SEMINAR ON MAPPING CLASS GROUPS OF 3-MANIFOLDS

ORGANISED BY RACHAEL BOYD AND MARK POWELL

Our goal is to understand what is known about mapping class groups, and more generally diffeomorphism groups, of compact 3-manifolds. To begin with, we will focus on mapping class groups, i.e. the connected components of the diffeomorphism groups. We will restrict ourselves to orientable 3-manifolds, in the first instance.

After the overview talk we will jump straight into examples of known mapping class groups. Talks should state the results, and then go into as much detail as time allows on the ideas in the proof.

An ancillary aim of the seminar is to begin to produce a survey on diffeomorphism groups of 3-manifolds. There has been exciting recent progress, meaning we have an excellent understanding of the mapping class groups of all 3-manifolds. But it is spread throughout the literature, and there is a need for a coherent localised description. As a first step towards this target, each speaker should type up the notes from their talk. We will share a communal Overleaf document for this purpose.

1. **Overview and basic definitions**, (Rachael Boyd, Oct 3):
Give some basic definitions: diffeomorphism groups, isometry groups, mapping class groups. Describe relation between them and state weak and strong generalised Smale conjecture. Mention that both maps in the composition $\mathrm{Diff}(M) \to \mathrm{PL}(M) \to \mathrm{Homeo}(M)$ are homotopy equivalences. This uses the Smale conjecture, and smoothing theory. State the goals of the seminar.

2. **Example:** S^3, (Isacco Nonino, Oct 10):
Give an overview of Cerf's proof that every orientation-preserving diffeomorphism of S^3 is isotopic to the identity, i.e. $\pi_0(\mathrm{Diff}^+(S^3)) = \{0\}$, and $\pi_0(\mathrm{Diff}(S^3)) \cong \mathbb{Z}/2$ [Cer68]. There are alternative proofs by Laudenbach and Eliashberg (see Geiges-Zehmisch [GZ10]). Discuss the Smale conjecture $\mathrm{Diff}(S^3) \simeq O(4)$ [Hat83].

3. **Example:** $S^1 \times S^2$, (Daniel Galvin, Oct 17):
Explain the computation of the mapping class group of $S^1 \times S^2$ by Gluck [Glu61; Glu62]. Discuss the homotopy type of $\mathrm{Diff}(S^1 \times S^2) \simeq O(2) \times O(3) \times \Omega O(3)$ due to Hatcher [Hat81]; note that he posted a rewrite of his original paper on his website.

4. **Example: Lens spaces**, (Brendan Owens, Oct 24):
Explain the computation of the mapping class groups of lens spaces due to Bonahon [Bon83].

5. **Example: Elliptic 3-manifolds**, (Philipp Bader, Oct 31):
Describe the classification of elliptic 3-manifolds. Discuss mapping class groups

of elliptic 3-manifolds, namely those with finite fundamental groups. See Hong–Kalliongis–McCullough–Rubinstein [Hon+12] for an overall discussion and references, which include work of Asano [Asa78], Rubinstein [Rub78; Rub79], Boileau–Otal [BO86], Rubinstein–Birman [RB84]. Describe how to compute the isometries of such 3-manifolds. Mention the generalised Smale conjecture for these manifolds, which has now been proven [Hon+12].

6. **Example: Haken 3-manifolds**, (Mark Powell, Nov 7):
Define Haken 3-manifolds (known as *sufficiently large* 3-manifolds in the 1970s literature). Present Waldhausen's results on mapping class groups of Haken 3-manifolds M, in particular that the map $\pi_0(\mathrm{Diff}(M)) \to \pi_0(\mathrm{hAut}(M))$ is an isomorphism [Wal68] (see also [Sco72]). Discuss that the latter coincides with $\mathrm{Out}(\pi_1(M))$. Mention the work of Hatcher [Hat76] and Ivanov [Iva76], who showed that in fact $\mathrm{Diff}(M) \to \mathrm{hAut}(M)$ is a homotopy equivalence. Johannson [Joh79] showed that 'simple' Haken 3-manifolds (those with trivial JSJ decomposition) have finite mapping class groups, and McCullough [McC91] showed in general they are finitely presented, and investigated other finiteness properties.

7. **Example: Hyperbolic 3-manifolds**, (John Nicholson, Nov 14):
Discuss Mostow rigidity [Mos68]. For Haken hyperbolic 3-manifolds, Waldhausen's results [Wal68] compute the mapping class groups. Discuss the work of Gabai [Gab97; Gab01] and Gabai–Meyerhoff–Thurston [GMT03] on completing the proof that $\pi_0(\mathrm{Diff}(M)) \cong \pi_0(\mathrm{Isom}(M))$.

8. **JSJ decompositions and irreducible 3-manifolds**, (Csaba Nagy, Nov 21):
Follow Jaco lectures [Jac80] and exposition in Aschenbrenner, Friedl and Wilton [AFW15], or Hatcher's notes on classification of 3-manifolds. State the JSJ Decomposition Theorem, and combine it with the Elliptisation Theorem and the Hyperbolisation Theorem to get the Geometrisation Theorem. Include examples. Discuss how to use the JSJ theorem together with the knowledge of the mapping class groups of the geometric pieces to compute the mapping class groups of irreducible 3-manifolds.

9. **Reducible 3-manifolds**, (Weizhe Niu, 28 Nov):
Discuss how to compute the mapping class group of a reducible 3-manifold in terms of the mapping class groups of the irreducible summands in the prime decomposition, plus the number of $S^1 \times S^2$ prime summands. The question of finite presentation is addressed in Hatcher–McCullough [HM90]. References include Cesar de Sa–Rourke [CR79], Hendricks–Laudenbach [HL84], Hendricks–McCullough [HM87], Brendle–Broaddus–Putman [BBP23]. Focus on the case of $\#^k S^1 \times S^2$, following [Lau73; Lau74; BBP23].

References

[AFW15] Matthias Aschenbrenner, Stefan Friedl, and Henry Wilton. *3-manifold groups*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2015, pp. xiv+215.

[Asa78] Kouhei Asano. "Homeomorphisms of prism manifolds". In: *Yokohama Math. J.* 26.1 (1978), pp. 19–25.

[BBP23] Tara Brendle, Nathan Broaddus, and Andrew Putman. "The mapping class group of connect sums of $S^2 \times S^1$". In: *Trans. Amer. Math. Soc.* 376.4 (2023), pp. 2557–2572.

[BO86] Michel Boileau and Jean-Pierre Otal. "Groupe des difféotopies de certaines variétés de Seifert". In: *C. R. Acad. Sci. Paris Sér. I Math.* 303.1 (1986), pp. 19–22.

[Bon83] Francis Bonahon. "Difféotopies des espaces lenticulaires". In: *Topology* 22.3 (1983), pp. 305–314.

[Cer68] Jean Cerf. *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$). Vol. No. 53. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1968, pp. xii+133.

[CR79] Eugénia César de Sá and Colin Rourke. "The homotopy type of homeomorphisms of 3-manifolds". In: *Bull. Amer. Math. Soc. (N.S.)* 1.1 (1979), pp. 251–254.

[Gab01] David Gabai. "The Smale conjecture for hyperbolic 3-manifolds: $\mathrm{Isom}(M^3) \simeq \mathrm{Diff}(M^3)$". In: *J. Differential Geom.* 58.1 (2001), pp. 113–149.

[Gab97] David Gabai. "On the geometric and topological rigidity of hyperbolic 3-manifolds". In: *J. Amer. Math. Soc.* 10.1 (1997), pp. 37–74.

[Glu61] Herman Gluck. "The embedding of two-spheres in the four-sphere". In: *Bull. Amer. Math. Soc.* 67 (1961), pp. 586–589.

[Glu62] Herman Gluck. "The embedding of two-spheres in the four-sphere". In: *Trans. Amer. Math. Soc.* 104 (1962), pp. 308–333.

[GMT03] David Gabai, G. Robert Meyerhoff, and Nathaniel Thurston. "Homotopy hyperbolic 3-manifolds are hyperbolic". In: *Ann. of Math. (2)* 157.2 (2003), pp. 335–431.

[GZ10] Hansjörg Geiges and Kai Zehmisch. "Eliashberg's proof of Cerf's theorem". In: *J. Topol. Anal.* 2.4 (2010), pp. 543–579.

[Hat76] Allen Hatcher. "Homeomorphisms of sufficiently large P^2-irreducible 3-manifolds". In: *Topology* 15.4 (1976), pp. 343–347.

[Hat81] A. Hatcher. "On the diffeomorphism group of $S^1 \times S^2$". In: *Proc. Amer. Math. Soc.* 83.2 (1981), pp. 427–430.

[Hat83] Allen E. Hatcher. "A proof of the Smale conjecture, $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$". In: *Ann. of Math. (2)* 117.3 (1983), pp. 553–607.

[HL84] Harrie Hendriks and François Laudenbach. "Difféomorphismes des sommes connexes en dimension trois". In: *Topology* 23.4 (1984), pp. 423–443.

[HM87] Harrie Hendriks and Darryl McCullough. "On the diffeomorphism group of a reducible 3-manifold". In: *Topology Appl.* 26.1 (1987), pp. 25–31.

[HM90] Allen Hatcher and Darryl McCullough. "Finite presentation of 3-manifold mapping class groups". In: *Groups of self-equivalences and related topics (Montreal, PQ, 1988)*. Vol. 1425. Lecture Notes in Math. Springer, Berlin, 1990, pp. 48–57.

[Hon+12] Sungbok Hong, John Kalliongis, Darryl McCullough, and J. Hyam Rubinstein. *Diffeomorphisms of elliptic 3-manifolds.* Vol. 2055. Lecture Notes in Mathematics. Springer, Heidelberg, 2012, pp. x+155.

[Iva76] N. V. Ivanov. "Groups of diffeomorphisms of Waldhausen manifolds". In: *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 66 (1976). Studies in topology, II, pp. 172–176, 209.

[Jac80] William Jaco. *Lectures on three-manifold topology.* Vol. 43. CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence, RI, 1980, pp. xii+251.

[Joh79] Klaus Johannson. "On the mapping class group of simple 3-manifolds". In: *Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977).* Vol. 722. Lecture Notes in Math. Springer, Berlin, 1979, pp. 48–66.

[Lau73] F. Laudenbach. "Sur les 2-sphères d'une variété de dimension 3". In: *Ann. of Math. (2)* 97 (1973), pp. 57–81.

[Lau74] François Laudenbach. *Topologie de la dimension trois: homotopie et isotopie.* Vol. No. 12. Astérisque. With an English summary and table of contents. Société Mathématique de France, Paris, 1974, pp. i+152.

[McC91] Darryl McCullough. "Virtually geometrically finite mapping class groups of 3-manifolds". In: *J. Differential Geom.* 33.1 (1991), pp. 1–65.

[Mos68] G. D. Mostow. "Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms". In: *Inst. Hautes Études Sci. Publ. Math.* 34 (1968), pp. 53–104.

[RB84] J. H. Rubinstein and J. S. Birman. "One-sided Heegaard splittings and homeotopy groups of some 3-manifolds". In: *Proc. London Math. Soc. (3)* 49.3 (1984), pp. 517–536.

[Rub78] J. H. Rubinstein. "One-sided Heegaard splittings of 3-manifolds". In: *Pacific J. Math.* 76.1 (1978), pp. 185–200.

[Rub79] J. H. Rubinstein. "On 3-manifolds that have finite fundamental group and contain Klein bottles". In: *Trans. Amer. Math. Soc.* 251 (1979), pp. 129–137.

[Sco72] G. P. Scott. "On sufficiently large 3-manifolds". In: *Quart. J. Math. Oxford Ser. (2)* 23 (1972), 159–172, correction, ibid. (2) **24 (1973), 527–529**.

[Wal68] Friedhelm Waldhausen. "On irreducible 3-manifolds which are sufficiently large". In: *Ann. of Math. (2)* 87 (1968), pp. 56–88.