M435: INTRODUCTION TO DIFFERENTIAL GEOMETRY

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Contents

6. Geodesics

We will give definitions of geodesics in terms of length minimising curves, in terms of the geodesic curvature vanishing, in terms of the covariant derivative of vector fields, and in terms of a set of equations. We will attempt to show that (some of) these approaches are equivalent.

6.1. Covariant Derivatives recap.

Let's begin by recalling covariant derivatives. A *vector field* on a surface Σ is a smooth function $\mathbf{v}\colon \Sigma \to T\Sigma$ such that $\mathbf{v}(p) = \mathbf{v}_p \in T_p\Sigma$ for all $p \in \Sigma$.

By considering a vector field as a vector field on $\mathbb{R}^3$ restricted to Σ, we defined the covariant derivative $\nabla_{\mathbf{v}}\mathbf{w}$ of $\mathbf{w}$ along $\mathbf{v}$, by computing $\nabla(\mathbf{w}) \cdot \mathbf{v}$.

In the special case that $\mathbf{w} = \mathbf{n}$ the normal vector, and when $\mathbf{v}$ is a tangent vector, the covariant derivative is again a tangent vector: since the normal vector is constant length, its derivative is orthogonal to it, whence is a tangent vector.

When $\mathbf{v} = a\mathbf{r}_u + b\mathbf{r}_v$, then $\nabla_{\mathbf{v}}(\mathbf{w}) = a\mathbf{w}_u + b\mathbf{w}_v$. This completes our recap.

6.2. Covariant derivative projected to tangent plane.

To define the covariant derivative along $\mathbf{v}$ as a function $T_p\Sigma \to T_p\Sigma$, we need to project onto the tangent plane.

Definition 6.1. Let $\mathbf{v}, \mathbf{w}$ be vector fields on Σ. The (intrinsic) *covariant derivative* at p of $\mathbf{w}$, $\widetilde{\nabla}_{\mathbf{v}}(\mathbf{w})$ is defined to be the projection of the covariant derivative at p of the vectors in $\mathbb{R}^3$ onto the tangent plane $T_p\Sigma$.

Example 6.2. We'll do an example on S^2. The parametrisation for today is

$$\mathbf{r}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v).$$

Consider the vector field $\mathbf{w} = (0,1)$ in the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$. we have:

$$\mathbf{r}_u = (-\sin u \sin v, \cos u \sin v, 0)$$

$$\mathbf{r}_v = (\cos u \cos v, \sin u \cos v, -\sin v)$$

$$\mathbf{n} = -\mathbf{r}(u,v)$$

In $\mathbb{R}^3$,

$$\nabla_{\mathbf{r}_u}(\mathbf{w}) = \nabla_{\mathbf{r}_u}(\mathbf{r}_v) = \mathbf{r}_{vu}$$

while

$$\nabla_{\mathbf{r}_v}(\mathbf{w}) = \nabla_{\mathbf{r}_v}(\mathbf{r}_v) = \mathbf{r}_{vv} = \mathbf{n}.$$

So

$$\widetilde{\nabla}_{\mathbf{r}_v}(\mathbf{w}) = 0$$

and $\mathbf{r}_{vu} \cdot \mathbf{n} = M = 0$, so $\mathbf{r}_{vu}$ is a tangent vector. Therefore

$$\widetilde{\nabla}_{\mathbf{r}_u}(\mathbf{w}) = \widetilde{\nabla}_{\mathbf{r}_u}(\mathbf{r}_v) = \mathbf{r}_{vu} = (-\sin u \cos v, \cos u \cos v, 0).$$

On the equator, $\cos v = 0$, and so $\widetilde{\nabla}_{\mathbf{v}}(\mathbf{w}) = 0$ for all $\mathbf{v} \in T_pS^2$. Off the equator, the covariant derivative $\widetilde{\nabla}_{\mathbf{v}}(\mathbf{w})$ can be nonzero in the u-coordinate direction.

If I walk east with my left arm always pointing north, then I have to turn, unless I am on the equator. (Imagine a tiny circle around the north pole.)

6.3. **Parallel vector fields.** Let $\alpha\colon I \to \Sigma$ be a smooth path.

Definition 6.3. A vector field $\mathbf{w}$ defined on α is called *parallel* along α if $\widetilde{\nabla}_{\alpha'(t)}(\mathbf{w}) = 0$ for all $t \in I$. We denote this covariant derivative by $\frac{D\mathbf{w}}{Dt}$, where the curve α is implicit.

The next proposition gives us a good idea of what parallel is about.

Proposition 6.4. *Let $\mathbf{w}, \mathbf{v}$, vector fields on Σ, be parallel along $\alpha\colon I \to \Sigma$. Then $\mathbf{w} \cdot \mathbf{v}$ is constant.*

So in particular angles are constant and the length of the vectors are constant, taking $\mathbf{w} = \mathbf{v}$.

Proof. We want to see that the derivative of $\mathbf{w} \cdot \mathbf{v}$ is zero:

$$(\mathbf{w} \cdot \mathbf{v})' = \mathbf{w}' \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v}'.$$

Since $\mathbf{w}'$ is parallel to $\mathbf{n}$, and $\mathbf{v}$ is a tangent vector, their dot product vanishes. Similarly $\mathbf{w} \cdot \mathbf{v}' = 0$. Thus $\mathbf{w} \cdot \mathbf{v}$ is constant as claimed. □

Theorem 6.5 (Parallel transport). *Let $\alpha\colon I \to \Sigma$ be a smooth path, and let $\mathbf{w}_0 \in T_{\alpha(t_0)}\Sigma$ be a vector, with $t_0 \in I$. There exists a unique vector field $\mathbf{w}(t)$ parallel along α. (i.e. $\widetilde{\nabla}_{\alpha'(t)}(\mathbf{w}) = \frac{D\mathbf{w}}{Dt} = 0$.) At $t_1 \in I$, the vector $\mathbf{w}_1 = \mathbf{w}(t_1)$ is called the* parallel transport *of $\mathbf{w}_0$ along α at $\alpha(t_1)$.*

We won't give the proof of this theorem.

Note that this was an extrinsic definition of the notion of a parallel vector field, since we used the covariant derivative in $\mathbb{R}^3$ and then projected to the tangent plane. One can make an intrinsic definition, depending only on the first fundamental form. (This is related to connections and the Levi-Civita connection, for those wanting to study Riemannian geometry.)

Exercise 6.6. Do Carmo, Section 4-4: 9, 15. (Read Example 1).

6.4. **Geodesics as parallel curves.** Geodesics are the trajectories of particles which move on Σ, subject to no forces other than the requirement to remain on Σ.

Definition 6.7 (Geodesic (1))**.** A smooth regular curve $\alpha\colon I \to \Sigma$ is a *geodesic* if $\frac{D}{Dt}(\alpha'(t)) = 0$.

i.e. the tangent vector to α is parallel along α.

Note that $\frac{D}{Dt}(\alpha'(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(\alpha'(t)) - \alpha'(t)\cdot\mathbf{n}$; recall that $\{\alpha', \mathbf{n}, \alpha' \wedge \mathbf{n}\}$ is an orthogonal basis for $T\mathbb{R}^3$ at each point of α. The condition that $\frac{D}{Dt}(\alpha'(t)) = 0$ implies that $\alpha'' \cdot \alpha' = 0$, so $\alpha' \cdot \alpha' = |\alpha'|$ is a constant. Therefore we can assume that α is parametrised by arc length.

Also $\alpha'' \cdot (\alpha' \wedge \mathbf{n}) = \kappa_g = 0$, so α *is a geodesic if and only if the geodesic curvature is zero.*

6.5. **The geodesic equations.** Let's find the intrinsic geodesic equations using the condition that $\alpha''(t)$ is parallel to $\mathbf{n}$. Assume $\alpha\colon I \to \Sigma$ is parametrised by arc length. Define $u(t)$ and $v(t)$ by the equation $\alpha(t) = \mathbf{r}(u(t), v(t))$. So

$$\alpha'(t) = \mathbf{r}_u u'(t) + \mathbf{r}_v v'(t).$$

So $\alpha''(t)$ is normal to Σ if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_u u' + \mathbf{r}_v v') \cdot \mathbf{r}_u = 0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_u u' + \mathbf{r}_v v') \cdot \mathbf{r}_v.$$

So

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_u u' + \mathbf{r}_v v') \cdot \mathbf{r}_u \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big((\mathbf{r}_u u' + \mathbf{r}_v v') \cdot \mathbf{r}_u\big) - \Big((\mathbf{r}_u u' + \mathbf{r}_v v') \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_u)\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(Eu' + Fv') - (\mathbf{r}_u u' + \mathbf{r}_v v') \cdot (\mathbf{r}_{uu} u' + \mathbf{r}_{uv} v') \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(Eu' + Fv') - (\mathbf{r}_u \cdot \mathbf{r}_{uu}(u')^2 + (\mathbf{r}_u \cdot \mathbf{r}_{uv} + \mathbf{r}_v \cdot \mathbf{r}_{uu})u'v' + \mathbf{r}_v \cdot \mathbf{r}_{uv}(v')^2) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(Eu' + Fv') - \frac{1}{2}(E_u(u')^2 + 2F_u u'v' + G_u(v')^2).
\end{aligned}$$

A similar calculation with $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_u u' + \mathbf{r}_v v') \cdot \mathbf{r}_v = 0$ yields the *geodesic equations.*

Proposition 6.8. *$\alpha\colon I \to \Sigma$ is a geodesic if and only if $\alpha(t) = \mathbf{r}(u(t), v(t))$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}(Eu' + Fv') = \frac{1}{2}(E_u(u')^2 + 2F_u u'v' + G_u(v')^2)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(Fu' + Gv') = \frac{1}{2}(E_v(u')^2 + 2F_v u'v' + G_v(v')^2)$$
$$E(u')^2 + 2Fu'v'G(v')^2 = 1$$

The last equation can be replaced with the condition that α is parametrised by arc length.

Note that these equations only depend on the first fundamental form, so we have an intrinsic description of geodesics.

Example 6.9. (i) Geodesics on the plane are (segments of) straight lines.
(ii) Geodesics on the sphere are (segments of) great circles.

Exercise 6.10. (1) Check that $\alpha\colon \mathbb{R} \to \mathbb{R}^2 \subset \mathbb{R}^3$, given by $\alpha(t) = \mathbf{a} + \mathbf{b}t$, where $\mathbf{a} = (a_1, a_2, 0)$ and $\mathbf{b} = (b_1, b_2, 0)$ with $b_1^2 + b_2^2 = 1$, satisfy the geodesic equations for the plane with first fundamental form $\mathrm{d}u^2 + \mathrm{d}v^2$.
(2) Consider the sphere $\mathbf{r}(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$ and let $\alpha\colon (0, 2\pi) \to S^2$ be defined by $u(t) = t$, $v(t) = c$ where $c \in (0, \pi)$ is a constant. For what value(s) of c does this curve satisfy the geodesic equations?

We will study geodesics of surfaces of revolution in more detail soon, after one more theorem.

6.6. **Length minimising curves.** We could have developed the theory of geodesics by starting with the definition as a curve which locally minimises arc length, in the following sense.

Definition 6.11 (Geodesic (2))**.** A smooth regular curve $\alpha\colon [a, b] \to \Sigma$ is a geodesic if for every smooth family of smooth curves

$$\{\alpha_s\colon [a, b] \to \Sigma \mid s \in (-\varepsilon, \varepsilon)\},$$

such that $\alpha_0 = \alpha$, $\alpha_s(a) = \alpha(a)$ and $\alpha_s(b) = \alpha(b)$ for all $s \in (-\varepsilon, \varepsilon)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}(L(\alpha_s))|_{s=0} = 0.$$

So roughly, a geodesic is a curve which is a critical point of the function $\{\text{curves}\} \to \mathbb{R}_{\geq 0}$ given by the length function. Since we can always perturb a curve to make it longer, critical points are *local* minima.

To see why we need to also say local, note that there are two geodesics between any two non-antipodal points on S^2, and they are of different lengths.

Theorem 6.12. *The two definitions of geodesic are equivalent.*

Proof. Suppose

$$\alpha_s = \mathbf{r}(u(s, t), v(s, t)).$$

The length function is

$$L(\alpha_s) = \int_a^b \sqrt{R(s, t)}\mathrm{d}t$$

where

$$R(s,t) = E(u')^2 + 2Fu'v' + G(v')^2.$$

Here, $u' = \frac{\partial u}{\partial t}$ and $v' = \frac{\partial v}{\partial t}$, since u and v are functions of two variables. The derivative of L is

$$\frac{\mathrm{d}}{\mathrm{d}s}L(\alpha_s) = \frac{1}{2}\int_a^b \frac{1}{\sqrt{R}}\frac{\partial R}{\partial s}\mathrm{d}t;$$

Since $[a,b]$ is compact and R and its partial derivatives are continuous, we may differentiate under the integral sign. Now we compute:

$$\frac{\partial R}{\partial s} = (E_u(u')^2 + 2F_u u'v' + G_u(v')^2)\frac{\partial u}{\partial s} + (E_v(u')^2 + 2F_v u'v' + G_v(v')^2)\frac{\partial v}{\partial s} + 2(Eu' + Fv')\frac{\partial u'}{\partial s} + 2(Fu' + Gv')\frac{\partial v'}{\partial s}.$$

Note that $\frac{\partial u'}{\partial s}$ is strange notation for $\frac{\partial^2 u}{\partial s \partial t}$ and $\frac{\partial v'}{\partial s} = \frac{\partial^2 v}{\partial s \partial t}$. Next we use integration by parts.

$$\int_a^b \frac{(Eu' + Fv')}{\sqrt{R}}\frac{\partial^2 u}{\partial s \partial t}\mathrm{d}t = \left[\left(\frac{(Eu' + Fv')}{\sqrt{R}}\right)\frac{\partial u}{\partial s}\right]_a^b - \int_a^b \frac{\partial}{\partial t}\left(\frac{(Eu' + Fv')}{\sqrt{R}}\right)\frac{\partial u}{\partial s}\mathrm{d}t.$$

Note that $\frac{\partial u}{\partial s}(a) = \frac{\partial u}{\partial s}(b) = 0$ since all $\alpha_s(a) = \alpha(a)$ and $\alpha_s(b) = \alpha(b)$ for all $s \in (-\varepsilon, \varepsilon)$. Therefore, performing a similar computation with

$$\int_a^b \frac{(Fu' + Gv')}{\sqrt{R}}\frac{\partial^2 v}{\partial s \partial t}\mathrm{d}t$$

we have:

$$\frac{\mathrm{d}}{\mathrm{d}s}(L(\alpha_s))|_{s=0} = \int_a^b \left\{P\frac{\partial u}{\partial s} + Q\frac{\partial v}{\partial s}\right\}|_{s=0}\mathrm{d}t$$

where

$$P = \frac{1}{2\sqrt{R}}(E_u(u')^2 + 2F_u u'v' + G_u(v')^2) - \frac{\partial}{\partial t}\left(\frac{(Eu' + Fv')}{\sqrt{R}}\right)$$

and

$$Q = \frac{1}{2\sqrt{R}}(E_v(u')^2 + 2F_v u'v' + G_v(v')^2) - \frac{\partial}{\partial t}\left(\frac{(Fu' + Gv')}{\sqrt{R}}\right).$$

If α is parametrised by arc length, then $R = 1$ and if the geodesic equations are satisfied, we have $P = Q = 0$, so $\frac{\mathrm{d}}{\mathrm{d}s}(L(\alpha_s))|_{s=0} = 0$. So Definition (1) implies Definition (2).

Conversely, suppose that $\frac{\mathrm{d}}{\mathrm{d}s}(L(\alpha_s))|_{s=0} = 0$ for *all* families α_s. We want to show that $P(t) = Q(t) = 0$ for all $t \in [a,b]$.

Suppose for a contradiction that $P(t_0) \neq 0$ for some $t_0 \in (a,b)$. If $P(t_0) \neq 0$ for some $t_0 \in (a,b)$. If $P(t_0) > 0$ then there exists an interval $J = (t_0 - \delta, t_0 + \delta)$, on which $P(t) > \frac{1}{2}P(t_0) > 0$ for all $t \in J$ by continuity.

Choose a smooth function $\Phi\colon [a,b] \to [0,1]$ with $\Phi(t_0) = 1$ and $\Phi(t) = 0$ for $t \ni (t_0 - \delta, t_0 + \delta)$. Define

$$u(s,t) = u(t) + s\Phi(t); \quad v(s,t) = v(t)$$

where $s \in (-\varepsilon, \varepsilon)$. So $\alpha_s(t) = \mathbf{r}(u,(s,t), v(s,t))$ as above. Then $\frac{\partial u}{\partial s} = \Phi(t)$ and $\frac{\partial v}{\partial s} = 0$. We therefore have

$$\frac{\mathrm{d}}{\mathrm{d}s}(L(\alpha_s))|_{s=0} = \int_a^b P\Phi(t)\mathrm{d}t \geq \frac{1}{2}P(t_0)\int_a^b \phi(t)\mathrm{d}t > 0.$$

This contradicts the hypothesis that $\frac{\mathrm{d}}{\mathrm{d}s}(L(\alpha_s))|_{s=0} = 0$. Therefore $P(t) = 0$ for all $t \in [a,b]$. Similarly we may argue that $Q(t) = 0$ for all $t \in [a,b]$. Therefore locally length minimising curves (Geodesic Definition (2)) satisfy the geodesic equations, so are geodesics in the sense of Definition (1), and the two definitions are equivalent as claimed. □

The following is a useful fact to know.

Theorem 6.13. *For each point p in Σ and for each direction $\mathbf{v} \in T_p\Sigma$ with $|\mathbf{v}| = 1$, there is a unique geodesic through p whose tangent vector at p is $\mathbf{v}$.*

The proof follows from an existence theorem for differential equations.

6.7. **Geodesics on surfaces of revolution.** We'll consider the surface of revolution defined by a single smooth positive function $f\colon \mathbb{R} \to \mathbb{R}_{>0}$.

$$\mathbf{r}(u,v) = (f(u)\cos v, f(u)\sin v, u).$$

We compute:

$$\mathbf{r}_u = (f'(u)\cos v, f'(u)\sin v, 1)$$
$$\mathbf{r}_v = (-f(u)\sin v, f(u)\cos v, 0)$$
$$E = f'(u)^2 + 1; \quad F = 0; \quad G = f(u)^2.$$

The geodesic equations:

$$(1)\ \frac{\mathrm{d}}{\mathrm{d}t}(Eu' + Fv') = \frac{1}{2}(E_u(u')^2 + 2F_u u'v' + G_u(v')^2)$$
$$(2)\ \frac{\mathrm{d}}{\mathrm{d}t}(Fu' + Gv') = \frac{1}{2}(E_v(u')^2 + 2F_v u'v' + G_v(v')^2)$$
$$(3)\ E(u')^2 + 2Fu'v'G(v')^2 = 1$$

become the following. The first (1) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(u' + f'^2u') = \frac{1}{2}(2f'f''u'^2 + 2ff'v'^2),$$

which becomes

$$u'' + f'^2u'' + 2f'f''u'^2 = f'f''u'^2 + ff'v'^2 \ (1).$$

The second equation (2) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(f^2v') = 0 \ (2)$$

so that $f^2v' = c$ is constant. The third equation (3) gives

$$(1 + f'^2)u'^2 + f^2v'^2 = 1 \ (3).$$

From now on when we refer to equations (1), (2) and (3) we mean their specific forms above for surfaces of revolution.

Claim. *If $u(t), v(t)$ solve equations (2) and (3) then they solve equation (1) too as long as $u'(t) \neq 0$.*

Exercise 6.14. Differentiate equation (3) and equation (2). Rearrange the derivative of equation (2) to obtain an expression for v''. Substitute this for v'' in the derivative of equation (3). Factor out u'. Then assuming that $u' \neq 0$, show that equation (1) is satisfied. This proves the claim.

We (you) deal with the case of parallels next.

Exercise 6.15. Show that the curves given by $u(t) = a$, where a is a constant, and $v(t) = t/f(a)$ are geodesics if and only if $f'(a) = 0$.

We can relate this to what we already know about geodesics on the sphere.

Also, note that if $c = 0$ we obtain from equation (2), we have $v' = 0$ so $v = A$ a constant. Thus the meridians of a surface of revolution are always geodesics.

We now compute formulae for general geodesics which are not meridians nor parallels. By equation (2), $f^2 v' = c$. Therefore

$$v'^2 = \frac{c^2}{f^4}.$$

Substituting this into (3), which is

$$(1 + f'^2)u'^2 + f^2 v'^2 = 1,$$

we have

$$(1 + f'^2)u'^2 + \frac{c^2}{f^2} = 1.$$

Rearranging we have

$$u' = \frac{1}{f}\sqrt{\frac{f^2 - c^2}{1 + f^2}}.$$

Suppose that we can solve this differential equation to find $u(t)$: Then, assuming we can perform this integration, we know $u(t)$. We can then write an integral for $v(t)$:

$$v(t) = \int \frac{c}{f(u(t))^2}\mathrm{d}t + B,$$

where again B is a constant of integration.

Exercise 6.16. For the cylinder with $f(u) = 1$ for all $u \in \mathbb{R}$, find $u(t)$ and $v(t)$ up to the constants c, A and B. Show that we have a helix, and find the slope in terms of c. Find the equations of all geodesics between $(1, 0, 0)$ and $(1, 0, 1)$.

Proposition 6.17 (Clairaut's relation)**.** *Let $\alpha \colon I \to \Sigma$ be a geodesic on a surface of revolution. Let ρ be the distance of $\alpha(t)$ from the z-axis, and let ϕ be the angle between $\alpha'(t)$ and the parallel u = constant through $\alpha(t)$. Then the quantity*

$$\rho \cos\phi$$

is a constant for all t.

Proof. The parallel is the curve with tangent vector $\mathbf{r}_v$. The angle of $\alpha'(t) = \mathbf{r}_u u' + \mathbf{r}_v v'$ (which has length 1 since we may assume α is parametrised by arc length) with $\mathbf{r}_v$ is given by

$$\cos\phi = \frac{(\mathbf{r}_u u' + \mathbf{r}_v v')\cdot \mathbf{r}_v}{\sqrt{\mathbf{r}_v \cdot \mathbf{r}_v}} = \frac{Fu' + Gv'}{\sqrt{G}} = f(u)v'.$$

Since $f(u) = \rho$ we have $\rho\cos\phi = f(u)^2 v'$ which is a constant by equation (2). □

In fact Clairaut's relation holds for more general surfaces of revolution

$$\mathbf{r}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

where f, g parametrise a smooth curve in the x-z plane which misses the z-axis.

Exercise 6.18. Prove that on an ellipsoid of revolution every geodesic which is not a meridian remains between two parallels.

Exercise 6.19. The catenoid Σ is the surface of revolution arising from $f(u) = \cosh u$. If a geodesic parametrised by arc length starts on the intersection of the catenoid with the plane $z = 1$ (which is the circle $\{u = 1\}$ on Σ), what is the minimal angle which its initial tangent vector must make with the circle $\{u = 1\}$ in order for the geodesic to cross below the $x - y$ plane at some time in the future.

This example shows that geodesics can get caught up in narrow necks if they don't have enough momentum.

Exercise 6.20. Do Carmo Section 4-4, questions: 5, 17, 18, 20. Clairaut's relation is very useful.

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