M435: INTRODUCTION TO DIFFERENTIAL GEOMETRY

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Contents

5. Second fundamental form and the curvature

5.1. **The second fundamental form.**

We will give several ways of motivating the definition of the second fundamental form. Like the first fundamental form, the second fundamental form is a symmetric bilinear form on each tangent space of a surface Σ. Unlike the first, it need not be positive definite.

The idea of the second fundamental form is to measure, in $\mathbb{R}^3$, how Σ curves away from its tangent plane at a given point. The first fundamental form is an *intrinsic* object whereas the second fundamental form is extrinsic. That is, it measures the surface as compared to the tangent plane *in* $\mathbb{R}^3$. By contrast the first fundamental form can be measured by a denizen of the surface, who does not possess 3 dimensional awareness.

Given a smooth patch $\mathbf{r} \colon U \to \Sigma$, let $\mathbf{n}$ be the unit normal vector as usual. Define

$$\mathbf{R}(u, v, t) := \mathbf{r}(u, v) - t\mathbf{n}(u, v),$$

with $t \in (-\varepsilon, \varepsilon)$. This is a 1-parameter family of smooth surface patches. How is the first fundamental form changing in this family? We can compute that:

$$\frac{1}{2}\frac{\partial}{\partial t}\Big|_{t=0}\big(E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2\big) = L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2$$

where

$$L := -\mathbf{r}_u \cdot \mathbf{n}_u$$
$$2M := -(\mathbf{r}_u \cdot \mathbf{n}_v + \mathbf{r}_v \cdot \mathbf{n}_u)$$
$$N := -\mathbf{r}_v \cdot \mathbf{n}_v.$$

The reason for the negative signs will become clear soon.

For example:

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\Big|_{t=0}\big(E(u,v,t)\big)\\
=&\frac{1}{2}\frac{\partial}{\partial t}\Big|_{t=0}\big((\mathbf{r}_u - t\mathbf{n}_u)\cdot(\mathbf{r}_u - t\mathbf{n}_u))\big)\\
=&\frac{1}{2}\frac{\partial}{\partial t}\Big|_{t=0}\big(\mathbf{r}_u\cdot\mathbf{r}_u - 2t\mathbf{r}_u\cdot\mathbf{n}_u + t^2\mathbf{n}_u\cdot\mathbf{n}_u\big)\\
=&\frac{1}{2}\big(-2\mathbf{r}_u\cdot\mathbf{n}_u + 2t\mathbf{n}_u\cdot\mathbf{n}_u\big)|_{t=0}\\
=&-\mathbf{r}_u\cdot\mathbf{n}_u
\end{aligned}$$

Exercise 5.1. Check the analogous computations for $2M$ and N.

The form:

$$L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2$$

is called the *second fundamental form*. Like the first fundamental form, it also defines a bilinear form on the tangent space:

$$(a\mathbf{r}_u + b\mathbf{r}_v, c\mathbf{r}_u + d\mathbf{r}_v) \mapsto \begin{pmatrix} a & b\end{pmatrix}\begin{pmatrix} L & M\\ M & N\end{pmatrix}\begin{pmatrix} c\\ d\end{pmatrix}.$$

5.2. **Second fundamental form alternative derivation.** We can think of the second fundamental form as measuring distance from the tangent plane. Consider the Taylor expansion around a point (u,v):

$$\mathbf{r}(u+\delta u, v+\delta v) = \mathbf{r}(u,v) + \delta u\mathbf{r}_u + \delta v\mathbf{r}_v + \frac{1}{2}(\mathbf{r}_{uu}\delta u^2 + 2\mathbf{r}_{uv}\delta u\delta v + \mathbf{r}_{vv}\delta v^2) + \text{h.o.t.}$$

Therefore

$$\begin{aligned}
&(\mathbf{r}(u+\delta u, v+\delta v) - \mathbf{r}(u,v))\cdot\mathbf{n}\\
=&\frac{1}{2}(\mathbf{r}_{uu}\cdot\mathbf{n}\delta u^2 + 2\mathbf{r}_{uv}\cdot\mathbf{n}\delta u\delta v + \mathbf{r}_{vv}\cdot\mathbf{n}\delta v^2) + \text{h.o.t}\\
=&\frac{1}{2}(L\delta u^2 + 2M\delta u\delta v + N\delta v^2) + \text{h.o.t}
\end{aligned}$$

where

$$\begin{aligned}
L &= \mathbf{r}_{uu}\cdot\mathbf{n}\\
M &= \mathbf{r}_{uv}\cdot\mathbf{n}\\
N &= \mathbf{r}_{vv}\cdot\mathbf{n}
\end{aligned}$$

We prefer this definition of L, M and N with second derivatives of $\mathbf{r}$, since usually $\mathbf{n}$ is tricky to differentiate, as there is a square root in the denominators. Note that since $\mathbf{r}_u\cdot\mathbf{n} = 0$, we have

$$\mathbf{r}_{uu}\cdot\mathbf{n} + \mathbf{r}_u\cdot\mathbf{n}_u = 0$$

so

$$\mathbf{r}_{uu}\cdot\mathbf{n} = -\mathbf{r}_u\cdot\mathbf{n}_u = L.$$

Also $\mathbf{r}_v\cdot\mathbf{n} = 0$, so

$$\mathbf{r}_{vv}\cdot\mathbf{n} + \mathbf{r}_v\cdot\mathbf{n}_v = 0$$

which implies that

$$\mathbf{r}_{vv} \cdot \mathbf{n} = -\mathbf{r}_v \cdot \mathbf{n}_v = N$$

and

$$\mathbf{r}_{uv} \cdot \mathbf{n} + \mathbf{r}_u \cdot \mathbf{n}_v = 0$$
$$\mathbf{r}_{vu} \cdot \mathbf{n} + \mathbf{r}_v \cdot \mathbf{n}_u = 0$$

which imply

$$\mathbf{r}_{uv} \cdot \mathbf{n} = -\mathbf{r}_u \cdot \mathbf{n}_v = -\mathbf{r}_v \cdot \mathbf{n}_u = M.$$

Example 5.2. (a) For the plane $L = M = N = 0$.

(b) For the sphere of radius a, $\mathbf{r} = a\mathbf{n}$, so that

$$\mathbf{r}_u \cdot \mathbf{n}_u = -a^{-1}\mathbf{r}_u \cdot \mathbf{r}_u = a^{-1}E.$$

Repeating this argument for F and M and for G and N, we have

$$L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2 = a^{-1}\big(E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2\big)$$

(c) Consider the graph $z = f(x, y)$ of a smooth function $f\colon \mathbb{R}^2 \to \mathbb{R}$. So the parametrisation is $\mathbf{r} = (x, y, f(x, y))$. The second fundamental form is a multiple of the Hessian:

$$\frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

(d) Fix a point $p \in \Sigma$. Since every surface is locally a graph (we didn't prove this but we could have: See Do Carmo Section 2-2, Proposition 3, Page 63, which uses the Inverse Function Theorem. We can parametrise by the inverse of projection onto the tangent plane $T_p\Sigma$. Our point is a critical point of $f(x, y)$, and we can understand the local behaviour: maximum, minimum or saddle point in terms of the Hessian matrix i.e. in terms of the second fundamental form. This will be described soon in detail using the *principal curvatures.*

5.3. **The shape operator.** The shape operator, also known as the Weingarten map, is a function

$$S\colon T_p\Sigma \to T_p\Sigma$$

which we can use to define the second fundamental form in terms of covariant derivatives of the normal vector; the Gauss map $\Sigma \to S^2$.

Define S by

$$\mathbf{v}_p \mapsto -\nabla_{\mathbf{v}_p}\mathbf{n}.$$

So

$$S(\mathbf{r}_u) = -\nabla_{\mathbf{r}_u}\mathbf{n} = -\mathbf{n}_u$$

and

$$S(\mathbf{r}_v) = -\nabla_{\mathbf{r}_v}\mathbf{n} = -\mathbf{n}_v.$$

This measures how the normal vector tilts as we move infinitesimally away from our point. All 3 versions of the second fundamental form are really the same idea in different guises.

Note that

$$S(\mathbf{r}_u) \cdot \mathbf{r}_u = L$$
$$S(\mathbf{r}_u) \cdot \mathbf{r}_v = M$$

$$S(\mathbf{r}_v) \cdot \mathbf{r}_v = N$$

We can define a bilinear form on each tangent space by

$$B(\mathbf{v}, \mathbf{w}) := S(\mathbf{v}) \cdot \mathbf{w}.$$

The matrix of this with respect to the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$ is that of the second fundamental form

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Now, for later, what is the matrix of the shape operator?

Let a, b, c, d be such that

$$S(\mathbf{r}_u) = a\mathbf{r}_u + b\mathbf{r}_v$$

and

$$S(\mathbf{r}_v) = c\mathbf{r}_u + d\mathbf{r}_v.$$

Then dotting with $\mathbf{r}_u$ and $\mathbf{r}_v$ we have:

$$L = S(\mathbf{r}_u) \cdot \mathbf{r}_u = aE + bF$$

$$M = S(\mathbf{r}_u) \cdot \mathbf{r}_v = aF + bG$$

$$M = S(\mathbf{r}_v) \cdot \mathbf{r}_u = cE + dF$$

$$N = S(\mathbf{r}_v) \cdot \mathbf{r}_v = cF + dG$$

Therefore

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

so that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

This matrix need not be symmetric.

Exercise 5.3. (1) Compute the second fundamental form of a surface of revolution.

$$\mathbf{r}(u, v) = \big(f(u) \cos v, f(u) \sin v, u\big).$$

(2) Compute the second fundamental form of the patch of the torus:

$$\mathbf{r}(u, v) = \big((a + b \cos u) \cos v, (a + b \cos u) \sin v, b \sin u\big).$$

The second fundamental form characterises the plane.

Theorem 5.4. *Suppose the second fundamental form of a surface is identically zero. Then the surface is a part of a plane.*

Proof. If $L = M = N = 0$ then $\mathbf{n}_u \cdot \mathbf{r}_u = \mathbf{n}_u \cdot \mathbf{r}_v = \mathbf{n}_v \cdot \mathbf{r}_u = \mathbf{n}_v \cdot \mathbf{r}_v = 0$ so $\mathbf{n}_u$ and $\mathbf{n}_\mathbf{v}$ are parallel to $\mathbf{n}$. Since $\mathbf{n}$ is of constant length we therefore have that $\mathbf{n}$ is constant. $\square$

5.4. **Principal curvature.** We want to consider the curvature of the set of curves on Σ through a fixed point $p \in \Sigma$.

Let $\gamma(t) = \mathbf{r}(u(t), v(t))$ such that $\mathbf{r}(u(0), v(0)) = p$. Suppose furthermore that $|\gamma'(t)| = 1$ i.e. γ is parametrised by arc length. Let $\tau := \gamma'(t)$. We have an orthonormal basis for $T\mathbb{R}^3$ at each point of the image of γ.

$$\{\mathbf{n}, \tau, \mathbf{n} \wedge \tau\}.$$

Recall this frame from the spherical curves section in Chapter 2. The curvature is $k(t) = |\gamma''(t)|$. Since $\gamma' \cdot \gamma' = 1$ we have

$$\gamma'' \cdot \gamma' = 0,$$

so that $\gamma'' \cdot \tau = 0$. Therefore

$$\gamma'' = \kappa_n \mathbf{n} + \kappa_g (\mathbf{n} \wedge \tau)$$

where $\kappa_n = \gamma'' \cdot \mathbf{n}$ is the *normal curvature* and $\kappa_g = \gamma'' \cdot (\mathbf{n} \wedge \tau)$ is the *geodesic curvature* of γ at p. Note that $\kappa_n^2 + \kappa_g^2 = k^2$.

Now, $\tau \cdot \mathbf{n} = 0 = \gamma' \cdot \mathbf{n}$ so $\gamma'' \cdot \mathbf{n} + \gamma' \cdot \mathbf{n}' = 0$. Thus

$$\kappa_n = -\gamma' \cdot \mathbf{n}'.$$

By the chain rule we have

$$\gamma' = u'\mathbf{r}_u + v'\mathbf{r}_v$$
$$\mathbf{n}' = u'\mathbf{n}_u + v'\mathbf{n}_v.$$

So taking the dot product

$$\kappa_n = -\gamma' \cdot \mathbf{n}' = L(u')^2 + 2Mu'v' + N(v')^2.$$

We want to compute the principal curvatures and directions.

Definition 5.5. The principal curvatures of Σ at $p \in \Sigma$ are the maximum and minimum values of the normal curvatures of curves through p.

Consider a plane Π through p which contains $\mathbf{n}, \tau$. Then $\Pi \cap \Sigma$ is a curve $\gamma(t)$ through p, which we will call $\gamma(t)$, parametrised as above by arc length.

Note that $\gamma''(t) \cdot (\mathbf{n} \wedge \tau) = 0$ since $\mathbf{n} \wedge \tau$ is normal to Π. Thus $\kappa_g = 0$ and $\kappa_n = k$. So the principal curvatures are the maximum and minimum curvatures of curves $\Pi \cap \Sigma$ at p where Π ranges over all planes through p containing $\mathbf{n}_p$.

So, we can compute the principal curvatures as the maximum and minimum values of

$$L\xi^2 + 2M\xi\eta + N\eta^2$$

for $(\xi, \eta) \in \mathbb{R}^2$ such that

$$E\xi^2 + 2F\xi\eta + G\eta^2 = 1.$$

This latter condition is that $|\gamma'(t)| = 1$.

Since the first fundamental form is positive definite, this is the maximum/minimum of a function on $\mathbb{R}^2$ restricted to an ellipse. To find the maximum and minimum values, we use simultaneous diagonalisation.

Recall that the spectral theorem from linear algebra says that any real symmetric matrix A can be diagonalised over the real numbers. This means that there is a orthogonal matrix P such that PAP^T is a diagonal matrix. Orthogonal means $PP^T = \mathrm{Id}$.

Apply this to the first fundamental form's matrix. There exists an orthogonal matrix R such that

$$R\begin{pmatrix} E & F \\ F & G \end{pmatrix} R^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1, λ_2 are the eigenvalues of the first fundamental form. The first fundamental form is positive definite so $\lambda_1, \lambda_2 > 0$. Let

$$Q := \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} \end{pmatrix} R.$$

Then

$$Q\begin{pmatrix} E & F \\ F & G \end{pmatrix} Q^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now the matrix $Q\begin{pmatrix} L & M \\ M & N \end{pmatrix} Q^T$ is a symmetric matrix, so there exists another orthogonal matrix P such that

$$PQ\begin{pmatrix} L & M \\ M & N \end{pmatrix} Q^T P^T = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

is diagonal and

$$PQ\begin{pmatrix} E & F \\ F & G \end{pmatrix} Q^T P^T = PP^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We therefore change coordinates to

$$\begin{pmatrix} X \\ Y \end{pmatrix} = PQ\begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

In these coordinates

$$E\xi^2 + 2F\xi\eta + G\eta^2 = X^2 + Y^2$$

and

$$L\xi^2 + 2M\xi\eta + N\eta^2 = \kappa_1 X^2 + \kappa_2 Y^2.$$

The maximum and minimum values of the latter subject to $X^2 + Y^2 = 1$ are κ_1 and κ_2. These are our principal curvatures.

The vectors $(1, 0)$ and $(0, 1)$ in the (X, Y) coordinates are the principal directions.

It turns out that we need to find the eigenvalues and eigenvectors of $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$. To check this:

$$\begin{aligned}
&PQ\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} Q^{-1}P^{-1} \\
=&PQ\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} Q^{T}P^{T} \\
=&PQ\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} Q^{-1}P^{-1}PQ \begin{pmatrix} L & M \\ M & N \end{pmatrix} Q^{T}P^{T} \\
=&\left(PQ\begin{pmatrix} E & F \\ F & G \end{pmatrix} Q^{-1}P^{-1}\right)^{-1} PQ \begin{pmatrix} L & M \\ M & N \end{pmatrix} Q^{T}P^{T} \\
=&\left(PQ\begin{pmatrix} E & F \\ F & G \end{pmatrix} Q^{T}P^{T}\right)^{-1} PQ \begin{pmatrix} L & M \\ M & N \end{pmatrix} Q^{T}P^{T} \\
=&\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)^{-1} PQ \begin{pmatrix} L & M \\ M & N \end{pmatrix} Q^{T}P^{T} \\
=&\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)^{-1} \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \\
=&\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.
\end{aligned}$$

The eigenvalues and eigenvectors are the same as above. Note that $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$ coincides with the matrix which earlier we computed to represent the shape operator. Therefore we can also define the principal curvatures and directions as follows.

Definition 5.6. The *principal curvatures* at p are the eigenvalues of the shape operator at p and the principal directions are the eigenvectors.

We also define two important quantities. The Gaussian curvature in particular is of central importance in the geometry of surfaces.

Definition 5.7. The *Gaussian curvature* K is the product of the principal curvatures $\kappa_1\kappa_2$ which is equal to

$$\det\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}\right) = \frac{LN - M^2}{EG - F^2}.$$

Definition 5.8. The *mean curvature* H is the average $\frac{\kappa_1+\kappa_2}{2}$ of the principal curvatures, which is equal to

$$\frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}\right) = \frac{LG - 2MF + NE}{2(EG - F^2)}.$$

Example 5.9. (a) The Gaussian and mean curvatures of the plane are zero.
(b) The principal curvatures of the sphere of radius a are both $-1/a$, since

$$L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2 = -a^{-1}\big(E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2\big)$$

and therefore

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} -a^{-1} & 0 \\ 0 & -a^{-1} \end{pmatrix}.$$

All directions are principal directions. The mean curvature is $-1/a$ and the Gaussian curvature is a constant $1/a^2$.
(c) The torus with parametrisation

$$\mathbf{r}(u,v) = \big((a+b\cos u)\cos v, (a+b\cos u)\sin v, b\sin u\big).$$

$(u,v) \in (0,2\pi)\times(0,2\pi)$. We compute:

$$\mathbf{r}_u = \big(-b\sin u\cos v, -b\sin u\sin v, b\cos u\big)$$
$$\mathbf{r}_v = \big(-(a+b\cos u)\sin v, (a+b\cos u)\cos v, 0\big),$$

so that

$$\mathbf{r}_u \wedge \mathbf{r}_v = (-b(a+b\cos u)\cos u\cos v, -b(a+b\cos u)\cos u\sin v, -b(a+b\cos u)\sin u).$$

We have

$$|\mathbf{r}_u \wedge \mathbf{r}_v| = b(a+b\cos u)$$

so

$$\mathbf{n} = -(\cos u\cos v, \cos u\sin v, \sin u).$$

Also

$$\mathbf{r}_{uu} = \big(-b\cos u\cos v, -b\cos u\sin v, -b\sin u\big)$$
$$\mathbf{r}_{uv} = \big(b\sin u\sin v, -b\sin u\cos v, 0\big)$$
$$\mathbf{r}_{vv} = \big(-(a+b\cos u)\cos v, -(a+b\cos u)\sin v, 0\big)$$

Therefore we take the relevant dot products to find:

$$E = b^2;\quad F = 0;\quad G = (a+b\cos u)^2$$

and

$$L = b;\quad M = 0;\quad N = (a+b\cos u)\cos u.$$

Thus the Gaussian curvature is given by

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\cos u}{b(a+b\cos u)}.$$

The mean curvature is given by

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{a + 2b\cos u}{2b(a+b\cos u)}.$$

Note that the curvatures are constant along lines of constant u. Topologically the torus is $S^1 \times S^1$, but geometrically the two copies of S^1 are not symmetric. In particular note that when $u = 0$, $K > 0$, when $u = \pm\pi/2$, we have $K = 0$, and when $u = \pi$ we have $K < 0$.

(d) For the catenoid, with parametrisation

$$\mathbf{r}(u,v) = (\cosh v \cos u, \cosh v \sin u, v)$$

with $u \in (0, 2\pi)$ and $v \in \mathbb{R}$, we have

$$\mathbf{r}_u = (-\cosh v \sin u, \cosh v \cos u, 0)$$
$$\mathbf{r}_v = (\sinh v, \cos u, \sinh v \sin u, 1)$$
$$\mathbf{n} = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v).$$

So

$$E = \cosh^2 v; \quad F = 0; \quad G = \cosh^2 v.$$

Also

$$\mathbf{r}_{uu} = -\cosh v(\cos u, \sin u, 0)$$
$$\mathbf{r}_{uv} = \sinh v(-\sin u, \cos u, 0)$$
$$\mathbf{r}_{vv} = \cosh v(\cos u, \sin u, 0)$$

Therefore

$$L = -1; \quad M = 0; \quad N = 1.$$

So the Gaussian curvature is given by

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{1}{\cosh^4 v}$$

The mean curvature is given by

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{-\cosh^2 v - 0 + \cosh^v}{\cosh^4 v} = 0.$$

The catenoid has mean curvature zero; such surfaces are called *minimal*.

Definition 5.10. An asymptotic direction is a unit vector (with respect to the first fundamental form) in the tangent space along which the normal curvature vanishes. To compute, find (a,b) such that

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

and scale so that

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1$$

An asymptotic curve through $p \in \Sigma$ is a curve in Σ along which the normal curvature is zero. So $\gamma\colon t \mapsto \mathbf{r}(u(t), v(t))$, $\gamma(0) = p$, such that $L(u')^2 + 2Mu'v' + N(v')^2 = 0$.

Exercise 5.11. (i) Find the Gaussian and mean curvatures of a graph $z = f(x,y)$.

(ii) Compute principal curvatures and principal directions of the saddle surface $z = xy$ at the origin, and of an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2$ at each of the points where two of x, y and z are zero. Also compute the Gaussian and mean curvatures everywhere of these surfaces.

(iii) Give an expression for the Gaussian curvature of a surface of revolution.

$$\mathbf{r}(u,v) = \big(f(u)\cos v, f(u)\sin v, u\big).$$

(iv) Do Carmo 3.3 Exercises 1–7, 20. Question 6 on the pseudo sphere is a particularly good example: a surface with Gaussian curvature constant and equal to -1. For question 20, an umbilical point is a point where the principal curvatures are equal.

Here is a cool interpretation of the Gaussian curvature in terms of how area is altered by the Gauss map. Recall that the Gaussian curvature is a determinant, so we shouldn't be too surprised that it has an interpretation in terms of area.

Theorem 5.12. *Let $x \in \Sigma$ and let $\{U_i\}$ with $U_i \ni x$ for all i be a collection of open sets in Σ contracting to x. Then*

$$\frac{\text{signed area}(\mathbf{n}(U))}{\text{area}(U)}$$

tends to the Gaussian curvature $K(x)$ as U contracts to x.

Here the area of $\mathbf{n}(U)$ has a sign according to whether $\mathbf{n}$ preserves orientation in S^2, with respect to the standard orientation on $S^2 = \mathbb{C} \cup \{\infty\}$.

Proof. Let $\mathbf{r}\colon V \to U \subseteq \Sigma$ be a parametrisation. The area of U is given by

$$\int_V |\mathbf{r}_u \wedge \mathbf{r}_v| \mathrm{d}u\mathrm{d}v = \int_V \sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v.$$

The area of $\mathbf{n}(U)$ is given by

$$\int_V |\mathbf{n}_u \wedge \mathbf{n}_v| \mathrm{d}u\mathrm{d}v,$$

however to find the signed area we instead compute

$$\int_V \mathbf{n} \cdot (\mathbf{n}_u \wedge \mathbf{n}_v)\mathrm{d}u\mathrm{d}v.$$

The vector $\mathbf{n}_u \wedge \mathbf{n}_v$ points in the direction of $\mathbf{n}$ or in the opposite direction, so taking the indicated dot product with the unit vector $\mathbf{n}$ gives the correct sign. Now

$$\begin{aligned}
\mathbf{n} \cdot (\mathbf{n}_u \wedge \mathbf{n}_v) =& \frac{(\mathbf{r}_u \wedge \mathbf{r}_v) \cdot (\mathbf{n}_u \wedge \mathbf{n}_v)}{|\mathbf{r}_u \wedge \mathbf{r}_v|} \\
=& \frac{(\mathbf{r}_u \cdot \mathbf{n}_u)(\mathbf{r}_v \cdot \mathbf{n}_v) - (\mathbf{r}_u \cdot \mathbf{n}_v)(\mathbf{r}_v \cdot \mathbf{n}_u)}{|\mathbf{r}_u \wedge \mathbf{r}_v|} \\
=& \frac{LN - M^2}{\sqrt{EG - F^2}}
\end{aligned}$$

Thus the stated quotient is

$$\frac{\int_V \frac{LN-M^2}{\sqrt{EG-F^2}}\mathrm{d}u\mathrm{d}v}{\int_V \sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v}.$$

In the limit, by continuity, as U shrinks to x, V shrinks to the preimage of x, and the integral tends to the value at x, which is

$$\frac{LN - M^2}{EG - F^2} = K$$

as claimed. $\square$

Corollary 5.13. *Suppose Σ is convex and compact. Then*

$$\int_\Sigma K \mathrm{d}A = 4\pi.$$

Proof. The Gauss map is a bijection in this case. $K\mathrm{d}A = \mathrm{area}(\mathbf{n}(\mathrm{d}A))$, $\int_\Sigma K\mathrm{d}A = \int_\Sigma \mathrm{area}(\mathbf{n}(\mathrm{d}A))$ which is the area of the unit sphere. $\square$

This is a remarkable corollary in itself. Think of all the possible convex compact surfaces one could make by deforming a 2-sphere. The curvature can be highly variable. Nevertheless the *integral* of the curvature always conspires to be 4π. Pretty-pretty-pretty good. (Said in Curb Your Enthusiasm style.)

5.5. **The Theorema Egregium of Gauss.** The theorem of this section is so remarkable that even Gauss thought so, and he called it the Theorema Egregium, which is latin for 'remarkable theorem'. Recall that the principal curvatures are defined in terms of both the first and second fundamental form. The second fundamental form was measuring how a surface curves away from its tangent plane near a point, and it seems like such a notion is vital for the study of curvature. Amazingly, the product of the principal curvatures is intrinsic.

Theorem 5.14 (Gauss)**.** *The Gaussian curvature is an intrinsic quantity. That is, it can be computed purely in terms of the coefficients of the first fundamental form and their derivatives. Another way to say this is that the Gaussian curvature is invariant under local isometries.*

The Gaussian curvature has less information than the two principal curvatures separately, but it has the great advantage that it only depends on the first fundamental form, so makes sense as a notion on an abstract smooth surface with a first fundamental form, i.e. if we defined a surface which is not embedded in $\mathbb{R}^3$, but exists as an abstract notion.

Proof. We begin by defining the Christoffel symbols $\Gamma^k_{ij} = \Gamma^k_{ji}$, where each of i, j, k is from $\{1, 2\}$. Recall that $\{\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}\}$ is a basis for $T\mathbb{R}^3$ at each point of a surface Σ. Therefore we can write the second derivatives at each point in terms of this basis.

$$\begin{aligned}
\mathbf{r}_{uu} &= \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + L\mathbf{n} \\
\mathbf{r}_{uv} &= \Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + M\mathbf{n} \\
\mathbf{r}_{vv} &= \Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v + N\mathbf{n}
\end{aligned}$$

Note the appearance of L, M and N as the coefficients of $\mathbf{n}$, by their definitions. Note that the Γ^k_{ij} are also functions of (u, v).

Our proof is in two steps:

(1) Express the Christoffel symbols in terms of the coefficients E, F and G of the first fundamental form.
(2) Express the Gaussian curvature in terms of the Christoffel symbols.

Combining these two completes the proof of the theorem. The first step is accomplished in the following lemma.

Lemma 5.15. *The Christoffel symbols can be expressed in terms of the coefficients of the first fundamental form, as follows.*

$$\begin{pmatrix}\Gamma^1_{11}\\ \Gamma^2_{11}\end{pmatrix} = \begin{pmatrix}E & F\\ F & G\end{pmatrix}^{-1} \begin{pmatrix}\frac{1}{2}E_u\\ F_u - \frac{1}{2}E_v\end{pmatrix}$$
$$\begin{pmatrix}\Gamma^1_{12}\\ \Gamma^2_{12}\end{pmatrix} = \begin{pmatrix}E & F\\ F & G\end{pmatrix}^{-1} \begin{pmatrix}\frac{1}{2}E_v\\ \frac{1}{2}G_u\end{pmatrix}$$
$$\begin{pmatrix}\Gamma^1_{22}\\ \Gamma^2_{22}\end{pmatrix} = \begin{pmatrix}E & F\\ F & G\end{pmatrix}^{-1} \begin{pmatrix}F_v - \frac{1}{2}G_u\\ \frac{1}{2}G_v\end{pmatrix}$$

Proof of Lemma 5.15. Start with $E = \mathbf{r}_u \cdot \mathbf{r}_u$. Differentiating yields

$$\frac{1}{2}E_u = \mathbf{r}_{uu} \cdot \mathbf{r}_u = \Gamma^1_{11}E + \Gamma^2_{11}F.$$

In a similar vein we compute the derivatives of the first fundamental form coefficients.

$$\begin{aligned}
\frac{1}{2}E_v &= \mathbf{r}_{uv} \cdot \mathbf{r}_u = \Gamma^1_{12}E + \Gamma^2_{12}F\\
F_u &= \mathbf{r}_{uu} \cdot \mathbf{r}_v + \mathbf{r}_u \cdot \mathbf{r}_{uv} = \Gamma^1_{11}F + \Gamma^2_{11}G + \Gamma^1_{12}E + \Gamma^2_{12}F\\
F_v &= \mathbf{r}_{uv} \cdot \mathbf{r}_v + \mathbf{r}_u \cdot \mathbf{r}_{vv} = \Gamma^1_{12}F + \Gamma^2_{12}G + \Gamma^1_{22}E + \Gamma^2_{22}F\\
\frac{1}{2}G_u &= \mathbf{r}_{uv} \cdot \mathbf{r}_v = \Gamma^1_{12}F + \Gamma^2_{12}G\\
\frac{1}{2}G_v &= \mathbf{r}_{vv} \cdot \mathbf{r}_v = \Gamma^1_{22}F + \Gamma^2_{22}G.
\end{aligned}$$

Rearranging yields the claimed equations:

$$\begin{aligned}
\begin{pmatrix}\frac{1}{2}E_u\\ F_u - \frac{1}{2}E_v\end{pmatrix} &= \begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}\Gamma^1_{11}\\ \Gamma^2_{11}\end{pmatrix}\\
\begin{pmatrix}\frac{1}{2}E_v\\ \frac{1}{2}G_u\end{pmatrix} &= \begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}\Gamma^1_{12}\\ \Gamma^2_{12}\end{pmatrix}\\
\begin{pmatrix}F_v - \frac{1}{2}G_u\\ \frac{1}{2}G_v\end{pmatrix} &= \begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}\Gamma^1_{22}\\ \Gamma^2_{22}\end{pmatrix}
\end{aligned}$$

This completes the proof of the lemma. □

We continue with the proof of the Theorema Egregium; we now need to show step (2), that the Gaussian curvature can be expressed in terms of the Christoffel symbols.

Next we recall the matrix of the negative of the shape operator $-S\colon \mathbf{v} \mapsto -\nabla_{\mathbf{v}}\mathbf{n}$, which acts on the tangent space of a surface, sending

$$\mathbf{r}_u \mapsto \mathbf{n}_u = a_{11}\mathbf{r}_u + a_{21}\mathbf{r}_v$$
$$\mathbf{r}_v \mapsto \mathbf{n}_v = a_{12}\mathbf{r}_u + a_{22}\mathbf{r}_v$$

So $\mathbf{n}_u = -S(\mathbf{r}_u)$, $\mathbf{n}_v = -S(\mathbf{r}_v)$. The matrix of the negative of the shape operator with respect to the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$ is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

We saw before that this matrix is

$$-Z = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} FM - GL & FN - GM \\ FL - EM & FM - EN \end{pmatrix}.$$

Therefore

$$\mathbf{n}_u = \frac{1}{EG - F^2}\big((FM - GL)\mathbf{r}_u + (FL - EM)\mathbf{r}_v\big)$$

and

$$\mathbf{n}_v = \frac{1}{EG - F^2}\big((FN - GM)\mathbf{r}_u + (FM - EN)\mathbf{r}_v\big).$$

Now for the coup de grace. We equate $(\mathbf{r}_{uu})_v = (\mathbf{r}_{uv})_u$, in particular the coefficients of $\mathbf{r}_u$ and $\mathbf{r}_v$ in this equation.

Take

$$\mathbf{r}_{uu} = \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + L\mathbf{n}$$

and differentiate:

$$\begin{aligned}(\mathbf{r}_{uu})_v =& (\Gamma^1_{11})_v\mathbf{r}_u + \Gamma^1_{11}\mathbf{r}_{uv} + (\Gamma^2_{11})_v\mathbf{r}_v + \Gamma^2_{11}\mathbf{r}_{vv} + L_v\mathbf{n} + L\mathbf{n}_v \\ =& (\Gamma^1_{11})_v\mathbf{r}_u + \Gamma^1_{11}(\Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + M\mathbf{n}) + (\Gamma^2_{11})_v\mathbf{r}_v \\ &+ \Gamma^2_{11}(\Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v + N\mathbf{n}) + L_v\mathbf{n} + L(-S(\mathbf{r}_v)).\end{aligned}$$

This is equal to $(\mathbf{r}_{uv})_u = (\Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + M\mathbf{n})_u$, which is:

$$\begin{aligned}(\mathbf{r}_{uv})_u =& (\Gamma^1_{12})_u\mathbf{r}_u + \Gamma^1_{12}\mathbf{r}_{uu} + (\Gamma^2_{12})_u\mathbf{r}_v + \Gamma^2_{12}\mathbf{r}_{uv} + M_u\mathbf{n} + M\mathbf{n}_u \\ =& (\Gamma^1_{12})_u\mathbf{r}_u + \Gamma^1_{12}(\Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + L\mathbf{n}) + (\Gamma^2_{12})_u\mathbf{r}_v \\ &+ \Gamma^2_{12}(\Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + M\mathbf{n}) + M_u\mathbf{n} + M(-S(\mathbf{r}_u)).\end{aligned}$$

Recall that $S(\mathbf{r}_u) = a_{11}\mathbf{r}_u + a_{21}\mathbf{r}_v$ and $S(\mathbf{r}_v) = a_{12}\mathbf{r}_u + a_{22}\mathbf{r}_v$. Equate coefficients of $\mathbf{r}_v$ to get

$$\Gamma^1_{11}\Gamma^2_{12} + (\Gamma^2_{11})_v + \Gamma^2_{11}\Gamma^2_{22} + La_{22} = \Gamma^1_{12}\Gamma^2_{11} + (\Gamma^2_{12})_u + \Gamma^2_{12}\Gamma^2_{12} + Ma_{21}.$$

Then note that

$$La_{22} - Ma_{21} = \frac{LFM - LEN - MFL + M^2E}{EG - F^2} = \frac{-E(LN - M^2)}{EG - F^2} = -EK.$$

So

$$EK = (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^2_{12}\Gamma^2_{12} - \Gamma^1_{12}\Gamma^2_{11}.$$

This expresses the Gaussian curvature K in terms of the Christoffel symbols and E, so completely in terms of the first fundamental form. This completes the proof.

Since, for any local isometry, we have coordinates with respect to which the first fundamental forms are the same, this proves that the Gaussian curvature is an invariant of local (and therefore global) isometries. □

Remark 5.16. The *Gauss equations* can be derived in a similar fashion to the equation above, by equating coefficients of $\mathbf{r}_u$ and $\mathbf{r}_v$ in $(\mathbf{r}_{uu})_v = (\mathbf{r}_{uv})_v$ and $(\mathbf{r}_{vv})_u = (\mathbf{r}_{uv})_v$. They are, in addition to the equation above:

$$FK = (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^1_{22}$$
$$FK = (\Gamma^2_{12})_v - (\Gamma^2_{22})_u + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11}$$
$$GK = (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{11}\Gamma^1_{22} + \Gamma^1_{12}\Gamma^1_{22} - \Gamma^2_{12}\Gamma^1_{22} - \Gamma^1_{12}\Gamma^1_{12}$$

Exercise 5.17. The formula for K becomes a lot simpler when $F = 0$. Show that in this case

$$K = \frac{-1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right),$$

and then use this formula to compute the Gaussian curvature of the sphere

$$\mathbf{r}(\theta, \phi) = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, a\sin\phi)$$

Note on exam question 4: Saying that two surfaces have different first fundamental forms with some coordinates does *not* imply that the surfaces are not isometric. One has to show that the surfaces could not have the same first fundamental forms after *any* change of coordinates. This is where the Gaussian curvature comes in. As an invariant, it is the same in all parametrisations. Therefore is one surface has constant $K = 0$ and another has constant $K = 1$, those surfaces cannot be locally isometric.

Exercise 5.18. Do Carmo Section 4-3, Questions 1, 3, 6, 8, 9.

References

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