M435: INTRODUCTION TO DIFFERENTIAL GEOMETRY

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Contents

7. The Gauss-Bonnet Theorem

The Gauss-Bonnet theorem combines (almost) everything we have learnt in one theorem.

- Curves in $\mathbb{R}^3$ (especially those living on surfaces in $\mathbb{R}^3$).
- Euler characteristic (topological invariant leading to topological classification of compact surfaces.)
- Gaussian curvature (intrinsic local isometry invariant defined from first and second fundamental forms, really only depends on 1st fundamental form.
- Lengths, angles and areas.
- Geodesics.

7.1. **Statement of local Gauss-Bonnet theorem.** Let $\mathbf{r}\colon U \to \Sigma$ be a smooth patch of a surface $\Sigma \subset \mathbb{R}^3$, where $U \subseteq \mathbb{R}^2$ is open, and suppose that $\alpha\colon [a,b] \to \Sigma$ is a piecewise smooth simple closed curve in Σ.

(This means: there exist $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$, with $\alpha(a) = \alpha(b)$, while for $t \neq s \in (a,b)$ we have $\alpha(t) \neq \alpha(s)$, such that α is continuous, and $\alpha|\colon [t_i, t_{i+1}] \to \Sigma$ is smooth for $i = 0, \dots, n-1$, with one-sided smoothness at the endpoints.)

So $\alpha(I)$ is a curvilinear polygon, and let $R \subset \Sigma$ be a region enclosed by $\alpha(I)$.

Theorem 7.1 (Local Gauss-Bonnet theorem)**.**

$$\int_R K \,\mathrm{d}A = (2-n)\pi - \int_\alpha \kappa_g \,\mathrm{d}s + \sum_{i=1}^{n} \theta_i,$$

where

- *K is the Gaussian curvature of Σ.*

- *κ_g is the geodesic curvature of α.*
- *s is the arc length.*
- *$\theta_1, \dots, \theta_n$ are the internal angles of the polygon (in radians!).*

7.2. **Area of geodesic triangles.**

Corollary 7.2. *Let $\triangle$ be a* geodesic *triangle (all the sides are geodesics). Then*

$$\pi + \int_R K \, \mathrm{d}A = \theta_1 + \theta_2 + \theta_3,$$

where $\theta_1, \theta_2, \theta_3$ are the internal angles of the triangle.

Example 7.3. (1) On a sphere of radius 1, we have

$$\sum_{i=1}^{3} \theta_i = \pi + \operatorname{area}(\triangle).$$

(2) In the plane, $\sum_{i=1}^{3} \theta_i = \pi$.
(3) On a surface of constant curvature -1 (like the pseudo sphere), we have

$$\sum_{i=1}^{3} \theta_i = \pi - \operatorname{area}(\triangle).$$

In negatively curved space, the largest area a triangle can have is π, and for that we would need all 3 angles to be zero. Even when the sides get a very long, geodesics are curved in towards each other, so the area stays quite small.

7.3. **Special case of the plane.** Consider the special case of the local Gauss-Bonnet theorem in the plane. Now $K \equiv 0$, so the theorem becomes:

$$(2-n)\pi - \int_\alpha \kappa_g \, \mathrm{d}s + \sum_{i=1}^{n} \theta_i = 0.$$

We rewrite it by defining ϑ_i to be the exterior angle of each vertex of the polygon. This is defined, for the vertex $v = \alpha(t_i)$, by the angle from $\lim_{t \to t_i^-} \frac{\alpha(t)-\alpha(t_0)}{t-t_0}$ to $\lim_{t \to t_i^+} \frac{\alpha(t)-\alpha(t_0)}{t-t_0}$, measured anticlockwise as positive. Then $\pi - \theta_i = \vartheta_i$, so angles greater than π are measured as negative. We therefore obtain

$$\int_\alpha \kappa_g \, \mathrm{d}s + \sum_{i=1}^{n} \vartheta_i = 2\pi.$$

To prove this, we use a lemma from Chapter 1. The geodesic curvature of a regular curve in the plane is the same as the planar curvature from Chapter 1. We showed, as part of the proof of the Fundamental Theorem of Planar Curves, that

$$\kappa_g(t) = k(t) = \phi'(t),$$

where ϕ is the angle of the tangent vector $\alpha'(s)$ to our curve α measured anticlockwise from the positive x-axis. Therefore

$$\int_{t_i}^{t_{i+1}} \kappa_g \,\mathrm{d}t = \int_{t_i}^{t_{i+1}} \phi'(t)\mathrm{d}t = \lim_{t\to t_{i+1}^-} \phi(t) - \lim_{t\to t_i^+} \phi(t).$$

The total change in angle for a simple closed curve which is parametrised in the anticlockwise direction is 2π. The change in angle over the smooth parts is the integral of the geodesic curvature:

$$\int_\alpha \kappa_g \,\mathrm{d}s = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \phi'(t)\mathrm{d}t = \sum_{i=0}^{n-1} \lim_{t\to t_{i+1}^-} \phi(t) - \lim_{t\to t_i^+} \phi(t).$$

We also have

$$\vartheta_i = \lim_{t\to t_i^+} \phi(t) - \lim_{t\to t_i^-} \phi(t).$$

These formulae should be correctly interpreted at t_0 and t_n, where $\alpha(t_0) = \alpha(t_n)$. Thus the sum

$$\int_\alpha \kappa_g \,\mathrm{d}s + \sum_{i=1}^{n} \vartheta_i$$

gives the total change in angle of the tangent vector around the entire simple closed piecewise smooth curve and we obtain 2π as required (it could be some other multiple of 2π if the orientation were changed or the curve had self intersections - consider a figure eight curve, for example. For that the total change in angle is 0.)

7.4. **Global Gauss-Bonnet theorem.** Here is our plan: 1) state the global Gauss-Bonnet theorem and deduce it from the local theorem. 2) Prove the local Gauss-Bonnet theorem.

Theorem 7.4 (Global Gauss-Bonnet)**.** *Let Σ be a compact smooth surface in $\mathbb{R}^3$. Then*

$$\int_\Sigma K \,\mathrm{d}A = 2\pi\chi(\Sigma).$$

Recall that $\chi(\Sigma) = V - E + F$ is the Euler characteristic of Σ, computing using a subdivision with V vertice, E edges and F faces. Recall that the Euler characteristic is independent of the choice of subdivision, and is invariant under homeomorphisms of surfaces.

Example 7.5. (i) *Any* smooth surface in $\mathbb{R}^3$ which is homeomorphic to S^2 satisfies

$$\int_\Sigma K \,\mathrm{d}A = 4\pi.$$

We already saw this for *convex* surfaces using the Gauss map. The surface need not be *isometric* to S^2 for this relation to hold.

(ii) We have

$$\int_{\mathbb{T}} K\mathrm{d}A = 0$$

for any smooth surface $\mathbb{T}$ which is homeomorphic to the torus.

The global Gauss-Bonnet theorem is a truly remarkable theorem! It was remarkable that K is an invariant of local isometries, when the principal curvatures are not. But K depends strongly, at a given point, on the first fundamental form. The fact that the integral of the Gaussian curvature conspires to be a purely topological invariant is somewhat incredible.

Homeomorphic surfaces can be very far from locally isometric. For example, imagine blowing a surface up like a balloon; lengths of curves get much bigger. However this "spreads out" the Gaussian curvature, so the integral is still the same. This is a nice way to think about it: there's a fixed amount of curvature and it can be smeared around the compact surface in different ways depending on how one deforms it, but one cannot create or destroy curvature without changing the topological type.

There are many more theorems of this type in mathematics, where local differential geometry concepts integrated over a manifold assemble to give a topological quantity, inspired by the archetypal Gauss-Bonnet theorem.

Example 7.6. The sphere of radius a, Σ_a.

$$\int_{\Sigma_a} K\,\mathrm{d}A = \frac{1}{a^2}\cdot 4\pi a^2 = 4\pi = 2\pi\chi(S^2).$$

Suppose $\mathbf{r}\colon U \to \Sigma$ is a patch which covers all but a union of curves. Then we can compute $\int_\Sigma K\mathrm{d}A$ directly by computing

$$\int_U \frac{LN-M^2}{EG-F^2}\cdot\sqrt{EG-F^2}\mathrm{d}u\mathrm{d}v = \int_U \frac{LN-M^2}{\sqrt{EG-F^2}}\mathrm{d}u\mathrm{d}v.$$

Example 7.7. Consider the torus with $\mathbf{r}\colon U \to \Sigma$, where

$$\mathbf{r}(u,v) = ((a+b\cos v)\cos u, (a+b\cos v)\sin u, b\sin v).$$

where $0 < b < a$. We computed before that

$$K = \frac{\cos v}{a+b\cos v}.$$

Also $\sqrt{EG-F^2} = a+b\cos v$, so the integral of the Gaussian curvature is

$$\int_0^{2\pi}\int_0^{2\pi} \cos v\,\mathrm{d}u\mathrm{d}v = 2\pi[\sin v]_0^2\pi = 0 = 2\pi\chi(\Sigma).$$

Proof of global Gauss-Bonnet theorem from local Gauss-Bonnet theorem. First, subdivide Σ into curvilinear polygons P_j, with $j = 1\ldots F$, such that each $P_j \subset \mathbf{r}_j(U_j)$ for some smooth patch $\mathbf{r}_j\colon U_j \to \Sigma$. Let V, E, F be the number of vertices, edges and faces respectively. We sum the local Gauss-Bonnet equation for each polygon, over all the polygons.

The Gaussian curvature component is easy:

$$\sum\int_{P_j} K\,\mathrm{d}A = \int_\Sigma K\,\mathrm{d}A.$$

Suppose that P_j has n_j edges. Then

$$\sum_j (2-n_j)\pi = 2\pi F - 2\pi E,$$

since each edge is counted twice in this summation, as it belongs to two polygons.

The terms $\int \kappa_g \mathrm{d}s$ sum to zero since each edge occurs twice with opposite orientation. (Here we use that any compact surface in $\mathbb{R}^3$ is orientable. But in fact we can subdivide a non-orientable surface using geodesic polygons, so the geodesic curvature component can be made to vanish in the non-orientable case too. The Gauss-Bonnet theorem holds for abstract compact surfaces, not just those contained in $\mathbb{R}^3$, which we have studied in this course.)

Finally

$$\sum_j \sum_{i=1}^{n} \theta_i^j = 2\pi V;$$

this is the sum of the internal angles at a vertex of all the polygons of which that vertex is part. For each vertex, the sum is 2π.

Thus, putting it all together, we have

$$\int_\Sigma K \,\mathrm{d}A = 2\pi F - 2\pi E + 0 + 2\pi V = 2\pi\chi(\Sigma).$$

□

It remains to prove the local Gauss-Bonnet theorem.

7.5. **Proof of local Gauss-Bonnet theorem.** We begin by proving the simpler version below.

Theorem 7.8. *Let α be a smooth simple closed curve on a patch $\mathbf{r}(U)$ of a surface Σ, with anticlockwise parametrisation, enclosing a region R. Then*

$$\int_\alpha \kappa_g \,\mathrm{d}s = 2\pi - \int_R K \,\mathrm{d}A.$$

Anticlockwise means if you put the thumb of your right hand on an interior point of R, aiming in the direction of the normal vector, then your fingers curl around in the direction of the orientation on α.

Example 7.9. We can use this to compute the area of the spherical cap $S^2 \cap \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 \,|\, z \geq d\}$ for some $d \in (0, 1)$.

We recall a computation from the chapter on geometry of curves. The boundary of such the spherical cap R is given by

$$\alpha(t) = \left(a\cos\left(\frac{t}{a}\right), a\sin\left(\frac{t}{a}\right), \sqrt{1-a^2}\right),$$

where $a \in (0, 1)$ such that $d = \sqrt{1-a^2}$, and $t \in [0, 2\pi a]$. Then

$$\alpha'(t) = \left(-\sin\left(\frac{t}{a}\right), \cos\left(\frac{t}{a}\right), 0\right).$$

Note that $|\alpha'(t)| = 1$, so we have an arc length parametrisation. This gives the first vector in an orthonormal basis for $T\mathbb{R}^3$ at each point on α. Next,

$$\alpha''(t) = \left(-\frac{1}{a}\cos\left(\frac{t}{a}\right), -\frac{1}{a}\sin\left(\frac{t}{a}\right), 0\right).$$

The normal vector to the sphere is $\alpha(t)$, and we compute

$$\begin{aligned}
&\alpha(t) \wedge \alpha'(t) \\
&\quad = \left(a\cos\left(\frac{t}{a}\right), a\sin\left(\frac{t}{a}\right), \sqrt{1-a^2}\right) \wedge \left(-\sin\left(\frac{t}{a}\right), \cos\left(\frac{t}{a}\right), 0\right) \\
&\quad = \left(-\sqrt{1-a^2}\cos\left(\frac{t}{a}\right), -\sqrt{1-a^2}\sin\left(\frac{t}{a}\right), a\right).
\end{aligned}$$

Recall $\{\alpha', \mathbf{n} = -\alpha, \alpha' \wedge \mathbf{n}\}$ is an orthonormal basis at each point of α, and that the geodesic curvature is the component of α'' in the direction of $\alpha' \wedge \mathbf{n}$. Thus

$$k_g(t) = \alpha''(t) \cdot (\alpha \wedge \alpha')(t) = \frac{\sqrt{1-a^2}}{a} = \frac{d}{\sqrt{1-d^2}}.$$

We therefore have

$$\int_\alpha \kappa_g \, \mathrm{d}s = \int_0^{2\pi a} \frac{\sqrt{1-a^2}}{a} \mathrm{d}t = 2\pi\sqrt{1-a^2} = 2\pi d.$$

since t is arc length. Then since the Gaussian curvature is constant and equal to one, the area of the spherical cap we seek is

$$\int_R \mathrm{d}A = \int_R K \, \mathrm{d}A = 2\pi - \int_\alpha \kappa_g \mathrm{d}s = 2\pi(1-d).$$

Before proving the theorem we recall Green's theorem, which is the planar specialisation of Stokes' theorem.

Theorem 7.10 (Stokes' theorem)**.** *Let $\alpha \colon [a,b] \to \mathbb{R}^3$ be a piecewise smooth simple closed curve which bounds a smooth regular surface $\Sigma \subset \mathbb{R}^3$ with normal vector $\mathbf{n}$, and let $\mathbf{v}$ be a smooth vector field on Σ. Then*

$$\int_\alpha \mathbf{v} \cdot \alpha' \, \mathrm{d}t = \int_\Sigma (\nabla \wedge \mathbf{v}) \cdot \mathbf{n} \, \mathrm{d}A.$$

Corollary 7.11 (Green's theorem)**.** *Let $\alpha \colon [a,b] \to \mathbb{R}^2$ be a piecewise smooth simple closed planar curve defined by $\alpha(t) = (u(t), v(t))$, which bounds a region $S \subset \mathbb{R}^2$, and let $(P(u,v)$ and $Q(u,v))$ be smooth functions $\mathbb{R}^2 \to \mathbb{R}$. Then*

$$\int_\alpha P\frac{\mathrm{d}u}{\mathrm{d}t} + Q\frac{\mathrm{d}v}{\mathrm{d}t} \, \mathrm{d}t = \int_S Q_u - P_v \, \mathrm{d}u \, \mathrm{d}v.$$

Proof. Define $\mathbf{v} := (P, Q, 0)$, $\alpha(t) := (u(t), v(t), 0)$, and $\Sigma = S$, so that we embed $\mathbb{R}^2$ in $\mathbb{R}^3$ as the x-y plane. Note that $\mathbf{n} = (0,0,1)$. Then apply Stokes' theorem, to obtain Green's theorem. □

Proof of Theorem 7.8. The first step is to find an orthonormal basis for the tangent space $T_{\mathbf{r}(u,v)}\Sigma$ at each point of our patch $\mathbf{r}(U)$. This is done using *Gram-Schmidt orthonormalisation.* You should have learnt this in linear algebra.

The process is as follows. Take a basis, and make the first vector of length one. Then make the second vector orthogonal to the first, by substracting its components in the direction of the first. Then make this length one. Now take the third vector, and make it orthogonal to the new first and second vectors by subtracting its components in the directions of the new first and second vectors. Now make the resulting vector of length one. This describes the process for 3-dimensional vector spaces: for n-dimensional vector spaces one proceeds inductively. We only need the process for 2 dimensional vector spaces, so we stop here.

Apply the Gram-Schmidt process to the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$. So, define

$$\mathbf{e}_1 = \frac{\mathbf{r}_u}{\sqrt{E}}.$$

We then replace $\mathbf{r}_v$ with $\mathbf{r}_v - (\mathbf{r}_v \cdot \mathbf{e}_1)\mathbf{e}_1$ to make it orthogonal to $\mathbf{e}_1$. We have

$$\mathbf{r}_v \cdot \mathbf{e}_1 = \frac{\mathbf{r}_v \cdot \mathbf{r}_u}{\sqrt{E}} = \frac{F}{\sqrt{E}},$$

so

$$\mathbf{r}_v - (\mathbf{r}_v \cdot \mathbf{e}_1)\mathbf{e}_1 = \mathbf{r}_v - \frac{F\mathbf{r}_u}{E}$$

and

$$|\mathbf{r}_v - (\mathbf{r}_v \cdot \mathbf{e}_1)\mathbf{e}_1| = |\mathbf{r}_v - \frac{F\mathbf{r}_u}{E}| = \sqrt{\mathbf{r}_v \cdot \mathbf{r}_v - \frac{2F\mathbf{r}_u \cdot \mathbf{r}_v}{E} + \frac{F^2\mathbf{r}_u \cdot \mathbf{r}_u}{E^2}} = \sqrt{G - \frac{F^2}{E}}.$$

Therefore we define

$$\mathbf{e}_2 = \frac{\mathbf{r}_v - \frac{F\mathbf{r}_u}{E}}{\sqrt{G - \frac{F^2}{E}}} = \frac{E\mathbf{r}_v - F\mathbf{r}_u}{\sqrt{E}\sqrt{EG - F^2}}.$$

Now we have an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for the tangent space at each point of $\mathbf{r}(U)$.

Let $\beta\colon [a,b] \to U$ be the curve in $\mathbb{R}^2$ whose image in the surface is α, i.e. $\mathbf{r} \circ \beta(s) = \alpha(s)$ for all $s \in [a,b]$, were s is arc length along α. We consider the integral

$$I := \int_\beta \mathbf{e}_1 \cdot \mathbf{e}_2' \, \mathrm{d}s.$$

We may think of $\mathbf{e}_1(u,v)$, $\mathbf{e}_2(u,v)$ as functions of (u,v). We may therefore evaluate them at a point $\beta(s) \in U$. By evaluating this integral in two different ways, one along the curve and one using Green's theorem, we will obtain the desired result.

First we investigate the integral I along β. Let $\delta(s)$ be the angle between $\alpha'(s)$ and the vector $\mathbf{e}_1$ at $\alpha(s)$, with the angle measured anticlockwise.

Then since $|\alpha'(s)| = 1$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal basis, we have

$$\alpha'(s) = \cos\delta\mathbf{e}_1 + \sin\delta\mathbf{e}_2.$$

Define $\eta := \mathbf{n} \wedge \alpha'$. So

$$\eta := -\sin\delta\mathbf{e}_1 + \cos\delta\mathbf{e}_2.$$

Then

$$\begin{aligned}\alpha'' &= \delta'(-\sin\delta\mathbf{e}_1 + \cos\delta\mathbf{e}_2) + \mathbf{e}_1'\cos\delta + \mathbf{e}_2'\sin\delta \\ &= \delta'\eta + \mathbf{e}_1'\cos\delta + \mathbf{e}_2'\sin\delta.\end{aligned}$$

Thus

$$\kappa_g = \alpha'' \cdot \eta = \delta' + (\mathbf{e}_1'\cos\delta + \mathbf{e}_2'\sin\delta) \cdot (-\sin\delta\mathbf{e}_1 + \cos\delta\mathbf{e}_2).$$

Note that $\mathbf{e}_i \cdot \mathbf{e}_i = 1$ so $\mathbf{e}_i \cdot \mathbf{e}_i' = 0$. Also $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ so $\mathbf{e}_1' \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}_2' = 0$. Therefore

$$\kappa_g = \delta' - \mathbf{e}_1 \cdot \mathbf{e}_2'.$$

Rearranging

$$\mathbf{e}_1 \cdot \mathbf{e}_2' = \delta' - \kappa_g,$$

so

$$I = \int_\beta \mathbf{e}_1 \cdot \mathbf{e}_2' \,\mathrm{d}s = \int_\beta \delta'(s)\,\mathrm{d}s - \int_\beta \kappa_g \,\mathrm{d}s.$$

We saw in the section above on the special case of the plane, that

$$\int_\beta \delta'(s)\,\mathrm{d}s = 2\pi.$$

Therefore

$$I = 2\pi - \int_\beta \kappa_g \,\mathrm{d}s = 2\pi - \int_\alpha \kappa_g \,\mathrm{d}s.$$

Next we want to evaluate I in a different way using Green's theorem, to obtain $\int_R K\,\mathrm{d}A$. This will prove the theorem. We have

$$\begin{aligned} I &= \int_\beta \mathbf{e}_1 \cdot \mathbf{e}_2' \,\mathrm{d}s \\ &= \int_\beta \mathbf{e}_1 \cdot (u'(\mathbf{e}_2)_u + v'(\mathbf{e}_2)_v)\,\mathrm{d}s \\ &= \int_\beta \mathbf{e}_1 \cdot (\mathbf{e}_2)_u u' + \mathbf{e}_1 \cdot (\mathbf{e}_2)_v) v' \,\mathrm{d}s \\ &= \int_\beta Pu' + Qv' \,\mathrm{d}s \end{aligned}$$

where $P = \mathbf{e}_1 \cdot (\mathbf{e}_2)_u$ and $Q = \mathbf{e}_1 \cdot (\mathbf{e}_2)_v$. By Green's theorem, we have that I is equal to the double integral below:

$$I = \int_{\mathbf{r}^{-1}(R)} (Q_u - P_v)\,\mathrm{d}u\mathrm{d}v.$$

Lemma 7.12. *With P and Q as above, we have:*

$$Q_u - P_v = K\sqrt{EG - F^2},$$

where K is the Gaussian curvature and E, F and G are the coefficients of the first fundamental form.

Using the lemma, we have that

$$I = \int_{\mathbf{r}^{-1}(R)} K\sqrt{EG - F^2}\,\mathrm{d}u\mathrm{d}v =: \int_R K\,\mathrm{d}A$$

Thus

$$\int_R K\,\mathrm{d}A = I = 2\pi - \int_\alpha \kappa_g \,\mathrm{d}s$$

as required. It remains to prove the lemma.

Proof of Lemma 7.12. This requires computing $Q_u - P_v$. We have:

$$Q_u - P_v = (\mathbf{e}_1)_u \cdot (\mathbf{e}_2)_v + \mathbf{e}_1 \cdot (\mathbf{e}_2)_{uv} - (\mathbf{e}_1)_v \cdot (\mathbf{e}_2)_u - \mathbf{e}_1 \cdot (\mathbf{e}_2)_u v = (\mathbf{e}_1)_u \cdot (\mathbf{e}_2)_v - (\mathbf{e}_1)_v \cdot (\mathbf{e}_2)_u.$$

Note that $K\sqrt{EG - F^2} = \frac{LN - M^2}{\sqrt{EG - F^2}}$, and recall that this is equal to $\mathbf{n} \cdot (\mathbf{n}_u \wedge \mathbf{n}_v)$.

Here was the computation, which appeared already in the chapter on Gaussian curvature and the second fundamental form:

$$\begin{aligned}
\mathbf{n} \cdot (\mathbf{n}_u \wedge \mathbf{n}_v) &= \frac{(\mathbf{r}_u \wedge \mathbf{r}_v) \cdot (\mathbf{n}_u \wedge \mathbf{n}_v)}{|\mathbf{r}_u \wedge \mathbf{r}_v|} \\
&= \frac{(\mathbf{r}_u \cdot \mathbf{n}_u)(\mathbf{r}_v \cdot \mathbf{n}_v) - (\mathbf{r}_u \cdot \mathbf{n}_v)(\mathbf{r}_v \cdot \mathbf{n}_u)}{|\mathbf{r}_u \wedge \mathbf{r}_v|} \\
&= \frac{LN - M^2}{\sqrt{EG - F^2}}
\end{aligned}$$

However we can also compute $\mathbf{n}$ as $\mathbf{e}_1 \wedge \mathbf{e}_2$. Therefore

$$\mathbf{n} \cdot (\mathbf{n}_u \wedge \mathbf{n}_v) = (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{n}_u \wedge \mathbf{n}_v) = (\mathbf{e}_1 \cdot \mathbf{n}_u)(\mathbf{e}_1 \cdot \mathbf{n}_v) - (\mathbf{e}_1 \cdot \mathbf{n}_v)(\mathbf{e}_2 \cdot \mathbf{n}_u).$$

Now $\mathbf{e}_1 \cdot \mathbf{n} = \mathbf{e}_2 \cdot \mathbf{n} = 0$, so

$$\begin{aligned}
\mathbf{e}_1 \cdot \mathbf{n}_u &= -(\mathbf{e}_1)_u \cdot \mathbf{n} \\
\mathbf{e}_1 \cdot \mathbf{n}_v &= -(\mathbf{e}_1)_v \cdot \mathbf{n} \\
\mathbf{e}_2 \cdot \mathbf{n}_u &= -(\mathbf{e}_2)_u \cdot \mathbf{n} \\
\mathbf{e}_2 \cdot \mathbf{n}_v &= -(\mathbf{e}_2)_v \cdot \mathbf{n}.
\end{aligned}$$

Making these substitutions yields

$$K\sqrt{EG - F^2} = ((\mathbf{e}_1)_u \cdot \mathbf{n})((\mathbf{e}_2)_v \cdot \mathbf{n}) - ((\mathbf{e}_1)_v \cdot \mathbf{n})((\mathbf{e}_2)_u \cdot \mathbf{n}).$$

This is now looking rather close to

$$Q_u - P_v = (\mathbf{e}_1)_u \cdot (\mathbf{e}_2)_v - (\mathbf{e}_1)_v \cdot (\mathbf{e}_2)_u.$$

To get rid of the $\mathbf{n}$s, note that $\mathbf{e}_i \cdot \mathbf{e}_i = 1$ for $i = 1, 2$, so $(\mathbf{e}_i)_u \cdot \mathbf{e}_i = 0$ and $(\mathbf{e}_i)_v \cdot \mathbf{e}_i = 0$ for $i = 1, 2$. Thus we may write, for $w = u, v$, the partial derivatives in coordinates with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$, as:

$$(\mathbf{e}_1)_w = ((\mathbf{e}_1)_w \cdot \mathbf{e}_2)\mathbf{e}_2 + ((\mathbf{e}_1)_w \cdot \mathbf{n})\mathbf{n}$$

and

$$(\mathbf{e}_2)_w = ((\mathbf{e}_2)_w \cdot \mathbf{e}_1)\mathbf{e}_1 + ((\mathbf{e}_1)_w \cdot \mathbf{n})\mathbf{n}.$$

Then substituting these into the equation to $Q_u - P_v$ above, the dot products

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{n} = \mathbf{e}_2 \cdot \mathbf{n} = 0,$$

so only the $\mathbf{n} \cdot \mathbf{n}$ terms survive. We arrive at the expression

$$((\mathbf{e}_1)_u \cdot \mathbf{n})((\mathbf{e}_2)_v \cdot \mathbf{n}) - ((\mathbf{e}_1)_v \cdot \mathbf{n})((\mathbf{e}_2)_u \cdot \mathbf{n}),$$

which as we saw above is equal to $\mathbf{n} \cdot (\mathbf{n}_u \wedge \mathbf{n}_v)$ which in turn is equal to $K\sqrt{EG - F^2}$. This completes the proof of the lemma. □

With the proof of Lemma 7.12 complete, this finishes the proof of Theorem 7.8 □

Now we show how to modify in the case of corners, so α may only be piecewise smooth and not smooth.

Proof of full local Gauss-Bonnet theorem. To adapt the proof to the case that α has corners, we revise our first evaluation of the integral I. We showed that

$$I = \int_\beta \delta'(s)\,\mathrm{d}s - \int_\beta \kappa_g\,\mathrm{d}s.$$

But instead of $\int_\beta \delta'(s)\,\mathrm{d}s = 2\pi$ we have

$$\int_\beta \delta'(s)\,\mathrm{d}s = 2\pi - \sum_{i=1}^n \vartheta_i,$$

where as in Section 7.3 above the ϑ_i are the exterior angles at the corners. Since $\theta_i = \pi - \vartheta_i$, this becomes

$$\int_\beta \delta'(s)\,\mathrm{d}s = (2-n)\pi + \sum_{i=1}^n \theta_i.$$

Therefore we have

$$I = (2-n)\pi - \int_\alpha \kappa_g\,\mathrm{d}s + \sum_{i=1}^n \theta_i,$$

which is the right hand side of the local Gauss-Bonne theorem as required. □

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