma_i$ to another component of $\mathrm{Fr}\,\Sigma_i$, then, by Claim 2, the unique component C_j of $M - \mathring{\Sigma}_j$ having $C_i \subset \mathring{C}_j$ is a regular neighborhood of C_i in M while $C_j \cap \partial M$ is a regular neighborhood of $C_i \cap \partial M$ in ∂M . It follows that there exists a unique component F_j of $\mathrm{Fr}_M\Sigma_j$ and a co-product Q from F_i to F_j with $Q \subset M - \mathring{\Sigma}_j$. However, Q is also contained in $\Sigma_i - \mathring{\Sigma}_j$. For otherwise, there is a component (σ_i, φ_i) of (Σ_i, Φ_i) contained in Q which has the property, by Lemma IX.1 and the fact that (Σ_i, Φ_i) is perfectly embedded, that σ_i itself must be a co-product from F_i to a component F_i' of $\operatorname{Fr}\Sigma_j$ and so, since $\Sigma_j \subset \overset{o}{\Sigma}_i$, there is still another component F_i'' of $\operatorname{Fr}\Sigma_i$ in Q with F_i, F_i' and F_i'' parallel in Q. It follows that (σ_i, φ_i) must be homotopic (as a pair) into $(\Sigma_i - \sigma_i, \Phi_i - \varphi_i)$.

But such a situation contradicts property (iii) for the pair (Σ_i, Φ_i) to be perfectly-embedded.

If C_i is a co-product from some component of $\operatorname{Fr}\Sigma_i$ to another component of $\operatorname{Fr}\Sigma_i$, then by Claim 2, the unique component C_j of $M - \overset{o}{\Sigma}_j$ having $C_i \subset \overset{o}{C}_j$ is a co-product from some component of $\operatorname{Fr}\Sigma_j$ to another component of $\operatorname{Fr}\Sigma_j$. In this case, however, there are two co-products Q and Q': one from F_i to a component F_j of $\operatorname{Fr}\Sigma_j$ and the other from F_i to a component $F_j' \neq F_j$ of $\operatorname{Fr}\Sigma_j$.

Both Q and Q' are contained in $M - \overset{o}{\Sigma}_j$ (and $Q \cap Q' = F_i$). Choose notation so that $C_i \subset Q'$. Then Q is contained in $\Sigma_i - \overset{o}{\Sigma}_j$; for otherwise, there is a component (σ_i, φ_i) of (Σ_i, Φ_i) contained in Q and, exactly as in the above argument, the pair (σ_i, φ_i) is homotopic (as a pair) into $(\Sigma_i - \sigma_i, \Phi_i - \varphi_i)$. Again such a situation contradicts property (iii) for the pair (Σ_i, Φ_i) to be perfectly-embedded.

This completes the proof of Lemma IX.2. ∎

Two well-embedded 3-manifold pairs $(\Sigma', \Phi') \subset (M, \partial M)$ and $(\Sigma, \Phi) \subset (M, \partial M)$ are <u>equivalent</u> if there is a homeomorphism $J : M \longrightarrow M$ isotopic to the identity on M such that $J(\Sigma') = \Sigma$ and $J(\Phi') = \Phi$; i.e., the pair (Σ, Φ) is equivalent to the pair (Σ', Φ') if they are ambiently isotopic. The well-embedded pair $(\Sigma', \Phi') \subset (M, \partial M)$ is "less than or equal to" the well-embedded pair $(\Sigma, \Phi) \subset (M, \partial M)$, written $(\Sigma', \Phi') \leq (\Sigma, \Phi)$, if there is a homeomorphism $J : M \longrightarrow M$ isotopic to the identity on M such that $J(\Sigma') \subset \mathrm{Int}_M(\Sigma)$ and $J(\Phi') \subset \mathrm{Int}_{\partial M}(\Phi)$.

IX.3. EXERCISE: The relation "less than or equal to" is a partial order on the collection of well-embedded pairs in $(M, \partial M)$.

The next exercise is just for fun and is not needed in these lectures. See V.2.2 and V.2.3 of [J-S_1] for variations.

IX.4. EXERCISE: Let M be a Haken-manifold. Suppose that $(\Sigma, \Phi) \subset (M, \partial M)$ and $(\Sigma', \Phi') \subset (M, \partial M)$ are perfectly-embedded 3-manifold pairs. If $(\Sigma, \Phi) \leq (\Sigma', \Phi')$ and $(\Sigma', \Phi') \leq (\Sigma, \Phi)$, then (Σ', Φ') is equivalent to (Σ, Φ). [Hint: Use Lemma VII.9.]

IX.5. LEMMA: Let M be a Haken-manifold and suppose that the collection of well-embedded 3-manifold pairs in $(M, \partial M)$ has the above partial order. Then any collection of perfectly-embedded 3-manifold pairs in $(M, \partial M)$ has maximal elements.

Proof: This follows immediately from Lemma IX.2 and Zorn's Lemma. ∎

IX.6. COROLLARY: Let M be a Haken-manifold. The collection of perfectly-embedded Seifert pairs in $(M, \partial M)$ has maximal elements.

In case the Haken-manifold M is closed or has incompressible boundary, I shall show that there is a unique (up to ambient isotopy of M) maximal perfectly-embedded Seifert pair in $(M, \partial M)$. This maximal perfectly-embedded Seifert pair is the canoncial Seifert pair discussed in the introduction to the chapter. Of course, in some cases, the collection of perfectly-embedded Seifert pairs in $(M, \partial M)$ is empty. Before beginning the sequence of lemmas establishing the uniqueness of a maximal perfectly-embedded Seifert pair, I need the next lemma and exercise, which are refinements of the work of

Chapter VIII and present the major results of that chapter in terms of perfectly-embedded pairs.

IX.7. LEMMA: Let M be a Haken-manifold that is closed or has incompressible boundary. The Essential Homotopy Theorem (VIII.4), the Homotopy Annulus Theorem (VIII.10) and the Homotopy Torus Theorem (VIII.11) remain true for the pair $(M, \partial M)$ if the word "well-embedded" in the conclusion is replaced by "perfectly-embedded."

Proof: The proof is straightforward. Namely, fix whichever theorem is to be refined. Then among all well-embedded Seifert pairs contained in $(M, \partial M)$ and satisfying the conclusion of the theorem, choose (Σ, Φ) so that the number of components of Σ is as small as possible; call this number n. Now, among all well-embedded Seifert pairs $(\Sigma, \Phi) \subset (M, \partial M)$, satisfying the conclusion of the theorem and such that Σ has n components, choose (Σ, Φ) so as to minimize the number of components of $\operatorname{Fr} \Sigma$. I claim that such a pair (Σ, Φ) is perfectly-embedded in $(M, \partial M)$.

Of the three properties that (Σ, Φ) must satisfy to be perfectly-embedded, only property (ii) needs any special attention. So, in this case suppose that some component F of $\operatorname{Fr} \Sigma$ is parallel into ∂M. Let (σ, φ) be the component of (Σ, Φ) such that F is a component of $\operatorname{Fr} \sigma$. If (σ, φ) is an S^1-pair, it can be easily argued that this situation contradicts the choice of (Σ, Φ) such that Σ has n components and the number of components of $\operatorname{Fr} \Sigma$ is minimal. So, suppose that (σ, φ) is an I-pair and not an S^1-pair. Then F is an annulus and there exists an annulus F' in ∂M and a co-product Q from F to F'. Consider $\sigma' = \sigma \cup Q$, which meets ∂M in $\varphi' = \varphi \cup F'$. The pair (σ', φ') is not an I-pair; however, there still is a contradiction. The 2-manifold $\varphi \cup F' \subset \partial M$ is incompressible in ∂M ; yet $\varphi \cup F'$ is compressible in $\sigma \cup Q$; and so, ∂M is compressible in M . This contradicts the hypothesis of the lemma. ∎

IX.8. EXERCISE: Let M be a Haken-manifold that is closed or has incompressible boundary. Suppose that $(\Sigma_1, \Phi_1), \ldots, (\Sigma_n, \Phi_n)$ are perfectly-embedded Seifert pairs in $(M, \partial M)$ such that $\partial\Phi_i \cap \partial\Phi_j' = \emptyset$, $i \neq j$. Show that there exists a perfectly-embedded Seifert pair $(\Sigma, \Phi) \subset (M, \partial M)$ and homeomorphisms $J_i \colon M \longrightarrow M$, each isotopic to the identity on M , such that $J_i(\Sigma_i) \subset \Sigma$ and $J_i(\Phi_i) \subset \Phi$. [Hint: See Lemma VIII.6 and Lemma IX.7. Caution: Remark VIII.7 is <u>not</u> true if the word "well-embedded" in the conclusion of Lemma VIII.6 is replaced by "perfectly-embedded."]

It follows from Theorem VII.16 that if M is a Haken-manifold and ∂M is incompressible, then the pair $(M, \partial M)$ has a characteristic pair factor, which is unique (up to ambient isotopy of M). (It may help the reader to review the definition of the characteristic pair factor for $(M, \partial M)$ and Theorem VII.16.) The characteristic pair factor for $(M, \partial M)$ has a very close relationship with the second factor of a maximal Seifert pair (a characteristic Seifert pair) for $(M, \partial M)$ (see Exercise IX.14); hence, the anticipated name given in Chapter VII. To make this relationship explicit in general, I need to augment the characteristic pair factor in some cases. If Φ is a characteristic pair factor for $(M, \partial M)$, then the union of Φ and all tori components of ∂M that have <u>nontrivial</u> intersection with Φ is the <u>augmented characteristic pair factor for</u> $(M, \partial M)$.

IX.9. REMARK: It follows from Theorem VII.16 (uniqueness of the characteristic pair factor) that the augmented characteristic pair factor is unique (up to ambient isotopy of M).

IX.10. LEMMA: *Let M be a Haken-manifold that is closed or has incompressible boundary. Then for any perfectly-embedded Seifert pair $(\Sigma, \Phi) \subset (M, \partial M)$ there exists a homeomorphism $J : M \longrightarrow M$, isotopic to the identity on M, such that $J(\Phi)$ is contained in the augmented characteristic pair factor for $(M, \partial M)$.*

Proof: If φ' is a component of Φ, then there is a component (σ, φ) of (Σ, Φ) such that φ' is a component of φ.

If the pair (σ, φ) is an I-pair, then since (Σ, Φ) is perfectly-embedded, the inclusion map of φ' into M is the initial-end of an essential homotopy $(\varphi' \times I, \varphi' \times \partial I) \longrightarrow (M, \partial M)$ (actually, there is a spatial deformation from φ' to a component φ'' of Q, possibly $\varphi'' = \varphi'$). Notice this is not true if the pair (Σ, Φ) is only well-embedded. So, if (σ, φ) is an I-pair it follows from defining property (ii) for a characteristic pair factor for $(M, \partial M)$ that φ' can be ambiently isotoped into the characteristic pair factor for $(M, \partial M)$.

If the pair (σ, φ) is an S^1-pair (and __not__ an I-pair), then φ' is either an annulus or a torus. If φ' is an annulus, then since (Σ, Φ) is perfectly-embedded, the inclusion map of φ' into M is the initial-end of an essential homotopy $(\varphi' \times I, \varphi' \times \partial I) \longrightarrow (M, \partial M)$. (Again, there exists a spatial deformation from φ' into a component φ'' of φ; however, φ'' may be a torus.) If φ' is a torus, then the inclusion map of φ' into M is __not__ the initial-end of an essential homotopy $(\varphi' \times I, \varphi' \times \partial I) \longrightarrow (M, \partial M)$. (I am assuming that the pair (σ, φ) is not an I-pair.) However, the pair (σ, φ) cannot be a fibered solid torus, since φ', as a component of ∂M, is incompressible, or the pair $(S^1 \times S^1 \times I, S^1 \times S^1 \times \partial I)$, since (σ, φ) is not an I-pair, or the pair $(S^1 \times S^1 \times I, S^1 \times S^1 \times \{0\})$, since (Σ, Φ) is perfectly-embedded; so, there exists an essential map from $(S^1 \times I, S^1 \times \partial I) \longrightarrow (\sigma, \varphi)$ with $S^1 \times \{0\}$ mapping to a fiber in φ'. Again I use that (Σ, Φ) is perfectly-embedded to conclude that this map considered as a map from $(S^1 \times I, S^1 \times \partial I) \longrightarrow (M, \partial M)$ is an essential map. So, in this case, it follows from defining property (ii) for a characteristic pair factor for $(M, \partial M)$ that φ' meets the characteristic pair factor for $(M, \partial M)$. So, φ', itself, is a component of the augmented characteristic pair factor for $(M, \partial M)$.

Therefore, the desired homeomorphism J exists. ■

IX.11. LEMMA: Let M be a Haken-manifold that is closed or has incompressible boundary. Then there exists a perfectly-embedded Seifert pair $(\Sigma, \Phi) \subset (M, \partial M)$ and a homeomorphism $J : M \longrightarrow M$, isotopic to the identity on M, such that the augmented characteristic pair factor for $(M, \partial M)$ is contained in $J(\Phi)$.

Proof: By defining property (i) for a characteristic pair factor for $(M, \partial M)$ and Lemma IX.7 (applied to the Essential Homotopy Theorem (VIII.4)), there exists a perfectly-embedded Seifert pair $(\Sigma', \Phi') \subset (M, \partial M)$ and a homeomorphism $J' : M \longrightarrow M$, isotopic to the identity on M, such that the characteristic pair factor for $(M, \partial M)$ is contained in $J'(\Phi')$.

If the characteristic pair factor for $(M, \partial M)$ is equal to the augmented characteristic pair factor for $(M, \partial M)$, then set $\Sigma = \Sigma'$, $\Phi = \Phi'$ and $J = J'$.

If the characteristic pair factor for $(M, \partial M)$ is not equal to the augmented characteristic pair factor for $(M, \partial M)$, then there are components of the augmented characteristic pair factor that are tori components of ∂M

and not components of the characteristic pair factor for $(M,\partial M)$. Of course, the characteristic pair factor for $(M,\partial M)$ must meet such a torus. In this case the pair (Σ',Φ') may need to be altered. I argue inductively on the number of tori boundary components of M that are contained in the augmented characteristic pair factor for $(M,\partial M)$ and are not in the characteristic pair factor for $(M,\partial M)$. It is sufficient to consider the case in which there is one such torus.

Let T be a torus component of the augmented characteristic pair factor for $(M,\partial M)$ and suppose that T is not in the characteristic pair factor for $(M,\partial M)$. Then there exists $(k \geq 1)$ components $(\sigma',\varphi'),\ldots,(\sigma'_k,\varphi'_k)$ of (Σ',Φ') such that $T \cap \varphi'_i \neq \emptyset$ $(1 \leq i \leq k)$. Necessarily, for $1 \leq i \leq k$ the pair (σ'_i,φ'_i) is an S^1-pair. Let V be a regular neighborhood of T in M. Set $\sigma' = V \cup \bigcup_{i=1}^{k}\sigma_i$ and set $\varphi' = T \cup \bigcup_{i=1}^{k}\varphi_i$. The pair (σ',φ') is an S^1-pair. However, (σ',φ') may not be perfectly-embedded (it may not even be well-embedded).

The pair (σ',φ') has the property that $\sigma' \cap \partial M = \varphi'$. If a component F of $\mathrm{Fr}\,\sigma'$ is compressible in M, then F must be a torus and F must bound a solid torus K in $M - \overset{o}{\sigma}'$. Since (Σ,Φ) is perfectly-embedded, F is not a component of $\mathrm{Fr}\,\Sigma$; so, the pair (σ'',φ'') where $\sigma'' = \sigma' \cup K$ and $\varphi'' = \varphi'$ is an S^1-pair. If I argue inductively on the number of components of $\mathrm{Fr}\,\sigma'$ which are compressible in M, it follows that there is no loss in assuming that $\mathrm{Fr}\,\sigma'$ is incompressible in M. So, I may assume that the pair (σ',φ') is well-embedded. Using a similar argument, I may assume that in addition to the above, there is no component of $\mathrm{Fr}\,\sigma'$ which is parallel into ∂M.

Set $\Sigma = \Sigma' \cup \sigma'$ and set $\Phi = \Phi' \cup \varphi'$. The pair (Σ,Φ) is a well-embedded Seifert pair in $(M, \partial M)$ such that no component of $\mathrm{Fr}\ \Sigma$ is parallel into ∂M and there exists a homeomorphism $J : M \longrightarrow M$, isotopic to the identity on M such that the augmented characteristic pair factor for $(M, \partial M)$ is contained in $J(\Phi)$. Among all such Seifert pairs, if I choose (Σ, Φ) so that the number of components of Σ is as small as possible, then (Σ, Φ) is perfectly-embedded in M and, along with the homeomorphism J, satisfies the conclusion of Lemma IX.11. ∎

IX.12. THEOREM: *Let M be a Haken-manifold that is closed or has incompressible boundary. Then there exists a unique* (up to ambient isotopy of M) *maximal, perfectly-embedded Seifert pair in* $(M, \partial M)$.

Proof: By Corollary IX.6, maximal perfectly-embedded Seifert pairs exist in $(M, \partial M)$. So, suppose that $(\Sigma_1, \Phi_1) \subset (M, \partial M)$ and $(\Sigma_2, \Phi_2) \subset (M, \partial M)$ are maximal perfectly-embedded Seifert pairs. I need to show that there exists a homeomorphism $J : M \longrightarrow M$, isotopic to the identity on M, such that $J(\Sigma_1) = \Sigma_2$ and $J(\Phi_1) = \Phi_2$. To do this, I shall compare both Φ_1 and Φ_2 to the augmented characteristic pair factor for $(M, \partial M)$.

Let Φ^* be the augmented characteristic pair factor for $(M, \partial M)$. It follows from Lemma IX.10 that I may assume that $\Phi_1 \subset \Phi^*$. (This does not use that $(\Sigma_1, \Phi_1) \subset (M, \partial M)$ is maximal.) Now, by Lemma IX.11 there exists a perfectly-embedded Seifert pair $(\Sigma', \Phi') \subset (M, \partial M)$ with $\Phi^* \subset \Phi'$. So, $(\Sigma_1, \Phi_1) \subset (M, \partial M)$ and $(\Sigma', \Phi') \subset (M, \partial M)$ are perfectly-embedded Seifert pairs with $\Phi \subset \Phi'$; hence, by Exercise IX.8, there exists a perfectly-embedded Seifert pair $(\Sigma'', \Phi'') \subset (M, \partial M)$ such that both $(\Sigma_1, \Phi_1) \subset (\Sigma'', \Phi'')$ and $(\Sigma', \Phi') \subset (\Sigma'', \Phi'')$. Now, since (Σ_1, Φ_1) is maximal, it follows that there exists a homeomorphism $J' : M \longrightarrow M$, isotopic to the identity on M, such that $J'(\Sigma_1) = \Sigma''$ and $J'(\Phi_1) = \Phi''$. I conclude that $\Phi^* \subset J'(\Phi_1)$; or, more generally, if $(\Sigma_1, \Phi_1) \subset (M, \partial M)$ is a maximal perfectly-embedded Seifert pair, then Φ_1 contains the augmented characteristic pair factor for $(M, \partial M)$ (well-defined up to ambient isotopy of M).

From the preceding argument, I may assume that I have chosen the augmented characteristic pair factor Φ^* so that $\Phi^* \subset \Phi_1$. Also, by Lemma IX.10, I may assume that $\Phi_2 \subset \Phi_*$. Hence, $\Phi_2 \subset \Phi_* \subset \Phi_1$. It follows from Exercise IX.8 that there exists a perfectly-embedded Seifert pair $(\Sigma_3, \Phi_3) \subset (M, \partial M)$ such that $(\Sigma_1, \Phi_1) \subset (\Sigma_3, \Phi_3)$ and $(\Sigma_2, \Phi_2) \subset (\Sigma_3, \Phi_3)$. By (Σ_1, Φ_1) and (Σ_2, Φ_2) both being maximal, perfectly-embedded Seifert pairs in $(M, \partial M)$, I have for $i = 1, 2$ homeomorphisms $J_i : M \longrightarrow M$, isotopic to the identity on M, such that $J_i(\Sigma_i) = \Sigma_3$ and $J_i(\Phi_i) = \Phi_3$. Set $J = J_2^{-1} \circ J_1$. Then $J : M \longrightarrow M$ is a homeomorphism, isotopic to the identity on M, such that $J(\Sigma_1) = \Sigma_2$ and $J(\Phi_1) = \Phi_2$. ∎

I have shown that if M is a Haken-manifold that is closed or has incompressible boundary, then there exists a unique (up to ambient isotopy of M) maximal perfectly-embedded Seifert pair $(\Sigma, \Phi) \subset (M, \partial M)$. This unique maximal perfectly-embedded Seifert pair is called the <u>characteristic Seifert pair for</u> M.

IX.13. REMARK: The characteristic Seifert pair for M is the canonical Seifert pair promised in the introduction of the chapter. This is formalized in Theorem IX.15 (where I show that the characteristic Seifert pair for M serves as the Seifert pair in the conclusion of both the Homotopy Annulus Theorem and the Homotopy Torus Theorem), in Exercise IX.16 (where it can be shown that the characteristic Seifert pair for M serves as the Seifert pair in the conclusion of the Essential Homotopy Theorem), and in Theorem IX.17

(where I generalize Theorem IX.15 to nondegenerate mappings of arbitrary Seifert pairs into $(M,\partial M)$).

IX.14. EXERCISE: Let M be a Haken-manifold with incompressible boundary. Let (Σ, Φ) be the characteristic Seifert pair for M. Then Φ is the augmented characteristic pair factor for $(M,\partial M)$.

IX.15. THEOREM: Let M be a Haken-manifold that is closed or has incompressible boundary. Let $(\Sigma, \Phi) \subset (M,\partial M)$ be the characteristic Seifert pair for M. Then any nondegenerate map from either $(S^1 \times I, S^1 \times \partial I)$ or $(S^1 \times S^1, \emptyset)$ into $(M,\partial M)$ is homotopic (as a map of pairs) to a map (from either $(S^1 \times I, S^1 \times \partial I)$ or $(S^1 \times S^1, \emptyset)$ into $(M,\partial M)$, respectively) with its image contained in (Σ, Φ).

Proof: Let f be a nondegenerate map from either $(S^1 \times I, S^1 \times \partial I)$ or $(S^1 \times S^1, \emptyset)$ into $(M,\partial M)$. Then, in either case, by Lemma IX.7 (Homotopy Annulus Theorem (VIII.10) or Homotopy Torus Theorem (VIII.11)) there exists a perfectly-embedded Seifert pair $(\Sigma', \Phi') \subset (M,\partial M)$ and the map f can be deformed (as a map of pairs) to a map having its image in (Σ', Φ'). Clearly, the perfectly-embedded Seifert pair (Σ', Φ') is contained in a maximal perfectly-embedded Seifert pair in $(M,\partial M)$; and so, by Theorem IX.12, I may assume that $(\Sigma', \Phi') \subset (\Sigma, \Phi)$. ∎

IX.16. EXERCISE: Let M be a Haken-manifold with incompressible boundary. Let (Σ, Φ) be the characteristic Seifert pair for M. Let K be a compact polyhedron. Suppose that $f : (K \times I, K \times \partial I) \longrightarrow (M,\partial M)$ is a map of pairs such that K has no component k for which the subgroup $(f|k \times I)_*(\pi_1(k \times I)) \subset \pi_1(M)$ is trivial. If f is essential, then there exists a map $g : (K \times I, K \times \partial I) \longrightarrow (M, \partial M)$ homotopic to f (as a map of pairs) such that $g(K \times I) \subset \Sigma$ and $g(K \times \partial I) \subset \Phi$.

Let (S,F) be a Seifert pair and let (M,T) be a 3-manifold pair. A map $f : (S,F) \longrightarrow (M,T)$ is _nondegenerate_ if f is essential (as a map of pairs) and $f_*: \pi_1(S) \longrightarrow \pi_1(M)$ is injective.

IX.17. THEOREM: _Let M be a Haken-manifold that is closed or has incompressible boundary. Let $(\Sigma, \Phi) \subset (M, \partial M)$ be the characteristic Seifert pair for M. Let (S,F) be a Seifert pair distinct from $(\mathbb{D}^2 \times I, \mathbb{D}^2 \times \partial I)$, $(\mathbb{D}^2 \times S^1, \emptyset)$, $(S^3, \emptyset)$ or $(S^2 \times S^1, \emptyset)$. Then any nondegenerate map f from (S,F) into $(M, \partial M)$ is homotopic (as a map of pairs) to a map g from (S,F) into $(M, \partial M)$ such that $g(S) \subset \Sigma$ and $g(F) \subset \Phi$._

Proof: This is another of those proofs that just seems to require the separate consideration of several situations. I have tried to organize the proof so that the reader can quickly understand the line of argument. To do this, I have broken the proof into a number of assertions, which can be used as an outline of the proof. I suggest that the reader first read through all of the statements of the numbered assertions and then, if desired, go to the detail of the proof.

Case 1. The pair (S,F) is an I-pair.

Assertion 1: _In this case, the version of the Essential Homotopy Theorem, given in Exercise IX.16, can be used to obtain the desired conclusion._

If (S,F) is a twisted I-pair, let $(\tilde{S}, \tilde{F})$ be the canonical two-sheeted covering of (S,F) such that $(\tilde{S}, \tilde{F})$ is a product I-pair. Let $p : (\tilde{S}, \tilde{F}) \longrightarrow (S,F)$ be the covering projection. If (S,F) is a product I-pair, set $\tilde{S} = S$, $\tilde{F} = F$ and $p = id$. Let $\tilde{f} : (\tilde{S}, \tilde{F}) \longrightarrow (M, \partial M)$ be the map $\tilde{f} = f \circ p$. Then, in this situation, it follows (see e.g. III.4.3 of [J-S_1]) that $\tilde{f}$ is essential. Since both p and f induce injections on π_1, $\tilde{f}$ induces an injection on π_1. So, $\tilde{f}$ is nondegenerate. By Exercise IX.16 (here I use that (S,F) is not $(\mathbb{D}^2 \times I, \mathbb{D}^2 \times \partial I)$), the map $\tilde{f}$ is homotopic, as a map of pairs, to a map $\tilde{g}$ taking $\tilde{S}$ into Σ and $\tilde{F}$ into Φ. Since $p|\tilde{F}$ is a homeomorphism on each component of $\tilde{F}$, I may use $\tilde{g}$ to prove that $f|F : F \longrightarrow \partial M$ is homotopic to a map taking F into Φ. So, there is no loss in generality to assume that $f|F: F \longrightarrow \partial M$ with its image already contained in Φ. Then $\tilde{f}(\tilde{F}) \subset \Phi$ and, by the homotopy extension property, I may assume that $\tilde{g}$ is homotopic to $\tilde{f}$ (rel $\tilde{F}$).

Now, the point of view is changed a bit. Using the above, consider f as a mapping of (S,F) into (M,Σ) $(f(F) \subset \Phi \subset \Sigma)$ and $\tilde{f}$ as a mapping of $(\tilde{S},\tilde{F})$ into (M,Σ). By the existence of $\tilde{g}$, $\tilde{f}$ is <u>not</u> essential as a map into (M,Σ). However, if f is essential as a map into (M,Σ), then again it follows, as above (III.4.3 of [J-S_1]), that $\tilde{f}$ is essential as a map into (M,Σ). This contradiction implies that there exists a map $g : (S,F) \longrightarrow (M,\partial M)$, homotopic to f (rel F), with $g(S) \subset \Sigma$. This is the desired conclusion in this case.

<u>Case</u> 2. The pair (S,F) is an S^1-pair and <u>not</u> an I-pair.

Assertion 2. <u>In this case, if</u> $\partial S = \emptyset$, <u>then</u> $\partial M = \emptyset$, M <u>is a Seifert fibered manifold and</u> $(M,\emptyset) = (\Sigma, \Phi)$.

Suppose that $\partial S = \emptyset$ ($(S,F) = (S,\emptyset)$ is an S^1-pair). By hypothesis and Remark III.19, the manifold M is aspherical. Since the map f is nondegenerate (and $(S,F) \neq (S^3,\emptyset)$ or $(S^2 \times S^1,\emptyset)$), S is not a lens space; and so it follows from Theorem VI.15 and Remark VI.19 that S is aspherical.

Let $\tilde{M}$ be the covering space of M, with projection map p, corresponding to the conjugacy class of $f_*(\pi_1(S))$ in $\pi_1(M)$. Then there is a lifting $\tilde{f} : (S,\emptyset) \longrightarrow (\tilde{M},\emptyset)$ of f such that $\tilde{f}$ induces an isomorphism on π_1. By the above, both S and $\tilde{M}$ are aspherical; hence, $\tilde{f}$ induces a homotopy equivalence. It follows that $\tilde{M}$ is closed and, by Corollary VII.6, that $\tilde{M}$ is a Haken-manifold. Now, by $[W_3]$ (as reported in Theorem 13.6 of $[He_1]$) $\tilde{f}$ is homotopic to a homeomorphism. Hence, $\tilde{M}$ is a Seifert fibered manifold. By VI.29, M is also a Seifert fibered manifold and $(M,\emptyset) = (\Sigma, \Phi)$ This establishes Assertion 2.

Assertion 3. <u>In this case, if</u> $\partial S \neq \emptyset$, <u>then there is no loss in generality to assume that the map</u> f <u>when restricted to each component of the closure of</u> $\partial S - F$ (a map of either $(S^1 \times I, S^1 \times \partial I)$ or $(S^1 \times S^1, \emptyset)$ into $(M, \partial M)$) <u>is nondegenerate</u>.

Suppose that A is a component of the closure of $\partial S - F$. If A is an annulus, then $f|A : (A,\partial A) \longrightarrow (M,\partial M)$ is a map of the pair $(S^1 \times I, S^1 \times \partial I)$ into $(M,\partial M)$. Since f_* induces an injection on π_1 and A is saturated, $(f|A)_*$ induces an injection on π_1. If $f|A : (A,\partial A) \longrightarrow (M,\partial M)$ is essential, then $f|A$ is nondegenerate; otherwise, $f|A$ is homotopic (rel ∂A) to a map taking A into ∂M. If the former is the case, then there is nothing to do. If the latter is the case, set $F' = F \cup A$; then f is homotopic (rel F) to a map $f' : (S,F') \longrightarrow (M,\partial M)$ with $f'(F') \subset \partial M$. If A is a torus, then $f|A : (A,\emptyset) \longrightarrow (M,\partial M)$ is a map of the pair $(S^1 \times S^1, \emptyset)$ into $(M,\partial M)$. Since f_* induces an injection on π_1, either $f|A$ induces an injection on π_1 or S is a solid torus and $A = \partial S$. I now use that the pair (S,F) is not $(\mathbb{D}^2 \times S^1, \emptyset)$; so, $f|A$ induces an injection on $\pi_1(A)$. If $f|A : (A,\emptyset) \longrightarrow (M,\partial M)$ is essential, then $f|A$ is nondegenerate; otherwise, $f|A$ is homotopic to a map taking A into ∂M. If the former is the case, then there is nothing to do. If the latter is the case, set $F' = F \cup A$; then f is homotopic (rel F) to a map $f': (S,F') \longrightarrow (M,\partial M)$ with $f'(F') \subset \partial M$. In any case, I have the map $f': (S,F') \longrightarrow (M,\partial M)$ nondegenerate; and if f' is homotopic (as a map of pairs) to a map $g : (S,F') \longrightarrow (M,\partial M)$ such that $g(S) \subset \Sigma$ and $g(F') \subset \Phi$, then f is homotopic (as a map of pairs) to g . It follows that by reasoning inductively on the number of components of the closure of $\partial S-F$, there is no loss in generality to assume, originally, that the map f restricted to each component of the closure of $\partial S-F$ is nondegenerate. This establishes Assertion 3.

Assertion 4. _In this case, if_ $\partial S \neq \emptyset$ _and_ $F = \emptyset$, _then_ $(S,F) = (S,\emptyset)$ _and there exists a map homotopic to_ f (as a map of pairs) _taking_ ∂S _into_ Σ . _It then follows that there exists a map_ $g : (S,\emptyset) \longrightarrow (M,\partial M)$ _homotopic to_ f (as a map of pairs) _such that_ $g(S) \subset \Sigma$.

Each component of ∂S is a torus; and by Assertion 3, if B is a component of ∂S , then $f|B : (B,\emptyset) \longrightarrow (M,\partial M)$ (a map of $(S^1 \times S^1,\emptyset)$ into $(M,\partial M)$) is nondegenerate. I can apply Theorem IX.15 to the map $f|B$ for each component B of ∂S and conclude that there is a map of $(S,\emptyset)$ into $(M,\partial M)$ homotopic to f (as a map of pairs) such that each component B of ∂S is mapped into Σ . I shall continue to call this map f.

I want to show that $f(S) \subset \Sigma$ (I have $f(\partial S) \subset \Sigma$). If this is not the case, then $f(S)$ meets $Fr(\Sigma)$. I may apply Lemma III.9, using the standard arguments, to conclude that there is a map $g : (S,\emptyset) \longrightarrow (M,\partial M)$ homotopic to f (rel ∂S) such that each component of $g^{-1}(Fr\ \Sigma)$ is a two-sided, incompressible surface in S ; and among all such maps, the number of components of $g^{-1}(Fr\ \Sigma)$ is a minimum. Since $g(\partial S) \subset \Sigma$, each component of $g^{-1}(\mathrm{Fr}\ \Sigma)$ is closed; and since a component of $\mathrm{Fr}\ \Sigma$ is either an annulus or a torus, a component of $g^{-1}(\mathrm{Fr}\ \Sigma)$ is either a 2-sphere or a torus. By the incompressibility of each component of $g^{-1}(\mathrm{Fr}\ \Sigma)$, no component is a 2-sphere.

Let $\tilde{C}$ be the closure of a component of $S - g^{-1}(\mathrm{Fr}\ \Sigma)$ such that $g|\tilde{C} : \tilde{C} \longrightarrow C$ where C is the closure of a component of $M - \Sigma$. Then $g|\tilde{C} : \tilde{C} \longrightarrow C$ is a boundary preserving map. Again, by $[W_3]$, as reported in Theorem 13.6 of $[He_1]$, either $\tilde{C}$ is an I-bundle and $g|\tilde{C}$ is homotopic (rel $\partial\tilde{C}$) to a map of $\tilde{C}$ into ∂C or $g|\tilde{C}$ is homotopic to a covering map of $\tilde{C}$ onto C.

If $\tilde{C}$ is an I-bundle and $g|\tilde{C}$ is homotopic (rel $\partial\tilde{C}$) to a map of $\tilde{C}$ into ∂C, then I can obtain a contradiction to the minimality of the number of components of $g^{-1}(\mathrm{Fr}\ \Sigma)$ in the selection of g. So, I shall assume that $g|\tilde{C} : \tilde{C} \longrightarrow C$ is homotopic to a covering map of $\tilde{C}$ onto C.

I first argue that $\tilde{C}$ is itself a Seifert fibered manifold. In order to do this, I shall prove that each component of $g^{-1}(\mathrm{Fr}\ \Sigma)$ (an incompressible torus in S) is saturated in the Seifert fibration of S. So, suppose that T is a component of $g^{-1}(\mathrm{Fr}\ \Sigma)$. By Theorem VI.34 either T does not separate S and T is a fiber in a fibration of S as a surface bundle over S^1, or T does separate S and $S = S_1 \cup S_2$ where $S_1 \cap S_2 = \partial S_1 = \partial S_2 = T$ and S_i $(i = 1,2)$ is a twisted I-bundle over a Klein bottle, or T is saturated in some Seifert fibration of S. However, in the first and second situations, S is closed; so $\partial S = \emptyset$. This contradicts the assumption that $\partial S \neq \emptyset$. In the third situation, it follows that each component of $g^{-1}(\Sigma)$ is saturated in the <u>same</u> Seifert fibration of S. So, $\tilde{C}$ is itself a Seifert fibered manifold. I again apply VI.29 to conclude that C is a Seifert fibered manifold. If C is a Seifert fibered manifold other than $S^1 \times S^1 \times I$, then, since C is the closure of a component of $M - \Sigma$, the pair (Σ, Φ) would not be a characteristic Seifert pair for M ((Σ, Φ) would not be maximal). I shall assume that C is $S^1 \times S^1 \times I$ and hence, $\widetilde{C}$ itself is $S^1 \times S^1 \times I$. In the argument for proving that $g(S) \subset \Sigma$, I have that the only situation possible is that the closure of some component C of $M - \Sigma$, which is homeomorphic to $S^1 \times S^1 \times I$, meets $g(S)$ and each component $\widetilde{C}$ of $g^{-1}(C)$ is homeomorphic to $S^1 \times S^1 \times I$ and $g|\widetilde{C}: \widetilde{C} \longrightarrow C$ is a covering map. I shall show that this too leads to a contradiction; and hence, $g(S) \subset \Sigma$.

Let $\widetilde{T}_0$ and $\widetilde{T}_1$ be the boundary components of $\widetilde{C}$. Let T_0 and T_1 be the boundary components of C such that $f|\widetilde{T}_i: \widetilde{T}_i \longrightarrow T_i$, $i = 0,1$. If $\widetilde{C}$ separates S, let $\widetilde{C}_0$ and $\widetilde{C}_1$ be the closures of the components of $S-\widetilde{C}$, indexed so that $\widetilde{T}_i \subset \partial\widetilde{C}_i$, $i = 0,1$. If $\widetilde{C}$ does not separate S, let $\widetilde{\widetilde{C}}$ denote the closure of the component of $S-\widetilde{C}$. If $\widetilde{C}$ separates S and for $i = 0$ or 1 one of the mappings $g|\widetilde{C}_i: (\widetilde{C}_i, \widetilde{T}_i) \longrightarrow (M-\overset{o}{C}, T_i)$ is <u>not</u> essential, then there is a contradiction to the minimality condition on the number of components of $g^{-1}(\mathrm{Fr}\ \Sigma)$. (It can be easily argued that this is the case iff for $i = 0$ or 1 one of $\widetilde{C}_i$ is homeomorphic to $S^1 \times S^1 \times I$.) Now, if $\widetilde{C}$ separates S and for $i = 0$ or 1 both of the mappings $g|\widetilde{C}_i: (\widetilde{C}_i, \widetilde{T}_i) \longrightarrow (M-\overset{o}{C}, T_i)$ are essential, then by Theorem VI.18 the Seifert fibration of the S^1-pair containing T_0 extends over C to the Seifert fibration of the S^1-pair containing T_1. This contradicts the fact that (Σ, Φ) is a characteristic Seifert pair for M ((Σ, Φ) is a maximal perfectly-embedded Seifert pair). If $\widetilde{C}$ does not separate S, then the mapping $g|\widetilde{\widetilde{C}}: (\widetilde{\widetilde{C}}, \widetilde{T}_0 \cup \widetilde{T}_1) \longrightarrow (M-\overset{o}{C}, T_0 \cup T_1)$ is essential and C does not separate M. Again, by Theorem VI.18, the Seifert fibration of the S^1-pair containing T_0 extends over C to the Seifert fibration of the S^1-pair containing T_1;

and this contradicts the fact that (Σ, Φ) is a characteristic Seifert pair for M. Hence, these contradictions lead to the conclusion that $g(S) \subset \Sigma$. This establishes Assertion 4.

REMARK: The remaining situation to be considered is the case that (S,F) is an S^1-pair (and <u>not</u> an I-pair) with both $\partial S \neq \emptyset$ and $\partial F \neq \emptyset$. I handle this situation by constructing a kind of two-dimensional "skeleton" K of S, which contains F, and by proving that there exists a <u>well-embedded</u> Seifert pair $(\Sigma', \Phi') \subset (M, \partial M)$ such that the map $f|K : (K,F) \longrightarrow (M,\partial M)$ is homotopic (as a map of pairs) to a map $f' : (K,F) \longrightarrow (M,\partial M)$ with $f'(K) \subset \Sigma'$ and $f'(F) \subset \Phi'$. The "skeleton" K of S is constructed in such a way that the map f' can be extended to a map $g : (S,F) \longrightarrow (M,\partial M)$ homotopic to f (as a map of pairs) such that $g(S) \subset \Sigma'$ and $g(F) \subset \Phi'$. The fact that (Σ, Φ) is a characteristic Seifert pair for M enables me to replace (Σ', Φ') by (Σ, Φ).

Assertion 5. <u>In this case, if $\partial S \neq \emptyset$ and $F \neq \emptyset$, then there is no loss in generality to assume that each component of ∂S intersects F.</u>

For suppose that B is a component of ∂S and $B \cap F = \emptyset$. Since $F \neq \emptyset$ and F is saturated, there is a two-sided, saturated annulus $A \subset S$ with $\partial A = a \cup b$ where a is a regular fiber in F and b is a regular fiber in B. The singular annulus $f|A$ is a homotopy (rel a) from $f|b$ to a map taking b into ∂M. Set $F' = F \cup U(b)$ where $U(b)$ is a saturated annular neighborhood of b in B. Then f is homotopic (rel F) to a map $f' : (S,F') \longrightarrow (M,\partial M)$. Notice that if A' is the annulus in B complementary to $U(b)$ (i.e., $A' = (\overline{B - U(b)})$), then $f'|A' : (A', \partial A') \longrightarrow (M, \partial M)$ is essential, since, by Assertion 3, $f|B$ is essential. Furthermore, if there exists a map $g : (S,F') \longrightarrow (M,\partial M)$ homotopic to f' (as a map of pairs) such that $g(S) \subset \Sigma$ and $g(F') \subset \Phi$, then g satisfies all of the requirements in the conclusion of Theorem IX.17. By reasoning inductively on the number of components of ∂S that do not meet F, it follows that there is no loss in generality to assume, originally, that for each component B of ∂S, $B \cap F \neq \emptyset$. This completes the proof of Assertion 5.

Assertion 6. _In this case, if_ $\partial S \neq \emptyset$ _and_ $F \neq \emptyset$ (each component of ∂S meets F), _then there is a two-sided_ 2-_manifold_ $\mathfrak{a} \subset S$ _such that each component of_ $\mathfrak{a}$ _is an annulus, saturated in the given Seifert fibration of_ (S,F), $\partial\mathfrak{a} \subset F$, _and each component of_ S _split at_ $\mathfrak{a}$ _is a fibered solid torus._

Since $\partial S \neq \emptyset$ and $F \neq \emptyset$, it follows from Assertion 5 that there is no loss in generality to assume that each component of ∂S meets F. By Remark VIII.1(b) the Seifert fibered manifold S has a Seifert fibration with orbit manifold N and projection map $p : S \longrightarrow N$ such that F is saturated (with respect to p) over some 1-manifold $\bar{F} \subset \partial N$. By the preceding sentence, $\bar{F}$ meets each component of ∂N. It is easy to argue that there exists a two-sided, pairwise-disjoint collection of arcs $\bar{\mathfrak{a}} \subset N$ such that each component of $\bar{\mathfrak{a}}$ has its end points in $\bar{F}$ and each component of N split at $\bar{\mathfrak{a}}$ is a disk having at most one exceptional point of the given Seifert fibration. Set $\mathfrak{a} = p^{-1}(\bar{\mathfrak{a}})$. Then $\mathfrak{a}$ satisfies the conclusions of Assertion 6.

Assertion 7. _In this case, if_ $\partial S \neq \emptyset$, $F \neq \emptyset$ _and_ K _is the two-complex equal to_ ∂S _union the_ 2-_manifold_ $\mathfrak{a}$ _of Assertion_ 6, _then there exists a map_ f': $(K,F) \longrightarrow (M,\partial M)$ _homotopic to_ $f|K$ (as a map of pairs) _such that_ $f'(K) \subset \Sigma$ _and_ $f'(F) \subset \Phi$.

If $K = \mathfrak{A} \cup \partial S$, where $\mathfrak{A}$ is the 2-manifold of Assertion 6, then each component of $K-F$ is an open annulus whose closure is an annulus with its boundary in F. I shall prove Assertion 7 by induction on the number of components of $K-F$. The induction step is more easily carried through if I prove a variant of Assertion 7 and then use it to prove Assertion 7 itself. I shall prove that if K is a subcomplex of S such that $F \subset K$ and each component of $K-F$ is an open annulus whose closure is an annulus with its boundary in F, then there exists a <u>well-embedded</u> S^1-<u>pair</u> $(\Sigma', \Phi') \subset (M, \partial M)$ and a map $f': (K,F) \longrightarrow (M, \partial M)$ homotopic to $f|K$ (as a map of pairs) such that $f'(K) \subset \Sigma'$ and $f'(F) \subset \Phi'$. (Notice that I only require that $(\Sigma', \Phi'$ be well-embedded; however, I do specify that (Σ', Φ') is an S^1-pair.)

The induction begins with $K = F$. Each component of F is either an annulus or a torus saturated in the given Seifert fibration of S. If C is a torus component of F, then C is incompressible (otherwise, $C = F = \partial S$ and S is a solid torus; but this cannot happen if $f: (S,F) \longrightarrow (M, \partial M)$ is essential). So, $f|C: C \longrightarrow \partial M$ induces an injection on π_1, $f(C)$ must be a torus component of ∂M and $f|C: C \longrightarrow f(C)$ is homotopic to a covering map.

If $F = \partial S$, then by [W_3] (reported as Theorem 13.6 in [He_1]), either (S,F) is an I-pair and f is not essential as a map of pairs or f is homotopic (as a map of pairs) to a covering map of S onto M. Clearly, the former does not happen. So, the latter happens and by VI.29, the pair $(M, \partial M)$ is itself an S^1-pair. In this case, set $\Sigma' = M$, $\Phi' = \partial M$ and $f' = f|F$.

If $F \neq \partial S$, then by Assertion 5 there is no loss in generality to assume that each component of ∂S meets F. Therefore, the closure of each component A of $\partial S-F$ is an annulus and by Assertion 3 there is no loss in generality to assume that the map $f|A : (A,\partial A) \longrightarrow (M,\partial M)$ is nondegenerate It follows that $f(F)$ is contained in the augmented characteristic pair factor for $(M,\partial M)$; and therefore, by Exercise IX.14, I may assume that $f(F) \subset \Phi$. (This seems like a lot and, of course, if $f(S) \subset \Sigma$, I would have the desired conclusion. It is also tempting to try to finish the proof at this stage by mocking the proof of Assertion 4. I believe this can be done but things are really messy then.) I actually can and want to prove more. Namely, I claim that if C is a component of F , either $f|C : C \longrightarrow \partial M$ has its image in a torus component of Φ or C is an annulus and $f|C$: $C \longrightarrow \partial M$ is homotopic to a map taking C into the boundary of some component of Φ . If C is a torus then $f|C : C \longrightarrow \partial M$ has its image in a torus component of Φ . If C is an annulus, I have that $f|C : C \longrightarrow \partial M$ such that $f(C) \subset \Phi$. If the component of Φ containing $f(C)$ is either an annulus or a torus, then the claim follows. Otherwise, the component of Φ containing $f(C)$ does not have Euler characteristic zero and there exists an I-pair (σ, φ) which is a component of (Σ, Φ) , such that (σ, φ) is not an S^1-pair and $f(C) \subset \varphi$. Now consider $f^{-1}(\mathrm{Fr}\ \sigma)$. This is a collection of pairwise disjoint two-sided surfaces in S , none of which meet F . I can modify f by a homotopy (rel F) so that there is no loss in generality to assume that each surface in $f^{-1}(\mathrm{Fr}\ \sigma)$ is incompressible. Since $f(C) \subset \varphi$ and (σ, φ) is an I-pair, it follows that either (S,F) is an I-pair (a contradiction in this case (Case 2)) or the map $f|C : C \longrightarrow \varphi$ is homotopic to a map taking C into $\mathrm{Fr}\ \sigma$ and so $f|C : C \longrightarrow \varphi$ is homotopic in φ to a map taking C into $\partial \varphi$. This establishes my claim.

I have established that $f|F : F \longrightarrow \partial M$ is homotopic to a map f': $F \longrightarrow \partial M$ such that $f'(F)$ is contained in the union of the tori components of Φ and the components of $\partial\Phi$. Let (Σ', Φ') be a regular neighborhood in M of the union of the tori components of Φ and the components of $\partial\Phi$. Then (Σ', Φ') is a well-embedded S^1-pair and $f'\colon F \longrightarrow \partial M$ satisfies the inductive step for $K = F$.

Now, assume that the number of components of $K-F$ is n $(n \geq 1)$. Let A be the closure of a component of $K-F$. Set $K' = (\overline{K-A})$. If I reason inductively, then there exists a well-embedded S^1-pair $(\Sigma'', \Phi'') \subset (M, \partial M)$ and a map $f''\colon (K', F) \longrightarrow (M, \partial M)$ homotopic to $f|K'$ (as a map of pairs such that $f''(K') \subset \Sigma''$, $f''(F) \subset \Phi''$. Consider the map $f|A : (A, \partial A) \longrightarrow (M, \partial M)$ with $f|A\,(\partial A) \subset \Phi''$.

If $f|A$ is not essential, then $f|A$ is homotopic (rel ∂A) to a map $f_A\colon (A, \partial A) \longrightarrow (M, \partial M)$ such that $f_A(A) \subset \partial M$. If $f_A(\partial A)$ is contained in just one component of Φ'', then I may assume that $f_A(A) \subset \Phi''$. In this situation, let $\Sigma' = \Sigma''$, $\Phi' = \Phi''$ and define $f'\colon (K, F) \longrightarrow (M, \partial M)$ as $f'|K' = f''$ and $f'|A = f_A$. If $f_A(\partial A)$ is contained in distinct components of Φ'', say φ_0'' and φ_1'', then there exists an incompressible annulus $\varphi' \subset \partial M$ such that $f_A(A) \subset \varphi'$, $\varphi_0'' \cup \varphi_1'' \subset \varphi'$ and $\partial\varphi' \cap \partial\Phi'' = \emptyset$. Set σ' equal to a regular neighborhood of φ' in M; then (σ', φ') is a well-embedded S^1-pair in $(M, \partial M)$. It follows from Lemma VIII.6 that there exists a well-embedded S^1-pair $(\Sigma', \Phi') \subset (M, \partial M)$ such that both $(\Sigma'', \Phi'') \subset (\Sigma', \Phi')$ and $(\sigma', \varphi') \subset (\Sigma', \Phi')$. In this situation let $f'\colon (K, F) \longrightarrow (M, \partial M)$ be defined as $f'|K' = f''$ and $f'|A = f_A$.

If $f|A$ is essential, then I set $T = \Phi''$ and apply the Essential Homotopy Theorem (VIII.4) to the pair (M, T) and the map $f|A : (A, \partial A) \longrightarrow (M, T)$. Hence, there exists an S^1-pair $(\sigma', \varphi') \subset (M, T)$ (in general, a Seifert pair; however, here each component of T has Euler characteristic zero and so the Seifert pair is an S^1-pair) and a map $f_A: (A, \partial A) \longrightarrow (M, T)$ homotopic to $f|A$ (as a map of pairs) such that $f_A(A) \subset \sigma'$ and $f_A(\partial A) \subset \varphi'$. In this situation I may actually take f_A homotopic to $f|A$ (rel ∂A); I shall assume that this is the case. Again by Lemma VIII.6 there exists a well-embedded S^1-pair $(\Sigma', \Phi') \subset (M, \partial M)$ such that both $(\Sigma'', \Phi'') \subset (\Sigma', \Phi')$ and $(\sigma', \varphi') \subset (\Sigma', \Phi')$. Define $f': (K, F) \longrightarrow (M, \partial M)$ as $f'|K' = f''$ and $f'|A = f_A$.

I now have that if K is the two-complex equal to ∂S union the 2-manifold $\mathfrak{A}$ of Assertion 6, then there exists a well-embedded Seifert pair $(\Sigma', \Phi') \subset (M, \partial M)$ and a map $f': (K, F) \longrightarrow (M, \partial M)$ homotopic to $f|K$ (as a map of pairs) such that $f'(K) \subset \Sigma'$ and $f'(F) \subset \Phi'$.

Now, among all well-embedded Seifert pairs in $(M, \partial M)$ satisfying the preceding, choose (Σ', Φ') so that the number of components of Σ' is as small as possible; call this number n. Then, among all well-embedded Seifert pairs $(\Sigma', \Phi') \subset (M, \partial M)$, satisfying the preceding and such that Σ' has n components, choose (Σ', Φ') so as to minimize the number of components of $\mathrm{Fr}\, \Sigma'$. Using that $f': (K, F) \longrightarrow (M, \partial M)$ is essential (as a map of pairs), it follows that I may choose (Σ', Φ') satisfying the preceding and such that (Σ', Φ') is a perfectly-embedded Seifert pair. Hence, I may assume that $\Sigma' \subset \Sigma$ and $\Phi' \subset \Phi$. This completes the proof of Assertion 7.

Assertion 8. _In this case, if_ $\partial S \neq \emptyset$ _and_ $F \neq \emptyset$, _then there exists a map_ $g: (S, F) \longrightarrow (M, \partial M)$ _homotopic to_ f (as a map of pairs) _such that_ $g(S) \subset \Sigma$ _and_ $g(F) \subset \Phi$.

Let K be the two-complex equal to ∂S union the 2-manifold $\mathfrak{A}$ of Assertion 6. Then by Assertion 7, there exists a map $f': (K, F) \longrightarrow (M, \partial M)$ homotopic to $f|K$ (as a map of pairs) such that $f'(K) \subset \Sigma$ and $f'(F) \subset \Phi$.

Let C be the closure of a component of $S-K$. Then by Assertion 6, C is a solid torus and $\partial C \subset K$. Since $f'|K$ is homotopic to $f|K$ and M is irreducible, it follows that f' can be extended over C. If I reason inductively on the number of components of $S-K$, then I may assume that f' has been extended so that $f': (S,F) \longrightarrow (M,\partial M)$ is a map homotopic to f (as a map of pairs) and $f'(K) \subset \Sigma$, $f'(F) \subset \Phi$. Among all maps of (S,F) into $(M,\partial M)$ homotopic to f (as a map of pairs) and mapping K into Σ and F into Φ, choose $g: (S,F) \longrightarrow (M,\partial M)$ such that $g^{-1}(\mathrm{Fr}\ \Sigma)$ has as small a number of components as possible.

I claim that $g^{-1}(\mathrm{Fr}\ \Sigma) = \emptyset$; hence $g(S) \subset \Sigma$. A component of $g^{-1}(\mathrm{Fr}\ \Sigma)$ is a 2-manifold contained in a component of $S-K$. By the choice of K such a component is a solid torus. It follows that by surgery on g, not changing g in a neighborhood of K, that I could eliminate such components. Hence, if $g^{-1}(\mathrm{Fr}\ \Sigma) \neq \emptyset$, there is a contradiction of my choice of g. This contradiction establishes the proof of Assertion 8.

This completes the proof of Theorem IX.17. ∎

I have defined the characteristic Seifert pair for M to be the unique maximal perfectly-embedded Seifert pair (Σ, Φ) contained in $(M,\partial M)$. It is interesting to notice that the characteristic Seifert pair for M is completely determined by the property given in the conclusion of Theorem IX.17. (This approach is the one used in [J-S_1].) On the other hand, the characteristic Seifert pair for M is not determined by the property given in the conclusion of Theorem IX.15 (see Examples IX.20(g) and IX.20(h)).

IX.18. PROPOSITION: Let M be a Haken-manifold that is closed or has incompressible boundary. Let $(\Sigma, \Phi) \subset (M,\partial M)$ be a Seifert pair. The following are equivalent:

(i) (Σ, Φ) <u>is the characteristic Seifert pair for</u> M.

(ii) (Σ, Φ) <u>is perfectly-embedded and any nondegenerate map from a Seifert pair</u> (S,F), <u>which is distinct from</u> $(\mathbb{D}^2 \times I, \mathbb{D}^2 \times \partial I)$, $(\mathbb{D}^2 \times S^1, \emptyset)$, $(S^2 \times S^1, \emptyset)$ <u>or</u> $(S^3, \emptyset)$, <u>into</u> $(M, \partial M)$ <u>is homotopic</u> (as a map of pairs) <u>to a map from</u> (S,F) <u>into</u> $(M, \partial M)$ <u>with the image of</u> S <u>in</u> Σ <u>and the image of</u> F <u>in</u> Φ.

Proof: That (i) implies (ii) is just Theorem IX.17. So, suppose that (ii) holds for the pair (Σ, Φ). Let (Σ', Φ') be the characteristic Seifert pair for M. Now, for each component (σ', φ') of (Σ', Φ') the inclusion induced map $(\sigma', \varphi') \subset (M, \partial M)$ is a nondegenerate map; and since Fr σ' is incompressible in M, the pair (σ', φ') is not equivalent to $(\mathbb{D}^2 \times I, \mathbb{D}^2 \times \partial I)$ or $(\mathbb{D}^2 \times S^1, \emptyset)$ (it is not possible for (σ', φ') to be either $(S^2 \times S^1, \emptyset)$ or $(S^3, \emptyset)$). Hence, (σ', φ') can be homotoped (as a pair) into (Σ, Φ). It follows from Lemma VII.9 that (Σ', Φ') can be ambiently isotoped into (Σ, Φ). Since (Σ, Φ) is perfectly-embedded and (Σ', Φ') is the characteristic Seifert pair for M (maximal perfectly-embedded Seifert pair), the pair $(\Sigma, \Phi) = (\Sigma', \Phi')$

A 3-manifold M is <u>simple</u> if every two-sided, incompressible torus in M is parallel into ∂M.

IX.19. REMARKS:

(a) This generalizes the notion of a simple knot space to an arbitrary 3-manifold.

(b) If a Haken-manifold M is simple, then it is either a special Seifert fibered manifold or every $\mathbb{Z} + \mathbb{Z}$ subgroup of $\pi_1(M)$ is peripheral. Among all Haken-manifolds, the special Seifert fibered manifolds are precisely those that are simple and Seifert fibered (Torus Theorem VIII.14).

(c) It is the class of simple Haken-manifolds that are not Seifert fibered manifolds that W. Thurston claims admit a hyperbolic structure ([Th_2]).

IX.20. COROLLARY: Let M be a Haken-manifold that is closed or has incompressible boundary. Let (Σ, Φ) be the characteristic Seifert pair for M. Each component of the closure of $M - \Sigma$ is simple.

Proof: This follows immediately from Theorem IX.15. ■

IX.21. EXAMPLES: There are many examples of Haken-manifolds with a non-trivial characteristic Seifert pair. However, I am going to give just a few examples, which are familiar. A couple of definitions make these examples more easily explained.

A cable space is a manifold obtained from a solid torus $S^1 \times \mathbb{D}^2$ by removing an open regular neighborhood in $S^1 \times \mathring{\mathbb{D}}^2$ of a simple closed curve k that lies in a torus $S^1 \times J$, where J is a simple closed curve in $\mathring{\mathbb{D}}^2$ and k is noncontractible in $S^1 \times J$. Notice that the complement of a cable knot in S^3 always contains a cable space with incompressible boundary. Indeed, to say that k' is a cable knot in S^3 means that there exists a knotted solid torus T in S^3 with $k' \subset \mathring{T}$ and a homeomorphism $h : S^1 \times \mathbb{D}^2 \longrightarrow T$ such that $h(k) = k'$ for some curve k as above. A cable space is a Seifert fibered manifold.

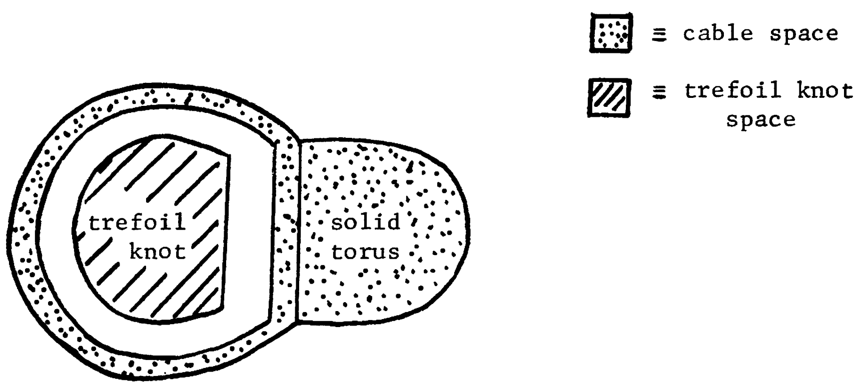
An (n-fold) composing space is a compact 3-manifold homeomorphic to $W \times S^1$, where W is a disk-with-(n-) holes. The complement of a knot in S^3 that is a composition of n non-trivial knots contains an n-fold composing space with incompressible boundary. An n-fold composing space is a Seifert fibered manifold.

IX.22. LEMMA: A Seifert-fibered manifold that is contained in a knot space in S^3 and has incompressible boundary is either a torus knot space, a cable space or a composing space (see Lemma VI.3.4 of [J-S_1]).

(a) The characteristic Seifert pair of the trefoil knot space is the trefoil knot space itself.

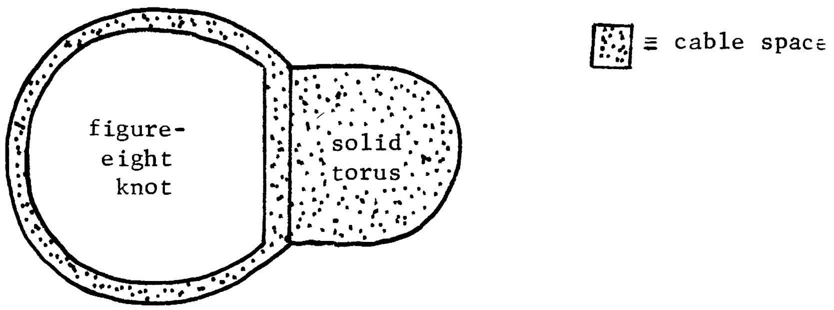
(b) The characteristic Seifert pair of the figure-eight knot space is the empty set. (Notice that the characteristic Seifert pair of any simple knot is the empty set. A regular neighborhood of a boundary torus is <u>not</u> perfectly-embedded.)

(c) The characteristic Seifert pair of a nontrivial cable knot about the trefoil-knot has two components: one is homeomorphic to the trefoil-knot space and the other is a cable space (see Schematic 9.1). The simple part is homeomorphic to $S^1 \times S^1 \times I$.



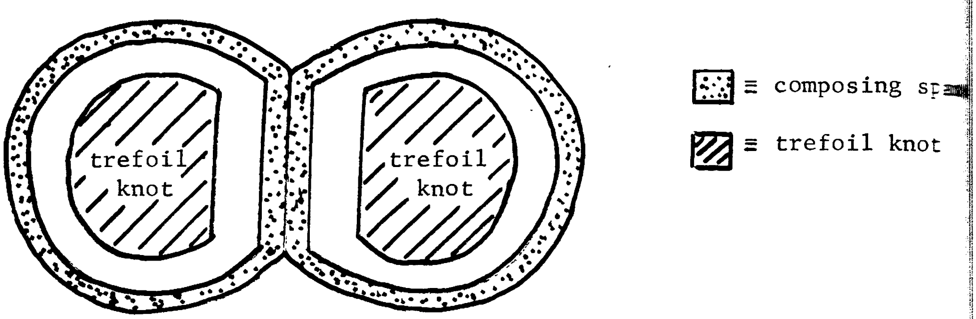
Schematic 9.1

(d) The characteristic Seifert pair of a nontrivial cable knot about the figure-eight knot has one component: a cable space (see Schematic 9.2). The simple part is homeomorphic to the figure-eight space.



Schematic 9.2

(e) The characteristic Seifert pair of the granny knot has three components: two components are each homeomorphic to the trefoil knot space and the third component is a 2-fold-composing-space (see Schematic 9.3). In this case, the simple part has two components, each homeomorphic to $S^1 \times S^1 \times I$.



Schematic 9.3

(f) Let $M = F \times_{\varphi} S^1$ be a closed surface bundle over S^1 with fiber F and sewing map φ.

The following lemma appears in [J-S_4].

IX.23. LEMMA: Let F be a compact surface and $\varphi : F \to F$ a homeomorphism. Then there exists a compact incompressible 2-manifold $F_0 \subset F$ and a homeomorphism $\varphi_0 : F \to F$ isotopic to φ such that

(i) $\varphi_0 | F_0 : F_0 \to F_0$ is periodic, and

(ii) if $f : K \to F$ is a map such that there exists $n > 0$ and $\varphi^n \circ f$ is homotopic to f, then f is homotopic to $f' : K \to F$ with $f'(K) \subset F_0$.

Now, for $M = F \times_\varphi S^1$, let F_0 and φ_0 be as in the conclusion of Lemma IX.23. Then $M = F \times_{\varphi_0} S^1$. Let Σ be the submanifold of M obtained by setting F_0' equal to the union of the components of F_0 which are not 2-cells and setting $\Sigma = F_0' \times I/\langle (x,0) = (\varphi_0(x),1) \rangle$. Then it can be shown by VI.27 that Σ is a Seifert fibered manifold.

As for the present example, the following is true:

(i) If $\chi(F) < 0$, then $(\Sigma, \emptyset)$ is the characteristic Seifert pair for M.

(ii) If $\chi(F) = 0$ and $F_0 = \emptyset$, then a regular neighborhood of F (homeomorphic to $S^1 \times S^1 \times I$) is the characteristic Seifert pair for M.

(iii) If $\chi(F) = 0$ and F_0 is an annulus, then M is a Seifert fibered manifold (an S^1-bundle over a torus or a Klein bottle); and so, $(M, \emptyset)$ is the characteristic Seifert pair for M.

(iv) If $\chi(F) = 0$ and $F_0 = F$, then M is a Seifert fibered manifold (one of the examples 12.3(1) or (2) or (5) or (6) or (7) on page 122 of [He_1]); and so, $(M, \emptyset)$ is the characteristic Seifert pair for M.

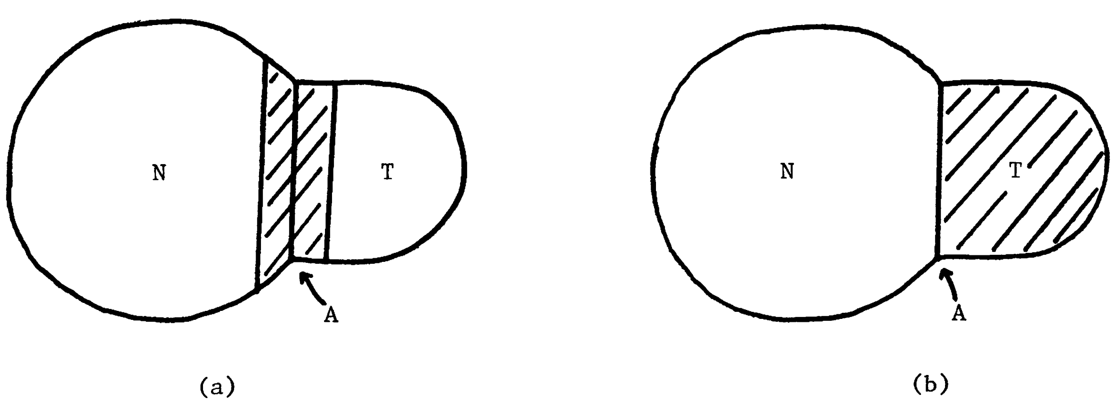
(g) Let N be a Haken-manifold with connected, incompressible boundary satisfying the following properties:

(1) $\chi(\partial N) < 0$, and

(2) N does not contain any essential, embedded annulus or any essential, embedded torus (see Exercise IX.23).

Let $h_1: S^1 \times I \longrightarrow \partial N$ be an embedding that is injective on $\pi_1(S^1 \times I)$ and let $h_2: S^1 \times I \longrightarrow \partial(\mathbb{D}^2 \times S^1)$ be an embedding that is injective, but <u>not</u> surjective, on $\pi_1(S^1 \times I)$. Let M be the manifold obtained from the disjoint union of N and $\mathbb{D}^2 \times S^1$ via the identification $h_1(x) = h_2(x)$ for $x \in S^1 \times I$; i.e., M is obtained from N by adding a nontrivial root along a simple closed curve in ∂N. Set $T = \mathbb{D}^2 \times S^1$. Then $M = N \cup T$; and $N \cap T = A$ is an essential annulus in M.

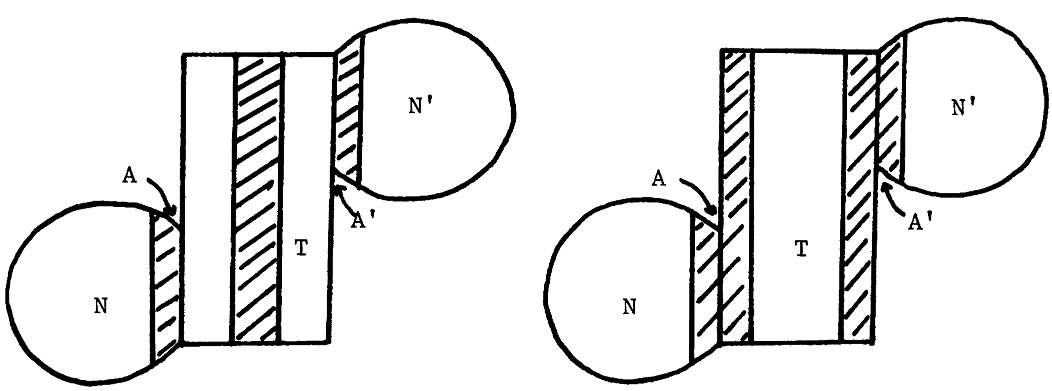
The regular neighborhood of A in M satisfies the conclusion of Theorem IX.15 (see Schematic 9.4(a)), yet, the pair $(T, T \cap \partial M)$ is the characteristic Seifert pair for M and N is the simple part (see Schematic 9.4(b)).



<u>Schematic 9.4</u>

(h) Let N be as in the preceding example (IX.20(g)). Let N' be a disjoint copy of N. Let $h : S^1 \times I \longrightarrow \partial N$ (respectively, $(h' : S^1 \times I \longrightarrow \partial N')$) be an embedding that is an injection on $\pi_1(S^1 \times I)$. Let T be the manifold $S^1 \times S^1 \times I$. Let $\alpha_i : (S^1 \times I) \longrightarrow S^1 \times S^1 \times \{i\}$ be an embedding that is an injection on $\pi_1(S^1 \times I)$ for $i = 0,1$ and assume that the homotopy class of α_0 is different from that of α_1 in $S^1 \times S^1 \times I$. Let M be the manifold obtained from the disjoint union of N, N' and T via the identifications $h(x) = \alpha_0(x)$ and $h'(x) = \alpha_1(x)$ for each $x \in S^1 \times I$. Then $M = N \cup N' \cup T$ where $N \cap T = A$ is an essential, embedded annulus in M and $N' \cap T = A'$ is an essential, embedded annulus in M. Of course, $N \cap N' = \emptyset$.

The manifold having three components: one a regular neighborhood of A in M, one a regular neighborhood of A' in M and one the product $S^1 \times S^1 \times [1/3, 2/3]$ in T satisfies the conclusion of Theorem IX.15 (see Schematic 9.5(a)); yet the pair (Σ, Φ) having two components (σ, φ), where σ is a regular neighborhood of $S^1 \times S^1 \times \{0\}$ in M and $\varphi = \sigma \cap \partial M$, and (σ', φ'), where σ' is a regular neighborhood of $S^1 \times S^1 \times \{1\}$ in M and $\varphi' = \sigma' \cap \partial M$, is the characteristic Seifert pair for M (see Schematic 9.5(b)).



(a) (b)

Schematic 9.5

IX.23. EXERCISE: Show that there exists a manifold $N \subset \mathbb{R}^3$ (S^3) that satisfies the properties for the manifold N in Example IX.23(g). [**Hint**: Consider the complement in S^3 of either of the graphs given in Figure 9.6.]

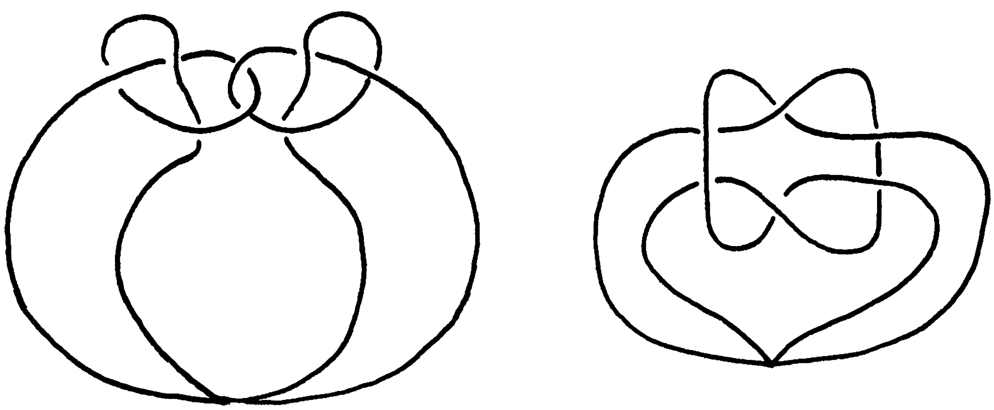


Figure 9.6

CHAPTER X. DEFORMING HOMOTOPY EQUIVALENCES

In this chapter I study the extent to which the fundamental group of a 3-manifold determines its homeomorphism type. Of course, there are many well-known examples of 3-manifolds that have isomorphic fundamental groups and are not homeomorphic. In fact, as early in these notes as Chapter II (Exercise II.14 and Lemma II.15), I gave the examples $L(p,q) \# L(p,q)$ and $L(p,q) \# -L(p,q)$, which have isomorphic fundamental groups and are not homeomorphic, unless $q^2 \equiv -1 (\text{mod } p)$. I shall give more examples of 3-manifolds with isomorphic fundamental groups and distinct homeomorphism types in this chapter. However, in the case of many Haken-manifolds (those that are closed or have incompressible boundary) it is remarkable that this situation can be completely described after the work of Waldhausen [W_3] and Johannson, [Jo_1] and [Jo_2]. I have used the theorem due to Waldhausen throughout these lectures, referring to the version given as Theorem 13.6 in [He_1]. In this chapter I will give a more general version of Waldhausen's theorem, which is due to Evans [E_1] and Tucker [Tu_3]. However, the main result of the chapter is a new proof of the theorem of Johannson [Jo_2]. The proof is based on observations of Swarup (Proposition VII.22 and Proposition VII.24).

In general, the problem that I have proposed in this chapter is considered from a more restrictive setting (hence, the title of this chapter) Namely, there are two basic questions.

X.1. EXISTENCE: Can a homotopy equivalence between two compact 3-manifolds be deformed to a homeomorphism?

X.2. UNIQUENESS: Are two homotopic homeomorphisms between compact 3-manifolds isotopic?

I will not consider the uniqueness problem, X.2, in these lectures. It has been answered in the affirmative for Haken-manifolds by Waldhausen [W_3]; and, I know of no counterexamples in the case of compact, orientable 3-manifolds.

I will consider the existence problem, X.1, for Haken-manifolds. The fact that Haken-manifolds are irreducible avoids the unresolved problem with the 3-dimensional Poincaré Conjecture and the technical difficulties encountered with the second homotopy group and splitting at 2-spheres. This latter problem was solved by Tally [Ta_1] and, independently, by Hendriks and Landenbach [H-L]. The book [La_1] by Landenbach concerns the study of both questions on existence (X.1) and uniqueness (X.2). By considering Haken-manifolds, I also immediately eliminate the classical examples furnished by Lens spaces.

X.3. EXAMPLE: The Lens spaces $L(p,q)$ and $L(p',q')$ are homeomorphic if and only if $|p| \equiv |p'|$ and $q' \equiv \pm q^{\pm 1} \pmod p$.

It is easy to construct a homeomorphism between $L(p,q)$ and $L(p',q')$ if the conditions are satisfied. So, suppose that $L(p,q)$ and $L(p',q')$ are homeomorphic.

Set $L = L(p,q)$ and set $L' = L(p',q')$. Both L and L' admit genus one Heegaard splittings, say $(L;F)$ and $(L';F')$, respectively. Let H_1 and H_2 denote the closures of $L-F$ and let H_1' and H_2' denote the closures of $L'-F'$. Since L and L' are assumed homeomorphic, there is a homeomorphism $h : L \longrightarrow L'$. Furthermore, I may assume that $p' = p > 0$.

I shall prove that the desired equation $q' \equiv \pm q^{\pm 1} \pmod p$ holds using the additional assumption that h is isotopic to a homeomorphism that takes the torus F onto the torus F' (see Question II.16). Now, F has a framing α, β such that β is contractible in H_1 and $p\alpha + q\beta$ is contractible in H_2. Similarly, F' has a framing α', β' such that β' is contractible in H_1' and $p\alpha' + q'\beta'$ is contractible in H_2'.

I may assume that $h(F) = F'$. There are two possible cases.

<u>Case 1</u>. $h(H_1) = H_1'$.

In this case the matrix of the homeomorphism $h|F : F \longrightarrow F'$ must have the form $\begin{pmatrix} \pm 1 & 0 \\ y & \pm 1 \end{pmatrix}$, since $h(\beta) = \pm\beta'$ and $h(p\alpha + q\beta) = \pm(p\alpha' + q'\beta')$

It follows in this case that $q' \equiv \pm q^{+1} \pmod p$.

<u>Case 2</u>. $h(H_1) = H_2'$.

In this case the matrix of the homeomorphism $h|F : F \longrightarrow F'$ must have the form $\begin{pmatrix} \mp q & \pm p \\ y & \pm q' \end{pmatrix}$, since $h(\beta) = \pm(p\alpha' + q'\beta')$ and $h(p\alpha + q\beta) = \pm\beta'$. It follows in this case that $q' \equiv \pm q^{-1} \pmod p$

<u>The Lens spaces</u> $L(p,q)$ <u>and</u> $L(p',q')$ <u>are homotopic equivalent if and only of</u> $|p| \equiv |p'|$ <u>and</u> $qq' = \pm \omega^2 \pmod p$; i.e., the product qq' is a quadratic residue modulo p.

A proof of this is given as Lemma 3.23 of [He_1].

In particular, the Lens spaces L(5,1) and L(5,2) have isomorphic fundamental groups and are not homeomorphic. The Lens spaces L(7,1) and L(7,2) are homotopy equivalent and not homeomorphic. This latter example yields a negative answer to the existence question X.1 in the general setting of compact 3-manifolds.

X.4. REMARK: There has been a lot of interesting work in the past year on irreducible 3-manifolds with finite fundamental group (at least in the case in which it is known that the universal cover is S^3). In particular, there is the work of Thomas [Th_1], Evans and Maxwell [E-M], Myers [My_1] and Rubenstein [Ru_1] and [Ru_2]. I will not be covering this work in these lectures. However, there are a couple of related questions that have intrigued me in the past, which I mention in parting from 3-manifolds with finite fundamental group.

X.5. QUESTION: If the irreducible 3-manifold M is homotopy equivalent to $\mathbb{R}P^3$, real projective 3-space, then is M homeomorphic to $\mathbb{R}P^3$?

X.6. QUESTION: Can $\mathbb{R}P^3$ be obtained by doing surgery on a nontrivial knot complement in S^3? More generally, what Lens spaces can be obtained by doing surgery on a nontrivial knot complement in S^3?

I shall now restrict my attention to Haken-manifolds. The next theorem is primarily due to Waldhausen. The version that I am giving is due to Evans [Ev_1] and Tucker [Tu_3]. See the remarks (X.9) after the proof of Corollary X.8.

X.7. THEOREM: Let M and M' be Haken-manifolds. Suppose that $\pi_1(M) \neq 1$ and $f : (M,\partial M) \longrightarrow (M',\partial M')$ is a map such that $f_*: \pi_1(M) \longrightarrow \pi_1(M')$ is an injection. Then, there is a homotopy $f_t: (M,\partial M) \longrightarrow (M',\partial M')$ such that $f_0 = f$ and either

(i) $f_1: M \longrightarrow M'$ is a covering map, or

(ii) M is a product I-bundle over a closed surface, and $f_1(M) \subset \partial M'$, or

(iii) M' (and hence M) _is a solid torus and_ $f_1: M \longrightarrow M'$ _is a branch covering with branching locus a circle, or_

(iv) M _is a cube-with-handles and_ $f_1(M) \subset \partial M'$.

Furthermore, if $f|\partial M$ _is a covering map, then_ f_t _may be chosen so that_ $f_t|\partial M = f|\partial M$, _for all_ t.

Proof: I shall prove the theorem by reasoning inductively on $-\chi(\partial M)$. Since $\pi_1(M) \neq 1$, $-\chi(\partial M) \geq 0$. First, I make an observation, originally due to T. Tucker [Tu_3]. This is the only place in the proof that I use the induction hypothesis.

Observation 1: Under the hypothesis of the theorem, if for some component B of ∂M there exists a nontrivial element of $\ker((f|B)_* : \pi_1(B) \longrightarrow \pi_1(C))$ represented by a simple closed curve in B, where C is the component of $\partial M'$ containing $f(B)$, then conclusion (iv) of the theorem holds.

For suppose that J is a noncontractible simple closed curve in B and $f(J)$ is contractible in C. Since f_* is an injection, the simple closed curve J is contractible in M; hence, by Dehn's Lemma, I.6, J bounds a disk $D \subset M$. Up to homotopy of f, I may assume that $f(D) \subset C$. Let M_1 be the manifold obtained by splitting M at D. Then $f|M_1 : (M_1, \partial M_1) \longrightarrow (M', \partial M')$ and $(f|M_1)_*$ is an injection. If some component of M_1 has trivial fundamental group, then M_1 is connected and is a 3-cell; it follows that M is a solid torus and f can be deformed (rel ∂M) so that $f(M) \subset \partial M'$. Thus possibility (iv) holds. Otherwise, if I reason by induction, using that the map $f|M_1$ satisfies the hypothesis of the theorem on each component of M_1, no component of M_1 has trivial fundamental group, and each component of M_1 has Euler characteristic less than or equal to $-\chi(\partial M) - 2$, then for each component of M_1, one of the conclusions of the theorem holds. If conclusion (i) or (iii) holds for a component M_1' of M_1, then $f_*(\pi_1(M_1'))$ has finite index in $\pi_1(M')$; however, $\pi_1(M_1')$ has infinite index in $\pi_1(M)$. So, I may assume for each component of M_1 either conclusion (ii) or conclusion (iv) holds. In both cases it follows that there exists a homotopy $f_t: M \longrightarrow M'$ such that $f_0 = f$ and $f_1(M) \subset C$. Now, if conclusion (ii) holds for a component M_1' of M_1, then M_1' is a product I-bundle and the map $f_1(M_1') \subset C$. Hence, C must be incompressible in M' and $(f_1|M_1')_*(\pi_1(M_1'))$ has finite index in $\pi_1(C)$. But $(f_1)_*(\pi_1(M))$ is isomorphic to a subgroup of $\pi_1(C)$ containing $(f_1|M_1')_*(\pi_1(M_1'))$. This gives a contradiction to the fact that $\pi_1(M_1')$ has infinite index in $\pi_1(M)$. So, the only possibility is that conclusion (iv) holds for each component of M_1; and therefore, conclusion (iv) holds for M.

I now break the proof into two cases. The first case uses Observation 1 and hence, it yields conclusion (iv).

Case 1: For some component B of ∂M the degree of the map $f|B$ is zero.

In this case I shall prove that there is a nontrivial element of $\ker((f|B)_*: \pi_1(B) \longrightarrow \pi_1(C))$ represented by a simple closed curve in B, where C is the component of $\partial M'$ containing $f(B)$. Since $f|B : B \longrightarrow C$ has degree zero, $(f|B)_*(\pi_1(B))$ is a free subgroup of $\pi_1(C)$. Hence, by either [St_5] or [J_4] the desired element of $\ker((f|B)_*)$ exists.

By Observation 1, conclusion (iv) holds.

Case 2: For each component B of ∂M, the degree of the map $f|B$ is not zero.

In this case I shall prove that either conclusion (ii) holds or for each component B of ∂M the map $(f|B)_*: \pi_1(B) \longrightarrow \pi_1(C)$ is an injection, where C is the component of $\partial M'$ containing $f(B)$. I need another observation.

Observation 2: Under the hypothesis of the theorem and in this case, if for distinct components B_0 and B_1 of ∂M there exists a component C of $\partial M'$ such that $f(B_i) \subset C$, $i = 0$ and 1, then conclusion (ii) of the theorem holds.

Since $f|B_i: B_i \longrightarrow C$ has nonzero degree, the subgroup $(f|B_i)_*(\pi(B_i))$ has finite index in $\pi_1(C)$, $i = 0$ and 1. Let $H = f_*^{-1}(\mathrm{Im}(\pi_1(C) \longrightarrow \pi_1(M')))$. Then for $i = 0$ and 1 (and up to conjugation) the subgroups $\mathrm{Im}(\pi_1(B_i) \longrightarrow \pi_1(M))$ have finite index in H. Let $\widetilde{M}_H$ be the covering space of M corresponding to the conjugacy class of H in $\pi_1(M)$. Then both B_0 and B_1 lift to components $\widetilde{B}_0$ and $\widetilde{B}_1$, respectively, of $\partial\widetilde{M}_H$ and $\mathrm{Im}(\pi_1(\widetilde{B}_i) \longrightarrow \pi_1(\widetilde{M}_H))$ has finite index in $\pi_1(\widetilde{M}_H)$. It follows (see for example Theorem 10.5 of [He_1]) that $\widetilde{M}_H$ is a product I-bundle over a closed surface; and so, M is a product I-bundle over a closed surface. Conclusion (ii) follows in this situation.

I can now assume that I am in Case 2 and the map of $\pi_0(\partial M)$ into $\pi_0(M')$ induced by f is an injection.

Let $\widetilde{M}'$ denote the covering space of M' corresponding to the conjugacy class of $f_*(\pi_1(M))$ in $\pi_1(M')$. Let $p : \widetilde{M}' \rightarrow M'$ denote the covering projection. There is a map $\widetilde{f} : (M, \partial M) \longrightarrow (\widetilde{M}', \partial\widetilde{M}')$ such that $f = p \circ \widetilde{f}$. I shall prove that for each component B of ∂M the map $(\widetilde{f}|B)_*: \pi_1(B) \longrightarrow \pi_1(\widetilde{C})$ is an injection, where $\widetilde{C}$ is the component of $\partial\widetilde{M}'$ containing $\widetilde{f}(B)$. Notice that the map of $\pi_0(\partial M)$ into $\pi_0(\partial\widetilde{M}')$ induced by $\widetilde{f}$ is a bijection and $\widetilde{M}'$ is compact.

Now, if B_i is a component of ∂M and $\widetilde{C}_i$ is the component of

$\partial\widetilde{M}'$ such that $\widetilde{f}(B_i) \subset \widetilde{C}_i$, then $\widetilde{C}_i$ is compact and $\deg(\widetilde{f}|B_i)$ is not zero. Set d_i equal to the absolute value of $\deg(\widetilde{f}|B_i)$. By the Kneser formula [Kn_2] (see Proposition 4.3 of [Ed_1]), $\chi(B_i) \leq d_i\chi(\widetilde{C}_i)$. Both M and M' are Haken-manifolds and therefore they are aspherical (Remark III.19). It follows that $\widetilde{M}'$ is aspherical and that the map $\widetilde{f} : M \longrightarrow \widetilde{M}'$ is a homotopy equivalence. I now have the equations:

$$2\chi(M) = 2\chi(\widetilde{M}'),$$

$$2\chi(M) = \chi(\partial M) = \Sigma_i\chi(B_i) \leq \Sigma_i d_i\chi(\widetilde{C}_i),$$

and $$2\chi(\widetilde{M}') = \chi(\partial\widetilde{M}') = \Sigma_i\chi(\widetilde{C}_i).$$

It follows that either $d_i = 1$ for $\chi(\widetilde{C}_i) \neq 0$ or $\chi(\widetilde{C}_i) = 0$. If some component $\widetilde{C}_i$ of $\partial\widetilde{M}'$ has $\chi(\widetilde{C}_i) \neq 0$, then $d_i = 1$; and since the map $\widetilde{f}$ induces a bijection from $\pi_0(\partial M)$ to $\pi_0(\partial\widetilde{M}')$, it follows that $d_i = 1$ for each component C_i of $\partial\widetilde{M}'$. By the same argument used in the proof of Theorem 3.1 of [E_1], I have that $(\widetilde{f}|B)_*$ is an injection for each component B of ∂M and so $(f|B)_*$ is an injection for each component B of ∂M. In this situation the hypothesis of Theorem 13.6 of [He_1] is satisfied and conclusions (i), (ii) or (iii) of the theorem follow.

So, I may assume that for each component $\widetilde{C}_i$ of $\partial\widetilde{M}'$, $\chi(\widetilde{C}_i) = 0$; i.e., each component of $\partial\widetilde{M}'$ is a torus. It follows that each component of $\partial M'$ is a torus. I claim that each component of ∂M is a torus. For suppose that some component B of ∂M is not a torus. Then $\ker(f|B)_*$ contains the commutator subgroup of $\pi_1(B)$ and as such contains a nontrivial element represented by a simple closed curve in B. If this were the case, then by Observation 1, conclusion (iv) of the theorem would hold. This would contradict that $\deg(f|B)$ is nonzero (as well as a number of other things). So, each component of ∂M is a torus.

Since $\deg(f|B)$ is nonzero for each component B of ∂M, (as I noted before) the index of $(f|B)_*(\pi_1(B))$ is finite in $\pi_1(C)$, where C is the component of $\partial M'$ containing $f(B)$. Since both $\pi_1(B)$ and $\pi_1(C)$ are free abelian of rank two, it follows that $(f|B)_*$ is an injection. Again, the hypothesis of Theorem 13.6 of [He_1] is satisfied and the proof of the theorem follows. ■

X.8. COROLLARY: Let M and M' be Haken-manifolds. Suppose that $f : (M, \partial M) \longrightarrow (M', \partial M')$ is a map such that $f_*: \pi_1(M) \longrightarrow \pi_1(M')$ is an isomorphism. Then f is homotopic (possibly not keeping the image of ∂M in $\partial M'$) to a homeomorphism. Furthermore, if $f|\partial M: \partial M \longrightarrow \partial M'$ is a homeomorphism, then f is homotopic (rel ∂M) to a homeomorphism.

Proof: If $\pi_1(M) = 1$, then both M and M' are 3-cells and the result follows. In this case the "furthermore" part of the corollary comes from [A_1].

If $\pi_1(M) \neq 1$, then by Theorem X.7, one of the situations (i) through (iv) occurs. If (i) occurs, then since f_* is an isomorphism (and so an epimorphism) the covering map f_1 is a homeomorphism. If (ii) occurs, then since both M and M' are orientable, they are both product I-bundles over the same closed surface and f_1 (and therefore f) can be deformed to a homeomorphism; of course, in this case the homotopy does not keep the image of ∂M in $\partial M'$. In case (iii) both M and M' are solid tori and f_1 (and therefore f) can be deformed to a homeomorphism; in this case if the boundary index is greater than one, then the homotopy does not keep the image of ∂M in $\partial M'$. Finally, in case (iv) it follows from f_* being an isomorphism that $\pi_1(M')$ is also a free group. By Exercise I.32 the manifold M' is a cube-with-handles. The remainder of the proof follows exactly as in the first paragraph of Theorem 3.2 of $[E_1]$; I shall repeat it here for completeness.

Both M and M' have the same genus; so, there is a homeomorphism $g : M' \longrightarrow M$. The map $g \circ f$ induces an automorphism α on $\pi_1(M)$. By applying a result of $[Z_1]$, there exists a homeomorphism $h : M \longrightarrow M$ such that $h_* = \alpha$. But now, both f and $g^{-1}h$ induce the same isomorphism from $\pi_1(M)$ into $\pi_1(M')$. Both M and M' are aspherical so f is homotopic to the homeomorphism $g^{-1}h$. In this case the homotopy cannot keep the image of ∂M in $\partial M'$. $\blacksquare$

X.9. REMARKS:

(a) It follows from Corollary X.8 that a closed Haken-manifold is completely determined by its fundamental group. This was first proved by Waldhausen $[W_3]$.

(b) I mentioned, just prior to the statement of Theorem X.7, that the primary contribution to Theorem X.7 was the work of Waldhausen $[W_3]$. He assumed in Theorem 6.1 of $[W_3]$ that both M and M' were either closed or had incompressible boundary. He concluded in this case that either (i) or (ii) of Theorem X.7 held. The version of the theorem that I have given can be said to be after Tucker $[Tu_3]$, Evans $[E_1]$ and Swarup $[Sw_4]$. This version of the theorem was found by Tucker $[Tu_3]$; however, Lemma 1 of $[Tu_3]$ is false. In the comments after Conjecture 13.10 of $[He_1]$, which I have answered here in the affirmative, it is implied that Lemma 1 of $[Tu_3]$ is false for closed, orientable surfaces; however, the only counterexamples known to me are for bounded manifolds (see also Theorem 4.4 of $[Ed_1]$). Tucker later found a proof of Theorem X.7, which avoided the error in $[Tu_3]$; his proof is unpublished. Corollary X.8 is the version due to Evans $[E_1]$; however, his methods require much more work than the methods that I have used. My method of proof is probably closest in spirit to that of [Sw_4]. In fact, Case 1 of the proof of Theorem X.7 is from [Sw_4]. I had difficulty with the latter part of the manuscript [Sw_4].

The previous theorem and its corollary dealt with "boundary-preserving" mappings. Since I am interested in the extent to which the fundamental group of a Haken-manifold determines its topological type, and since Haken-manifolds with isomorphic fundamental group are homotopy equivalent (Remark III.19), I am interested in relaxing the requirement that the homotopy equivalence be prescribed as boundary preserving. The following examples give one some conditions to reflect upon. These are the "standard" examples and coincide with those given in [W_6].

X. 10. EXAMPLE: Let N be a Haken-manifold with connected, nonempty boundary such that the inclusion $\pi_1(\partial N) \longrightarrow \pi_1(N)$ is not onto. Let M be the manifold obtained from N by adding a one-handle to N; i.e. M is a "disk-sum" of N and $\mathbb{D}^2 \times S^1$. The manifold M is a Haken-manifold; and so, by Remark III. 19, M is aspherical. Now, the natural map from the group of self-homotopy equivalences of M to the group of outer automorphisms of $\pi_1(M)$ is an isomorphism. The group $\pi_1(M)$ is isomorphic to $\pi_1(N) * \mathbb{Z}$. Hence, for t a generator of $\mathbb{Z}$ and g any element of $\pi_1(N)$ (based in ∂N, say) there is an automorphism $\varphi_g\colon \pi_1(M) \longrightarrow \pi_1(M)$ that is the identity on $\pi_1(N)$ and takes t to gt. By the preceding, any automorphism is induced by a self-homotopy equivalence of M. On the other hand, if g is not peripheral in $\pi_1(N)$, then gt is not peripheral in $\pi_1(M)$. Since any automorphism induced by a homeomorphism must take peripheral elements to peripheral elements, it follows that for g any nonperipheral element in $\pi_1(N)$, the automorphism φ_g of

$\pi_1(M)$ can not be realized by a homeomorphism of M. This gives a self-homotopy equivalence of M that can not be realized by a self-homeomorphism of M.

Another example is to take $S^1 \times S^1 \times I$ and let M be the Haken-manifold obtained by adding a one-handle to $S^1 \times S^1 \times I$, which meets only $S^1 \times S^1 \times \{1\}$, and let M' be the Haken-manifold obtained by adding a one-handle to $S^1 \times S^1 \times I$, which meets both $S^1 \times S^1 \times \{1\}$ and $S^1 \times S^1 \times \{0\}$. Then M and M' are homotopy equivalent Haken-manifold; but they are not homeomorphic.

In these examples one obtains a homotopy equivalence by letting a one-handle "drag" about a nonperipheral loop. I wish to thank R. K. Miller for correcting a mistake in an earlier version of this example.

X.11. EXAMPLE: Let M be a cube-with-handles. Then any homotopy equivalence of M can be deformed to a homeomorphism. Of course, this is just Corollary X.8, since any map into a cube-with-handles is homotopic to a "boundary-preserving" map. This example comes from $[Z_1]$, where it is shown that each automorphism of $\pi_1(M)$ can be realized by a homeomorphism. In fact, the results of $[Z_1]$ were used in the proof of Corollary X.8.

For the above reasons I shall now restrict my attention to Haken-manifolds with incompressible boundary (in view of Remark X.9(a), I only need to consider manifolds with nonempty boundary).

X.12. EXAMPLE: Let M be the manifold $F \times S^1$ where F is a disk-with-two-holes and let M' be the manifold $F' \times S^1$ where F' is the once-punctured torus. Then M and M' are homotopy equivalent; however, they are clearly not homeomorphic (M has three boundary components while M' has connected boundary).

This example is not surprising, since the disk-with-two-holes is homotopy equivalent to the once-punctured torus. In the 2-dimensional case, orientable surfaces are analogous (see Chapter IV) to Haken-manifolds that are closed or have incompressible boundary. It is true in this 2-dimensional case that the "peripheral structure" must be taken into account. In fact, if "peripheral structure" is defined (in the 2-dimensional case) as the number of boundary components, then the pair (fundamental group, peripheral structure) completely determines the topological type of a compact, orientable surface. The next examples for 3-manifolds show that "peripheral structure" in the 3-dimensional case does not lend itself to such a simple definition.

X.13. EXAMPLE: Let M_1 and M_2 be Haken-manifolds with nonempty incompressible boundary. Let $h_i: S^1 \times I \to \partial M_i$ $(i = 1,2)$ be embeddings that are injective on $\pi_1(S^1 \times I)$. Let $\tau : S^1 \times I \to S^1 \times I$ be defined as $\tau(s,t) = (s,1-t)$; i.e., "flip in the I-factor." Let M (respectively, M') be obtained from the disjoint union of M_1 and M_2 via the identification $h_1(x) \sim h_2(x)$ (respectively, $h_1(x) \sim h_2 \circ \tau(x)$), $x \in S^1 \times I$; i.e., $M = M_1 \underset{h_1=h_2}{\cup} M_2$ and $M' = M_1 \underset{h_1=h_2\circ\tau}{\cup} M_2$. Then M and M' are always homotopy equivalent Haken-manifolds with incompressible boundary; yet, in general, M and M' are <u>not</u> homeomorphic. For example, the classical case of this is when M is the "granny knot" complement and M' is the "square knot" complement. Here both M and M' have only one boundary component.

However, in this dimension there is an even more curious example. Namely, let M_1 be a Haken-manifold with incompressible boundary being a genus three surface. (Many such examples exist in $\mathbb{R}^3(S^3)$ as the complement of a wedge of three simple closed curves.) Let $h_1: S^1 \times I \to \partial M_1$ be an embedding such that $h_1(S^1 \times I)$ separates ∂M_1 into two components with the closure of one a once-puncture torus and the closure of the other a once-punctured genus two surface. Let M_2 be a disjoint copy of M_1 and let $h : M_1 \to M_2$ be a homeomorphism. Set $h_2 = h \circ h_1: S^1 \times I \to \partial M_2$. Now, let $M = M_1 \underset{h_1=h_2}{\cup} M_2$ and let $M' = M_1 \underset{h_1=h_2\circ\tau}{\cup} M_2$. The manifolds M and

M' are homotopic equivalent; they are not homeomorphic; and moreover, the homotopy types of the components of ∂M (two surfaces, one having genus two and the other having genus four) are distinct from the homotopy types of the components of $\partial M'$ (two surfaces, each having genus three).

The family of Seifert fibered manifolds with nonempty boundary (see Exercise VI.8), in general, furnish many examples of homotopy equivalent manifolds that are not homeomorphic (in particular, see Example X.12). These examples have been analyzed by Waldhausen ([W_4] and Example 1.4 of [W_6]) and others ([O-V-Z] and [O_1]). I think one collection of such examples is particularly interesting, the "homotopy torus knot spaces."

X.14. EXAMPLE: Let $\mathcal{T}$ be the collection of Seifert fibered manifolds with orbit manifold $B = \mathbb{D}^2$ and with precisely two exceptional fibers. The two exceptional fiber types can be normalized to be of type (α_1, β_1) and (α_2, β_2) where $0 < \beta_i < \alpha_i$, $i = 1$ or 2. Up to homeomorphism, the manifolds are classified by the set of pairs of integers $\{(\alpha_1, \beta_1), (\alpha_2, \beta_2): 0 < \beta_i < \alpha_i, \ i = 1 \text{ or } 2\}$ and the equivalence relation $\{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\} \sim \{(\alpha_1, \alpha_1 - \beta_1), (\alpha_2, \alpha_2 - \beta_2)\}$. Up to homotopy equivalence, the manifolds are classified by the set of integers $\{\alpha_1, \alpha_2\}$. Each of these manifolds can be obtained by removing a small open tube about a simple closed curve on a genus one Heegaard surface in a Lens space. Hence, the term "torus knot." I prefer to use the term "homotopy torus knot" to distinguish them (topologically) from the classical torus knots in S^3 (see for example Exercise 13.11 in [He_1]).

In each of the examples X.12 through X.14 there is an essential, embedded annulus in the manifold. A quick consideration of some homotopy equivalences in each of these examples, and the obstruction to deforming them to homeomorphisms, exhibits exotic behavior near such an annulus. This is particularly evident in Examples X.13 and X.14. It is fair to say that such behavior is the only exotic behavior. This is formalized in the next theorem, which follows from the work of K. Johannson [Jo_2]. The elegant and short proof given here is due to A. Swarup.

X.15. THEOREM (Johannson [Jo_2]). _Let_ M _and_ M' _be Haken-manifolds with incompressible boundary. Suppose that_ $f : M \to M'$ _is a map such that_ $f_*: \pi_1(M) \to \pi_1(M')$ _is an isomorphism. If_ M _does not contain an embedded, essential annulus, then there is a homotopy_ $f_t: M \to M'$ _such that_ $f_0 = f$ _and_ $f_1: M \to M'$ _is a homeomorphism._

Proof: By Corollary X.8, it is sufficient to prove that f can be deformed to a boundary preserving map, $f : (M,\partial M) \to (M',\partial M')$.

Let B be a component of ∂M. Since M does not contain an embedded, essential annulus, it follows from the Annulus Theorem, VIII.13, that there is no nondegenerate map of $(S^1 \times I, S^1 \times \partial I)$ into $(M,\partial M)$. Hence the characteristic pair factor of $(M,\partial M)$ is empty. So, by Proposition VII.24, $e(\pi_1(M),\pi_1(B)) = 1$. Since f_* is an isomorphism, $e(\pi_1(M'),f_*(\pi_1(B))) = 1$; and so, by Proposition VII.22, the subgroup $f_*(\pi_1(B))$ is peripheral in $\pi_1(M')$. The deformation of f can now be established, since for each component B of ∂M the map $f|B$ is homotopic to a map taking B into a component of $\partial M'$. ∎

Using the characteristic Seifert pair, the analysis of homotopy equivalences between Haken-manifolds with incompressible boundary has been described by Johannson [Jo_2]. I shall give a new proof here based on generalizations of Propositions VII.22 and VII.24.

Before stating the theorem and giving its proof, I need some terminology and notation.

Let (X, Y) be a 3-manifold pair. Let W be a two-sided surface in X with $\partial W \subset \text{Int } Y$. The surface W is parallel into Y if there exists an embedding $h : W \times I \longrightarrow X$ such that $h|W \times \{0\}: W \times \{0\} \longrightarrow W$ is a homeomorphism and $h|(\partial W \times I) \cup (W \times \{1\}):(\partial W \times I) \cup (W \times \{1\}) \longrightarrow Y$ is an embedding. The surface W is parallel into $\overline{(\partial X - Y)}$ if there exists an embedding $h : W \times I \longrightarrow X$ such that $h|W \times \{0\}: W \times \{0\} \longrightarrow Y$ is a homeomorphism, $h|\partial W \times I : \partial W \times I \longrightarrow Y$ and $h|W \times \{1\}: W \times \{1\} \longrightarrow \overline{(\partial X - Y)}$ is an embedding. The 3-manifold pair (X, Y) is simple if every incompressible annulus or torus W in X with $\partial W \subset \text{Int } Y$ is either parallel into Y or parallel into $\overline{(\partial X - Y)}$.

X.16. REMARKS. (a) The 3-manifold M is simple iff the pair $(M, \emptyset)$ is simple; i.e., each incompressible torus W in M is parallel into ∂M. This is precisely the definition given in Chapter IX just prior to Remark IX.19.

(b) The 3-manifold pair $(M, \partial M)$ is simple iff there are no essential, embedded annuli or tori in M. This is sometimes called atoroidal in the literature.

(c) If the 3-manifold pair (M, T) is simple, then M is simple.

I need the following generalization of Corollary IX.20.

X.17. LEMMA: Let M be a Haken-manifold that is closed or has incompressible boundary. Let (Σ, Φ) be the characteristic Seifert pair for M. Let X be a component of $\overline{(M - \Sigma)}$ and set $Y = X \cap \partial M$. The 3-manifold pair (X, Y) is simple.

Proof: This result follows immediately from Theorem IX.15.

X.18. REMARK: If M is a Haken-manifold that is closed or has incompressible boundary and (Σ, Φ) is the characteristic Seifert pair for M, I call a component (σ, φ) of (Σ, Φ) a <u>Seifert factor of</u> M; and if X is a component of $\overline{(M - \Sigma)}$ and $Y = X \cap \partial M$, I call (X, Y) a <u>simple factor of</u> M. Both the simple factors of M and the Seifert factors of M are uniquely determined up to ambient isotopy of M.

The collection of simple factors of M is called the <u>characteristic simple pair of</u> M. Like the characteristic Seifert pair of M, the characteristic simple pair is unique up to ambient isotopy of M.

I shall describe some of the useful properties of both the Seifert factors and the simple factors of a Haken-manifold, shortly. However, I do point out now a common misunderstanding about the simple factors. It is often believed that a simple factor has no essential, embedded annuli. Quite often to the contrary, <u>a simple factor may have essential, embedded annuli</u>. But, if (X, Y) is a simple factor and there is an essential (rel ∂X), embedded annulus A in X, then the boundary of A must meet ∂Y in a nontrivial fashion. The typical example here can be obtained by starting with a 3-manifold M' having one Seifert factor that is a regular neighborhood of an essential, embedded annulus $A \subset M'$. Then let A' be an annulus in $\partial M'$ such that $\partial A'$ and ∂A can not be made disjoint with an ambient isotopy of M'. Let M be the manifold obtained by adding a solid torus T to M' by identifying an annulus t in ∂T, which has winding number greater than one, to A' in $\partial M'$. Now, the Seifert factor of M is $(T, \overline{(\partial T - t)})$ and the simple factor of M is $(M', \overline{(\partial M' - A')})$. By choice, the manifold M' contains the essential annulus A.

Each Seifert factor of M that meets ∂M contains an essential, embedded annulus in M; conversely, if M contains an essential, embedded annulus, then there is a Seifert factor of M that meets ∂M. If M is a Haken-manifold with incompressible boundary, the *peripheral characteristic Seifert pair of* M is the collection of Seifert factors of M that meet ∂M; i.e., the components of the characteristic Seifert pair of M that meet ∂M. *The peripheral characteristic Seifert pair of M is unique up to ambient isotopy of* M.

X.19. REMARK: If M is a Haken-manifold with incompressible boundary, then the following properties are equivalent:

(i) M does not contain an essential, embedded annulus.

(ii) The characteristic pair factor for $(M,\partial M)$ is empty.

(iii) The peripheral characteristic Seifert pair of M is empty.

(iv) $\pi_1(M)$ does not split as a nontrivial free product $A \underset{C}{*} B$ or an HNN group $A \underset{C}{*}$ where C is a cyclic group.

This immediately gives, as a corollary to Theorem X.15, the following result, which is a combination of the work of Waldhausen [W_3] and Johannson [Jo_2].

X.20. COROLLARY: *Let* M *be a Haken-manifold that is closed or has incompressible boundary. If it is known that* M *satisfies one of the properties* (i) *through* (iv) *of Remark* X.19, *then the topological type of* M *is completely determined by* $\pi_1(M)$. ■

X.21. THEOREM (Johannson [Jo_2]): *Let* M *and* M' *be Haken-manifolds with incompressible boundary. Let* (Λ,Ψ) *and* (Λ',Ψ') *denote the peripheral characteristic Seifert pairs for* M *and* M', *respectively. Suppose that* $f : M \to M'$ *is a homotopy equivalence. Then there is a homotopy* $f_t : M \to M'$ *such that*

(i) $f_0 = f$,

(ii) $f_1 | \overline{(M - \Lambda)}: \overline{(M - \Lambda)} \longrightarrow \overline{(M' - \Lambda')}$ is a homeomorphism, and

(iii) $f_1 | \Lambda : \Lambda \longrightarrow \Lambda'$ is a homotopy equivalence.

Proof: I shall first make some observations and notational conventions. The proof will be carried out by establishing a number of assertions. These assertions may be used as an "outline" of the proof.

X.22. OBSERVATIONS AND NOTATION: Let (Σ, Φ) (respectively, (Σ', Φ')) be the characteristic Seifert pair of M (respectively, M').

(1) Suppose that either $M = \Sigma$ or $M' = \Sigma'$. Say $M = \Sigma$. If $M = \Sigma$ is a Seifert fibered manifold, then, by Observation VI.11(a), the group $\pi_1(M)$ has a nontrivial cyclic normal subgroup. The map $f_*: \pi_1(M) \longrightarrow \pi_1(M')$ is an isomorphism; so, $\pi_1(M')$ has a nontrivial cyclic normal subgroup. Now, by Theorem VI.24, M' is a Seifert fibered manifold. Thus $M' = \Sigma'$ and there is nothing to prove. If M is an I-bundle, then since ∂M is irreducible, M is an I-bundle over a closed surface. It follows in this case (for example, see Theorem 10.5 of [He_1]) that M' is an I-bundle over a closed surface; so, $M' = \Sigma'$ and, again, there is nothing to prove.

(2) Suppose that either $\Lambda = \emptyset$ or $\Lambda' = \emptyset$. Say $\Lambda = \emptyset$. By the implication (iii) implies (iv) of Remark X.19, $\pi_1(M)$ does not split as a nontrivial free product $A \underset{C}{*} B$ or an HNN group $A \underset{C}{*}$, where C is a cyclic group. So, $\pi_1(M')$ has the same property; and now, by the implication (iv) implies (iii) of Remark X.19, $\Lambda' = \emptyset$. Thus, Theorem X.21 follows in this case from Theorem X.15.

(3) The Seifert factors of M (respectively, M') partition into three classes.

(i) The Seifert factors that have fundamental group isomorphic to $\mathbb{Z}$ are <u>tubes</u>; I denote this collection by $\mathcal{T}$ (respectively, $\mathcal{T}'$).

(ii) The Seifert factors that are S^1-pairs and <u>not</u> tubes; I denote this collection by $\mathcal{S}$ (respectively, $\mathcal{S}'$).

(iii) The Seifert factors that are I-pairs and not tubes; I denote this collection by $\mathcal{B}$ (respectively, $\mathcal{B}'$).

(By the preceding observations, I may assume that $M \neq \Sigma$ (respectively, $M' \neq \Sigma'$) and $\Lambda \neq \emptyset$ (respectively, $\Lambda' \neq \emptyset$); hence $\mathcal{B}$ (respectively, $\mathcal{B}'$) is the collection of I-pairs of (Σ, Φ) (respectively, $(\Sigma' \Phi')$) that are not S^1-pairs.)

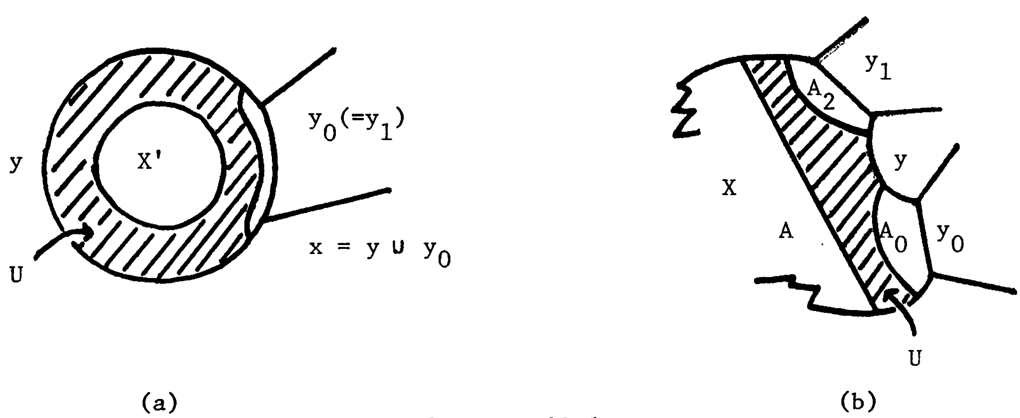
The next observation is very important and is used throughout the proof of Theorem X.21.

(4) <u>If</u> (X,Y) <u>is a simple factor of</u> M (M'), <u>where</u> $Y = X \cap \partial M$ $(Y = X \cap \partial M')$, <u>then either there is no component of</u> Y <u>that is an annulus or</u> Y <u>is the union of precisely two annuli and there is a pair isomorphism from</u> (X,Y) <u>to the product I-pair</u> $(S^1 \times I \times I, S^1 \times I \times \partial I)$.

<u>Proof of</u> (4): Suppose y is a component of Y and y is an annulus. Let $\partial_0 y$ and $\partial_1 y$ denote the components of ∂y. Let y_0 and y_1 be the components of $\operatorname{Fr} X$ (y_0 and y_1 are annuli) such that $\partial_i y \subset y_i$, $i = 0$ or 1. There are the two possibilities that either $y_0 = y_1$ or $y_0 \neq y_1$.

If $y_0 = y_1$ (see Schematic 10.1(a)), let x be the component of ∂X containing y. Set U equal to a small regular neighborhood of x minus an even smaller regular neighborhood of y_0. Consider the S^1-pair $(U, U \cap \partial X)$. The set $\operatorname{Fr} U$ contains an annulus, parallel into y_0, and a torus. The torus component must be incompressible, for otherwise the pair (X,Y) is an S^1-pair $(X = \mathbb{D}^2 \times S^1)$, perfectly embedded in M and not homotopic (as a pair) into (Σ, Φ). This contradicts Theorem IX.17. The torus component must not be essential in M; for otherwise, the pair (X,Y) is an S^1-pair $(X = S^1 \times S^1 \times I)$, perfectly embedded in M and not homotopic (as a pair) into (Σ, Φ). This contradicts Theorem IX.17. Now, the only possibility left is that the S^1-pair $(U, U \cap \partial X)$ is perfectly embedded in M and not homotopic (as a pair) into (Σ, Φ). This also contradicts Theorem IX.17. The conclusion is that no such annulus, y, exists.

If $y_0 \neq y_1$ (see Schematic 10.1(b)), then $y_0 \cup y \cup y_1$ is an annulus in ∂X. Set U equal to a small regular neighborhood of $y_0 \cup y \cup y_1$ in X minus even smaller regular neighborhoods of both y_0 and y_1 in X. Consider the S^1-pair $(U, U \cap \partial X)$. The set $\mathrm{Fr}\, U$ contains three annuli: the annulus A, parallel into $y_0 \cup y \cup y_1$; the annulus A_0, parallel into y_0; and the annulus A_1, parallel into y_1. If A is essential in M, then the pair $(U, U \cap \partial X)$ is a perfectly embedded S^1-pair in M and is not homotopic (as a pair) into (Σ, Φ). This contradicts Theorem IX.17. The conclusion then is that no such annulus, y, exists. So, the only possibility is that A is not essential. This gives the other possibility of (4); namely, $Y = y_0 \cup y_1$ and the pair (X,Y) is pair isomorphic to $(S^1 \times I \times I, S^1 \times I \times \partial I)$.



Schematic 10.1

(5) The simple factors of M (respectively, M') partition into three classes.

(i) The simple factors of M (respectively, M') that have fundamental group isomorphic to $\mathbb{Z}$ are simple tubes; i.e., if $(Q, Q \cap \partial M)$ is a simple factor of M and $\pi_1(Q) \approx \mathbb{Z}$, then the pair $(Q, Q \cap \partial M)$ is pair isomorphic to the product I-pair $(S^1 \times I \times I, S^1 \times I \times \partial I)$. Furthermore, each simple tube meets precisely two components of (Σ, Φ); one is an I-pair and not an S^1-pair, while the other is an S^1-pair and not an I-pair.

To see this, observe that Q is homeomorphic to a solid torus; hence, $Q \cap \partial M \neq \emptyset$ and each component of $Q \cap \partial M$ must be an annulus. The conclusion follows from Observation (4) above. I denote this collection by $\mathcal{Q}$ (respectively, $\mathcal{Q}'$).

(ii) The simple factors of M (respectively, M') that have fundamental group isomorphic to $\mathbb{Z} \times \mathbb{Z}$ are simple shells; i.e., if $(P, P \cap \partial M)$ is a simple factor of M and $\pi_1(P) \approx \mathbb{Z} \times \mathbb{Z}$, then the pair $(P, P \cap \partial M)$ is pair isomorphic to the product I-pair $(S^1 \times S^1 \times I, \emptyset)$. Furthermore, each simple shell meets only components of (Σ, Φ) that are S^1-pairs and not I-pairs. (I am assuming that $\partial M \neq \emptyset$; so, M is not a torus bundle over S^1.)

To see this, observe that P is homeomorphic to $S^1 \times S^1 \times I$. Hence, if $P \cap \partial M \neq \emptyset$, a component of $P \cap \partial M$ is either an annulus or a torus. The former cannot happen by Observation (4) above. The latter cannot happen since $\mathrm{Fr}\, P$ is essential. I denote this collection by $\mathcal{P}$ (respectively, $\mathcal{P}'$).

(iii) The simple factors of M (respectively, M') that have non-abelian fundamental group. I denote this collection by $\mathcal{N}$ (respectively, $\mathcal{N}'$).

I shall first prove that <u>there is a homotopy</u> $f_t: M \longrightarrow M'$ <u>such that</u>

(i) $f_0 = f$,

(ii) $f_1|\overline{(M - \Sigma)}:\overline{(M' - \Sigma')} \to (M' - \Sigma')$ <u>is a homeomorphism</u>, and

(iii) $f_1|\Sigma : \Sigma \longrightarrow \Sigma'$ <u>is a homotopy equivalence</u>.

As I stated above, this is done by establishing a number of assertions. As I establish assertions, it is to be implicitly understood that each of the preceding assertions remains true; if I state an assertion about f, then it is to be implicitly understood that the same is true about f' (with the appropriate use of notation), where f' is the homotopy inverse of f (and vice-versa); and to avoid too much notation, after each deformation, I continue to call the deformed map f (respectively, f').

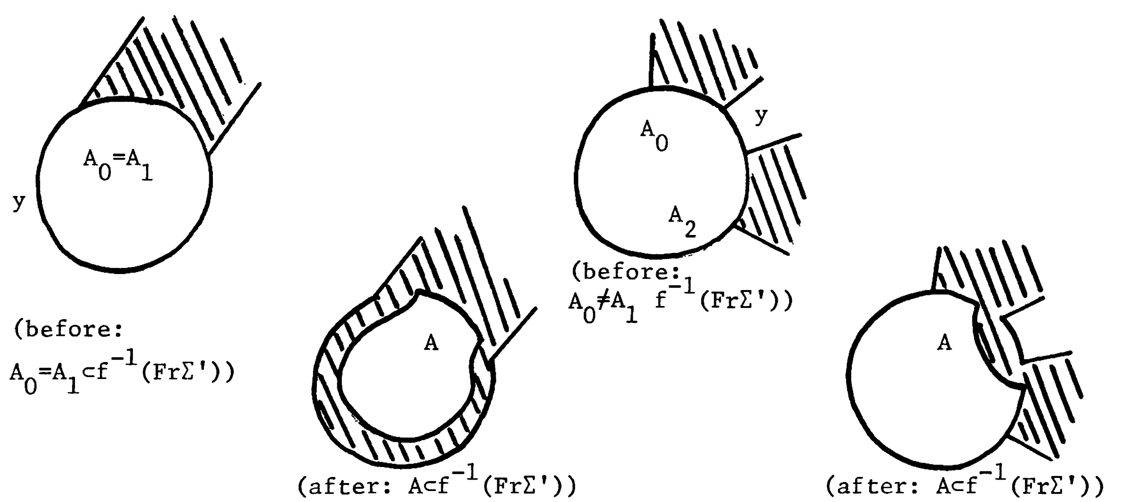
ASSERTION 1. <u>There is a deformation such that each component of</u> $f^{-1}(\mathrm{Fr}\Sigma')$ <u>is an essential, incompressible embedded annulus or torus and the number of components of</u> $f^{-1}(\mathrm{Fr}\Sigma')$ <u>that are annuli is as small as possible</u>.

Proof of Assertion 1: It follows from Lemma III.9 that there is a deformation such that each component of $f^{-1}(\mathrm{Fr}\Sigma')$ is incompressible; hence, a component of $f^{-1}(\mathrm{Fr}\Sigma')$ is either a disk, or incompressible annulus or an incompressible torus. Choose f such that $f^{-1}(\mathrm{Fr}\Sigma')$ is incompressible and the number of components of $f^{-1}(\mathrm{Fr}\Sigma')$ that are annuli is as small as possible, say $f^{-1}(\mathrm{Fr}\Sigma'$ has n annuli. Now, among all deformations of f such that $f^{-1}(\mathrm{Fr}\Sigma')$ is incompressible and the number of components of $f^{-1}(\mathrm{Fr}\Sigma')$ that are annuli is precisely n, choose f such that the total number of components of $f^{-1}(\mathrm{Fr}\Sigma')$ is as small as possible. Such an f satisfies the desired properties in

Assertion 1.

ASSERTION 2. *It follows that if* y *is the closure of a component of* $\partial M - f^{-1}(\mathrm{Fr}\Sigma')$ *and* y *is an annulus, then the map* $f|y:(y,\partial y) \to (M',\mathrm{Fr}\Sigma')$ *is essential; and therefore, nondegenerate.*

Proof of Assertion 2.: Suppose that y is the closure of a component of $\partial M - f^{-1}(\mathrm{Fr}\Sigma')$ and y is an annulus. Then there exists annuli A_0 and A_1 in $f^{-1}(\mathrm{Fr}\Sigma')$ such that $y \cap A_i = y \cap \partial A_i = \partial_i y$ is a component of ∂y for $i = 0$ and 1. It is possible that $A_0 = A_1$. If $A_0 = A_1$, then there is a deformation moving points only in a neighborhood of y in M such that the annulus component $A_0 = A_1$ of $f^{-1}(\mathrm{Fr}\Sigma')$ is replaced by a torus component A (see Schematic 10.2(a)). If $A_0 \neq A_1$, then there is a deformation moving points only in a neighborhood of y in M such that the annuli A_0 and A_1 of $f^{-1}(\mathrm{Fr}\Sigma')$ are replaced by a single annulus component A (see Schematic 10.2(b)). In either case, there is a contradiction to the choice of f in Assertion 1.



(a) (b)

Schematic 10.2

ASSERTION 3. <u>There is a deformation such that if</u> X <u>is a component of</u> $f^{-1}(\Sigma')$ <u>and</u> $Y = X \cap \partial M$, <u>then the pair</u> (X,Y) <u>is a perfectly-embedded Seifert pair</u>. (In fact, if (σ',φ') is a component of (Σ',Φ') and X is a component of $f^{-1}(\sigma')$, then for (σ',φ') an I-pair and <u>not</u> an S^1-pair, the pair (X,Y) is a perfectly-embedded I-pair and <u>not</u> an S^1-pair and for (σ',φ') an S^1-pair, the pair (X,Y) is a perfectly-embedded S^1-pair.)

Proof of Assertion 3: Let (σ',φ') be a component of (Σ',Φ') and let X be a component of $f^{-1}(\sigma')$. Set $Y = X \cap \partial M$.

Suppose (σ',φ') is an I-pair and <u>not</u> an S^1-pair. I shall show that the pair (X,Y) is an I-pair and <u>not</u> an S^1-pair. Since (σ',φ') is an I-pair there is a section W of the I-bundle σ' such that $(W,\partial W) \subset (\sigma',\varphi')$ where W is a compact, incompressible surface with $\partial W \neq \emptyset$ (Observation X.22(1)) and the inclusion $\pi_1(W) \hookrightarrow \pi_1(\sigma')$ is an isomorphism. Let y be a component of Y. The map $f|y$ is (up to homotopy) a map $f|y:(y,\partial y) \to (W,\partial W)$ and $(f|y)_*$ is an injection. Hence, ((see for example Theorem 13.1 of [He_1]) there is a homotopy $g_t:(y,\partial y) \to (W,\partial W)$ such that $g_0 = f|y$ and either $g_1:y \to W$ is a covering map or y is an annulus and $g_1(y) \subset \partial W$. By Assertion 2, the latter situation cannot happen. It follows that up to homotopy $f|y:y \to \partial W$ is a covering map; so, $(f|y)_*(\pi_1(y))$ has finite index in $\pi_1(\sigma')$. Therefore, $\pi_1(y)$ has finite index in $\pi_1(X)$. It follows (see e.g. Theorem 10.5 of [He_1]) that either X is an I-bundle with y a component of the corresponding ∂I-bundle or $\pi_1(X) \approx \mathbb{Z}$. But, now this latter situation can not happen; since, $(f|y)_*\pi_1(y)$ is of finite index in $\pi_1(W)$ and if $\pi_1(y)$ is cyclic, then $\pi_1(W)$ would be cyclic. This contradicts the assumption that (σ',φ') is an I-pair and <u>not</u> an S^1-pair. Since y is an arbitrary component of Y, I have that (X,Y) is an I-pair and not an S^1-pair. (Of course, if $\pi_1(X) \approx \mathbb{Z}$, I could conclude that (X,Y) is a Seifert pair; however, the more accurate observation of Assertion 3 saves work later.)

Suppose (σ',φ') is an S^1-pair. Here I need to use that I have a homotopy equivalence. (I do not know if the first sentence of the assertion is false if, say, I only have an injection on π_1; but I do know that this weaker hypothesis allows I-pairs that are not S^1-pairs to be in the inverse image of an S^1-pair.)

First, if $\pi_1(\sigma')$ is infinite cyclic, then $\pi_1(X)$ is infinite cyclic and the pair (X,Y) is an S^1-pair. So, suppose that $\pi_1(\sigma')$ is not cyclic.

Now, $f'|\sigma':\sigma' \to M$ and $(f'|\sigma')_*$ is an injection. It follows that $f'|\sigma':(\sigma',\phi) \to (M,\partial M)$ is a nondegenerate map. By Theorem IX.17 the map $f'|\sigma'$ is homotopic to a map $g:(\sigma',\phi) \to (M,\partial M)$ such that $g(\sigma') \subset \Sigma$. Say, $g(\sigma') \subset \sigma$, where (σ,φ) is a component of (Σ,Φ). Since f' is the homotopy inverse of f, it follows that X homotopes into σ (however, the pair (X,Y) need not homotop, as a pair, into (σ,φ)). The covering space, $\tilde{M}_\sigma$, of M corresponding to $\pi_1(\sigma)$ compactifies (by Theorem VII.9) to a Seifert fibered manifold (by Theorem VI.24), say S_σ, and the pair (X,Y) lifts to $\tilde{M}_\sigma$. Denote the lifting of X to $\tilde{M}_\sigma$ by $\tilde{X}$. The components of $\mathrm{Fr}\tilde{X}$ may not be essential in the compactified manifold, S_σ. However, by checking the possibilities, for $\mathrm{Fr}\tilde{X}$ in S_σ, from the statement of Theorem VI.34, I at least can conclude that $\tilde{X}$, and therefore X, is a Seifert fibered manifold. It remains for me to show that Y is saturated in some Seifert fibration of X. Since X is Seifert fibered and each component of $\mathrm{Fr}X$ is a component of $f^{-1}(\mathrm{Fr}\Sigma')$, each component of Y is either an incompressible annulus or an incompressible torus in ∂X. This is a start in showing that Y is saturated. The problem, of course, is in trying to prove that every component y of Y that is an annulus is saturated in the same Seifert fibration of X. By Assertion 2 and Theorem VI.20 each is saturated in some Seifert fibration of X and the only possible discrepancies occur when X is homeomorphic to either $\mathbb{D}^2 \times S^1$, $S^1 \times S^1 \times I$ or the twisted I-bundle over the Klein bottle. Since $\mathbb{D}^2 \times S^1$ and the twisted I-bundle over the Klein bottle have connected boundary, if X is homeomorphic to either of these, then (X,Y) is an S^1-pair. There is a real problem in the case that X is homeomorphic to $S^1 \times S^1 \times I$.

So, suppose that X is homeomorphic to $S^1 \times S^1 \times I$. Partition Y into two subsets $Y_0 = Y \cap S^1 \times S^1 \times \{0\}$ and $Y_1 = Y \cap S^1 \times S^1 \times \{1\}$. If either $Y_0 = \phi$ or $Y_1 = \phi$, then (X,Y) is an S^1-pair. If $Y_0 \neq \phi \neq Y_1$ and if the homotopy class in X represented by a component of Y_0 is the same as the homotopy class in X represented by a component of Y_1, then (X,Y) is an S^1-pair. I shall prove that there is a deformation, not moving points outside a small neighborhood of X, such that one of these two cases occurs.

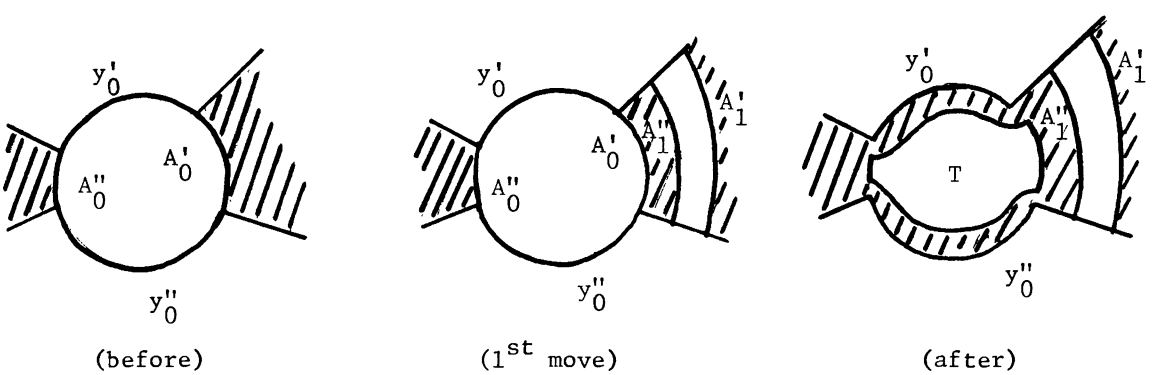
By Assertion 2, for each component y_i of $Y_i (i = 0 \text{ or } 1)$, the map $f|y_i : (y_i, \partial y_i) \to (\sigma', \varphi')$ is essential. So, by Theorem VI.20 and Observation VI.11(e), either the cyclic subgroup $(f|y_i)_*(\pi_1(y_i)) (i = 0,1)$ are in a unique normal, cyclic subgroup of $\pi_1(\sigma')$ or σ' is homeomorphic to one of $\mathbb{D}^2 \times S^1$, $S^1 \times S^1 \times I$ or the twisted I-bundle over the Klein bottle. But, if $(f|y_i)_*(\pi_1(y_i))$ $(i = 0,1)$ are in the same normal, cyclic subgroup of $\pi_1(\sigma')$, then the homotopy class in X represented by y_0 is the same as the homotopy class in X represented by y_1; and, I have proved that one of the above cases occurs. So, I must consider the situation where σ' is homeomorphic to one of $\mathbb{D}^2 \times S^1$, $S^1 \times S^1 \times I$ or the twisted I-bundle over the Klein bottle. Recall that X is homeomorphic to $S^1 \times S^1 \times I$; so σ' is not homeomorphic to $\mathbb{D}^2 \times S^1$. (Actually, it is not hard to show that σ' is not homeomorphic to the twisted I-bundle over the Klein bottle; however, this is not necessary.)

The situation is like this: σ' is homeomorphic to either $S^1 \times S^1 \times I$ or the twisted I-bundle over the Klein bottle, X is homeomorphic to $S^1 \times S^1 \times I$ and for each component y of Y the map $f|y:(y,\partial y) \to (\sigma',\mathrm{Fr}\sigma')$ is essential.

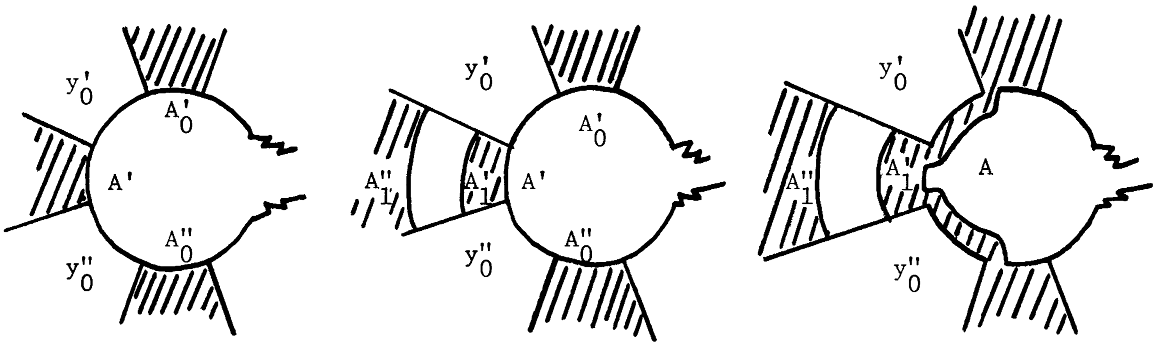
So, let W be a section in σ' such that $\pi_1(W) \hookrightarrow \pi_1(\sigma')$ is an isomorphism (i.e. if σ' is homeomorphic to $S^1 \times S^1 \times I$, then $W = S^1 \times S^1 \times \{\tfrac{1}{2}\}$; and if σ' is homeomorphic to the twisted I-bundel over the Klein bottle, then W is the natural embedding of the Klein bottle in σ'). The map $f|X$ can be deformed (rel $\mathrm{Fr}X$) such that $f^{-1}(W)$ is a collection of incompressible tori in X and incomressible two-sided annuli in X, where each annulus has its boundary in Y and each component of Y meets one and only one annulus in a single simple closed curve. Now, if any annulus of $f^{-1}(W)$ has one component of its boundary in y_0, a component of Y_0, and one component of its boundary in y_1, a component of Y_1, then the homotopy class in X represented by y_0 is the same as the homotopy class in X represented by y_1; and so, one of the above cases occurs. The only other possibility is that each annulus in $f^{-1}(W)$ has its boundary components in distinct components of Y_0 or in distinct components of Y_1. In this case both Y_0 and Y_1 must have an even number of components. I shall describe a deformation such that this situation does not occur; and hence, I will have that (X,Y) is an S^1-pair.

To this end, suppose that Y_0 has two components y_0' and y_0''. Then the component of ∂X corresponding to $S^1 \times S^1 \times \{0\}$, in the chosen parametrization, can be written as the union of four annuli: y_0', y_0'' and two annuli A_0' and A_0'' in $\mathrm{Fr}X \subset f^{-1}(\mathrm{Fr}\Sigma')$ (see Schematic 10.3(a)). So, by a deformation, only moving points in a neighborhood of A_0', I can shove $f|A_0'$ through σ' so that both the maps $f|y_0':(y_0',\partial y_0') \to (\sigma',\mathrm{Fr}\sigma')$ and $f|y_0'':(y_0'',\partial y_0'') \to (\sigma',\mathrm{Fr}\sigma')$ are inessential. The result is that a tube, U, in a neighborhood of A_0' in M appears in $f^{-1}(\sigma')$ and $\mathrm{Fr}U$ consists of two annuli A_1' and A_1''. Now, the number of annuli in $f^{-1}(\mathrm{Fr}\Sigma')$ has increased by two; however, both maps $f|y_0'$ and $f|y_1'$ are not essential. It follows, as above (see Schematic 10.2), that by a further deformation of f, only moving points in a neighborhood of $y_0' \cup y_0''$, the annuli components A_0' and A_0'' in $f^{-1}(\mathrm{Fr}\Sigma')$ are exchanged for the annuli components A_1' and A_1'' and the torus component T (see Schematic 10.3(a)). The number of annuli in $f^{-1}(\mathrm{Fr}\Sigma')$ has not changed and all previous assertions are clearly still satisfied (Assertion 2 only being dependent on the fact that the number of components of $f^{-1}(\mathrm{Fr}\Sigma')$ that are annuli is as small as possible). The new pairs in $f^{-1}(\sigma')$ obtained are S^1-pairs.

If Y_0 has more than two components, the same type of deformations can be made so that by reasoning inductively on the number of components of Y_0, I obtain new pairs in $f^{-1}(\sigma')$ each of which is an S^1-pair (see Schematic 10.3(b)).



(a)



(before) (1^{st} move) (after)

(b)

Schematic 10.3

ASSERTION 4. <u>There is a deformation such that</u> $f(\overline{(M-\Sigma)}) \subset \overline{(M'-\Sigma')}$.

Proof of Assertion 4. It follows from Assertion 3 that if X is a component of $f^{-1}(\Sigma')$ and $Y = X \cap \partial M$, then the pair (X,Y) is a perfectly embedded Seifert pair. So, by Theorem IX.17, there is an ambient isotopy of M taking (X,Y) into (Σ,Φ). Therefore, there is an ambient isotopy of M such that $f^{-1}(\Sigma') \subset \Sigma$. Hence, $f(\overline{(M-\Sigma)}) \subset \overline{(M'-\Sigma')}$.

It is easier to express the next three assertions if I introduce some terminology. Let $f: M \to M'$ be a homotopy equivalence between the 3-manifolds M and M'. Suppose that F' is a two-sided 2-manifold embedded in M'. The homotopy equivalence f splits at F' if

(i) f is transverse on F' (set $F = f^{-1}(F')$),

(ii) $f|F:F \to F'$ is a homotopy equivalence and

(iii) $f|M-F:M-F \to M'-F'$ is a homotopy equivalence.

ASSERTION 5. There is a deformation such that f splits at $\mathrm{Fr}\mathcal{N}'$.

Proof of Assertion 5: Recall that $\mathcal{N}$ (respectively, $\mathcal{N}'$) is the collection of nonabelian simple factors of M (respectively, M').

Let N be a component of $\mathcal{N}$. By Assertion 4, there is a component N' of $\overline{(M'-\Sigma')}$ such that $f(N) \subset N'$. Since N is a nonabelian simple factor of M and $f|N$ induces an injection on $\pi_1(N)$, the component N' has nonabelian fundamental group. Hence $N' \in \mathcal{N}'$. Similarly, there is a component $\tilde{N}$ of $\mathcal{N}$ such that $f'(N') \subset \tilde{N}$. (Notice that this is the first time that I have used the implicit fact that the assertions apply equally well to f and f'.) I claim that $\tilde{N} = N$.

For, assume that $\tilde{N} \neq N$. The map $f' \circ f \simeq \mathrm{id}_M$ and $(f' \circ f)(N) \subset \tilde{N}$; hence, N is homotopic into $\tilde{N}$. By Theorem VII.9 the covering space $\tilde{M}_{\tilde{N}}$ of M corresponding to the conjugacy class of $\pi_1(\tilde{N})$ in $\pi_1(M)$ admits a manifold compactification to a Haken-manifold. Since N homotops into $\tilde{N}$, the manifold N lifts to $\tilde{M}_{\tilde{N}}$. If $N \neq \tilde{N}$, then $N \cap \tilde{N} = \phi$; so, the lifting of N to $\tilde{M}_{\tilde{N}}$ shows that N homotops into $\mathrm{Fr}\tilde{N}$. But if this is the case, then $\pi_1(N)$ must be abelian. Thus the assumption that $N \neq \tilde{N}$ leads to a contradiction.

Now, consider the components of $f^{-1}(N')$. If C is a component of $f^{-1}(N')$ and $C \cap N = \phi$, then, as above, C homotops into $\mathrm{Fr}N$; so, $\pi_1(C)$ must be abelian. It follows that C is either a tube $(C \approx \mathbb{D}^2 \times S^1)$ or a shell $(C \approx S^1 \times S^1 \times I)$. Set $D = C \cap \partial M$.

I claim that no component of D is an annulus. For, assume that d is a component of D and d is an annulus. By Assertion 2 the map $f|d:(d,\partial d) \to (N',FrN')$ is essential. Since $(N',N' \cap \partial M')$ is a simple pair, either N' is homeomorphic to $(S^1 \times S^1 \times I)$ and $N' \cap \partial M' = \phi$ or $f|d:(d,\partial d) \to (N',FrN')$ is homotopic as a map of pairs to a map taking d into $N' \cap \partial M'$. The former cannot happen, since N' is a nonabelian simple factor. On the other hand, the latter implies that some component of $N' \cap \partial M$ is an annulus. By Observation X.22(4) this only happens if N' is a simple tube. Thus both possibilities contradict the fact that N' is a nonabelian simple factor. This establishes the claim.

Now, I claim that no component of D is a torus. For assume that d is a component of D and d is a torus. In this case C is a shell; and since no component of D is an annulus either $D = \partial C$ or $D = d$. The possibility that $D = \partial C$ contradicts Observation X.22(1); and the possibility that $D = d$ contradicts Assertion 1 (for if $D = d$ then some component of $f^{-1}(Fr\Sigma')$ is not essential). These contradictions establish this claim.

The conclusion, for C and D as above, is that $D = \phi$ and C is a simple shell $(C \approx S^1 \times S^1 \times I)$. Now, I shall show that there is a deformation removing C from $f^{-1}(N')$. The map $f|C:(C,\partial C) \to (N',FrN')$ is boundary preserving and $(f|C)_*$ is an injection on $\pi_1(C)$. By Theorem X.7 either $f|C$ is a covering map onto N' or $f|C$ is homotopic (rel ∂C) to a map taking C into FrN'. In this situation the former case cannot happen; for, then either N' has abelian fundamental group or N' is the twisted I-bundle over the Klein bottle. The first contradicts the fact that N' is a nonabelian simple factor and the second contradicts the fact that a simple factor cannot be Seifert fibered unless it is a simple tube or a simple shell. Now, if $f|C$ is homotopic (rel ∂C) to a map taking C into FrN', then there is a deformation of f, not moving points outside a neighborhood of C, removing C from $f^{-1}(N')$. The conclusion is that, (possibly after a deformation) I have $\tilde{N} = f^{-1}(N')$ connected and $N \subset \tilde{N}$. I claim that there is a further deformation such that $\tilde{N} = f^{-1}(N')$ is a regular neighborhood of N in M and $\tilde{N} \cap \partial M$ is a regular neighborhood of $N \cap \partial M$ in ∂M.

First, I may assume (possibly after a small deformation) that $N \cap Fr\tilde{N} = \phi$. Let F be a component of $Fr\tilde{N}$. If F is a torus and C is the component of $\tilde{N} - \mathring{N}$ containing F, then C is a shell ($C \approx S^1 \times S^1 \times I$). In this case, it follows, as above, that $C \cap \partial M = \phi$. So, C is a coproduct from F to a component of FrN. If F is an annulus and C is the component of $\tilde{N} - \mathring{N}$ containing F, then C is a tube ($C \approx \mathbb{D}^2 \times S^1$). In this case, I claim that C is a co-product from F to an annulus component of FrN. For, otherwise, there exists an annulus component d of $C \cap \partial M$ with $\partial d \subset f^{-1}(FrN')$; and, as above, the essential map $f|d:(d,\partial d) \to (N',FrN')$ gives a contradiction.

Having established the preceding claim, it follows that there is a deformation such that $N = f^{-1}(N')$; and so, $FrN = f^{-1}(FrN')$. Now, $f|N:N \to N'$ (and $f'|N':N' \to N$). By lifting the homotopy in M between the map $f'|N' \circ f|N$ and id_N to $\tilde{M}_N$, it follows that, $f|N:N \to N'$ is a homotopy equivalence with homotopy inverse $f'|N'$. Furthermore, $f|FrN:FrN \to FrN'$ is a homotopy equivalence with homotopy inverse $f'|FrN'$. Since N is an arbitrary component of $\mathcal{N}$ and each of the above deformations does not move points close to components of $\mathcal{N}$ distinct from N, the desired conclusions follow.

ASSERTION 6. <u>There is a deformation such that</u> f <u>splits at</u> $\mathrm{Fr}(\mathcal{H}' \cup \mathcal{P}')$.

Proof of Assertion 6: Recall that $\mathcal{P}$ (respectively, $\mathcal{P}'$) is the collection of simple factors of M (respectively, M') that have $\mathbf{Z} + \mathbf{Z}$ fundamental group; hence, each component of $\mathcal{P}$ (respectively, $\mathcal{P}'$) is a simple shell. The argument in this case is similar to the argument used in the proof of Assertion 5.

Let P be a component of $\mathcal{P}$. By Assertion 4 $f(P)$ is contained in a component of $\overline{(M'-\Sigma')}$. After Assertion 5, the only possibility is that $f(P)$ is contained in a component of $\mathcal{P}'$ or $\mathcal{Q}'$. Since each component of $\mathcal{Q}'$ has cyclic fundamental group, the only possibility is that there exists a component P' of $\mathcal{P}'$ and $f(P) \subset P'$. Similarly there exists a component $\tilde{P}$ of $\mathcal{P}$ and $f'(P') \subset \tilde{P}$. I claim that $\tilde{P} = P$.

For assume that $\tilde{P} \neq P$. The map $f' \circ f \simeq \mathrm{id}_M$ and $(f' \circ f)(P) \subset \tilde{P}$; hence, P is homotopic into $\tilde{P}$. Using a covering space argument, as in the analogous situation for Assertion 5, I conclude that the simple shells P and $\tilde{P}$ are separated in M by a shell component (σ,ϕ) of (Σ,Φ). However, by Observation X.22(5) part (ii), each component of Σ meeting either P or P' is an S^1-pair. But this allows (σ,ϕ) to homotop into $(\Sigma-\sigma,\Phi)$, which contradicts that (Σ,Φ) is the characteristic Seifert pair of M. The assumption $\tilde{P} \neq P$ leads to a contradiction; so $\tilde{P} = P$.

Now, consider the components of $f^{-1}(P')$. If C is a component of $f^{-1}(P')$, then C is a tube or a shell. Set $D = C \cap \partial M$.

I claim that $D = \phi$. First, assume that d is a component of D and d is an annulus. (The argument here is a bit more subtle than in the similar situation in the proof of Assertion 5. In fact, the situation is much like the horrible situation encountered in the proof of Assertion 3;

fortunately, here there is a lot more to use.) By Assertion 2, the map $f|d:(d,\partial d) \to (P',\mathrm{Fr}P')$ is essential. So, there exists annuli A_0 and A_1 in FrC such that $A_i \cap d = \partial_i d$ $(i = 0,1)$, where $\partial_i d$ is a component of ∂d. Notice that $A_0 \neq A_1$. By Observation X.22 (5) part (ii) each component of (Σ',Φ') meeting P' is an S^1-pair. Index the S^1-pairs meeting P' by (σ_0',φ_0') and (σ_1',φ_1') so that $f|d:\partial_i d \to \mathrm{Fr}\sigma_i'$, $i = 0,1$. Now, I apply the refined statement of Assertion 3 to observe that A_0 and A_1 are on the frontiers of perfectly-embedded S^1-pairs in $(M,\partial M)$. So, there exist distinct annuli d_0 and d_1 in $\overline{(}\partial M - f^{-1}(\mathrm{Fr}\Sigma')$ such that $f|d_i:(d_i,\partial d_i) \longrightarrow (\sigma_i', \mathrm{Fr}\sigma_i')$, $i = 0,1$, are essential maps. So, $f(\partial d_i)$ maps to a power of a fiber in some Seifert fibration of σ_i'. The connecting annulus d between d_0 and d_1, implies that the Seifert fibration of (σ_0',φ_0') extends over P' to the Seifert fibration of (σ_1',φ_1'). This contradicts the fact that (Σ',Φ') is the characteristic Seifert pair of M'.

The assumption that d is a component of D and d is a torus leads to a contradiction just as in the argument in the proof of Assertion 5. The only possibility is that $D = \phi$ and C is a simple shell.

The map $f|C:(C,\partial C) \to (P',\partial P')$ is boundary preserving and $(f|C)_*$ is an injection. By Theorem X.7 either $f|C$ homotops (as a map of pairs) to a covering map or $f|C$ homotops (as a map of pairs) to a map taking C into FrP'. If the former occurs, then by arguing as above (using Observation X.22 (5) part (ii)) there is a contradiction to (Σ',Φ') being the characteristic Seifert pair of M'. If the latter occurs, then there is a deformation removing C from $f^{-1}(P')$.

I conclude that $\tilde{P} = f^{-1}(P')$ is connected and $P \subset \tilde{P}$. Again, as above, no component of $\mathrm{Fr}\tilde{P}$ can be an annulus; so, there is a deformation such that $\tilde{P} = f^{-1}(P')$ is a regular neighborhood of P. Now, as in Assertion 5, there is a deformation so that f splits at $\mathrm{Fr}\mathcal{P}'$. This, together with Assertion 5, implies that the map f splits at $\mathfrak{N}' \cup \mathfrak{P}'$.

ASSERTION 7. <u>There is a deformation such that</u> f <u>splits at</u> $\mathrm{Fr}\Sigma'$.

Proof of Assertion 7: Recall that $\mathfrak{Q}$ (respectively, $\mathfrak{Q}'$) is the collection of simple factors of M (respectively, M') that have infinite cyclic fundamental group; hence, by Observation X.22 (5) part (i), a component of $\mathfrak{Q}$ (respectively, $\mathfrak{Q}'$) is a simple tube.

I need to show that f splits at $\mathrm{Fr}(\mathfrak{N}' \cup \mathfrak{P}' \cup \mathfrak{Q}')$. By Assertions 5 and 6, it is sufficient to show that f splits at $\mathfrak{Q}'$.

The argument here is not parallel to the arguments given in Assertions 5 and 6. Indeed, there is a good deal of control on the situation at this point.

First, I make an observation; I shall label it in sequence with the earlier observations (Observations X.22 (1) through (5)).

(6) <u>If</u> $\tilde{\tilde{Q}}$ <u>is a component of</u> $f^{-1}(\mathfrak{Q}')$, y <u>is a component of</u> $\tilde{\tilde{Q}} \cap \partial M$ <u>and</u> A_1 <u>and</u> A_2 <u>are distinct component of</u> $\mathrm{Fr}\tilde{\tilde{Q}}$ <u>such that</u> $\partial y \cap \partial A_i \neq \phi$, $i = 1,2$, <u>then notation may be chosen so that</u> $A_1 \subset \mathrm{Fr}X_1$, <u>where</u> X_1 <u>is a component of</u> $f^{-1}(\Sigma')$ <u>and for</u> $Y_1 = X_1 \cap \partial M$, <u>the pair</u> (X_1,Y_1) <u>is an I-pair and not an</u> S^1<u>-pair</u>, <u>and</u> $A_2 \subset \mathrm{Fr}X_2$, <u>where</u> X_2 <u>is a component of</u> $f^{-1}(\Sigma')$ <u>and for</u> $Y_2 = X_2 \cap \partial M$, <u>the pair</u> (X_2,Y_2) <u>is an</u> S^1<u>-pair</u>.

Proof of (6): Let $\tilde{\tilde{Q}}$ be a component of $f^{-1}(Q')$, where Q' is a component of $\mathfrak{Q}'$. By Observation X.22 (5) part (i), Q' is a simple tube and Q' meets precisely two components of (Σ',Φ'); one, say (σ_1',φ_1'), is an I-pair and <u>not</u> an S^1-pair, while the other, say (σ_2',φ_2'), is an S^1-pair and <u>not</u> an I-pair. Now, $\tilde{\tilde{Q}}$ is a tube. By Assertion 1, $\mathrm{Fr}\tilde{\tilde{Q}}$ is incompressible; and so, $\approx{Q} \cap \partial M \neq \phi$ and each component of $\approx{Q} \cap \partial M$ is an annulus. Let y be a component of $\approx{Q} \cap \partial M$. By Assertion 2, $f|y:(y,\partial y) \to (Q',\mathrm{Fr}Q')$ is essential; since the pair $(Q',Q'\cap\partial M')$ is a simple tube, it follows that there exists distinct components A_1 and A_2 of $\mathrm{Fr}\approx{Q}$ such that $\partial y = \partial_1 y \cup \partial_2 y$, where $\partial_1 y \subset A_1$ and $\partial_2 y \subset A_2$. Furthermore, notation may be chosen so that $A_i \subset \mathrm{Fr}X_i$, where X_i is the component of $f^{-1}(\sigma_i')$ that meets $\approx{Q}$ in A_i, $i = 1,2$. By the refined part of Assertion 3, if $Y_i = X_i \cap \partial M$, $i = 1,2$, then the pair (X_1,Y_1) is an I-pair and <u>not</u> an S^1-pair, while the pair (X_2,Y_2) is an S^1-pair. This establishes Observation (6).

Consider, the components of $f^{-1}(\mathfrak{Q}')$. If $\tilde{Q}$ is such a component, then <u>a priori</u> the possibilities for $\tilde{Q}$ are: $\tilde{Q}$ is contained in an I-pair of (Σ,Φ) that is <u>not</u> an S^1-pair, or $\tilde{Q}$ is contained in an S^1-pair of (Σ,Φ), or neither of the preceding (and hence, $\tilde{Q}$ contains a component Q of $\mathfrak{Q}$). I claim that the first two of these three possibilities contradicts Observation 6.

For suppose that $\tilde{Q} \subset \sigma$, where (σ,φ) is an I-pair of (Σ,Φ) that is <u>not</u> an S^1pair. Each component of $\mathrm{Fr}\tilde{Q}$ is an essential, embedded annulus; and so, each component of $\mathrm{Fr}\tilde{Q}$ is "vertical" in the I-bundle structure of σ; i.e., $\mathrm{Fr}\tilde{Q}$ can be isotoped so that it is saturated with respect to the given bundle projection for σ. On the other hand, there is a component Q' of $\mathfrak{Q}'$ such that $f(\tilde{Q}) \subset Q'$; and by Assertion 4, $f'(Q') \subset \overline{(M-\Sigma)}$. Since $f'\circ f \simeq \mathrm{id}_M$, it follows that $\tilde{Q}$ homotopes into $\overline{(M-\Sigma)}$. The conclusion is that for some component F of $\mathrm{Fr}\sigma$, $\tilde{Q}$ is "parallel" into F and is contained in some product neighborhood over F.

I claim that F is in $\mathrm{Fr}Q$ for some component Q of $\mathfrak{Q}$. The only other possibility is that F is in $\mathrm{Fr}N$ for some component N of $\mathfrak{N}$. But F is parallel into $\tilde{Q}$, which maps into Q' under f; so, by Observation X.22 (3) part (i) and the fact that f is a homotopy equivalence, it follows that

F freely homotopes into an S^1-pair in (Σ,Φ) that is <u>not</u> an I-pair. This is impossible if F is a component of $\mathrm{Fr}\sigma$, and a component of $\mathrm{Fr}N$, where (σ,φ) is an I-pair and <u>not</u> an S^1-pair and N is a component of $\mathcal{N}$.

Now, let $\tilde{\tilde{Q}}$ be the component of $f^{-1}(\mathrm{Fr}\mathcal{Q}')$ that contains Q (Q is the component of $\mathcal{Q}$ such that F is a component of $\mathrm{Fr}Q$). The existence of $\tilde{Q}$ as above forces each component of $f^{-1}(\Sigma')$ that meets $\tilde{\tilde{Q}}$ to be an S^1-pair This contradicts Observation (6).

Suppose that $\tilde{Q} \subset \sigma$ where (σ,φ) is an S^1-pair. Since each component of $\mathrm{Fr}\tilde{Q}$ is an essential, embedded annulus, it follows from Theorem VI.34 that each component of $\mathrm{Fr}\tilde{Q}$ is saturated in some Seifert fibration of σ; however, since (σ,φ) is an S^1-pair and each component of $\mathrm{Fr}\tilde{Q}$ has its boundary contained in φ, it follows that $\mathrm{Fr}\tilde{Q}$ is saturated in the given Seifert fibration of σ. So, each component of $f^{-1}(\Sigma')$ that meets $\mathrm{Fr}\tilde{Q}$ is an S^1-pair. This contradicts Observation (6). This contradiction and the preceding contradiction establish my claim that each component of $f^{-1}(\mathcal{Q}')$ contains a component of $\mathcal{Q}$. Similarly, each component of $f'^{-1}(\mathcal{Q})$ contains a component of $\mathcal{Q}'$.

The conclusion is that for each component Q' of $\mathcal{Q}'$, I have $\tilde{Q} = f^{-1}(Q')$ connected and there exists a component Q of $\mathcal{Q}$ such that $Q \subset \tilde{Q}$. I claim that there is a deformation such that $\tilde{Q} = f^{-1}(Q')$ is a regular neighborhood of Q in M and $\tilde{Q} \cap \partial M$ is a regular neighborhood of $Q \cap \partial M$ in ∂M. To see this, first observe that possibly after a small deformation, I may assume that $Q \cap \mathrm{Fr}\tilde{Q} = \phi$. If F is a component of $\mathrm{Fr}\tilde{Q}$ then F is an annulus. If C is the component of $\tilde{Q} - \mathring{Q}$ containing F, then C is a tube ($C \approx \mathbb{D}^2 \times S^1$) and, in fact, by Observation (6) C is a co-product from F to an annulus component of $\mathrm{Fr}Q$. This establishes the claim. It follows that there is a deformation such that $Q = f^{-1}(Q')$.

Similarly, if $\tilde{Q}$ is a component of $\mathcal{Q}$, there is a component Q' of $\mathcal{Q}'$ such that $Q' = f'^{-1}(\tilde{Q})$.

I claim that $\tilde{Q} = Q$. To see this recall (Observation X.22 (5) part (ii)) that for Q a component of $\mathcal{Q}$, then there exists an I-pair (σ,φ) of (Σ,Φ) that is not an S^1-pair and $\mathrm{Fr}Q \cap \mathrm{Fr}\sigma \neq \phi$. By Assertion 3, the I-pair (σ,φ) is mapped by f into an I-pair (σ',φ') of (Σ',Φ') that is *not* an S^1-pair. Similarly (σ',φ') is mapped by f' into an I-pair $(\tilde{\sigma},\tilde{\varphi})$ of (Σ,Φ) that is not an S^1-pair. Since $f'\circ f \simeq \mathrm{id}_M$, σ must homotop into $\tilde{\sigma}$. A covering space argument similar to those used above, gives that this is only possible if $\sigma = \tilde{\sigma}$. Furthermore, this rigidity forces $\tilde{Q} = Q$.

As in Assertions 5 and 6, there is a deformation such that f splits at $\mathcal{Q}'$; so, f splits at $\mathcal{N}' \cup \mathcal{P}' \cup \mathcal{Q}' = \mathrm{Fr}\Sigma'$.

ASSERTION 8. *There is a deformation such that* (i) $f|\overline{(M-\Sigma)}:\overline{(M-\Sigma)} \to (M'-\Sigma')$ *is a homeomorphism and*

(ii) $f|\Sigma:\Sigma \to \Sigma'$ *is a homotopy equivalence*.

Proof of Assertion 8: First, I shall prove that there is a deformation moving points only in a small neighborhood of $\mathcal{Q}$ such that $f|\mathcal{Q}:\mathcal{Q}\to\mathcal{Q}'$ is a homeomorphism. I can do this component at a time. So, let Q be a component of $\mathcal{Q}$. Then there exists a unique component Q' of $\mathcal{Q}'$ such that $f|Q:Q \to Q'$ is a homotopy equivalence with homotopy inverse $f'|Q'$. Since Q and Q' are simple tubes, they admit a parametrization as $S^1 \times I \times I$ so that their frontiers correspond to $S^1 \times \{0,1\} \times I$ and their meets with ∂M and $\partial M'$, respectively, correspond to $S^1 \times I \times \{0,1\}$. By Assertion 7, I may choose the parametrization so that $f|S^1 \times \{i\} \times I$: $S^1 \times \{i\} \times I \to S^1 \times \{i\} \times I$ is a homotopy equivalence for $i = 0,1$. I may deform f in a neighborhood of $S^1 \times \{0\} \times I$ so that $f|S^1 \times \{0\} \times I$: $S^1 \times \{0\} \times I \to S^1 \times \{0\} \times I$ is a homeomorphism. Having established this, I can deform f in a neighborhood of $S^1 \times \{1\} \times I$, using possibly a "flip" if necessary, to have $f|S^1 \times \{1\} \times I: S^1 \times \{1\} \times I \to S^1 \times \{1\} \times I$ a homeomorphism and $f|S^1 \times \{i\} \times \{j\}: S^1 \times \{i\} \times \{j\} \to S^1 \times \{i\} \times \{j\}$, $i = 0,1$ and $j = 0,1$. Using this and the fixed parametrization, I can deform f again, not moving points on $S^1 \times \{0,1\} \times I$, to obtain the desired homeomorphism.

Next, I shall prove that there is a deformation moving points only in a small neighborhood of $\mathcal{P}$ such that $f|\mathcal{P}: \mathcal{P} \to \mathcal{P}'$ is a homeomorphism. Hence, $f|\mathcal{P} \cup \mathcal{Q}: \mathcal{P} \cup \mathcal{Q} \to \mathcal{P}' \cup \mathcal{Q}'$ will be a homeomorphism. Again, I can do this component at a time. So, let P be a component of $\mathcal{P}$. Then there exists a unique component P' of $\mathcal{P}'$ such that $f|P:(P,\partial P) \to (P',\partial P')$ is a homotopy equivalence. Now, by Theorem X.7 and the fact that $P = f^{-1}(P'$ and f cannot be deformed so that $f^{-1}(P') = \phi$, the map f can be deformed in a neighborhood of P so that $f|P:P \to P'$ is a homeomorphism.

Now, I must prove that there is a deformation moving points only in a small neighborhood of $\mathcal{N}$ such that $f|\mathcal{N}: \mathcal{N} \to \mathcal{N}'$ is a homeomorphism. This is the most interesting part of this assertion (and of this proof); and establishing it will complete the proof of Assertion 8. To do this I need the promised generalization of Lemmas VII.22 and VII.24. I am probably overstating the similarities by calling what I do here a generalization of Lemmas VII.22 and VII.24. In fact, a generalization of VII.24 (to this relative situation) using the idea of ends of pairs of groups, is impossible (see Remark X.25). However, a method similar to the one that I use here does allow a generalization of VII.22, using the idea of ends of pairs of groups (see Exercies X.26).

As in the first two parts of the proof of this assertion, to show that there is a deformation moving points only in a small neighborhood of $\mathfrak{N}$ such that $f|\mathfrak{N}:\mathfrak{N} \to \mathfrak{N}'$ is a homeomorphism, I can do this component at a time. So, let N be a component of $\mathfrak{N}$. Then there exists a unique component N' of $\mathfrak{N}'$ such that $f|N:N \to N'$ is a homotopy equivalence with homotopy inverse $f'|N'$.

Let $T = N \cap \partial M$ and let $T' = N' \cap \partial M'$. The pairs (N,T) and (N',T') are Haken-manifold pairs such that no component of T or T' is an annulus. Now, by Assertion 5 (Assertion 7), $f|FrN:FrN \to FrN'$ is a homotopy equivalence with homotopy inverse $f'|FrN'$. So, after a deformation moving points only in a small neighborhood of FrN, I can assume that $f|FrN:FrN \to FrN'$ is a homeomorphism. Notice that $FrN = \overline{(\partial N-T)}$, $FrN' = \overline{(\partial N'-T')}$ and each component of $\overline{(\partial N-T)}$ and each component of $\overline{(\partial N'-T')}$ is either an incompressible annulus or torus. By Lemma X.17 the pairs (N,T) and (N',T') are simple. I formalize the situation that I have and show that there is the desired deformation of $f|N$ to a homeomorphism in the next lemma. First, however, I make one notational convenience.

Recall the "flip" of Example X.13. I shall use a homotopy version of it in the statement and proof of the next lemma. Namely, if A is an annulus parametrized as $A = S^1 \times I$, then I define the homotopy $\tau_s:S^1 \times I \to S^1 \times I$ as $\tau_s(x,r) = (x, (1-r)s + r(1-s))$, $0 \le s \le 1$.

X.23 LEMMA: *Suppose that* (N,T) *and* (N',T') *are Haken-manifold pairs such that no component of* T *or* T' *is an annulus and each component of* $\overline{(\partial N'-T')}$ *is either an incompressible annulus or torus. Suppose that* $f:(N,\overline{(\partial N-T)}) \to (N',\overline{(\partial N'-T')})$ *is a map of pairs such that* $f|\overline{(\partial N-T)}:\overline{(\partial N-T)} \to \overline{(\partial N'-T')}$ *is a homeomorphism. If* f *is a homotopy equivalence and* (N,T)

is a simple pair, then there exists a homotopy $f_s:(N,\overline{(\partial N-T)}) \to (N',\overline{(\partial N'-T')})$ $0 \le s \le 1$, *such that*

(i) $f_0 = f$,

(ii) f_1 *is a homeomorphism,*

and (iii) $f_s|\overline{(\partial N-T)}$ *is equal to* f *on each component that is not an annulus and is equal to* $f\circ\zeta_s$ *on each component that is an annulus, where* ζ_s *is either identity or* τ_s, $0 \le s \le 1$.

PROOF: Let t be a component of T. I shall first prove that there is a deformation (rel ∂t) of $f|t$ to a map taking t into $\partial N'$. Of course, it will then follow that there is a deformation (rel$\overline{(\partial N-T)}$) of f to a boundary preserving map; however, I will need more than this to complete the proof of Lemma X.23.

Let $\tilde{N}'$ be the covering space of N' corresponding to the conjugacy class of $f_*(\pi_1(t))$ in $\pi_1(N')$ and let $q:\tilde{N}' \to N'$ denote the covering projection. There is a lifting $\widetilde{f|t}:(t,\partial t) \to (\tilde{N}',\partial\tilde{N}')$ of $f|t$ (and $\widetilde{f|t}:\partial t \to \partial\tilde{N}'$ is an embedding). In order to prove that $f|t$ deforms (rel ∂t) to a map taking t into $\partial N'$, I shall prove that $\widetilde{f|t}$ deforms (rel ∂t) to a map taking t into $\partial\tilde{N}'$. I do this by constructing a particularly nice neighborhood of $\widetilde{f|t}(t)$ in $\tilde{N}'$, which is itself "parallel" into $\partial\tilde{N}'$.

To begin, let U be a regular neighborhood of $\widetilde{f|t}(t)$ in $\tilde{N}'$ and choose U so that $U \cap \partial\tilde{N}'$ is a regular neighborhood of $\widetilde{f|t}(\partial t)$ in $\partial\tilde{N}'$ (hence, a collection of pairwise disjoint annuli). Now, suppose that $\mathrm{Fr}U$ is not incompressible in $\tilde{N}'$. Then by the Loop Theorem (I.1) and Dehn's Lemma (I.6) there exist a disk $D \subset \tilde{N}'$ such that $D \cap F_rU = \partial D$ is a noncontractible curve in $\mathrm{Fr}U$. If $D \cap U = D \cap \mathrm{Fr}U = \partial D$, then add a two-handle to U with core D. The result is a neighborhood $\tilde{U}$ of $\widetilde{f|t}(t)$ such that $\pi_1(\tilde{U}) \hookrightarrow \pi_1(\tilde{N}')$ is an epimorphism and $\tilde{U} \cap \partial\tilde{N}' = U \cap \partial\tilde{N}'$ is a regular neighborhood of $\widetilde{f|t}(\partial t)$ in $\partial\tilde{N}'$. If $D \subset U$, then upto homotopy (rel ∂t) of $f|t$, I can assume that $D \cap \widetilde{f|t}(t) = \phi$. So, by a surgery on U at D, I obtain a neighborhood $\tilde{U}$ of $\widetilde{f|t}(t)$ such that $\pi_1(\tilde{U}) \hookrightarrow \pi_1(\tilde{N}')$ is an epimorphism and $U \cap \partial\tilde{N}' = U \cap \partial\tilde{N}'$ is a regular neighborhood of $\widetilde{f|t}(\partial t)$ in $\partial\tilde{N}'$. So, if I reason inductively on the Euler characteristic of $\mathrm{Fr}U$, there exists a neighborhood $\tilde{U}$ of $\widetilde{f|t}(t)$ such that $\pi_1(\tilde{U}) \hookrightarrow \pi_1(\tilde{N}')$ is an epimorphism, $\tilde{U} \cap \partial\tilde{N}' = U \cap \partial\tilde{N}'$ is a regular neighborhood of $\widetilde{f|t}(\partial t)$ in $\partial\tilde{N}'$, and $\mathrm{Fr}\tilde{U}$ is incompressible in $\tilde{N}'$. It follows that $\pi_1(\tilde{U}) \hookrightarrow \pi_1(\tilde{N}')$ is an isomorphism.

I claim that if $\tilde{L}$ is a component of $\mathrm{Fr}\tilde{U}$, then $\pi_1(\tilde{L}) \hookrightarrow \pi_1(\tilde{N}')$ is an isomorphism.

If $\tilde{L}$ is a component of $\mathrm{Fr}\tilde{U}$ and $\tilde{L}$ is closed, then $\pi_1(\tilde{U}) \approx \pi_1(\tilde{N}')$ is not a free group. So, t is a closed surface. It follows that $\pi_1(\tilde{U})$ is isomorphic to the fundamental group of a closed, orientable surface. Since $\tilde{U}$ is itself orientable, then, for example, by Theorem 10.5 of [He_1], $\tilde{U}$ is a product I-bundle and $\tilde{L}$ is a component of the associated ∂I-bundle. This proves that $\pi_1(\tilde{L}) \hookrightarrow \pi_1(\tilde{N}')$ is an isomorphism in the case that $\tilde{L}$ is closed.

If $\tilde{L}$ is a component of $\mathrm{Fr}\tilde{U}$ and $\tilde{L}$ is not closed, then $\partial\tilde{L}$ (up to isotopy in $\partial\tilde{N}'$) consists of a collection of curves in $\widetilde{f|t}(\partial t)$ (recall that $\tilde{U} \cap \partial\tilde{N}'$ is a regular neighborhood of $\widetilde{f|t}(\partial t)$ in $\partial\tilde{N}'$). Let K denote the curves of ∂t such that $\widetilde{f|t}(K)$ is homologous in $\partial\tilde{N}'$ to $\tilde{L}$. Then $\widetilde{f|t}(K)$ bounds in $\tilde{N}'$. The map $\widetilde{f|t}: t \to \tilde{N}'$ induces an isomorphism on homology; so, K bounds in t. It follows that $K = \partial t$. Now, let $\tilde{N}'(\tilde{L})$ be the covering space of $\tilde{N}'$ corresponding to the conjugacy class of $\pi_1(\tilde{L})$ in $\pi_1(\tilde{N}')$ and let $q': \tilde{N}'(\tilde{L}) \to \tilde{N}'$ denote the covering projection. Let $\tilde{t}$ be the covering space of t corresponding to the conjugacy class of $(\widetilde{f|t})_*^{-1}(\pi_1(\tilde{L}))$ in $\pi_1(t)$ and let $p:\tilde{t} \to t$ denote the covering projection. The map $\widetilde{f|t} \circ p$ lifts to $\widehat{\widetilde{f|t}}:\tilde{t} \to \tilde{N}'(\tilde{L})$ and since $(\widetilde{f|t})_*$ is an isomorphism, $\widehat{\widetilde{f|t}}(\tilde{t}) = (q')^{-1}(\widetilde{f|t})(t)$. The map $\widehat{\widetilde{f|t}}$ induces a homotopy equivalency so, in particular, $\widehat{\widetilde{f|t}}$ induces an isomorphism on homology. Since $\tilde{L}$ lifts to $\tilde{N}'(\tilde{L})$ and $K = \partial t$, it follows that ∂t lifts to $\tilde{t}$ and bounds in $\tilde{t}$. So, the degree of p is one and $(\widetilde{f|t})_*^{-1}(\pi_1(\tilde{L})) = \pi_1(t)$. This proves that $\pi_1(\tilde{L}) \hookrightarrow \pi_1(\tilde{N}')$ is an isomorphism in the case that $\partial\tilde{L} \neq \phi$; and thus in any case $\pi_1(\tilde{L}) \hookrightarrow \pi_1(\tilde{N}')$ is an isomorphism.

Since $\pi_1(\tilde{U}) \hookrightarrow \pi_1(\tilde{N}')$ is an isomorphism, each component of $\mathrm{Fr}\tilde{U}$ separates $\tilde{N}'$. I shall next prove that if $\tilde{L}$ is a component of $\mathrm{Fr}\tilde{U}$, then $\tilde{L}$ separates $\tilde{N}'$ into two components and one of them has compact closure. In order to do this, I have to bring information about t and N into play.

Let $\tilde{N}_t$ be the covering space of N corresponding to the conjugacy class of $\pi_1(t)$ in $\pi_1(N)$ and let $p:\tilde{N}_t \to N$ denote the covering projection. There is a component $\tilde{t}$ of $p^{-1}(t)$ such that $p|\tilde{t}:\tilde{t} \to t$ is a homeomorphism. By Corollary VII.8, $\tilde{N}_t$ admits a manifold compactification to the product $t \times I$ via a homeomorphism with $\tilde{t}$ corresponding to $t \times \{0\}$. I shall assume that such a parametrization of $\tilde{N}_t$ has been chosen so that $\tilde{N}_t$ corresponds to $t \times I$ with a closed subset of $t \times \{1\}$ missing and $\tilde{t}$ corresponds to $t \times \{0\}$. It follows that there is a lifting $\tilde{f}:\tilde{N}_t \to \tilde{N}'$ of $f \circ p$ such that $\tilde{f}|\tilde{t} = \widetilde{f|t} \circ p|\tilde{t}$ and $\tilde{f}$ is a proper homotopy equivalence.

Now, let $\tilde{L}$ be a component of $\mathrm{Fr}\tilde{U}$. Since $\tilde{f}$ is a proper map

$(\tilde{f})^{-1}(\tilde{L})$ is a compact 2-manifold. So, by Lemma III.9 there is a deformation (rel $\partial\tilde{N}_t$) (and so, in particular (rel $p^{-1}\overline{(}\partial N\text{-}T))$) such that $(\tilde{f})^{-1}(\tilde{L})$ is a compact, incompressible, two-sided 2-manifold. Suppose that $\tilde{F}$ is a component of $(\tilde{f})^{-1}(\tilde{L})$. Then $\partial\tilde{F} \cap \partial\tilde{N}_t \subset \partial t \times I \cup t \times \{1\}$. However, what is more important is the fact that $\partial\tilde{F} \cap p^{-1}\overline{(}\partial N - T)$ is contained in a neighborhood of $\partial\tilde{t}$ in $\partial t \times I$. This follows from the hypothesis that $f|\overline{(}\partial N - T)$: $\overline{(}\partial N - T) \longrightarrow \overline{(}\partial N' - T')$ is a homeomorphism; and so, in the case of the coverings $\tilde{N}_t$, with projection map $p\colon \tilde{N}_t \longrightarrow N$, and $\tilde{N}'$, with projection map $q\colon\tilde{N}'\longrightarrow N'$, the map $\tilde{f}|p^{-1}\overline{(}\partial N - T)$: $p^{-1}\overline{(}\partial N - T) \longrightarrow q^{-1}\overline{(}\partial N' - T')$ is an embedding into $\partial\tilde{N}'$. In particular, if a component of $\partial\tilde{F} \cap \partial\tilde{N}_t$ is contained in $t \times \{1\}$, then it is contained in $p^{-1}(T)$ and if a component of $\partial\tilde{F}$ is contained in $\partial t \times I$ it is contained in a component of $p^{-1}\overline{(}\partial N - T)$ that is a neighborhood of $\partial\tilde{t}$ in $\partial t \times I$ (and is parallel in $\partial t \times I$ into $\partial\tilde{t}$).

The components of $p^{-1}(T)$ are very easy to describe. Since the pair (N, T) is simple, there are two possibilities for a component of $p^{-1}(T)$ that is distinct from $\tilde{t}$. Namely, a component of $p^{-1}(T)$ that is distinct from $\tilde{t}$ is contained in $t \times \{1\}$ and is either simply connected or has infinite cyclic fundamental group, is in the same component of $\partial\tilde{N}_t$ as $\tilde{t}$ and can be deformed through $\partial\tilde{N}_t$ into $\partial\tilde{t}$; i.e., in the latter case such a component of $p^{-1}(T)$ can be parametrized as $S^1 \times I$ minus a closed subset of $S^1 \times \{1\}$, where $S^1 \times \{0\}$ is parallel in $\partial\tilde{N}_t$ (through a component of $p^{-1}\overline{(}\partial N - T))$ to a component of $\partial\tilde{t}$. Of course, this occurs only if $\partial\tilde{t} \neq \phi$.

Now, from the fact $\partial\tilde{F} \cap \partial\tilde{N}_t \subset \partial t \times I \cup t \times \{1\}$, it follows that the surface $\tilde{F}$ is parallel into a surface in $\partial t \times I \cup t \times \partial I$ in the compactification of $\tilde{N}_t$ to $t \times I$. On the other hand, from the above description of the components of $p^{-1}(T)$ that are distinct from $\tilde{t}$ and the fact $\partial\tilde{F} \cap p^{-1}(\overline{\partial N - T})$ is contained in a neighborhood of $\partial\tilde{t}$ in $\partial t \times I$, it follows that the surface $\tilde{F}$ is either a disk and is parallel in $\tilde{N}_t$ into a component of $p^{-1}(T) - \tilde{t}$ or an annulus and is parallel in $\tilde{N}_t$ into either a component of $p^{-1}(T) - \tilde{t}$ or a component of $p^{-1}(\overline{\partial N - T})$ meeting $\partial\tilde{t} \times I$, or a homeomorphic copy of t and is parallel in $\tilde{N}_t$ into $\tilde{t}$.

What I have proven is that $\tilde{f}^{-1}(\tilde{L})$ is compact and separates $\tilde{N}_t$ into a finite number of components having compact closures and one component whose closure is not compact. However, if both components of $\tilde{N}' - \tilde{L}$ have noncompact closure, then this fact contradicts that the map $\tilde{f}$ is a proper homotopy equivalence. (I am sure that this is well-known; and it probably has a reference somewhere. However, I could not find a reference. A quick proof can be obtained as follows: suppose that V^- and V^+ are the components of $\tilde{N}' - \tilde{L}$. If both V^- and V^+ have noncompact closures, then it follows from the above that notation can be chosen so that $\tilde{f}:\tilde{N}_t \to \tilde{N}'$ has the property that $\tilde{f}(\tilde{N}_t)$ misses a point $v \in V^-$. Suppose that $\tilde{f}'$ is the proper homotopy inverse of $\tilde{f}$. Then $\tilde{f}\circ\tilde{f}':\tilde{N}' \to \tilde{N}'$ has the property that $(\tilde{f}\circ\tilde{f}')(\tilde{N}')$ misses the point $v \in V^-$. However, if this is the case, then the homotopy between $(\tilde{f}\circ\tilde{f}')$ and $id_{\tilde{N}'}$ cannot be proper.)

It follows that $\tilde{L}$ separates $\tilde{N}'$ into two components and one of them has compact closure, as was to be shown. Hence, by $\pi_1(\tilde{L}) \hookrightarrow \pi_1(\tilde{N}')$ an isomorphism, it follows that $\tilde{L}$ is parallel into $\partial\tilde{N}'$. But since $\tilde{L}$ was an arbitrary component of $Fr\tilde{U}$, the neighborhood $\tilde{U}$ is "parallel" into $\partial\tilde{N}'$ and so $\widetilde{f|t}$ deforms (rel ∂t) into $\partial\tilde{N}'$.

So, I have shown that for any component t of T, there is a deformation (rel ∂t) of $f|t$ to a map taking t into $\partial N'$. I shall keep notation and assume that f itself has the property that $f|\partial N:\partial N \to \partial N'$ and $f|\overline{(\partial N-T)}:\overline{(\partial N-T)} \to \overline{(\partial N'-T')}$ is a homeomorphism.

Now, if t is a component of T, then $f|t$ induces an injection on $\pi_1(t)$; so, for example, by Theorem 13.1 of $[He_1]$, either the map $f|t$ deforms (rel ∂t) to a covering map of t onto a surface in $\partial N'$ or t is an annulus and $f|t$ collapses. However, the latter cannot happen since no component of T is an annulus. Again by t being an arbitrary component of T, I can assume that for any component t of T the map $f|t$ is a covering map of t onto a surface in $\partial N'$.

I want to claim that $f|\partial N:\partial N \to \partial N'$ is a covering map; however, this may not be the case. The point is that it is very close; and in fact, this only fails because of the behavior of f in neighborhoods of certain components of $\overline{(\partial N-T)}$ that are annuli. What I have is that $f|t$ is a covering map for each component t of T and $f|\overline{(\partial N-T)}$ is an embedding (actually, $f|t$ is an embedding for each component t of T; but this does not make any difference to the argument at this stage). So, suppose that t_1 and t_2 are distinct components of T and $f(t_1) \cap f(t_2) \neq \phi$. It follows that $f(t_1) \cap f(t_2)$ contains a component of T' or a component of $\overline{(\partial N'-T')}$. I claim that $f(t_1) \cap f(t_2)$ cannot contain a component of T'.

For suppose that t' is a component of T' and $t' \subset f(t_1) \cap f(t_2)$. Let t_i' be a component of $f^{-1}(t') \subset t_i$, $i = 1,2$. Since $f|t_i$ is a covering map, it follows that $f|t_i':t_i' \to t'$ is a covering map, $i = 1,2$. Set $H = f_*^{-1}(\pi_1(t'))$. Then both $\pi_1(t_1')$ and $\pi_1(t_2')$ have finite index in H. But then either H is infinite cyclic or there exists an essential annulus embedded in (N,T) with one boundary component in t_1' and one boundary component in t_2'. The former is not possible since no component of T' is an annulus; and the latter is not possible since the pair (N,T) is simple. So, $f(t_1) \cap f(t_2)$ does not contain a component of T'. Now, by arguing inductively and using the "flip" homotopy τ_s, $0 \le s \le 1$, I may assume that for distinct components t_1 and t_2 of T, $f(t_1) \cap f(t_2) =$ But since $f|\overline{(\partial N-T)}:\overline{(\partial N-T)} \to \overline{(\partial N'-T')}$ is a homeomorphism, it follows that there is a homotopy f_s, $0 \le s \le 1$, such that $f_0 = f$, $f_1|\partial N:\partial N \to \partial N'$ is a homeomorphism and $f_s|\overline{(\partial N-T)}$ is equal to f on components that are not annuli and equal to $f\circ\varepsilon_s$ on components that are annuli, where ε_s is either the identity or the "flip" homotopy τ_s, $0 \le s \le 1$.

The conclusion of Lemma X.23 follows from Theorem X.7. Hence, I have established Assertion 8.

ASSERTION 9. <u>There is a deformation such that</u> (i) $f|\overline{(M-\Lambda)}:\overline{(M-\Lambda)} \to \overline{(M'-\Lambda')}$ <u>is a homeomorphism and</u>

(ii) $f|\Lambda:\Lambda \to \Lambda'$ <u>is a homotopy equivalence</u>.

Proof of Assertion 9: It follows from Assertion 8 that for each component (σ,φ) of (Σ,Φ) that is not a component of (Λ,Ψ), the map $f|\sigma:\sigma \to \sigma'$ is a homotopy equivalence and $f|\partial\sigma:\partial\sigma \to \partial\sigma'$ is a homeomorphism. So, by Theorem X.7, $f|\sigma$ can be deformed (rel $\partial\sigma$) to a homeomorphism.

This completes the proof of Assertion 9 and establishes the proof of Theorem X.21. ∎

X.24 REMARK: By Remark X.19 it can be seen that Theorem X.15 follows immediately as a Corollary to Theorem X.21. So, the language of ends of pairs of groups is not at all needed. However, it is necessary to establish Theorem X.15 in some fashion (Observation X.22 (2)); and I particularly like the way that ends of pairs of groups capture the peripheral structure in the case of Theorem X.15. Furthermore, Theorem X.15 is remarkably easy to establish after Theorem X.7.

X. 25. REMARK: It follows from Remark VII.23 that there is no way to capture, in the relative situation, a generalization of VII.24 using the language of ends of pairs of groups. For example, in the case that (N,T) is a simple pair, t is a component of T and $H = \pi_1(t)$, one may think that if $G = \pi_1(N)$, then $e(G,H)$ may be one. However, by Remark VII.23 this would imply that ∂N is incompressible, which is not necessarily the case. But even if ∂N is incompressible, $e(G,H)$ need not be one. It is true that VII.22 does have a generalization to the relative situation using the language of ends of pairs of groups.

X.26. EXERCISE: Let N be a Haken-manifold. Let F be a compact, orientable surface with $\partial F \neq \phi$ and suppose that $\alpha:(F,\partial F) \to (N,\partial N)$ is a map such that $\alpha_*:\pi_1(F) \to \pi_1(N)$ is an injection. Set $G = \pi_1(N)$ and set $H = \alpha_*(\pi_1(F))$. If $e(G,H) = 1$, then α is inessential (as a map of pairs).

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Lectures on Three-Manifold Topology

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American Mathematical Society
with support from the
National Science Foundation