M435: INTRODUCTION TO DIFFERENTIAL GEOMETRY

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Contents

8. Hyperbolic Geometry

The hyperbolic plane is a very interesting and important example of a surface, and we will study its geometry for the remainder of the semester.

We need a paradigm shift! Instead of looking at a surface in $\mathbb{R}^3$, and computing its first fundamental form, we will take a surface and *specify* the first fundamental form.

There will be no 2nd fundamental form, no principal curvatures, however intrinsic notions like length, angle, area, Gaussian curvature and geodesics all make sense still, since we derived formula and equations for these in terms of the first fundamental form coefficients only. These formulae can now become the *definitions*.

8.1. **The Poincaré unit disc model.**

Define

$$\mathbb{D} := \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\} \cong \{z = x+iy \in \mathbb{C} \mid z\overline{z} = 1\}$$

with first fundamental form

$$\frac{4(\mathrm{d}x^2+\mathrm{d}y^2)}{(1-x^2-y^2)^2}.$$

That is $E = G = 4/(1-x^2-y^2)^2$ and $F = 0$.

This is the Poincaré unit disc model for the hyperbolic plane.

8.2. **The upper half plane model.** Define

$$\mathbb{H} := \{(u,v) \in \mathbb{R}^2 \,|\, v > 0\} = \{w = u + iv \in \mathbb{C} \,|\, v > 0\},$$

with first fundamental form

$$\frac{\mathrm{d}u^2 + \mathrm{d}v^2}{v^2},$$

so $E = G = 1/v^2$ and $F = 0$.

This is the *upper half plane model* for the hyperbolic plane.

We use complex numbers in these definitions because they enable us to simplify several of the formulae and calculations which are coming up.

Our aim for this last small section is to understand the following things in $\mathbb{D}$ and $\mathbb{H}$:

- Gaussian curvature.
- Isometries.
- Geodesics.
- Lengths, angles and areas.

8.3. **Isometries.** Recall that in geometry we consider surfaces up to isometry, that is modulo the equivalence relation given by $\Sigma_1 \sim \Sigma_2$ if and only if there is an isometry $g\colon \Sigma_1 \to \Sigma_2$.

Lemma 8.1. *$\mathbb{H}$ and $\mathbb{D}$ are isometric.*

Proof. Define a function

$$\begin{array}{rcl} f\colon \mathbb{H} & \to & \mathbb{D} \\ w & \mapsto & \frac{w-i}{w+i} = z. \end{array}$$

Recall that $w = u + iv \in \mathbb{H}$ and $z = x + iy \in \mathbb{D}$.

First note that $|w - i| < |w + i|$ if and only if $|z| < 1$ i.e. $z \in \mathbb{D}$. In particular note that this function on all of $\mathbb{C}$ maps the real axis to the unit circle.

In coordinates, we can write

$$z = \frac{u + iv - i}{u + iv + i} = \frac{u^2 + v^2 - 1 - 2ui}{u^2 + (v+1)^2}.$$

So

$$x = \frac{u^2 + v^2 - 1}{u^2 + (v+1)^2}; \;\; y = \frac{-2u}{u^2 + (v+1)^2}.$$

When restricted to $v > 0$, this gives a bijection to $\mathbb{D}$. To see this last claim more clearly, note that there is an inverse function in complex coordinates

$$w = i\left(\frac{1+z}{1-z}\right).$$

So when $|z| < 1$ this is defined and gives an inverse to $z = \frac{w-i}{w+i}$.

Formally we can write

$$\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$$
$$\mathrm{d}w = \mathrm{d}u + i\mathrm{d}v$$

so that

$$|\mathrm{d}z|^2 = \mathrm{d}x^2 + \mathrm{d}y^2$$

$$|\mathrm{d}w|^2 = \mathrm{d}u^2 + \mathrm{d}v^2.$$

By the chain rule

$$|\mathrm{d}w|^2 = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 |\mathrm{d}z|^2.$$

So the first fundamental form of the unit disc model $\mathbb{D}$ is expressed as

$$\frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}$$

and the first fundamental form of $\mathbb{H}$ is given by

$$\frac{|\mathrm{d}w|^2}{v^2} = \frac{-4|\mathrm{d}w|^2}{(w-\overline{w})^2}.$$

For the last equality note that

$$w - \overline{w} = u + iv - (u - iv) = 2iv$$

so

$$v^2 = (w-\overline{w})^2/(-4).$$

Now we compute that since

$$w = i\left(\frac{1+z}{1-z}\right)$$

we have

$$w - \overline{w} = \frac{i(1+z)}{1-z} + \frac{i(1+\overline{z})}{1-\overline{z}} = \frac{2i(1-|z|^2)}{(z-1)(\overline{z}-1)}$$

using that $z\overline{z} = |z|^2$. Therefore

$$\frac{1}{(w-\overline{w})^2} = \frac{(z-1)^2(\overline{z}-1)^2}{-4(1-|z|^2)^2}.$$

Next we compute $\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2$, since we also need to substitute this into

$$\frac{|\mathrm{d}w|^2}{v^2} = -4|\mathrm{d}z|^2\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 \frac{(z-1)^2(\overline{z}-1)^2}{-4(1-|z|^2)^2}.$$

Differentiating $w = \frac{i(1+z)}{1-z}$ yields

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{2i}{(z-1)^2}.$$

Thus

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 = \frac{\mathrm{d}w}{\mathrm{d}z}\overline{\frac{\mathrm{d}w}{\mathrm{d}z}} = \frac{2i}{(z-1)^2}\cdot\frac{-2i}{(\overline{z}-1)^2} = \frac{4}{(z-1)^2(\overline{z}-1)^2}.$$

Substituting this into the expression above yields

$$\frac{|\mathrm{d}w|^2}{v^2} = -4\cdot|\mathrm{d}z|^2\cdot\frac{4}{(z-1)^2(\overline{z}-1)^2}\cdot\frac{(z-1)^2(\overline{z}-1)^2}{-4(1-|z|^2)^2} = \frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}$$

as required. Thus since the map $f\colon \mathbb{H} \to \mathbb{D}$ given is a diffeomorphism which preserves first fundamental form, f is an isometry and $\mathbb{H}$ and $\mathbb{D}$ are isometric. $\square$

We now focus on the model given by $\mathbb{H}$ for a while. We investigated an isometry already. In fact we can understand *all* possible isometries $\mathbb{H} \to \mathbb{H}$.

Proposition 8.2. *All isometries of $\mathbb{H}$ are generated by (i.e. are compositions of) the Möbius transformations*

$$w \mapsto \frac{aw+b}{cw+d} = z,$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$, and the map $w \mapsto -\overline{w}$, or $u + iv \mapsto -(\overline{u+iv}) = -u + iv$.

The last map is reflection in the imaginary axis. The Möbius transformations have inverse functions

$$w = \frac{dz - b}{-cz + a}.$$

One ought to think of 2×2 matrices over $\mathbb{R}$, and their inverses.

We don't have time to do many proofs in this section. The check that these are isometries is straightforward and similar to the computation above showing that $\mathbb{D}$ and $\mathbb{H}$ are isometric. The fact that all isometries arise from compositions of these two is harder.

The isometries of $\mathbb{H}$ (in fact of any space) form a group, with multiplication by composition.

The isometries act on $\mathbb{H}$, in fact they act transitively, which means that for any $x, y \in \mathbb{H}$ there exists an isometry $f\colon \mathbb{H} \to \mathbb{H}$ such that $f(x) = y$. To see this note that $z \mapsto bz + a$ sends i to $a + bi$. Thus if $x = a + bi$ and $y = c + di$ we choose $f_1\colon \mathbb{H} \to \mathbb{H}$ so that $x \mapsto i$, i.e. $z \mapsto (z - a)/b$. Next we choose $f_2\colon \mathbb{H} \to \mathbb{H}$ which sends $i \to y$, i.e. $z \mapsto dz + c$. Then $f_2 \circ f_1 = \frac{dz+bc-ad}{b}$ is a Möbius transformation so is an isometry and it maps x to y as desired.

8.4. **Gaussian curvature.** Recall, from the proof of the theorema egregium, that Gaussian curvature is intrinsic, we derived the formula for K, the Gaussian curvature, when $F = 0$. This formula now serves as the definition for K.

$$K = \frac{-1}{2\sqrt{EG}} \left(\frac{\partial}{\partial u} \left(\frac{G_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) \right).$$

Into this formula we substitute $E = G = \frac{1}{v^2}$. We have:

$$G_u = 0;\ E_v = \frac{-2}{v^3};\ \frac{1}{\sqrt{EG}} = v^2.$$

Therefore

$$K = \frac{-v^2}{2} \left(\frac{\partial}{\partial v} \left(\frac{-2}{v^3} \cdot v^2 \right) \right) = v^2 \cdot \frac{\partial}{\partial v} \left(\frac{1}{v} \right) = v^2 \cdot -\frac{1}{v^2} = -1.$$

The Gaussian curvature is constant and equal to -1.

8.5. **Geodesics.** Recall the geodesic equations once again:

$$(1)\ \frac{\mathrm{d}}{\mathrm{d}t}(Eu' + Fv') = \frac{1}{2}(E_u(u')^2 + 2F_u u'v' + G_u(v')^2)$$
$$(2)\ \frac{\mathrm{d}}{\mathrm{d}t}(Fu' + Gv') = \frac{1}{2}(E_v(u')^2 + 2F_v u'v' + G_v(v')^2)$$
$$(3)\ E(u')^2 + 2Fu'v'G(v')^2 = 1$$

With $E = G = 1/v^2$ and $F = 0$ we have $E_u = G_u = 0$. Equation (1) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u'}{v^2}\right) = 0$$

Equation (2) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v'}{v^2}\right) = -\frac{1}{v^3}(u'^2 + v'^2).$$

Equation (3) becomes

$$\frac{1}{v^2}(u'^2 + v'^2) = 1.$$

Substituting (3) into (2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v'}{v^2}\right) == \frac{1}{v}.$$

Then we perform the differentiations on the left hand side:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v'}{v^2}\right) = \frac{v''v - 2v'^2}{v^3},$$

so that we have an equation

$$v''v + v^2 - 2v'^2 = 0.$$

from equations (2) and (3) Similarly, turning back to equation (1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u'}{v^2}\right) = \frac{u''v^2 - 2vu'v'}{v^4},$$

so that equation (1) becomes

$$u''v^2 - 2vv'u'.$$

We will see that any geodesic in $\mathbb{H}$ is of two forms, which are given below. Type (I) is defined by

$$u(t) = \frac{1}{C}\tanh(t) + A; \quad v(t) = \frac{1}{C}\operatorname{sech}(t).$$

where $C > 0$ and A are constants.
Type (II) is a curve of the form

$$u(t) = A; \quad v(t) = e^t.$$

Exercise 8.3. Check that the curves specified by these functions u and v above give solutions to the geodesic equations.

The type (I) geodesics are semicircles with centre $(A, 0)$ on the real axis and radius $1/C$, since

$$(u - A)^2 + v^2 = \frac{1}{C^2}(\tanh^2(t) + \operatorname{sech}^2(t)) = \frac{1}{C^2}.$$

They are just semicircles because only the upper halves lie in the upper half plane.

The type (II) curves are vertical lines above $u = A$. Recall that through any point there is a unique geodesic in a given direction. We therefore have *all* the geodesics of $\mathbb{H}$, since one can always find one of curves of either type (I) or (II) going through a given point with a given tangent direction. So these must be the unique curves.

All geodesics are semicircles with centre on the real axis or vertical half-lines, or segments of these two. Note that horizontal lines are not geodesics.

8.6. **Hyperbolic triangles.** These are triangles in $\mathbb{H}$ whose sides are geodesics. We allow the vertices to lie on the real axis, and call these (singly, doubly, triply) asymptotic triangles.

The local Gauss Bonnet theorem implies, since $K \equiv -1$, that the area of a triangle with interior angles θ_i, $i = 1, 2, 3$, is given by $\pi - \sum_{i=1}^{3} \theta_i$. For a triply asymptotic triangle all angles are zero and the area of the triangle is always π. Thus we can have triangles with infinite side length but finite volume.

Here is another cool fact from hyperbolic triangles.

Theorem 8.4 (Hyperbolic sine rule)**.** *Let $\triangle$ be a hyperbolic triangle in $\mathbb{H}$ with sides of length A, B, C and let their opposite angles be α, β, γ. Then*

$$\frac{\sin A}{\sinh \alpha} = \frac{\sin B}{\sinh \beta} = \frac{\sin C}{\sinh \gamma}.$$

Unfortunately we have no time for the proof.

8.7. **Parallel postulate.** In the plane, the parallel postulate states:

Given a straight line L and a point p in $\mathbb{R}^2$ not on that line, there is a unique straight line L' through P missing L. i.e. $P \in L'$ and $L \cap L' = \emptyset$.

It was long debated whether this was self-evident and should be an axiom, or if it required proof, and whether it could be proven from the other Euclidean axioms of geometry.

The discovery of hyperbolic geometry answered this question in the negative. Hyperbolic geometry satisfies all the other axioms, but does not satisfy the parallel postulate. One should replace $\mathbb{R}^2$ with $\mathbb{H}$ and "straight line" with "geodesic."

In spherical geometry, there does not exist any such line L' with the desired properties. So existence fails. In hyperbolic geometry, there are infinitely many such lines. So uniqueness fails.

We have already discussed spherical geometry; note that no two great circles are disjoint. To see the claim in hyperbolic geometry, take L to be a vertical line. Then for any P, there are (infinitely) many semicircles with centre on the real axis which pass through P, all of which miss the first vertical line.

Thus the parallel postulate is not self-evident, and it certainly cannot be proven from the other axioms.

8.8. **Area of a subset of $\mathbb{D}$.** For a change we go back to the Poincaré unit disc model $\mathbb{D}$. Let us compute the area of a disc in $\mathbb{D}$ whose radius is a, where $0 < a < 1$.

First convert to polars:

$$\frac{4(\mathrm{d}x^2 + \mathrm{d}y^2)}{(1 - x^2 - y^2)^2}$$

becomes

$$\frac{4(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2)}{(1 - r^2)^2}$$

Since $E = \frac{4}{(1-r^2)^2}$ and $G = \frac{4r^2}{(1-r^2)^2}$, we obtain

$$\sqrt{EG - F^2} = \frac{4r}{(1 - r^2)^2}.$$

Integrating this over the desired areas leaves us with the integral

$$A = \int_0^{2\pi} \int_0^a \frac{4r}{(1 - r^2)^2}\,\mathrm{d}r\mathrm{d}\theta = 4\pi \int_0^a \frac{2r}{(1 - r^2)^2}\,\mathrm{d}r.$$

Using the substitution $u = 1 - r^2$ this gives

$$4\pi \int_{1-a^2}^1 \frac{1}{u^2}\,\mathrm{d}u = \frac{4\pi a^2}{1 - a^2}.$$

Note that as $a \to 1$, $A \to \infty$. Although the Poincaré unit disc might look to have finite area, the first fundamental form we imposed on it means it has infinite area.

Computing the area of subsets of $\mathbb{H}$ is very similar.

8.9. **Computing length in $\mathbb{H}$.** One way to compute lengths in $\mathbb{H}$ is using isometries. The distance between any two points is defined to be the length of the shortest path between them. So find a segment of a geodesic which goes through both points and compute its length. This could be done by finding an arc length parametrisation (as given in the section above on geodesics) and looking at the change in the parameter. On the other hand, we can make use of isometries. Given $x, y \in \mathbb{H}$, find an isometry which sends x to i, and sends y to the imaginary axis, say to λi for some $\lambda \in \mathbb{R}_{>0}$.

The distance between i and λi along the path $u(t) = 0$, $v(t) = t$, with $t \in [1, \lambda]$ (or $[\lambda, 1]$, but assume for now that $1 < \lambda$). Since $u' = 0$ and $v' = 1$, the length is:

$$\int_1^\lambda \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\,\mathrm{d}t = \int_1^\lambda \sqrt{\frac{1}{v^2}}\,\mathrm{d}t = \int_1^\lambda \sqrt{\frac{1}{t^2}}\,\mathrm{d}t = [\log(t)]_1^\lambda = \log \lambda.$$

If $0 < \lambda < 1$ then the answer is $-\log \lambda$, but in that case $\log \lambda < 0$.

So then one must find an isometry of $\mathbb{H}$, namely a Möbius transformation of the form

$$z \mapsto \frac{az + b}{cz + d}$$

with $ad - bc > 0$, such that $x \mapsto i$ and $y \mapsto \lambda i$. Then the distance from x to y is $|\log \lambda|$.

8.10. **Angles in $\mathbb{H}$.** Since we have $E = G$ and $F = 0$, the first fundamental form is conformal, and angles in $\mathbb{H}$ are the same as angles in the underlying model open subset of $\mathbb{R}^2$.

References

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