M435: INTRODUCTION TO DIFFERENTIAL GEOMETRY

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Contents

1. Introduction

The plan is as follows, although this is the maximum we will do, we may cover less, depending on time. Extra reading can be found in the bibliography, [DoCa] and [Hit] are recommended sources.

(1) Geometry of curves.
(2) Topology of surfaces.
(3) Geometry of surfaces.
(4) Intrinsic vs. extrinsic. Theorema Egregium.
(5) Geodesics.
(6) Gauss-Bonnet theorem, Poincaré's theorem (Morse theory on surfaces).
(7) Hyperbolic Geometry.
(8) Back to curves - Fenchel and Fary-Milnor theorems, Four Vertex theorem.

After a warm up on curves, in topology of surfaces we will construct surfaces using glueing methods, and we will classify all possible closed surfaces up to homeomorphism.

In the geometry of surfaces we will work on understanding the effect of intrinsic geometry, measured in particular by the curvature, for example on whether the angles of a triangle add up to π, or more than π, or less than π. The remarkable theorem, or Theorema Egregium of Gauss says that the curvature, which at first glance appears to depend on the embedding of a surface in 3-space, is in fact an intrinsic property. That is, we don't need pictures of the Earth from space in order to tell that it is a ball. We will be particularly interested in surfaces with constant curvature.

The highlight of the course is undoubtedly the Gauss-Bonnet theorem, which is the archetypal local-global theorem. It is shown that for a surface a global topological quantity, the Euler characteristic, can be computed from local information at each point, namely the Gaussian curvature of the surface.

Now that your appetite is whetted, let us begin.

2. Geometry of Curves

Definition 2.1. A differentiable curve in the plane is a differentiable map $\alpha\colon I \to \mathbb{R}^2$ where $I = (a, b) \subseteq \mathbb{R}$ is an interval. (We can have $a = -\infty$ or $b = \infty$, or both.)

A differentiable curve in $\mathbb{R}^3$ is a differentiable map $\alpha\colon I \to \mathbb{R}^3$ where $I = (a, b) \subseteq \mathbb{R}$ is an interval. (We can have $a = -\infty$ or $b = \infty$, or both.)

Here differentiable means that $\alpha(t) = (x(t), y(t), z(t))$, where x, y, z are differentiable functions $(a, b) \to \mathbb{R}$.

A curve α is regular if $\alpha'(t) \neq 0$ for all $t \in I$.

Curves in the plane can also be given as the level sets of two variable functions $f(x, y) = c$. This is *implicit form*. A function whose image coincides with such a level set is sometimes called the *parametric form*.

Example 2.2. (i) $\alpha(t) = \mathbf{a} + \mathbf{b}t$, $t \in \mathbb{R}$. A straight line through the point determined by the vector $\mathbf{a}$ parallel to the vector $\mathbf{b}$.

(ii) $\alpha(t) = (\cos t, \sin t)$ $t \in \mathbb{R}$. The unit circle in $\mathbb{R}^2 \subset \mathbb{R}^3$.

(iii) $\alpha(t) = (a \cos t, a \sin t, bt)$ $t \in \mathbb{R}$, $a, b > 0$ constants. A helix, lying on a cylinder of radius a.

(iv) $\alpha(t) = (t^3, t^2)$. A non-regular differentiable curve. The derivative vanishes at $t = 0$. Exercise: sketch the curve.

(v) Non-example. $\alpha(t) = (t, |t|)$, $t \in \mathbb{R}$. Not a differentiable curve since $y(t)$ is not a differentiable function.

(vi) $\beta(t) = (\cos 2t, \sin 2t)$ $t \in \mathbb{R}$. Same circle as example (i), i.e. same subset of $\mathbb{R}^3$, but different parametrisation. In both cases the image is $\{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2 = 1\}$, however $\beta(t)$ is twice as fast as $\alpha(t)$.

$$|\beta'(t)| = \sqrt{(-2 \sin 2t)^2 + (2 \cos 2t)^2} = 2;$$

$$|\alpha'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1.$$

(vii) Given a function $f\colon \mathbb{R} \to \mathbb{R}$, its graph is $\{(x, y) \in \mathbb{R}^2 \mid y - f(x) = 0\}$. We can parametrise this set as a differntiable curve by $\alpha(t) = (t, f(t))$.

Exercise 2.3. Find a parametrisation for the curves in the plane given as the level sets of the following equations

(i) Ellipse $x^2/a^2 + y^2/b^2 = 1$.

(ii) Hyperbola $x^2/a^2 - y^2/b^2 = 1$.

(iii) A figure 8 curve, or an ∞ symbol, with the self intersection point at the origin of $\mathbb{R}^2$. Here is one implicit form $x^4 + 2x^2y^2 + y^4 - x^2 + y^2 = 0$. Here is one method: deduce this equation from the polar equation $r = \sqrt{\cos 2\theta}$, and then using $x = r\cos\theta$ and $y = r\sin\theta$ make θ the parameter. This can be made to give an α which is piecewise defined. Another implicit formula which describes a curve with the right qualitative shape is $x^4/16 + y^2 - x^2 = 0$. Can you find parametric equations for this implicit curve? Try typing the equations into Wolfram Alpha, it will draw them for you. You can also look up "lemniscate" on wikipedia, if you want to cheat.

Exercise 2.4. Show that the following formula gives a parametrisation of the subset of $\mathbb{R}^2$ from Example (v) above which is differentiable but which is not regular.

$$\alpha\colon t \mapsto \begin{cases} (t^2, -t^2) & t \leq 0 \\ (t^2, t^2) & t > 0 \end{cases}$$

In particular you need to check that this function is differentiable at $t = 0$.

The vector $\alpha'(t)$ is the *tangent vector* to α at $t \in I$. $|\alpha'(t)|$ is the speed of α at t (think of t as time). We consider regular curves only from now on. Points where $\alpha'(t) = 0$ are called *singular points.*

A regular curve is also often called an *immersion.*

2.1. **Tangent vectors and arc length.** For a curve without singular points we have the *unit tangent vector*

$$\mathbf{T}(t) := \frac{\alpha'(t)}{|\alpha'(t)|}.$$

Note that we do not have to have any special notation to indicate a vector in this course. It is allowed but not required to over- or underline or bold-face vector functions, but it will be apparent what are vector functions because we will always specify clearly the domain and codomain of all functions. $f\colon A \to B$, A is the *domain* and B is the *codomain.* The latter is not to be confused with $f(A)$, which is the *image* or *range* of f.

Definition 2.5. The *Tangent line* to a curve α at a point $p = \alpha(t_0)$ is the line through p which is parallel to $\alpha'(t_0)$.

Example 2.6. Suppose $\alpha(t) = (\cos t, \sin t, t)$. Then $\alpha'(t) = (-\sin t, \cos t, 1)$. So $|\alpha'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. At $p = (-1, 0, \pi)$ we have $t = \pi$ and $\alpha'(\pi) = (0, 1, 1)$. Therefore the tangent line to α at p is given by $\gamma(t) = (-1, 0, \pi) + t(0, -1, 1)$.

Exercise 2.7. Find the tangent line to the curve $\alpha(t) = (t, t^2, \log t)$ at the point $(1, 1, 0)$.

Using the tangent vector we can compute the arc length of a curve between two of its points. We know that in a constant speed situation the distance covered is the speed multiplied by the time for which this speed is maintained. When the speed is variable with time we take an integral instead of multiplying.

Definition 2.8. For a differentiable curve $\alpha\colon I = (a, b) \to \mathbb{R}^3$ (or $\mathbb{R}^2$) the arc length between $t = t_0$ and $t = t_1$, with $a \leq t_0 \leq t_1 \leq b$ is:

$$\int_{t_0}^{t_1} |\alpha'(u)|\mathrm{d}u.$$

We can *reparametrise a curve by arc length.* We can always do this in theory, although in practice it is often not worth the effort. It will be useful for the theory to assume that we have a curve parametrised by arc length. The point is that parametrisations are highly nonunique. By fixing a starting point and an orientation of a curve, namely a choice of preferred direction, we obtain a *canonical parametrisation* — the arc length one. If we switch the orientation, then we switch the canonical parametrisation $s \to -s$.

Definition 2.9. Let $\alpha\colon I \to \mathbb{R}^3$ be a curve. Fix a starting point $t_0 \in I$. The arc length function is

$$s(t) := \pm \int_{t_0}^{t} |\alpha'(u)|\mathrm{d}u.$$

The sign $\pm$ is decided according to whether increasing t from t_0 moves along the orientation of α or against it.

In order to reparametrise a regular curve by arc length first write $s = f(t)$. Since $|\alpha'(t)| > 0$ for all t, f is a one-one function (mean value theorem). Therefore f is a bijection between (t_0, b) and $f(t_0, b)$, which is another interval. so f admits an inverse. Substitute $t = f^{-1}(s)$ in $\alpha(t)$ to obtain the parametrisation by arc length.

Note that $\frac{\mathrm{d}s}{\mathrm{d}t} = |\alpha'(t)|$. If $t = s$, then $\frac{\mathrm{d}s}{\mathrm{d}t} = 1$ and $\mathbf{T}(t) = \alpha'(t) = \alpha'(s)$.

Example 2.10. Suppose that $\alpha(t) = (\cos t, \sin t, t)$, $t \in \mathbb{R}$. We already saw that $|\alpha'(t)| = \sqrt{2}$. Therefore the arc length function, starting at $t_0 = 0$, is

$$s(t) := \int_0^t |\alpha'(u)|\mathrm{d}u = \int_0^t \sqrt{2}\mathrm{d}u = [\sqrt{2}u]_0^t = \sqrt{2}t.$$

Inverting this function we have $t = s/\sqrt{2} = s\sqrt{2}/2$. This gives the reparametrisation by arc length of α:

$$\alpha(s) = (\cos(s\sqrt{2}/2), \sin(s\sqrt{2}/2), s\sqrt{2}/2).$$

Theorem 2.11. *The length of a curve is independent of the parametrisation.*

Proof. Suppose $f\colon (c, d) \to (g, h)$ is the reparametrisation function: $t = f(s)$, f is a bijection and $f'(s) > 0$ for all $s \in (c, d)$. For a curve $\alpha\colon I \to \mathbb{R}^3$, $(c, d) \subset I$, we obtain a new curve $\beta(s) = \alpha(f(s))$. The length of the new curve is given by:

$$\int_c^d |\beta'(s)|\mathrm{d}s = \int_c^d |\alpha'(f(s))f'(s)|\mathrm{d}s$$

Now let $t = f(s)$. Then since $f'(s) > 0$ we have

$$\int_c^d |\alpha'(f(s))|f'(s)\mathrm{d}s = \int_c^d |\alpha'(f(s))|\frac{\mathrm{d}t}{\mathrm{d}s}\mathrm{d}s = \int_g^h |\alpha'(t)|\mathrm{d}t,$$

where the last equality used integration by substitution. The final quantity is the length of α. This completes the proof. $\square$

Exercise 2.12. Let (p, q) be coprime integers; that is, $p, q > 1$ and there is no number other than one which divides both p and q. Consider the curve

$$\alpha_{p,q}(t) = ((2 + \cos pt) \cos qt, (2 + \cos pt) \sin qt, \sin pt) \;\; t \in \mathbb{R}.$$

Can you describe this curve? Note that only $t \in [0, 2\pi)$ matters, by periodicity. First, consider $(p, q) = (3, 2)$. Sketch this curve. What is interesting about this curve? Can you make something similar with a piece of string or your shoelace?

Exercise 2.13. Do Carmo page 7, section 1-3. Exercises 2 and 4.

2.2. **Curvature of plane curves.** For this section you need to recall the dot product. From now on all curves are regular/immersions.

Suppose we have a curve $\alpha\colon I \to \mathbb{R}^2$, such that $|\alpha'(s)| = 1$ i.e. the curve is parametrised by arc length.

Lemma 2.14. *The second derivative $\alpha''(s)$ is perpendicular to $\alpha'(s) = \mathbf{T}(s)$.*

Proof.

$$1 = |\alpha'(s)|^2 = \alpha'(s) \cdot \alpha'(s).$$

Therefore, differentiating, by the product rule we have

$$0 = 2\alpha'(s) \cdot \alpha''(s),$$

so that $\alpha'(s)$ and $\alpha''(s)$ are orthogonal. □

Note that this proof needs $|\alpha'(s)|$ to be constant.

The matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acts on $\mathbb{R}^2$, rotating by $\pi/2$ anti-clockwise. Fix $s \in I$. By the lemma,

$$\alpha''(s) = kJ(\alpha'(s))$$

for some $k \in \mathbb{R}$. This constant is $k(s)$, the curvature of α at s.

Definition 2.15. Let $\alpha\colon I \to \mathbb{R}^2$ be a curve parametrised by arc length. The *curvature* is defined by:

$$k(s) := \alpha''(s) \cdot J(\alpha'(s)).$$

Example 2.16.

(i) A straight line. Suppose $\alpha(s) = \mathbf{a} + \mathbf{b}s$ where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ are vectors with $|\mathbf{b}| = 1$. Then $\alpha'(s) = \mathbf{b}$ and $\alpha''(s) = \mathbf{0}$. Therefore

$$k(s) := \alpha''(s) \cdot J(\alpha'(s)) = \mathbf{0} \cdot J(\mathbf{b}) = 0.$$

A straight line has zero curvature.

(ii) A circle of radius a. Let $\alpha(s) = (a \cos(s/a), a \sin(s/a))$. Then

$$\alpha'(s) = (-\sin(s/a), \cos(s/a))$$

and

$$J(\alpha'(s)) = (-\cos(s/a), -\sin(s/a)).$$

Note that $|\alpha'(s)| = 1$ so indeed we have a parametrisation by arc length. Also

$$\alpha''(s) = (-\frac{1}{a} \cos(s/a), -\frac{1}{a} \sin(s/a))$$

The curvature is computed by

$$k(s) = \alpha''(s)\cdot J(\alpha'(s)) = (-\frac{1}{a}\cos(s/a), -\frac{1}{a}\sin(s/a))\cdot(-\cos(s/a), -\sin(s/a)) = 1/a.$$

A circle of radius a has curvature $1/a$.

Now consider the curve $\alpha(t) = (t\cos t, t\sin t)$. This is a spiral. The velocity vector at time t is

$$\alpha'(t) = (\cos t - t\sin t, \sin t + t\cos t).$$

Therefore $|\alpha'(t)| = \sqrt{1+t^2}$. The arc length function

$$s(t) = \int_0^t \sqrt{1+u^2}\mathrm{d}u$$

is hard to invert, so it is not feasible to write down the arc length parametrisation. We need a new computational technique.

We consider a general curve $\alpha\colon I \to \mathbb{R}^2$ for which we don't know the arc length parametrisation. We know that one exists, however. Suppose that it is given by the substitution $t = f(s)$, where f is a function with $f'(s) > 0$. Then $\beta(s) = \alpha(f(s))$ is the arc length parametrisation, i.e. $|\beta'(s)| = |\alpha'(f(s))f'(s)| = 1$. Now

$$\beta'(s) = \alpha'(f(s))f'(s)$$

so

$$\beta''(s) = \alpha''(f(s))(f'(s))^2 + \alpha'(f(s))f''(s).$$

We have

$$J(\beta'(s)) = J(\alpha'(f(s)))f'(s)$$

since $f'(s)$ is a scalar function and J is a linear map. Then

$$k(s) = \beta''(s)\cdot J(\beta'(s)) = \alpha''(f(s))\cdot J(\alpha'(f(s)))(f'(s))^3 + \alpha'(f(s))\cdot J(\alpha'(f(s)))f''(s)f'(s).$$

Then observe that $\mathbf{x}\cdot J(\mathbf{x}) = 0$ for any vector $\mathbf{x} \in \mathbb{R}^2$. Thus

$$k(s) = \alpha''(f(s))\cdot J(\alpha'(f(s)))(f'(s))^3 = \alpha''(t)\cdot J(\alpha'(t))(f'(s))^3.$$

Now since $t = f(s)$, we have

$$\mathrm{d}t/\mathrm{d}s = f'(s) = \frac{1}{\mathrm{d}s/\mathrm{d}t} = \frac{1}{|\alpha'(t)|}.$$

We arrive at our final formula:

$$k(t) = \frac{\alpha''(t)\cdot J(\alpha'(t))}{|\alpha'(t)|^3}$$

Let us apply this to the spiral $\alpha(t) = (t\cos t, t\sin t)$ from above. We computed

$$\alpha'(t) = (\cos t - t\sin t, \sin t + t\cos t)$$

and $|\alpha'(t)| = \sqrt{1+t^2}$. We also compute

$$\alpha''(t) = (-2\sin t - t\cos t, 2\cos t - t\sin t)$$

and

$$J(\alpha'(t) = (-\sin t - t\cos t, \cos t - t\sin t).$$

Therefore

$$k(t) = \frac{\alpha''(t) \cdot J(\alpha'(t))}{|\alpha'(t)|^3} = \frac{2+t^2}{(1+t^2)^{\frac{3}{2}}}.$$

Exercise 2.17. Compute the curvature the following curves.

(i) The graph of the curve $y = f(x)$, for a general function $f \colon \mathbb{R} \to \mathbb{R}$.
(ii) The curve given by $x^2 - y^2 = 1$.
(iii) The cycloid of Do Carmo, Section 1-3 Exercise 2.
(iv) Do Carmo section 1.5 Exercise 8 (a). $\alpha(t) = (t, \cosh t)$, $t \in \mathbb{R}$.

We can understand planar curvature better by considering the direction in which the tangent vector points.

Definition 2.18. Suppose that $\alpha \colon I \to \mathbb{R}^2$ is parametrised by arc length. Let $\phi(s)$ be the angle which $\alpha'(s)$ makes with the positive x axis, measured anti-clockwise from the positive x-axis to $\alpha'(s)$.

Since $|\alpha'(s)| = 1$, the vector is determined by the angle.

Proposition 2.19. *We have an equality $k(s) = \phi'(s)$.*

Proof. We can write $\alpha'(s) = (\cos(\phi(s)), \sin(\phi(s)))$. Therefore

$$\alpha''(s) = (-\sin(\phi(s))\phi'(s), \cos(\phi(s))\phi'(s)).$$

Also

$$J(\alpha'(s)) = (-\sin(\phi(s)), \cos(\phi(s))),$$

so that

$$k(s) = \alpha''(s) \cdot J(\alpha'(s)) = \phi'(s)$$

as claimed. □

When a curve turns to the left, the curvature is positive, and when it turns to the right the curvature is negative. Note that this is a convention, we could have defined J to be $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, a clockwise rotation by $\pi/2$, which would switch the sign of the curvature.

The following beautiful theorem shows the power of the curvature function. Essentially, the theorem says that if know where you started and what direction you were pointing in, and you drive your car around an open car park with constant speed, and you know at all times where your steering wheel is, then you can recover the path that you traced out.

Theorem 2.20 (Fundamental theorem of planar curves)**.** *Suppose we are given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, $\mathbf{b} \neq \mathbf{0}$, a continuous function $k \colon I \to \mathbb{R}$ and a point $t_0 \in I$. Then there exists a unique curve $\alpha \colon I \to \mathbb{R}^2$ parametrised by arc length with curvature function k, initial position $\alpha(t_0) = \mathbf{a}$ and initial direction $\alpha'(t_0) = \mathbf{b}$.*

Proof. Start with the equation $k(s) = \phi'(s)$. Let $\mathbf{b} = (b_1, b_2)$. Then we can compute $\phi(t_0)$ as $\tan^{-1}(b_2/b_1)$ if $b_1 \neq 0$, and $\frac{\pi b_2}{2|b_2|}$ if $b_1 = 0$. Thus we have

$$\phi(s) = \phi(t_0) + \int_{t_0}^{s} k(u) \mathrm{d}u.$$

Note that $\phi(s)$ need not be in the interval $[0, 2\pi)$ here. Also note that $\phi(s)$ is uniquely determined by the derivative $\phi' = k$ and the initial condition; this follows from the mean value theorem. Now

$$\alpha'(s) = (\cos(\phi(s)), \sin(\phi(s))),$$

so that

$$\alpha(s) = \mathbf{a} + \left(\int_{t_0}^{s} \cos(\phi(u))\mathrm{d}u, \int_{t_0}^{s} \sin(\phi(u))\mathrm{d}u \right)$$

is the required curve. Again, α is uniquely determined by its derivative and the initial condition.

□

An alternative statement would be that the curvature function determines a unique curve *up to a rigid motion of the plane.* We can make a translation followed by a rotation of $\mathbb{R}^2$ to match up any two choices of initial position and direction.

Exercise 2.21. Do Carmo page 25, Section 1-5, Exercise 11.

Note that page numbers are always book page numbers not the numbers in the pdf file.

2.3. Curvature of curves in $\mathbb{R}^3$.

Definition 2.22. Suppose that we have a curve $\alpha\colon I \to \mathbb{R}^3$, which is parametrised by arc length s. The *curvature* is defined to be

$$k(s) := \left| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \right| = |\alpha''(s)|.$$

One main difference is that curvature is now a non-negative quantity.

If we have a curve not parametrised by arc length then by the chain rule

$$k(t) = \left| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \right| = \frac{|\mathrm{d}\mathbf{T}/\mathrm{d}t|}{|\mathrm{d}s/\mathrm{d}t|} = \frac{|\mathbf{T}'(t)|}{|\alpha'(t)|}.$$

Example 2.23. The first two examples here are rather similar to the examples above, but are included for completeness.

(i) A straight line. Suppose $\alpha(s) = \mathbf{a} + \mathbf{b}s$ where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are vectors with $|\mathbf{b}| = 1$. Then

$$\alpha'(s) = \mathbf{b} = \mathbf{T}(s)$$

and

$$k(s) = |\mathbf{T}'(s)| = 0.$$

(ii) A circle of radius a. Let $\alpha(s) = (a\cos(s/a), a\sin(s/a), 0)$. Then

$$\alpha'(s) = (-\sin(s/a), \cos(s/a), 0).$$

Note that $|\alpha'(s)| = 1$ so indeed we have a parametrisation by arc length. Therefore $\mathbf{T}(s) = \alpha'(s)$. The curvature is computed by

$$k(s) = |\mathbf{T}'(s)| = |\alpha''(s)| = |(-\frac{1}{a}\cos(s/a), -\frac{1}{a}\sin(s/a), 0)| = 1/a.$$

A circle of radius a has curvature $1/a$.

(iii) A helix on a cylinder of radius 1. Let $\alpha(s) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2})$. Then

$$\alpha'(s) = (-\frac{1}{\sqrt{2}}\sin(s/\sqrt{2}), \frac{1}{\sqrt{2}}\cos(s/\sqrt{2}), 1/\sqrt{2}).$$

Note that $|\alpha'(s)| = 1$ so indeed we have a parametrisation by arc length. Therefore $\mathbf{T}(s) = \alpha'(s)$. The curvature is computed by

$$k(s) = |\mathbf{T}'(s)| = |\alpha''(s)| = |(-\frac{1}{2}\cos(s/\sqrt{2}), -\frac{1}{2}\sin(s/\sqrt{2}), 0)| = 1/2.$$

The helix has the same curvature as a circle of radius 2 in the plane.

The last example shows that in order to fully describe the behaviour of curves in $\mathbb{R}^3$, we need another quantity. This will appear soon. First we need to recall the wedge product.

2.4. **The wedge product.** The wedge product is also known as the cross product. The formula is often remembered in terms of the 3×3 matrix determinant.

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

Given 3 vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ let $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ denote the matrix with $\mathbf{a}$ as its first column, $\mathbf{b}$ as its second column and $\mathbf{c}$ as its third column.

Definition 2.24. The vector $\mathbf{a} \wedge \mathbf{b}$ is the unique vector such that

$$(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = \det(\mathbf{a}, \mathbf{b}, \mathbf{c})$$

for all $\mathbf{c} \in \mathbb{R}^3$.

Recall also the following properties of the wedge product.

Proposition 2.25. *(a)*

$$|\mathbf{a} \wedge \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta,$$

where θ is the acute angle between $\mathbf{a}$ and $\mathbf{b}$.

(b) $\mathbf{a} \wedge \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ if $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$.

(c) $\mathbf{a} \wedge \mathbf{0} = \mathbf{0}$ for all $\mathbf{a}$.

(d) $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$ (this implies that $\mathbf{a} \wedge \mathbf{a} = \mathbf{0}$ for all $\mathbf{a} \in \mathbb{R}^3$).

Exercise 2.26. (i) Compute the wedge product of $(3, 4, 5)$ and $(-2, -1, 1)$.

(ii) If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis of $\mathbb{R}^3$ satisfying the right hand rule, find all relations of the form $\mathbf{e}_i \wedge \mathbf{e}_j = \pm\mathbf{e}_k$.

(iii) Given three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ define

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] := (\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c}.$$

Use the definition of the wedge product above and your linear algebra knowledge to show that

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}].$$

2.5. **Formula for curvature without computing arc length.**

Proposition 2.27. *Let $\alpha\colon I \to \mathbb{R}^3$ be a regular curve. The parametrisation need not be the arc length parametrisation. The curvature is computed as*

$$k(t) = \frac{|\alpha''(t) \wedge \alpha'(t)|}{|\alpha'(t)|^3}.$$

Proof. We know that an arc length parametrisation exists. Suppose that it is given by the substitution $t = f(s)$, where f is a function with $f'(s) > 0$. Then $\beta(s) = \alpha(f(s))$ is the arc length parametrisation, i.e. $|\beta'(s)| = |\alpha'(f(s))f'(s)| = 1$. Now we begin similarly to as in the planar curve case.

$$\beta'(s) = \alpha'(f(s))f'(s)$$

so

$$\beta''(s) = \alpha''(f(s))(f'(s))^2 + \alpha'(f(s))f''(s).$$

Now since $|\beta'(s)| = 1$, we know that $\beta'(s)$ is orthogonal to $\beta''(s)$. Therefore

$$k(s) = |\beta''(s)| = |\beta''(s)||\beta'(s)| = |\beta''(s) \wedge \beta'(s)|.$$

So we compute $\beta''(s) \wedge \beta'(s)$, to be

$$(\alpha''(f(s)) \wedge \alpha'(f(s)))(f'(s))^3 + (\alpha'(f(s)) \wedge \alpha'(f(s)))f''(s)f'(s)$$

which, since $\alpha'(f(s)) \wedge \alpha'(f(s)) = 0$, is equal to

$$(\alpha''(f(s)) \wedge \alpha'(f(s)))(f'(s))^3.$$

As in the planar case, we have $f'(s) = 1/|\alpha'(t)|$, so that substituting this and $s = f^{-1}(t)$ gives the claimed formula. □

2.6. **The normal and binormal vectors and the torsion.**

Definition 2.28. For a curve $\alpha\colon I \to \mathbb{R}^3$ parametrised by arc length with $k(s) \neq 0$, we define the *normal vector* to be the unit vector:

$$\mathbf{N}(s) := \frac{\mathbf{T}'(s)}{|\mathbf{T}'(s)|} = \frac{\mathbf{T}'(s)}{k(s)} = \frac{\alpha''(s)}{|\alpha''(s)|}.$$

The normal vector $\mathbf{N}(s)$ is characterised as the unique unit vector at each point of $\alpha(I)$ such that $k(s)\mathbf{N}(s) = \alpha''(s) = \mathbf{T}'(s)$. By Lemma 2.14 we have that $\mathbf{T}(s) \perp \mathbf{N}(s)$.

All curves will be parametrised by arc length when the parameter is s. When the parameter is t, it may not be the arc length parametrisation.

Definition 2.29. The plane through $p = \alpha(s)$ determined by $\mathbf{T}(s)$ and $\mathbf{N}(s)$ is called the *osculating plane* of α at p.

We need that $k(s) \neq 0$ in order for $\mathbf{N}$ and the osculating plane to be well-defined. So far we have two orthogonal vectors at each point of a curve. We complete this to a basis for $\mathbb{R}^3$.

Definition 2.30. At each $s \in I$, the *binormal vector* is the unit normal to the osculating plane given by

$$\mathbf{B}(s) := \mathbf{T}(s) \wedge \mathbf{N}(s).$$

Next we note some easy facts about the binormal vector $\mathbf{B}(s)$.

Proposition 2.31. *The following hold for all $s \in I$.*

(a) We have $|\mathbf{B}| = 1$.
(b) The vectors $\mathbf{B}$ and $\mathbf{B}'$ are orthogonal.
(c) The vectors $\mathbf{B}'$ and $\mathbf{T}$ are orthogonal.

Proof. (a) *We have $|\mathbf{B}| = 1$.*

$$|\mathbf{B}| = |\mathbf{T}||\mathbf{N}| \sin\theta,$$

by Proposition 2.25 (a), where θ is the angle between $\mathbf{T}$ and $\mathbf{N}$. In fact we know $\theta = \pi/2$, $|\mathbf{T}| = 1 = |\mathbf{N}|$, so we have $|\mathbf{B}| = 1$ as claimed.

(b) *The vectors $\mathbf{B}$ and $\mathbf{B}'$ are orthogonal.* Differentiate

$$1 = |\mathbf{B}| = \mathbf{B} \cdot \mathbf{B},$$

to obtain $2\mathbf{B} \cdot \mathbf{B}' = 0$.

(c) *The vectors $\mathbf{B}'$ and $\mathbf{T}$ are orthogonal.* We compute:

$$\mathbf{B}' = \mathbf{T}' \wedge \mathbf{N} + \mathbf{T} \wedge \mathbf{N}' = k\mathbf{N} \wedge \mathbf{N} + \mathbf{T} \wedge \mathbf{N}' = \mathbf{T} \wedge \mathbf{N}'.$$

By Proposition 2.25 (b), we see that $\mathbf{B}'$ and $\mathbf{T}$ are orthogonal.

□

Since $\mathbf{B}'$ is orthogonal to both $\mathbf{T}$ and $\mathbf{B}$, we must have that $\mathbf{B}'$ is some multiple of the normal vector $\mathbf{N}$. We can therefore define the torsion.

Definition 2.32. Let $\alpha \colon I \to \mathbb{R}^3$ be a regular curve parametrised by arc length with $k(s) \neq 0$ for all s. The *torsion* of α at $p = \alpha(s)$ is

$$\tau(s) := \mathbf{B}' \cdot \mathbf{N}.$$

Warning: many authors use a minus sign in this definition. We are following Do Carmo. For each s, we can characterise the torsion $\tau(s)$ as the unique real number such that $\mathbf{B}'(s) = \tau(s)\mathbf{N}(s)$.

Exercise 2.33. (1) Do Carmo, page 22, Section 1-5 Exercises 1 and 2.
(2) Compute the curvature and torsion at all time t for the curve given by

$$\alpha(t) = (t, t^2, t^3) \ (t \in \mathbb{R}).$$

2.7. **The Serret-Frenet formulae.** So far, at each point of a curve parametrised by arc length with $\alpha'(s) \neq 0$ and $\alpha''(s) \neq 0$, we have 3 vectors $\mathbf{T}$,$\mathbf{N}$ and $\mathbf{B}$, giving an *orthonormal frame*; that is, a basis of $\mathbb{R}^3$ for which the vectors are mutually orthogonal and of unit length. We can express the *derivative of this frame*, in terms of the curvature and torsion.

Theorem 2.34 (Serret-Frenet formulae)**.** *Let $\alpha \colon I \to \mathbb{R}^3$ be a regular curve with $k(s) \neq 0$ for all s, and let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the local frame given by the tangent, normal and binormal vectors respectively. The derivative of the frame at s is given in terms of the curvature $k(s)$ and the torsion $\tau(s)$ by:*

$$\begin{array}{rcrcr} \mathbf{T}'(s) & = & & k(s)\mathbf{N}(s) & \\ \mathbf{N}'(s) & = & -k(s)\mathbf{T}(s) & & -\tau(s)\mathbf{B}(s) \\ \mathbf{B}'(s) & = & & \tau(s)\mathbf{N}(s) & \end{array}$$

The display of the above equations is intended to evoke a matrix. The frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is called the *Serret-Frenet frame*, or by some authors the *Serret-Frenet trihedron*. Differentiating the Serret-Frenet frame can be written formally in terms of matrices as follows:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}'\end{pmatrix} = \begin{pmatrix}0 & k & 0\\ -k & 0 & -\tau\\ 0 & \tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}.$$

Each entry of a vector is itself a vector, so we could think of these vectors as 3×3 matrices, where each of the vectors in an entry of the original larger vector becomes a row of the matrix. The 3×3 matrix in the equation above, which we will call A, is a *skew symmetric* matrix, since $A = -A^T$.

Proof of Serret-Frenet formulae. The first and third equations are by definition, or computations that we did above. By the Leibniz rule trick in Lemma 2.14, any vector function with constant length will always be orthogonal to its derivative; hence diagonal entries of A must be zero.

It remains to compute $\mathbf{N}'$. We have that $\mathbf{N} = \mathbf{B}\wedge\mathbf{T}$ so

$$\begin{aligned}\mathbf{N}' &= \mathbf{B}'\wedge\mathbf{T} + \mathbf{B}\wedge\mathbf{T}'\\ &= \tau\mathbf{N}\wedge\mathbf{T} + \mathbf{B}\wedge k\mathbf{N}\\ &= -\tau(\mathbf{T}\wedge\mathbf{N}) - k(\mathbf{N}\wedge\mathbf{B})\\ &= -\tau\mathbf{B} - k\mathbf{T}\end{aligned}$$

as claimed. □

Alternative proof using matrices. Write the vectors $\mathbf{T},\mathbf{N},\mathbf{B}$ as the rows of a 3×3 matrix:

$$P := \begin{pmatrix}T_1 & T_2 & T_3\\ N_1 & N_2 & N_3\\ B_1 & B_2 & B_3\end{pmatrix}.$$

The fact that these rows are mutually orthogonal implies that $PP^T = I = P^TP$, where I is the 3×3 identity matrix. Differentiating, we obtain

$$P'P^T + P(P^T)' = 0 \tag{2.35}$$

where 0 denotes the zero matrix. Multiplying on the right by P we obtain

$$P'P^TP + P(P^T)'P = 0,$$

which since $P^TP = I$ yields

$$P' = -P(P^T)'P.$$

Thus $A := -P(P^T)'$ is the matrix we seek i.e. $P' = AP$. Note that $A^T = (-P(P^T)')^T = -P'P^T$. Thus equation 2.35 implies that $A + A^T = 0$, that is A is skew symmetric. We know by definition that the first row of A is $(0,k,0)$ and that the last row is $(0,\tau,0)$. The middle row is filled in by the skew-symmetry property, so that

$$A = \begin{pmatrix}0 & k & 0\\ -k & 0 & -\tau\\ 0 & \tau & 0\end{pmatrix}$$

as claimed. □

Exercise 2.36. Verify the Serret-Frenet formulae (by differentiating the unit normal vector) for the helix

$$\alpha(s) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}).$$

In order to understand curvature and torsion, the following proposition is helpful. It says what happens when the quantities are identically zero.

Proposition 2.37. *(a) If $k(s) = 0$ for all $s \in I$, then $\alpha(s)$ is part of a straight line.*
(b) If $\tau(s) = 0$ for all $s \in I$ then $\alpha(s)$ is a planar curve, that is it lies inside some plane in $\mathbb{R}^3$.

Proof. (a) *If $k(s) = 0$ for all $s \in I$, then $\alpha(s)$ is part of a straight line.*
If $k(s) = 0$ then $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \mathbf{0}$ which implies that $\mathbf{T} = \mathbf{a}$, for some constant vector $\mathbf{a}$ with $|\mathbf{a}| = 1$. Then since $\mathbf{T} = \alpha'(s)$. Integrating again, we obtain $\alpha(s) = \mathbf{a}s + \mathbf{b}$ for some constant vector b. This is the parametric form of a straight line.

(b) *If $\tau(s) = 0$ for all $s \in I$ then $\alpha(s)$ is a planar curve, that is it lies inside some plane in $\mathbb{R}^3$.*
If $\tau(s) = 0$ then we have

$$\mathbf{B}' = \mathbf{0}.$$

This implies that $\mathbf{B}(s)$ is a constant vector which we just denote as $\mathbf{B}$. We claim that $\alpha(s)$ lies in a plane of which $\mathbf{B}$ is a normal vector. In particular proving the claim will finish the proof of the proposition. To prove the claim, we need to see that $\mathbf{B} \cdot \alpha(s)$ is a constant. So we differentiate $\mathbf{B} \cdot \alpha(s)$ to obtain

$$\mathbf{B}' \cdot \alpha(s) + \mathbf{B} \cdot \alpha'(s) = \mathbf{B} \cdot \alpha'(s) = \mathbf{B} \cdot \mathbf{T}(s) = 0$$

since $\mathbf{B} = \mathbf{T}(s) \wedge \mathbf{N}(s)$. Then $(\mathbf{B} \cdot \alpha(s))' = 0$ implies $\mathbf{B} \cdot \alpha(s)$ is a constant, c say, so that $\alpha(s)$ lies in the plane $\mathbf{r} \cdot \mathbf{B} = c$, as claimed.

□

We can therefore think about the curvature as representing the failure of a curve to be a straight line and the torsion as representing the failure of a curve to be planar.

Exercise 2.38. (a) Show that if the curvature of $\alpha(s)$ is a constant and $\tau(s) = 0$ then $\alpha(s)$ is part of a circle.
(b) Do Carmo, Section 1-5, Exercise 13, 15, 17, 18.

The final topic in our study of curves is the fundamental theorem for curves in R^3. This is the 3-dimensional analogue of the fundamental theorem for planar curves. It says that knowledge of initial position and direction, together with knowledge of the curvature and torsion functions, determine a unique curve parametrised by arc length. This is analogous to flying a plane at constant speed and always knowing the position of your flaps, then being able to recover your flight path.

Theorem 2.39 (Fundamental theorem for curves in $\mathbb{R}^3$)**.** *Suppose we are given a vector* $\mathbf{a}$ *and a vector* $\mathbf{b}$ *with $|\mathbf{b}| = 1$, an interval $I \subseteq \mathbb{R}$, a point $t_0 \in I$ together* *with a positive differentiable function* $k\colon I \to \mathbb{R}_{>0}$ *(so* $k(s) > 0$ *for all* s*) and a differentiable function* $\tau\colon I \to \mathbb{R}$*. Then there exists a unique curve* $\alpha\colon I \to \mathbb{R}^3$ *with* $\alpha(t_0) = \mathbf{a}$*,* $\alpha'(t_0) = \mathbf{b}$*, curvature at* $s \in I$ *given by* $k(s)$ *and torsion given by* $\tau(s)$*.*

We will omit the proof of this theorem, since it is quite long, we have seen the general idea in the planar case, and you can (and should) read the proof in Do Carmo pages 20–21 for the uniqueness statement and pages 309–311 for the existence statement. Note that the Serret-Frenet formulae are a crucial part of the proof.

2.8. **Spherical curves.** Next we will study curves on a surface, which is not the plane, namely the 2-sphere S^2. Since the 2-sphere is curved (we will see this precisely in later chapters, but intuitively it seems reasonable). We will study the geometry of "straight lines," or *geodesics* on the sphere.

Definition 2.40. The *2-sphere* S^2 is the subset of $\mathbb{R}^3$ defined by

$$\{\mathbf{x} \in \mathbb{R}^3 \,|\, \mathbf{x}\cdot\mathbf{x} = 1\}.$$

That is, the set of points at distance one from the origin.

Definition 2.41 (Spherical curves)**.** A curve $\sigma\colon I \to \mathbb{R}^3$ is a *spherical curve* if $\sigma(I) \subseteq S^2$ i.e. $\sigma(t)\cdot\sigma(t) = 1$ for all $t \in I$.

Example 2.42. (i) $\sigma(t) = (\cos t, \sin t, 0)$, $t \in \mathbb{R}$.
(ii) $\sigma(t) = (\sin t, 0, \cos t)$, $t \in \mathbb{R}$.

We can construct curves on the sphere by intersecting it with a plane. Let $\mathbf{n} \in \mathbb{R}^2$ be a unit vector. Suppose

$$\mathbf{a} \in D^3 := \{\mathbf{x} \in \mathbb{R}^3 \,|\, \mathbf{x}\cdot\mathbf{x} \le 1\}.$$

Let P be the plane defined by $\mathbf{x}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ i.e. the plane through $\mathbf{a}$ with normal vector $\mathbf{n}$. Then the intersection $P \cap S^2$ is a circle. To see this assume that $\mathbf{n} = (0,0,1)$; this can always be arranged by a rotation of coordinates. Then The equation of P reduces to $z = a_3$, where $\mathbf{a} = (a_1, a_2, a_3)$. Thus on the sphere $x^2 + y^2 + z^2 = 1$ we have the equation $x^2 + y^2 = 1 - a_3^2$ for the equation of the intersection $P \cap S^2$. This is the equation of the circle of radius $\sqrt{1 - a_3^2}$, in the plane $z = a_3$, with centre $(0, 0, a_3)$. One possible parametrisation of this curve is

$$\sigma(t) = \left(\left(\sqrt{1 - a_3^2}\right)\cos t, \left(\sqrt{1 - a_3^2}\right)\sin t, a_3\right), \; t \in \mathbb{R}.$$

If the origin lies in P, then $P \cap S^2$ is called a *great circle.* The radius of a great circle is one.

Now we return to a general (regular) spherical curve, parametrised by arc length s. We wish to obtain a moving frame which gives a basis for $\mathbb{R}^3$ at each point of $\sigma(I)$. We will then study spherical Serret Frenet formulae, detailing the time evolution of the frame.

Lemma 2.43. $\sigma(s)$ *is orthogonal to* $\sigma'(s)$ *for all* $s \in I$*.*

Proof. Since $|\sigma(s)| = 1$ for all s, we have $\sigma(s) \cdot \sigma(s) = 1$ so $2\sigma(s) \cdot \sigma'(s) = 0$ for all s, which implies $\sigma(s) \cdot \sigma'(s) = 0$ as claimed. □

In fact, any tangent vector to S^2 at $p \in S^2$ is orthogonal to the vector for p. Recall that $|\sigma'(s) = 1|$ so $\sigma'(s) = \mathbf{T}(s)$.

Definition 2.44. For a spherical curve $\sigma\colon I \to S^2$, denote $\tau := \sigma'(s)$. Also define the spherical unit normal

$$\nu(s) := \sigma(s) \wedge \tau(s).$$

Since $|s| = 1 = |\tau|$ and the two are orthogonal, we see that $|\nu| = 1$ too.

A *geodesic* is the equivalent of a straight line — the shortest distance between two points, at least it minimises arc length under small perturbations. But in other geometries, like the spherical geometry we consider here, the shortest path may not look like a straight line. Also on S^2 there are two geodesics between any two points, one longer than the other; unless the points are antipodal, in which case there are many geodesics, which are half a diameter, between any two points, and they are all of equal length e.g. the lines of constant longitude on the earth, between the north and south poles.

The *geodesic curvature* measures the failure of a curve to be a geodesic. Since $|\sigma'(s)| = 1$, we again have $\sigma''(s) \cdot \sigma'(s) = 0$ for all s.

Definition 2.45. Define the *geodesic curvature* of a spherical curve $\sigma\colon I \to S^2$ parametrised by arc length to be

$$k_g(s) := \sigma''(s) \cdot \nu(s).$$

Example 2.46.
(i) Let $\sigma(s) = (\cos s, 0, \sin s)$. Then

$$\sigma'(s) = (-\sin s, 0, \cos s) = \tau(s).$$

Thus $\nu = \sigma \wedge \tau = (0, -1, 0)$ and $\sigma''(s) = (-\cos s, 0, -\sin s)$. We compute $\sigma'' \cdot \nu = 0$. Great circles have geodesic curvature zero.

(ii) Next we consider the intersection of S^2 with any plane, as above. Let

$$\sigma(t) = \left(a\cos\left(\frac{t}{a}\right), a\sin\left(\frac{t}{a}\right), \sqrt{1-a^2}\right).$$

Then

$$\tau(t) = \sigma'(t) = \left(-\sin\left(\frac{t}{a}\right), \cos\left(\frac{t}{a}\right), 0\right).$$

Note that $|\sigma'(t)| = 1$, so we have an arc length parametrisation. Next,

$$\sigma''(t) = \left(-\frac{1}{a}\cos\left(\frac{t}{a}\right), -\frac{1}{a}\sin\left(\frac{t}{a}\right), 0\right).$$

The normal vector is

$$\begin{aligned}\nu(t) = \sigma(t) \wedge \tau(t) &= \left(a\cos\left(\frac{t}{a}\right), a\sin\left(\frac{t}{a}\right), \sqrt{1-a^2}\right) \wedge \left(-\sin\left(\frac{t}{a}\right), \cos\left(\frac{t}{a}\right), 0\right) \\ &= \left(-\sqrt{1-a^2}\cos\left(\frac{t}{a}\right), -\sqrt{1-a^2}\sin\left(\frac{t}{a}\right), a\right).\end{aligned}$$

Thus

$$k_g(t) = \sigma''(t) \cdot \nu(t) = \frac{\sqrt{1-a^2}}{a}.$$

The geodesic curvature vanishes if and only if $a = 1$ i.e. the circle is a great circle.

Definition 2.47. A curve on S^2 with $k_g(s) = 0$ for all s is called a *geodesic.*

A geodesic triangle on S^2 is a triangle whose sides are geodesics. Given a geodesic triangle $\triangle$ with angles α, β and γ, the following formula holds:

$$\alpha + \beta + \gamma - \pi = \text{area}(\triangle).$$

This can be contrasted with the formula for triangles in the plane, for which $\alpha + \beta + \gamma - \pi = 0$. It turns out that the difference in the sum of the angles of a triangle from π is a measure of curvature. As an example of the formula above, consider a triangle with all 3 angles $\pi/2$ on the 2 sphere. One vertex is at the north pole, and the other two vertices are on the equator, one on the Greenwich meridian and one on the meridian which runs through the middle of US central time; i.e. 6 hours behind Greenwich, one quarter of the way around the equator (pretend the earth has radius one). The left hand side of the formula is $\pi/2+\pi/2+\pi/2-\pi = \pi/2$. The right hand side is the area of this triangle; note that this is one eighth the area of the sphere, $4\pi/8 = \pi/2$. We verify the formula in this special case. We will prove this formula holds in general later in the course.

Next we have the Serret-Frenet formulae for spherical frames.

Theorem 2.48. *Let $\sigma\colon I \to S^2$ be a spherical curve parametrised by arc length, and let $\{\sigma, \tau\nu\}$ be the spherical frame. The derivative of the frame at $s \in I$ is given in terms of the geodesic curvature $k_g(s)$ by:*

$$\begin{array}{rcccc} \sigma'(s) &=& & \tau(s) & \\ \tau'(s) &=& -\sigma(s) & & +k_g(s)\nu(s) \\ \nu'(s) &=& & -k_g(s)\tau(s) & \end{array}.$$

As in class this can be proven using the skew symmetric matrix proof of the previous Serret-Frenet formulae. A great example of a more abstract proof that turns out to have a much wider applicability. Below we also give the less elegant proof via direct computation.

Proof. The first equation is by definition. Since τ has constant length we know it is orthogonal to its derivative. Differentiating τ we obtain $\tau' = \sigma''$. Since $\sigma'' \cdot \nu = k_g$ by definition of k_g, the component of τ' in the ν direction is k_g. Then $\tau' \cdot \sigma = \sigma'' \cdot \sigma$. We know that $\sigma \cdot \sigma = 1$, so $\sigma \cdot \sigma' = 0$ at all $s \in I$. Differentiating this again with the Leibniz rule gives

$$0 = \sigma' \cdot \sigma' + \sigma \cdot \sigma'' = 1 + \sigma \cdot \sigma'',$$

so that $\sigma \cdot \sigma'' = -1$ as claimed.

Finally we want to compute ν'. Starting with $\nu \cdot \sigma = 0$ and differentiating we obtain

$$0 = \nu' \cdot \sigma + \nu \cdot \sigma' = \nu' \cdot \sigma + \nu \cdot \tau\nu' \cdot \sigma.$$

We also know that ν is orthogonal to ν'. Therefore we just need to compute $\nu' \cdot \tau$. Start with $\nu \cdot \tau = 0$. Then

$$0 = \nu' \cdot \tau + \nu \cdot \tau' = \nu' \cdot \tau + \nu \cdot (-\sigma + k_g \nu) = \nu' \cdot \tau - \nu \cdot \sigma + k_g \nu \cdot \nu = \nu' \cdot \tau - k_g.$$

Therefore $\nu' \cdot \tau - k_g$ as claimed. This completes the proof of the Serret-Frenet formulae for spherical curves. □

Corollary 2.49. *Any geodesic on S^2 is a part of a great circle.*

Proof. A geodesic is a curve with $k_g = 0$. Thus $\nu'(s) = 0$ so $\nu(s)$ is a constant vector, $\mathbf{n}$ say. Then $\sigma(s) \cdot \mathbf{n} = 0$ for all s. That is, $\sigma(s)$ lies in the plane $\mathbf{x} \cdot \mathbf{n} = 0$ for all s. This is a plane through the origin with normal vector $\mathbf{n}$. The intersection of such a plane with S^2 is a great circle, as we saw above. So $\sigma(I)$ is part of this great circle. □

So planes try to fly on great circles. We didn't prove it, but it turns out that geodesics are distance minimising, at least under small enough perturbations. Geodesics on S^2 are the analogue of straight lines in $\mathbb{R}^2$.

Remark 2.50. There is also a fundamental theorem for spherical curves too. If one knows the initial position and direction, a given geodesic curvature function $k_g(s)$ determines a unique spherical curve parametrised by arc length.

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