M435: INTRODUCTION TO DIFFERENTIAL GEOMETRY

MARK POWELL

Contents

3. Topology of Surfaces 1
3.1. Review of continuity for functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$ 1
3.2. Topological spaces and continuity 4
3.3. Topological surfaces 6
3.4. Compact connected surfaces 8
3.5. The Euler characteristic 10
References 13

3. Topology of Surfaces

Now we move on from curves to surfaces. Topology is about understanding the large scale structure of surfaces. We will construct surfaces with glueing techniques. Roughly speaking, if our surfaces are all made of stretchy rubber, two are considered to be "the same," or homeomorphic in topology if one can be deformed into the other. In the next section, on geometry of surfaces, we also want to be able to measure distances, angles, areas, and *curvature*; this latter will be the key quantity as with curves. So in geometry surfaces are much more rigid than in topology. Thus topology has the disadvantage that we throw away potentially important geometric information – only deformations which preserve distances and angles and areas are allowed in geometry. The advantage in topology is that we can understand a complete answer to the problem of classifying closed bounded surfaces up to homeomorphism. This remarkable solution is the subject of this chapter. We need some background and some terminology to begin with.

3.1. Review of continuity for functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$.

Recall the definition of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$. We want to generalise the notion of continuity. First we generalise it to define continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, then we define continuous functions between any pair of sets, provided these sets are endowed with some extra information. This extra information is called a *topology* on a set.

In our case, since we consider surfaces, which are sets that can be constructed from glueing together subsets of $\mathbb{R}^2$, if you understand the notion of continuity for functions $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, then you are very close to understanding continuity for functions between surfaces.

We are aiming towards understanding the following definition.

Definition 3.1. A *homeomorphism* $f\colon A \to B$ between two sets is a continuous bijection whose inverse $f^{-1}\colon B \to A$ is also continuous.

The only problem with this definition is that we don't yet know what continuity means for functions between two arbitrary sets. Let's begin by recalling the definition of continuity for functions from $\mathbb{R}$ to $\mathbb{R}$, then functions from $\mathbb{R}^n \to \mathbb{R}^m$, then we will recast these definitions in a form which generalises. This is a common occurrence in math: the most obvious definition is not always the one which generalises, but by defining a concept in a seemingly more complicated way, we find a definition which can be easily adjusted to apply to some more general setting.

Definition 3.2. A function $f\colon I \to \mathbb{R}$ is continuous at $a \in I$ if for all $\varepsilon > 0$, there exists a $\delta > 0$, depending on ε, such that

$$|f(x) - f(a)| < \varepsilon$$

whenever $|x - a| < \delta$. The function f is *continuous* if it is continuous at a for all $a \in I$.

Equivalently:

Definition 3.3. A function $f\colon I \to \mathbb{R}$ is continuous at $a \in I$ if for all $\varepsilon > 0$, there exists a $\delta > 0$, depending on ε, such that

$$f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$$

whenever $x \in (a - \delta, a + \delta)$. The function f is *continuous* if it is continuous at a for all $a \in I$.

Finally:

Definition 3.4. A function $f\colon I \to \mathbb{R}$ is continuous at $a \in I$ if for all $\varepsilon > 0$, there exists a $\delta > 0$, depending on ε, such that

$$(a - \delta, a + \delta) \subseteq f^{-1}\big((f(a) - \varepsilon, f(a) + \varepsilon)\big).$$

The function f is *continuous* if it is continuous at a for all $a \in I$.

Now we consider functions on $\mathbb{R}^n$. First we define open and closed balls.

Definition 3.5. Let $\mathbf{a} \in \mathbb{R}^n$ and let $\varepsilon \in \mathbb{R}_{>0}$. Define the *open ball* centred on $\mathbf{a}$ with radius ε:

$$B_\varepsilon(\mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\| < \varepsilon\}.$$

Also define the *closed ball* centred on $\mathbf{a}$ with radius ε:

$$\overline{B_\varepsilon}(\mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\| \leq \varepsilon\}.$$

Next, continuity in $\mathbb{R}^n$. We give the definition for functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ but the domain and codomain can be arbitrary Euclidean spaces without needing any change in the definition.

Definition 3.6. Let $U \subseteq \mathbb{R}^2$ be a subset of $\mathbb{R}^2$. A function $f\colon U \to \mathbb{R}^2$ is continuous at $\mathbf{a} \in U$ if for all $\varepsilon > 0$, there exists a $\delta > 0$, depending on ε, such that

$$B_\delta(\mathbf{a}) \subseteq f^{-1}\big(B_\varepsilon(f(\mathbf{a}))\big).$$

The function f is *continuous* if it is continuous at $\mathbf{a}$ for all $\mathbf{a} \in U$.

Similarly to above, the definition above is equivalent to: $\|f(\mathbf{x} - f(\mathbf{a})\| < \varepsilon$ whenever $\|\mathbf{x} - \mathbf{a}\| < \delta$. We have the same definition for $\mathbb{R}^2$ as $\mathbb{R}$, except the meaning of the distance function $\|\cdot\|$ changes.

Example 3.7. We prove that a function from $\mathbb{R}^2 \to \mathbb{R}$ is continuous on $\mathbb{R}^2$. Possibly you managed to get through earlier courses without learning this.(?) We claim that the function

$$\begin{array}{rcl} f\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x^2 + y^2 - 2x \end{array}$$

is continuous everywhere.

Let $\varepsilon > 0$. Consider generic $\mathbf{a} = (a,b) \in \mathbb{R}^2$. We begin by noting that

$$\|(x,y) - (a,b)\| = \sqrt{(x-a)^2 + (y-b)^2} \geq \sqrt{(x-a)^2} = |x-a|.$$

Similarly

$$\|(x,y) - (a,b)\| \geq |y-b|.$$

Define

$$\delta := \frac{\varepsilon}{2|a| + 2|b| + 4}.$$

Note that this definition was chosen after I did the computation below, but for writing up the solution I brought the definition of δ forwards as if I have the prescience of Guild navigator. Suppose $\|(x,y) - (a,b)\| < \delta$ i.e. $(x,y) \in B_\delta(\mathbf{a})$. We need to show that $\|f(x,y) - f(a,b)\| < \varepsilon\|$. We have:

$$\begin{aligned} \|(x^2+y^2-2x) - (a^2+b^2-2a)\| &= \|(x-a)(x+a) + (y-b)(y+b) - 2(x-a)\| \\ &\leq |x+a||x-a| + |y+b||y-b| + 2|x-a| \\ &\leq |x+a|\delta + |y+b|\delta + 2\delta \\ &\leq (2|a| + \delta)\delta + (2|b| + \delta)\delta + 2\delta. \end{aligned}$$

If we assume that $\delta \leq 1$, then $\delta^2 \leq \delta$, so:

$$(2|a| + \delta)\delta + (2|b| + \delta)\delta + 2\delta \leq 2|a|\delta + 2|b|\delta + 4\delta = (2|a| + 2|b| + 4)\delta = \varepsilon.$$

To deal with the (largely irrelevant) case that $\delta > 1$, we instead define

$$\delta := \min\left\{\frac{\varepsilon}{2|a| + 2|b| + 4}, 1\right\}.$$

Then in the case that $\varepsilon > 2|a| + 2|b| + 4$, the proof proceeds as before, but with $\delta = 1$.

Exercise 3.8. (a) Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $(x,y) \mapsto 3x^2 - 4y^2$ is continuous everywhere.

(b) Show that the function $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x,y) \mapsto (3x^2 - 4y^2, 2x)$ is continuous everywhere.

3.2. **Topological spaces and continuity.** We want to define continuity for maps between arbitrary sets. To do this, we need extra structure on the sets. After all $\mathbb{R}^n$ is not just a set, it has a lot of interesting structure, such as the ability to measure the distance between any two points.

Definition 3.9. A *topological space* is a set X together with a set of subsets $\mathcal{T} = \{U_i\}$, $U_i \subseteq X$, called a *topology* on X, which must satisfy the axioms:
(T1) $\emptyset, X \in \mathcal{T}$.
(T2) If $U_i \in \mathcal{T}$ for all $i \in J$, where J is some indexing set, then $\bigcup\{U_i\}_{i\in I} \in \mathcal{T}$.
(T3) If $U_i \in \mathcal{T}$ for all $i = 1, \ldots, n$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.
The U_i are called the *open sets*.

The last two axioms (T2) and (T3) can be paraphrases as "closure under arbitrary unions" and "closure under finite intersections," respectively. An open set $U \in \mathcal{T}$ containing a given point $x \in X$ should be thought of as a set of points in X which are "close to" x. A general topological space does not need to have a precise numerical measure of distance, although many do. But the more open sets containing x that a point $y \in X$ belongs to, the closer to x we should think of y as being. Different topologies on a fixed set correspond to different ways of measuring this "closeness."

Example 3.10. (i) The discrete topology. For any set X, all subsets are declared to be open. i.e. $\mathcal{T}$ is the set of all subsets of X.
(ii) The trivial topology. Again, X any set. $\mathcal{T} = \{\emptyset, X\}$.
(iii) Important example. $X = \mathbb{R}^n$. We take $U \in \mathcal{T}$ if and only if there is an indexing set J and collections $\{\mathbf{a}_i\}_{i\in J}$, $\{\varepsilon_i\}_{i\in J}$ such that $U = \bigcup_{i\in J} B_{\varepsilon_i}(\mathbf{a}_i)$ i.e. U is a union of a (possibly uncountable) collection of open balls.

We will be concerned with surfaces as topological spaces. Surfaces can be constructed from glueing together subsets of $\mathbb{R}^2$, therefore understanding $\mathbb{R}^2$ with the topology of (iii), usually called the *standard topology*, will be the most important for us. Further study of topological spaces can be experienced in a topology course.

Exercise 3.11. (i) Check that the examples of discrete, trivial topology for arbitrary X, and standard topology on $\mathbb{R}^n$, from the example above, satisfy the axioms (T1 – 3).
(ii) Show that the rule: $U \in T$ if $X \setminus U$ is finite, or $U = \emptyset$, defines a topology on a set X. (Prove that the axioms hold).

Definition 3.12. Let $f\colon X \to Y$ be a map between topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$. We say that f is *continuous* if for all open sets $U \in \mathcal{T}'$, we have $f^{-1}(U) \in \mathcal{T}$.

The paraphrase is: "the inverse image of every open set is open."
Recall the following definition.

Definition 3.13. We say that a function $f\colon X \to Y$ between topological spaces X and Y is a *homeomorphism* if f is a continuous bijection whose inverse is also continuous. We say that two spaces X and Y are *homeomorphic* if there exists a homeomorphism $f\colon X \to Y$.

Example 3.14. (i) Let X have the discrete topology and let Y be any topological space. Then any map $f\colon X \to Y$ is continuous.
(ii) Let X be any topological space and let Y have the trivial topology. Then $f\colon X \to Y$ is continuous.

if the codomain had the discrete topology, or the domain had the trivial topology, it is much harder for f to be continuous.

Exercise 3.15. Let $X = \{1, 2, 3\}$ and let $Y = \{1, 2\}$. Suppose that Y has the discrete topology and let f be defined by $f(1) = 1$, $f(2) = 1$ and $f(3) = 2$. What is the smallest set $\mathcal{T}$, defining a topology on X, for which f is continuous?

Definition 3.16 (Subspace topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. Then A inherits the *subspace topology* from X by declaring the open subsets of A to be $\{U \cap A \,|U \in \mathcal{T}\}$.

Any subset can be thought of as an abstract set in its own right, independently from its embedding in a larger set. We often use the embedding to define a topology on this set, using the subspace topology, but once the topology has been defined we can forget the embedding.

Example 3.17. (i) Any two open intervals (a, b) and (c, d) are homeomorphic. An open interval has a subspace topology from being a subset of $\mathbb{R}$.
(ii) A planar curve has a topology from being a subset of $\mathbb{R}^2$. Any two ellipses are homeomorphic.

At this point, review the concept of an equivalence relation.

Exercise 3.18. Group the letters of the alphabet and the numbers with sans serif font into their homeomorphism classes without proof. Consider them as infinitely thin, abstract topological spaces in their own right, with the subspace topologies.

ABCDEFGHIJKLMNOPQRSTUVWXYZ1234567890

"Homeomorphic" defines an equivalence relation, so homeomorphism classes are the equivalence classes with respect to this equivalence relation. A good way to start: homeomorphic spaces will have the same number of boundary points.

The last exercise did not require proofs in the solutions. But suppose that you had been asked to give proofs. To show that two spaces are homeomorpic, one has to construct a homeomorphism explicitly i.e. define a bijection and show it is continuous with continuous inverse. To show that two spaces are not homeomorphic, we typically need an *invariant*. That is, some function from topological spaces, or the subset of spaces we are interested in, to some more tractable object, like the integers. The function should have the property that homeomorphic spaces have the same value of the function. The contrapositive of this then allows us to infer that two spaces are not homeomorphic, if we compute the output of our function on them to be different.

Given a topological space X, let $F_{\max}(X)$ be the maximum number of connected components (connected defined later) of the complement of a point in X, taken over all possible points in X. For example $F_{\max}(\mathsf{I}) = 2$ but $F_{\max}(\mathsf{H}) = 3$. Similarly let

$F_{\min}(X)$ be the minimum number of connected components of the complement of a point in X, taken over all possible points in X.

We then claim that two homeomorphic spaces have the same value of $F_{\min}$ and $F_{\max}$. The proof of this claim follows essentially from the fact that connectedness is preserved by homeomorphisms.

Definition 3.19 (Quotient topology)**.** Suppose we start with a topological space X and we define an equivalence relation $\sim$ on X. Let $q\colon X \to X/\sim$ be the quotient map sending $x \in X$ to its equivalence class $[x]$. The *quotient topology* on $X/\sim$ is defined by $U \subseteq X/\sim$ is open if and only if $q^{-1}(U)$ is open.

The quotient topology is the largest possible topology for which the quotient map q is continuous.

Example 3.20. Let $[0,1] \subset \mathbb{R}$ be the unit interval. Define an equivalence relation on $[0,1]$ by $0 \sim 1$ (and $x \sim x$ for all $x \in [0,1]$, and $1 \sim 0$; these are unavoidable by reflexivity and symmetry of an equivalence relation. The topological space $X/\sim$ is called the *circle*, and denoted S^1. It is homeomorphic to any ellipse in $\mathbb{R}^2$ with the subspace topology.

Exercise 3.21. Draw the space given by the quotient $X_1 \cup X_2 \cup X_3/\sim$ where $X_1 = X_2 = X_3 = [0,1]$ and where $x_i \sim x_j$, for $x_i \in X_i$ and $x_j \in X$, $i \neq j$, if and only if $x_i = x_j = 0$ or $x_i = x_j = 1$.

3.3. **Topological surfaces.** We give the definition of a surface and we construct surfaces using glueing. We will work in particular with surfaces which are closed and bounded, in this chapter; our aim is to classify such surfaces.

The following definition should not be worried about too much, topological spaces which do not satisfy the condition are rather pathological.

Definition 3.22. A topological space $(X, \mathcal{T})$ is *Hausdorff* if for any points $u, v \in X$, there exist open subsets $U, V \in \mathcal{T}$, such that $U \ni u$ and $V \ni v$ and $U \cap V = \emptyset$.

To paraphrase, any two points can be separated by open sets.

Definition 3.23. A *topological surface* is a Hausdorff topological space X which is locally homeomorphic to $\mathbb{R}^2$. Locally homeomorphic to $\mathbb{R}^2$ means that for each $x \in X$, there exists an open set $U \ni x$, sometimes called an open neighbourhood of x, together with a homeomorphism $\Phi_U\colon U \xrightarrow{\approx} \mathbb{R}^2$.

A surface is a topological space which locally looks like $\mathbb{R}^2$. Again, Hausdorff is included in the definition for technical reasons, but don't pay too much attention to it.

Recall that an open disc in $\mathbb{R}^2$ or more generally a convex union of open balls is homeomorphic to $\mathbb{R}^2$. So in the definition above it is enough to find a homeomorphism to some open disc in $\mathbb{R}^2$ from a neighbourhood of each point $x \in X$.

We will give some examples without proof. Later we will see how to show that something is a topological surface. For the first two it's rather easy.

Example 3.24. (i) The plane $\mathbb{R}^2$ is a topological surface.

(ii) The graph of a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a topological surface.

$$\{(x, y, f(x, y)) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$$

(iii) The sphere $S^2 = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \cdot \mathbf{x} = 1\}$ is a topological surface.

(iv) The torus $\mathbb{T} := \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - a)^2 + z^2 = r^2\}$, where $0 < r < a$.

(v) A regular parametrised surface in $\mathbb{R}^3$. Let $U \subset \mathbb{R}^2$ be an open subset and let $\mathbf{r}\colon U \to \mathbb{R}^3$ be a differentiable function with injective derivative (to be explained below). Then $\mathbf{r}(U)$ is topological surface. Let (u, v) be coordinates on U. $\mathbf{r}$ has injective derivative if the 3×2 matrix with columns $\mathbf{r}_u$ and $\mathbf{r}_v$ has zero kernel for all $(u, v) \in U$. Equivalently, $|\mathbf{r}_u \wedge \mathbf{r}_v| \neq 0$ for all $(u, v) \in U$. The vector $\mathbf{n} := \mathbf{r}_u \wedge \mathbf{r}_v$ is called the *unit normal*. For more examples of parametrised surfaces see below.

(vi) We can construct interesting surfaces by glueing together edges of polygons. For example, let $X = Y = [0, 1]$. Consider the quotient space:

$$\frac{X \times Y}{(x, 0) \sim (x, 1), (0, y) \sim (1, y), x \in X, y \in Y}.$$

The opposite edges of the square are identified. The resulting space is a topological surface, which is homeomorphic to the torus $\mathbb{T}$. Why is an identification space locally homeomorphic to $\mathbb{R}^2$?

(vii) Another famous surface is the Möbius band. Let $X = [0, 1]$ and let $Y = (0, 1)$. Then

$$M := \frac{X \times Y}{(0, y) \sim (1, 1 - y), y \in Y}.$$

Take a strip of paper, put half a twist in, and glue the ends together.

Next some examples of parametrised surfaces.

Example 3.25. In all the examples let $U = \mathbb{R}^2$. Later we will need to restrict U to patches where $\mathbf{r}$ is a homeomorphism, but for now we don't need this.

(i) A plane.

$$\mathbf{r}(u, v) = \mathbf{a} + u\mathbf{b} + v\mathbf{c},$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are constant vectors in $\mathbb{R}^3$, with $\mathbf{b}$, $\mathbf{c}$ linearly independent.

(ii) A sphere.

$$\mathbf{r}(u, v) := (\cos u \sin v, \sin u \sin v, \cos v).$$

(iii) A torus.

$$\mathbf{r}(u, v) = \big((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u\big)$$

(iv) A cylinder.

$$\mathbf{r}(u, v) = \big(a\cos v, a\sin v, u\big).$$

(v) A cone.

$$\mathbf{r}(u, v) = \big(au\cos v, au\sin v, u\big).$$

(vi) A helicoid.

$$\mathbf{r}(u, v) = \big(au\cos v, au\sin v, v\big).$$

(vii) A surface of revolution. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function.

$$\mathbf{r}(u, v) = \big(f(u)\cos v, f(u)\sin v, u\big).$$

(viii) More generally, a surface of revolution can arise as from rotating a regular plane curve $\alpha(t) = (x(t), y(t))$ about a line in the plane it does not touch.
$$\mathbf{r}(u, v) = (y(u)\cos v, y(u)\sin v, x(u)).$$
The previous example was the special case $\alpha(t) = (t, f(t))$.

Exercise 3.26. Compute the normal vector to the surfaces above at a generic point (u, v) and say where the parametrisations are regular.

Exercise 3.27. Express examples (i) - (v) as surfaces of revolution. You might have to miss out one or two points of the desired surface which would be on the line of rotation.

3.4. **Compact connected surfaces.**

Definition 3.28. A topological space X is compact if every open cover of X contains a finite subcover. An open cover is a collection $\{U_i\}$ where $U_i \in \mathcal{T}$ is open for all i and $\bigcup\{U_i\} = X$. A subcover is a subset of $\{U_i\}$ whose union still is the whole of X. A finite subcover is a subcover containing a finite number of subsets.

A subset of a topological space is said to be a compact subset if it is compact when considered as a topological space in its own right under the subspace topology.

Theorem 3.29 (Heine-Borel)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Here a *closed set* V means that the complement $\mathbb{R}^n \smallsetminus V$ is open. Equivalently closed subsets are subsets which contain all of their limit points, where a limit point of a subset $U \subset X$ is a point $x \in X$ such that $W \cap U \neq \emptyset$ for all open sets $W \ni x$. If $x \in U$, then x is a limit point of U, but also points outside of U can be limit points of U. For example the points of S^2 are limit points of the open unit ball $B_1(\mathbf{0})$ in $\mathbb{R}^3$. The points 0, 1 are limit points of the set $(0, 1)$.

Example 3.30. (i) Any closed bounded interval $[a, b]$, where $a < b \in \mathbb{R}$, is a compact topological space under the subspace topology.
(ii) The subspace $[a, b] \times [c, d] \subset \mathbb{R}^2$ is compact.
(iii) The 2-sphere S^2 is a compact topological surface.
(iv) The torus is a compact topological surface.
(v) An open interval (a, b) is a bounded subset, for $a, b \in \mathbb{R}$, but it is *not* compact.
(vi) An interval $[a, \infty)$ is a closed subset of $\mathbb{R}$ but it is *not* compact.
(vii) The examples of cylinder, cone, helicoid and surface of revolution above are compact if the u coordinate is restricted to a compact subset of $\mathbb{R}$, like a closed bounded interval $[a, b]$. These will not be topological surfaces due to the boundary points.
Alternatively, restrict u to an open interval (a, b) (which could be all of $\mathbb{R}$): then we have topological surfaces which are not compact.

We will consider *connected surfaces*. The technical definition is that a space X is connected if it does not admit a continuous *surjective* function $f\colon X \to \{0, 1\}$ to the 2 point set with the discrete topology. An easier concept is *path connected*: a space is path connected if any two points $x, y \in X$ have a path between them, that is there is a continuous function $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Theorem 3.31. *The image of a compact set under a continuous function is again compact.*

The proof uses the same technique your pet uses to find food.

Proof. Let $f\colon X \to Y$ be a continuous function from a compact space X to Y. Let $f(X)$ be the image of f. Let $\{U_i\}_{i\in J}$ be an open cover of $f(X)$ with indexing set J. Then we claim that $\{f^{-1}(U_i)\}_i \in J$ is an open cover of X. Certainly each $f^{-1}(U_i)$ is open, since f is continuous. Also each $x \in X$ belongs to some $f^{-1}(U_i)$; simply take the $i \in J$ for which $f(x) \in U_i$. Such an i always exists since $\{U_i\}_{i\in J}$ is an open cover.

Now use compactness of X to see that $\{f^{-1}(U_i)\}_i \in J$ has a finite subcover of X which after reindexing we write as $\{f^{-1}(U_i)\}_{i=1}^n$. Now $\{U_i\}_{i=1}^n$ is a finite subcover of the original open cover of $f(X)$. To see this we need to check that it is a cover. Let $y \in f(X)$. Then choose $x \in X$ such that $f(x) = y$. Since $\{f^{-1}(U_i)\}_i \in J$ is a cover of X, there is an i such that $y \in f^{-1}(U_i)$. Therefore $f(y) = x \in U_i$ as required. This completes the proof. □

As a corollary, note that if one space X is compact, and Y is homeomorphic to X, then Y is also compact.

Exercise 3.32. Show that the image of a connected space under a continuous map is again connected. Show that the image of a path connected space under a continuous map is again path connected.

An important decision one must make when seeking to decide how a surface fits into the classification of surfaces is whether or not the surface is *orientable.*

Definition 3.33. A topological surface is said to be *orientable* if it does not contain any subset homeomorphic to the Möbius band.

As well as the torus, we can construct two other interesting nonorientable topological surfaces using identification of the edges of a square. These are the Klein bottle $\mathbb{K}$ and the projective plane $\mathbb{P}$. In both cases let $X = Y = [0, 1]$.

$$\mathbb{K} := \frac{X \times Y}{(x,0) \sim (x,1), (0,y) \sim (1,1-y), x \in X, y \in Y}$$

$$\mathbb{P} := \frac{X \times Y}{(x,0) \sim (1-x,1), (0,y) \sim (1,1-y), x \in X, y \in Y}$$

These are compact topological surfaces. Neither of these surfaces can be embedded in $\mathbb{R}^3$.

Exercise 3.34. Show that both the Klein bottle $\mathbb{K}$ and the projective plane $\mathbb{P}$ are nonorientable, as claimed above.

Introduce notation for labelling edges. Torus from $aba^{-1}b^{-1}$. Klein bottle from $abab^{-1}$. Projective plane from $abab$.

We can also construct higher genus orientable surfaces. Start with an octogon and identify edges according to $aba^{-1}b^{-1}cdc^{-1}d^{-1}$. The resulting surface is called a genus 2 surface. The genus of a surface is the number of holes: the maximum number of simple closed curves which can be removed from the surface still leaving a connected space.

Definition 3.35 (Connected sum)**.** We form the connected sum $\Sigma_1 \# \Sigma_2$ of two topological surface Σ_1 and Σ_2 as follows. Choose open balls $D_1 \subset \Sigma_1$ and $D_2 \subset \Sigma^2$ and delete these from their respective surfaces. The open balls have circle boundaries S_1 and S_2. Let $f\colon S_1 \to S_2$ be a homeomorphism. Define

$$\Sigma_1 \# \Sigma_2 := \frac{(\Sigma_1 \smallsetminus D_1) \cup (\Sigma_2 \smallsetminus D_2)}{x \sim f(x), x \in S_1}.$$

Example 3.36. (i) $\mathbb{T} \# \mathbb{T}$ is homeomorphic to the genus 2 surface formed from an octogon above. This is also called a sphere with two handles.
(ii) $\mathbb{P} \# \mathbb{P} \approx \mathbb{K}$. This needs some slightly tricky picture drawing to prove.

Connect summing with a torus is also called *attaching a handle*. The official definition is below, a picture definition is easier.

Definition 3.37. Let Σ be a topological surface. Consider the disc $\Delta = B_4((0,0)) \subset \mathbb{R}^2$ and the sub-discs $D_+ = B_1((+2,0)), D_- = B_1((-2,0)) \subset \Delta$. Let $f\colon \Delta \to \Sigma$ be a map which is a homeomorphism onto $f(\Delta)$. Let $S_\pm$ be the image under f of the boundary circle of $D_\pm$. Form the quotient space

$$\frac{\Sigma \smallsetminus f(D_- \cup D_+)}{S_+ \sim S_-}$$

where the identification of S_+ and S_- is via $f(x,y) \sim f(-x,y)$

We have to be specific about the identification to ensure we get an orientable surface if we began with one.

The closed Möbius band is defined as follows. Let $X = Y = [0,1]$. Then

$$M := \frac{X \times Y}{(0,y) \sim (1, 1-y), y \in Y}.$$

Definition 3.38. We attach a crosscap to a surface Σ by removing a disc from Σ and identifying the boundary circle with the boundary circle of a copy of M.

The projective plane arises from attaching one crosscap to a sphere. The Klein bottle comes from attaching 2 crosscaps.

Exercise 3.39. Prove these two assertions.

Theorem 3.40 (Classification of closed surfaces)**.** *Any connected compact topological surface Σ is homeomorphic to precisely one of the following.*

- *A sphere with g handles attached, if Σ is orientable.*
- *A sphere with k crosscaps attached, if Σ is nonorientable.*

3.5. **The Euler characteristic.** A subdivision of a compact surface X is a partition of X into finitely many *cells* of dimension 0, 1 or 2, where an i-cell e^i is a subset homeomorphic to the i-disc $D^i = \{\mathbf{x} \in \mathbb{R}^i \mid \|\mathbf{x}\| < 1\}$. $\mathbb{R}^0 = D^0$ is a point. $D^1 \cong (-1,1)$. So 0-cells are points, or vertices. 1-cells are edges. Each 1-cell must have a 0-cell at each of its endpoints. Each 0-cell must have at least two 1-cells, or both ends of the same 1-cell, attached to it. The 2-cells are called faces; these are the connected components of the complement of the vertices and the edges.

Theorem 3.41. *Every surface has a subdivision.*

Given a subdivision P of a surface Σ let V be the number of vertices, let E be the number of edges and let F be the number of faces.

Definition 3.42. The Euler characteristic of a surface Σ with subdivision P is $\chi(\Sigma, P) = V - E + F$.

Proposition 3.43. *Let $f\colon \Sigma \to \Sigma'$ be a homeomorphism. Let $f(P)$ be the induced subdivision of Σ'. Then $\chi(\Sigma, P) = \chi(\Sigma', f(P))$.*

Proof. Since f is a homeomorphism, it is a homeomorphism restricted to each of the cells of P. Therefore they are still cells in $f(P)$. Since f is a bijection, $f(P)$ partitions Σ'. The number of cells of each type is the same, so the Euler characteristic is the same. □

Theorem 3.44. *The Euler characteristic is independent of the choice of subdivision of a surface. That is, the Euler characteristic is an invariant of the homeomorphism class of a surface.*

Example 3.45. The Euler characteristic of the sphere is 2. For the torus and the Klein bottle, the Euler characteristic is 0. For a genus 2 surface, the Euler characteristic is -2.

Adding a handle reduces the Euler characteristic by 2. To see this, we remove two 2-cells initially. This lowers χ by 2. Then we are left with 2 circles, the boundaries of these two 2-cells, which we need to identify. A circle has the same number of 0- and 1-cells. When identifying the 2 circles, we subdivide further so they have the same subdivisions, and then identify. The result is to reduce both the number of vertices and the number of edges by those comprising one of the circles. But since a circle has the same number of each, the Euler characteristic is not changed. Thus the effect of adding a handle is to reduce by 2 as claimed above.

Exercise 3.46. What is the effect on the Euler characteristic of a surface of attaching a crosscap? What is the Euler characteristic of the Projective plane?

We can use the Euler characteristic to decide precisely where in the homeomorphism classification a surface sits. First decide whether a surface is orientable or not. Then compute the Euler characteristic.

Theorem 3.47. *Let Σ be a compact connected orientable topological surface. Then Σ is homeomorphic to a sphere with g handles attached where $\chi(\Sigma) = 2 - 2g$.*

Theorem 3.48. *Let Σ be a compact connected nonorientable topological surface. Then Σ is homeomorphic to a sphere with k crosscaps attached where $\chi(\Sigma) = 2 - k$.*

Exercise 3.49. Derive a formula for the Euler characteristic of a connected sum in terms of the Euler characteristic of the summands. Use the second theorem above and your formula to prove that $\mathbb{P}\#\mathbb{P} = \mathbb{K}$. Prove that $\mathbb{P}\#\mathbb{K} = \mathbb{P}\#\mathbb{T}$ twice, once using the theorems above and once using a direct cut and paste method.

Exercise 3.50. A Platonic solid is a subdivision of (a surface homeomorphic to) the sphere into regular flat polygons, where each polygon is congruent and the same number of polygons meet at each vertex. Show that there are only 5 Platonic solids, identify the type of polygons used, the number of them and the number of them incident at each vertex. An example is the tetrahedron (pyramid with triangular base), made from 4 triangles, or the cube, made from 6 squares. Write down an equation computing the Euler characteristic, and show that there are only 5 sensible solutions.

Solution: Let n be the number of sides of each of the polygons. Let m be the number of polygons and let k be the number of polygons incident at each vertex. We compute the Euler characteristic of the 2-sphere $\chi(S^2) = 2$ using the subdivision of a Platonic solid. Then $F = m$. There are $nm/2$ edges and nm/k vertices, from the given conditions. Therefore

$$2 = \frac{nm}{k} - \frac{nm}{2} + m = m\left(\frac{n}{k} - \frac{n}{2} + 1\right).$$

Note that for physically sensible solutions, $n, m, k \geq 3$. (We could have $k = 2 = m$, and any n, but then the polygons would have to be curved, and these do not count as Platonic solids.) Rearranging the equation we obtain:

$$m = \frac{4k}{2n - nk + 2k}.$$

We must have $2n - nk + 2k > 0$ which implies that $\frac{2n}{n-2} > k \geq 3$. Solving the inequality with 3 on the right hand side yields $6 \geq n$. So we try $n = 3, 4, 5, 6$ in our search for solutions. Putting $n = 3$ gives

$$m = \frac{4k}{6 - k},$$

so we can have positive solutions for $k = 3, 4$ or 5, in which cases $m = 4, 8$ and 20 respectively. These correspond to the tetrahedron, the octahedron and the icosahedron. Next, with $n = 4$ we have

$$m = \frac{2k}{4 - k}$$

so we must have $k = 3$ and therefore $m = 6$. This is a cube. When $n = 5$, we have

$$m = \frac{4k}{10 - 3k},$$

so $k = 3$ and $m = 12$. Twelve pentagon make a dodecahedron. Finally when $n = 6$ we have $m = 4k/(12 - 4k)$ where even $k = 3$ is not allowed. We have found the 5 Platonic solids and proved that there cannot be any others.

Exercise 3.51. An engineer constructs a vessel in the shape of a torus from a finite number of steel plates. Each plate is in the form of a not necessarily regular curvilinear polygon with n edges. The plates are welded together along the edges so that at each vertex n distinct plates are joined together, and no plate is welded to itself. What is the number n? Give justification.

We have seen that when one restricts to compact surfaces and considers surfaces up to homeomorphism, we can understand the resulting surfaces very well. We will now move on to geometry, where there will be much more structure on a surface.

References

[DoCa] Manfredo P. DoCarmo *Differential Geometry of Curves and Surfaces*
[Hit] Nigel Hitchens *Lecture notes for B3: Geometry of Surfaces* http://www.maths.ox.ac.uk/system/files/coursematerial/2013/2377/9/notes2013.pdf

Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
E-mail address: macp@indiana.edu