MAPPING CLASS GROUPS OF 3-MANIFOLDS

PHILIPP BADER, RACHAEL BOYD, GIULIA CARFORA, DANIEL GALVIN, RICCARDO GIANNINI, CSABA NAGY, JOHN NICHOLSON, WEIZHE NIU, ISACCO NONINO, MARK PENCOVITCH, AND MARK POWELL

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1. Introduction

Let M be a compact 3-manifold. For most of this survey, M will be closed (i.e. $\partial M = \emptyset$) and irreducible (every 2-sphere bounds a 3-ball).

1.1. **Symmetries of M.** We can consider smooth, PL, and continuous homeomorphisms $M \to M$, and these form topological groups which sit in the following sequence:

$$\mathrm{Diff}(M) \to \mathrm{PL}(M) \to \mathrm{Homeo}(M)$$

Note: the left hand map exists but it is not natural, to define it one must introduce the notion of a piecewise differentiable map. These form a group $\mathrm{PDiff}(M)$ and there are maps $\mathrm{Diff}(M) \to \mathrm{PDiff}(M)$ and $\mathrm{PL}(M) \xrightarrow{\simeq} \mathrm{PDiff}(M)$ induced by inclusion. The map $\mathrm{Diff}(M) \to \mathrm{PL}(M)$ is then the composite of the first map with the homotopy inverse of the second map.

The resulting map $\mathrm{Diff}(M) \to \mathrm{PL}(M)$ is a homotopy equivalence by smoothing theory, and Hatcher's proof of the Smale conjecture [Hat83] and the map $\mathrm{Diff}(M) \to \mathrm{Homeo}(M)$ is an equivalence by work of Cerf [Cer59].

We therefore restrict ourselves to looking at $\mathrm{Diff}(M)$ equipped with the C^∞ topology/Whitney topology – see Kupers [Kup19] for a detailed introduction to this topology.

Definition 1.1. The *mapping class group* of M is:

$$\Gamma(M) \coloneqq \pi_0(\mathrm{Diff}(M)) = \mathrm{Diff}(M)/\text{isotopy}.$$

Recall f is isotopic to g if there exists a smooth map $F\colon M \times I \to M$ such that $F_0 = f$, $F_1 = g$ and $F_t : M \to M \in \mathrm{Diff}(M)$ for all $t \in I$.

Let $\mathrm{Diff}_0(M)$ be the connected component of $\mathrm{Diff}(M)$ which contains the identity (i.e. all diffeomorphisms of M which are isotopic to the identity). We will frequently refer to the following exact sequence:

$$\mathrm{Diff}_0(M) \to \mathrm{Diff}(M) \to \pi_0(\mathrm{Diff}(M)) = \Gamma(M). \tag{1.2}$$

We will also make use to the natural map

$$\Gamma(M) \to \mathrm{Out}(\pi_1(M)) \tag{1.3}$$

which comes from the connecting map in the long exact sequence of homotopy groups associated to the fibration

$$\mathrm{Diff}_p(M) \to \mathrm{Diff}(M) \to \mathrm{Emb}(\{x\}, M) \simeq M,$$

where $\{x\}$ is the one point space, and p is a chosen point in M. This map is not necessarily onto or injective, but we will study situations in which it is an isomorphism.

There are two more groups of symmetries that we will study. These are:

- $\mathrm{hAut}(M) := \{f\colon M \to M \mid f \text{ is a homotopy equivalence }\}$, the group of homotopy auto-equivalences of M. Note that $\mathrm{Homeo}(M) \hookrightarrow \mathrm{hAut}(M)$.
- $\mathrm{Isom}(M)$: if M has a Riemannian metric g, then $f \in \mathrm{Diff}(M)$ is an *isometry* if $f_*(g) = g$. This has the consequence that f is *distance preserving* with respect to g. It follows $\mathrm{Isom}(M) \hookrightarrow \mathrm{Diff}(M)$.

Putting together all the inclusions, we get the following sequence

$$\mathrm{Isom}(M) \hookrightarrow \mathrm{Diff}(M) \hookrightarrow \mathrm{Homeo}(M) \hookrightarrow \mathrm{hAut}(M). \tag{1.4}$$

1.2. **Geometrisation.** A metric g on M is *locally homogenous* if for all $x, y \in M$, there exist neighbourhoods $x \in U_x$, $y \in U_y$, and an isometry $U_x \to U_y$. If M has such a g, then it follows that $\mathrm{Isom}(\widetilde{M})$ acts transitively on $\widetilde{M}$ with the metric inherited from (M, g). We say that M "admits a geometric structure modelled on $\widetilde{M}$".

Theorem 1.5 (Thurston's geometrisation conjecture, proved by Perelman)**.** *After cutting M along a canonical system of tori and annuli (JSJ decomposition), the pieces all admit a geometric structure that is either Seifert fibred or atoroidal (the only embedded tori are boundary parallel).*

Moreover, the possible geometric structures are classified. There are eight of them: six are Seifert fibred and there are also hyperbolic and Sol manifolds. Of these eight, the Sol manifolds are the only ones with nontrivial JSJ decomposition.

We say that M is *geometric* if it admits one of the eight geometric structures.

1.3. **Smale conjecture.** The original Smale conjecture was the following statement.

Theorem 1.6 (Smale conjecture, [Hat83])**.** $\mathrm{Diff}(S^3) \simeq \mathrm{Isom}(S^3) \simeq O_4$.

Conjecture 1.7 (Generalised Smale conjecture)**.** *The inclusion* $\mathrm{Isom}(M) \hookrightarrow \mathrm{Diff}(M)$ *is a homotopy equivalence.*

This does not hold in general: for example $\mathrm{Diff}(S^1 \times S^2) \simeq O_3 \times O_2 \times \Omega O_3$, and $\mathrm{Isom}(S^1 \times S^2) \simeq O_3 \times O_2$.

We can instead consider the following diagram

$$\begin{array}{ccccc}
\mathrm{Diff}_0(M) & \longrightarrow & \mathrm{Diff}(M) & \longrightarrow & \pi_0(\mathrm{Diff}(M)) = \Gamma(M) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Isom}_0(M) & \longrightarrow & \mathrm{Isom}(M) & \longrightarrow & \pi_0(\mathrm{Isom}(M))
\end{array}$$

where $\mathrm{Isom}_0(M)$ is the connected component of the isometry group $\mathrm{Isom}(M)$ containing the identity. Restricting our attention to these connected components, a version of the Smale conjecture does hold in general:

Theorem 1.8 (Weak Smale conjecture)**.** *If M is geometric, then* $\mathrm{Isom}_0(M) \hookrightarrow \mathrm{Diff}_0(M)$ *is a homotopy equivalence. Moreover the homotopy type is known.*

Note: this theorem is due to many people in different cases, and one of our goals could be to track down all of the references.

The upshot of this theorem is that we can understand $\mathrm{Diff}(M)$ in terms of the mapping class group $\Gamma(M)$ and the connected component $\mathrm{Diff}_0(M) \simeq \mathrm{Isom}_0(M)$. Sometimes (but not always) we also get that the map on the right of the above diagram $(\pi_0(\mathrm{Isom}(M)) \to \Gamma(M))$ is a bijection, which can be very useful.

1.4. **Overview of this survey.**

1.4.1. *The 3-manifold S^3.* By work of Cerf, $\Gamma(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ [Cer68], i.e. every orientation-preserving diffeomorphism of S^3 is isotopic to the identity. We also know the Smale conjecture $\mathrm{Diff}(S^3) \simeq \mathrm{Isom}(S^3) \simeq O_3$ by Hatcher [Hat83].

1.4.2. *The 3-manifold $S^1 \times S^2$.* By work of Gluck [Glu61, Glu62], $\Gamma(S^1 \times S^2) \cong (\mathbb{Z}/2\mathbb{Z})^3$. We also know that $\mathrm{Diff}(S^1 \times S^2) \simeq O(2) \times O(3) \times \Omega O(3)$ due to Hatcher [Hat81].

1.4.3. *Lens spaces.* The mapping class groups $\Gamma(L(p,q))$ were computed by Bonahon [Bon83]. The generalised Smale conjecture 1.7 is also true for lens spaces by work of Waldhausen [Wal68]. The group of path components of the isometry groups of lens spaces are therefore in bijection with the mapping class groups, and these are shown in Tables 1 and 2 of [HKMR12].

1.4.4. *Haken's sufficiently large 3-manifolds.*

Definition 1.9. A 3-manifold M is *Haken* when M contains an incompressible surface Σ, i.e. $\pi_1(\Sigma) \hookrightarrow \pi_1(M)$ is injective.

Note that the definition of Haken does not fit nicely with the eight geometries in geometrisation. In older literature, Haken 3-manifolds are called *sufficiently large*.

Johannson [Joh79] showed that 'simple' Haken 3-manifolds (those with trivial JSJ decomposition) have finite mapping class groups, and McCullough [McC91] showed in general they are finitely presented, and investigated other finiteness properties.

Waldhausen showed that $\pi_0(\mathrm{Diff}(M)) \to \pi_0(\mathrm{hAut}(M))$ is an isomorphism [Wal68] (see also [Sco72]). The latter coincides with $\mathrm{Out}(\pi_1(M))$. Hatcher [Hat76] and Ivanov [Iva76], showed that in fact $\mathrm{Diff}(M) \to \mathrm{hAut}(M)$ is a homotopy equivalence.

1.4.5. *Elliptic 3-manifolds.*

Definition 1.10. A 3-manifold M is *Elliptic* if it admits spherical geometry ($\widetilde{M}$ is isometric to S^3 with the standard spherical metric). This is true if and only if $\pi_1(M)$ is finite.

Elliptic manifolds are one of the 6 Seifert fibred geometries. Note that lens spaces are elliptic. Elliptic manifolds are the focus of the Hong – Kalliongis – McCullough – Rubinstein [HKMR12] book, where they prove the generalised Smale conjecture 1.7 in detail for all the cases. Tables of $\Gamma(M) \cong \pi_0(\mathrm{Isom}(M))$ appear in the introduction.

1.4.6. *Hyperbolic 3-manifolds.*

Definition 1.11. M is *hyperbolic* if it admits hyperbolic geometry ($\widetilde{M}$ is isometric to $\mathbb{H}^3$ with the standard hyperbolic metric).

For finite volume hyperbolic manifolds, Mostow rigidity [Mos68] tells us that

$$\pi_0(\mathrm{Isom}(M)) \to \pi_0(\mathrm{hAut}(M)) \cong \mathrm{Out}(\pi_1(M))$$

where the first map is a surjective map, which factors through $\pi_0(\mathrm{Diff}(M)) = \Gamma(M)$.

For Haken hyperbolic 3-manifolds, Waldhausen's results [Wal68] compute the mapping class groups. Gabai [Gab97, Gab01] showed that $\mathrm{Diff}_0(M) \simeq *$, and Gabai–Meyerhoff–Thurston [GMT03] showed that $\Gamma(M) \hookrightarrow \pi_0(\mathrm{hAut}(M))$ is injective. It follows that $\Gamma(M) \cong \mathrm{Out}(\pi_1(M))$.

1.5. **Other topics.** We will also have two talks on cutting up 3-manifolds, and what we can say about the mapping class group in these cases.

We will first talk about the *JSJ* decomposition of a 3-manifold into Seifert fibred and atoroidal pieces (see e.g. [Jac80]). We will discuss how to use the JSJ theorem together with the knowledge of the mapping class groups of the geometric pieces to compute the mapping class groups of irreducible 3-manifolds.

We will then think about reducible manifolds, that is manifolds with nontrivial prime decomposition. This talk will focus on the case of $\#^k(S^1 \times S^2)$, following [Lau73, Lau74, BBP23].

2. The mapping class group of S^3

The aim of this section is to provide a broad overview of the study of $\pi_0(\mathrm{Diff}\, S^3)$. We start by discussing Cerf's celebrated $\Gamma_4 = 0$ Theorem [Cer68], giving a sketch of the original proof and showing how $\pi_0(\mathrm{Diff}\, S^3)$ comes into play. We then slide into contact geometry, explaining Eliashberg's modern proof of $\Gamma_4 = 0$ [GZ10]. This second approach is somehow more direct, and has a beautiful idea at its core. Lastly, we briefly mention Hatcher's result [Hat83] $\mathrm{Diff}\, S^3 \simeq O(4)$, which gives a complete answer to the original problem of understanding the homotopy type of $\mathrm{Diff}\, S^3$, or at least reducing it to the homotopy type of a standard space.

2.1. **Cerf's Theorem $\Gamma_4 = 0$.**

Theorem 2.1 (Cerf's Theorem)**.** $\Gamma_4 = 0$.

We start by explaining the meaning of $\Gamma_4 = 0$.

Definition 2.2. Let $\alpha_n \colon \mathrm{Diff}\, D^n \to \mathrm{Diff}\, S^{n-1}$ be the natural map given by the restriction to the boundary. We define Γ_n as $\mathrm{coker}\, \alpha_n$, i.e. $\mathrm{Diff}\, S^{n-1} / \mathrm{Im}\, \alpha_n$.

The group Γ_n carries the structure of an abelian group [Cer68]. To see this first note that $\mathrm{Diff}\, S^{n-1} \to \Gamma_n$ factors through $\pi_0(\mathrm{Diff}\, S^{n-1})$, so it suffices to see that $\pi_0(\mathrm{Diff}\, S^{n-1})$ is abelian. To see this, note that up to isotopy we can assume each element of $\mathrm{Diff}\, S^{n-1}$ is supported on a ball $D^{n-1} \subseteq S^{n-1}$. For two such diffeomorphisms, we can isotope their supports to be disjoint, and hence up to isotopy they commute.

Roughly speaking, Γ_n measures the failure of diffeomorphisms of $\partial D^n = S^{n-1}$ to extend over the whole D^n.

We give some motivation behind the study of this group. By work of Kervaire-Milnor on the h-cobordisms groups θ_n of homotopy n-spheres and Smale's proof of the generalised Poincaré Conjecture ($n \geq 5$), we know that $\Gamma_n = \theta_n$ and the latter corresponds to the group of oriented smoothings of S^n (again, $n \geq 5$). The identification $\Gamma_n \cong \{\text{Smoothings of } S^n\}/\text{isotopy}$ is given by taking a class $[F]$ and associating to it the smoothing obtained by gluing two n-discs along S^{n-1} via F.

The above correspondence is extremely helpful for computations, since a lot is known about Γ_n. For example $\Gamma_5, \Gamma_6 = 0$, which means that there are no exotic spheres in dimension five and six. On the other hand, $\Gamma_7 = \mathbb{Z}/28\mathbb{Z}$, giving us the first examples of exotic spheres, which live in dimension 7.

Cerf's result *does not* imply that there are no exotic spheres in dimension 4. However, it tells us that there are no exotic spheres that arise as the above construction, i.e. by gluing two 4-discs along an extendable diffeomorphism of S^3. There might still be other ways to get exotic phenomena, but at least we know that the classic construction (which exhausts all smoothings in bigger dimensions) does not create any weird creatures in dimension 4.

We provide a sketch of the proof of $\Gamma_4 = 0$. We will show a series of equivalent, if not stronger reformulations of the problem. Amidst all of them, $\mathrm{Diff}\, S^3$ appears quite naturally.

Sketch proof of Theorem 2.1. First of all, we note that

$$\Gamma_n \cong \operatorname{coker} \pi_0(\mathrm{Diff}\, D^n) \to \pi_0(\mathrm{Diff}\, S^{n-1}) \tag{2.3}$$

where map is induced by α_n on π_0. This can be interpreted as a consequence of S^{n-1} being collared in D^n. In fact, let $\phi \in \mathrm{Diff}\, S^{n-1}$ and ψ isotopic to ϕ via some isotopy H. Then we can think of the "time" coordinate t of the isotopy as the interval coordinate of the $S^{n-1} \times I$ collar inside D^n. Then we can push the isotopy inside the collar and force the map on the boundary of the disc $D^n \setminus (S^{n-1} \times I)$ to be ψ. Now if ψ extends to some map Ψ, it is also true that ϕ extends to some map Φ. Since they differ by some isotopy on the collar, Ψ and Φ are also isotopic. This shows that Γ_n can be equivalently defined at the level of π_0.

By 2.3, we see that if we can show that $\pi_0(\mathrm{Diff}\, S^3) = 0$, then we immediately obtain $\Gamma_4 = 0$. This is when the mapping class group of S^3 comes into the picture. Note that the statement $\pi_0(\mathrm{Diff}\, S^3) = 0$ is stronger than $\Gamma_4 = 0$.

We now reformulate the problem. We know that:

$$\pi_i(\mathrm{Diff}\, S^n) \cong \pi_i(\mathrm{Diff}\, (D^n, S^{n-1})) \oplus \pi_i(SO(n+1)) \tag{2.4}$$

for $i \geq 0$. Since we want to show $\pi_0(\text{Diff } S^3) = 0$, and we know that $\pi_0(SO(4)) = 0$, it suffices to show by 2.4 that $\pi_0(\text{Diff } D^3, S^2) = 0$.

Let G, H be Diff D^3 and Diff S^2 respectively. The map α_3 gives a locally trivial fibration:

$$G \to H$$

whose fibre is Diff (D^3, S^2). The long exact sequence in homotopy groups of the fibration yields an exact sequence:

$$\pi_1(G) \to \pi_1(H) \to \pi_0(\text{Diff } (D^3, S^2)) \to \pi_0(G) \to \pi_0(H) \to 0 \tag{2.5}$$

Now by Smale, we know $H \simeq SO(3)$ and the composition:

$$\pi_i(SO(3)) \to \pi_i(G) \to \pi_i(H) \tag{2.6}$$

is an isomorphism for all $i \geq 0$. 2.6 implies that:

(1) $\pi_0(H) = 0$ and
(2) $\pi_i(G) \to \pi_i(H)$ is surjective for all $i \geq 1$.

Combining 2.5 with the above implications, we obtain a short exact sequence as:

$$0 \mapsto \pi_0(\text{Diff } (D^3, S^2)) \to \pi_0(G) \to 0 \tag{2.7}$$

We can hence rephrase the question as showing the triviality of $\pi_0(\text{Diff } D^3)$.

Let now $\mathcal{E}$ be $\text{Emb}(D^3 \to \mathbb{R}^3)$, G_e be the identity component in G and $\mathcal{F}$ be $\text{Emb}(S^2 \to \mathbb{R}^3)$. G acts on $\mathcal{E}$ on the right using pre-composition, i.e. $j\phi = j \circ \phi$. This action gives rise to a locally trivial fibration:

$$G \to \mathcal{E} \to \mathcal{E}/G \tag{2.8}$$

where the latter is the *space of 3-discs* in $\mathbb{R}^3$. Let now $\mathcal{R} = \mathcal{E}/G_e$. This is the space of *parametrised 3-discs* in $\mathbb{R}^3$. We have a locally trivial fibration:

$$\mathcal{R} \to \mathcal{E}/G \tag{2.9}$$

whose fibre is G/G_e.

Now G/G_e is the mapping class group of D^3. Since G is locally path-connected, the quotient space is discrete (every connected component is open and we are shrinking them to points). Since 2.9 is a locally trivial fibration with discrete fibre, it is a *covering* of $\mathcal{E}/G$.

The strategy now is to show that this cover is trivial. This would imply that $G/G_e = \pi_0(D^3) = 0$ as follows. $\mathcal{E}$ is connected and hence $\mathcal{R}$ is connected as well, being its quotient. If 2.9 is a trivial cover, then $\mathcal{R} \cong (G/G_e \times \mathcal{E}/G)$. For this to hold, we need that $G/G_e \cong *$, since both spaces have to be connected and G/G_e is trivial.

To summarise what we obtained so far: the triviality of $\mathcal{R} \to \mathcal{E}/G$ implies $\pi_0(\text{Diff } D^3) = 0$ which in turn implies $\Gamma_4 = 0$.

There is a last equality that comes from Cerf's proof of the weak Schoenflies conjecture (smooth). More precisely, we can identify the base space $\mathcal{E}/G$ with $\mathcal{F}/H$. So in the end, what we really want to show is the triviality of $\mathcal{R} \to \mathcal{F}/H$.

To show this, Cerf finds a continuous section p of the covering map. His argument is rather long and complicated, involving Thom's Transversality theorems and a stratification of the base space $\mathcal{F}/H$ [Cer68]. It is worth to mention that the section

p is originally defined on a subset $(\mathcal{F}/H)^0 \cup (\mathcal{F}/H)^1$, and then extended using some general topology results over the whole $\mathcal{F}/H$. □

2.2. **Eliashberg's contact geometry approach.** We now present a proof of 2.1 that is somehow quite direct and does not require the chain-of-reformulations approach that Cerf uses in his seminal work (see [Eli92]). The powerful tools Eliashberg used come from the world of contact geometry. We cannot give a full introduction to the topic here, so we refer the reader to [Etn03] for a complete discussion of basic results in contact geometry.

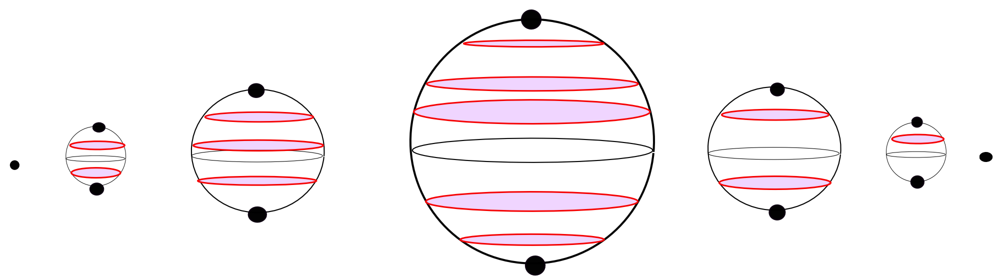

FIGURE 1. A movie presentation of S^3 in $\mathbb{C}^2$. The red circles are ∂D^t_s and they bound holomorphic discs. The black dots are the poles, which alltogether form K.

Sketch of the proof. Before actually working on the proof, we need to set up some notation and make some important observations. Let $S^3 \hookrightarrow \mathbb{C}^2$ be the unit sphere in $\mathbb{C}^2$, where we endow the complex plane with coordinates $z_i = x_i + iy_i$ for $i = 1, 2$. Let $H\colon S^3 \to \mathbb{R}$ be the height function given by the projection to the y_2 coordinate. For $t \in (-1, 1)$, the level sets $S^t\colon = H^{-1}(t)$ define a smooth foliation of $S^3 \setminus \{0, 0, 0, \pm 1\}$.

Let $q^t_\pm = (0, 0, \pm\sqrt{1 - t^2}, t)$ be the north and south poles of the spheres S^t and let K the unknot in S^3 given by the union of all the poles (including the points $(0, 0, 0, \pm 1)$). Then $S^3 \setminus K$ is foliated by circles spanning holomorphic discs $D^t_s\colon = D^4 \cap (\mathbb{C} \times \{x_2 = s, y_2 = t\})$ for $|t| < 1$, $|s| < \sqrt{1 - t^2}$. ∂D^t_s foliate $S^t \setminus \{q^t_\pm\}$, we call this foliation $\mathcal{F}_t$ (see Figure 1).

Let now $\varphi \in \operatorname{Diff} S^3$ an orientation preserving diffeomorphism. Recall that we can always move freely in its isotopy class if we want to understand whether φ extends to D^4, we discussed this in the previous proof. So it will be more convenient for us to choose a representative φ that possesses nice properties. The first one is "local triviality". Due to the disc theorem, we know that given a D^3 in S^3 and a self-diffeomorphism φ, there is always a diffeomorphism φ' s.t. $\varphi'|_D = id$.

The second property we want is φ being a *contactomorphism*. This notion comes from contact geometry and, roughly speaking, means that φ preserves the contact structure. To give a bit more detail, we can endow S^3 with its *standard tight contact structure* ξ_{std} and require that $\varphi_*\xi_{std} = \xi_{std}$. The reason this additional requirement on φ is helpful will become clear soon.

Eliashberg beautiful idea originates from the fact that S^3 has a *unique* tight contact structure up to isotopy. Using this extremely powerful result, we can show that each o.p. diffeomorphism φ can be isotoped to be a contactomorphism.

More precisely, let $\varphi\colon (S^3, \xi_{std}) \to S^3$. Then $\varphi_* \xi_{std}$ defines a tight contact structure on S^3. Using Eliashberg uniqueness result, we know there is an isotopy H of S^3 such that $H_0 = \mathrm{Id}$ and $(H_1)_* \varphi_* \xi_{std} = \xi_{std}$. We can then use $H \circ \varphi$ to istope φ to $H_1 \varphi$, where the latter is now a contactomorphism of (S^3, ξ_{std}).

Now to show that we every o.p. diffeomorphism of S^3 extends to the whole D^4, we can assume that our starting map is a contactomorphism of (S^3, ξ_{std}) fixing a neighborhood of K. We will show that such maps always extend, concluding our proof.

So let f be a contactomorphism of (S^3, ξ_{std}) that fixes a neighborhood of K. The standard tight contact structure gives a foliation $\mathcal{F}_{std}$ of each 2-sphere S^t by "spiraling" leaves that emanate from the north pole q^t_+ and terminate at the south pole q^t_-. This is the characteristic foliation obtained by intersecting the tangent planes with the contact planes. The two poles are elliptic singularities.

We can C^0-perturb the singular foliation so that the leaves are now meridians of S^t. Note that this foliation is now transverse to $\mathcal{F}_t$. Since f is a contactomorphism, it preserves the foliation $\mathcal{F}_{std}$. Hence, each sphere $f(S^t)$ is foliated by $f(\mathcal{F}_{std})$, which is isotopic to $\mathcal{F}_{std}$ (note that the singular points are fixed since f is the identity near K). By another C^0-perturbation, we can make sure that $f(S^t)$ is also foliated by meridians. Now both S^t and $f(S^t)$ are foliated by $\mathcal{F}_{std}$ and the leaves are preserved

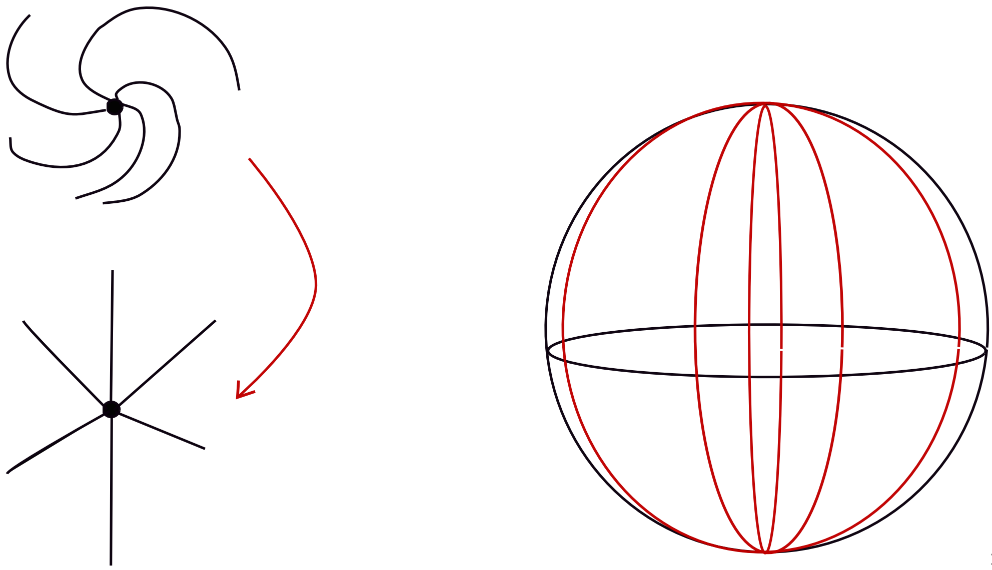

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FIGURE 2. Perturbing the characteristic foliation we can make sure each S^t is foliated by meridians.

by f. Since both foliations are transverse to $\mathcal{F}_t$, we can reparametrise f along the leaves so that it preserves $\mathcal{F}_t$.

To recap, we have a smooth foliation of $S^3 \setminus K$ by circles that span holomorphic discs, and this foliation is preserved by the map f. Thus, extending f over $D^4 \setminus K$ is equivalent to extend $f|_{\partial D^t_s}$ to D^t_s. This is a famous result by Alexander: we know that every o.p. diffeomorphism of S^1 extends over D^2. Thus f can be extended "disc by disc" as a smooth map $F : D^4 \setminus K \to D^{\setminus} K$. Since we started by assuming f was the identity near the unknot K, we can further extend F via the identity to obtain the desired diffeomorphism $D^4 \to D^4$. □

2.3. **Hatcher's Theorem.** We briefly discuss the homotopy type of Diff S^3.

Theorem 2.10 (Hatcher, [Hat83])**.** Diff $S^3 \simeq O(4)$.

Note this immediately recovers the triviality of the mapping class group of S^3.

While this is the more concise and direct way to state the result, Hatcher's formulation was rather different. For the sake of completion, we will state the original theorem.

Theorem 2.11 (Hatcher, [Hat83])**.** *Let $g_t \colon S^2 \to \mathbb{R}^3$ be a smooth family of C^∞ embeddings, $t \in S^k$. Then this extends to $\hat{g}_t : D^3 \to \mathbb{R}^3$ for all $k \geq 0$.*

We show that the two formulations are indeed equivalent.

First, we know that:

$$\text{Diff}\, S^n \simeq O(n+1) \times \text{Diff}(D^n, S^{n-1}) \tag{2.12}$$

for all n. The statement Diff $S^3 \simeq O(4)$ is then equivalent to $\text{Diff}(D^3, S^2) \simeq *$. Why is the latter statement equivalent to the one in Hatcher's original work? Theorem 2.11 says that the natural map $\rho : \mathcal{E} \to \mathcal{F}$ is surjective on π_k for all $k \geq 0$.

Consider the following commutative diagram:

$$\begin{array}{ccc} \text{Emb}(D^n, \mathbb{R}^n) & \xrightarrow{\quad\rho\quad} & \text{Emb}(S^{n-1}, \mathbb{R}^n) \\ & {\scriptstyle\simeq}\searrow \qquad \swarrow & \\ & GL(n, \mathbb{R}) & \end{array}$$

Here the lower left map is given by evaluating the derivative at a point (giving us an homotopy equivalence), and the lower right map is evaluating the derivative at a point and adjoining the normal vector. From the diagram we see that ρ is injective on π_k for all k. Moreover, ρ is a fibration whose fibre is just $\text{Diff}(D^n, S^{n-1})$. Hence we see that $\text{Diff}(D^3, S^2) \simeq *$ iff ρ is also surjective on all π_k. This shows that the two statements are indeed equivalent.

3. The mapping class group of $S^1 \times S^2$

In this section we will compute the mapping class group of $S^1 \times S^2$. This computation is due to Glück [Glu61, Glu62]. We will begin by presenting the original motivation behind studying this space, and then move on to the computation, which we will break down into several stages.

3.1. **Motivation.**

3.2. **Statement of the theorem, and overview of the proof.** First, let us describe some self-homeomorphisms of the manifold in question, $S^1 \times S^2$. Let $a\colon S^2 \times S^2$ be the antipodal map and let $s\colon S^1 \times S^1$ be the conjugation map, i.e. if we parameterise S^1 as the unit complex numbers, then s is the map sending $z \in \mathbb{C}$ to its complex conjugate z^*. In an abuse of notation, we will then define maps $a, s\colon S^1 \times S^2 \to S^1 \times S^2$ as $\mathrm{Id}_{S^1} \times a$ and $s \times \mathrm{Id}_{S^2}$, respectively. We define a final map, the so-called *Glück twist*, as

$$\begin{aligned} T\colon S^1 \times S^2 \to& S^1 \times S^2, \\ (\theta, x) \to& (\theta, r_\theta(x)) \end{aligned}$$

where $r_\theta\colon S^2 \to S^2$ is the map given by a positive rotation by angle θ about the vertical axis (from south pole to north pole).

We can now state the theorem, i.e. the computation of the mapping class group of $S^1 \times S^2$.

Theorem 3.1 (Glück, [Glu61, Glu62])**.** *The mapping class group of $S^1 \times S^2$ is computed to be*

$$\pi_0(\mathrm{Homeo}(S^1 \times S^2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

where the three generators of the $\mathbb{Z}/2$-summands are a, s and T.

We now lay out the strategy of the proof of this theorem. First, note that we have a homomorphism

$$\varphi\colon \pi_0(\mathrm{Homeo}(S^1 \times S^2)) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

which sends a representative homeomorphism to its induced map on $H_1(S^1 \times S^2) \cong \mathbb{Z}$ paired with its induced map on $H_2(S^1 \times S^2) \cong \mathbb{Z}$. Given a homeomorphism, both of these induced maps are the ± 1 maps on these homology groups and, since isotopic homeomorphisms must induce the same maps, this gives us the well defined homomorphism φ above.

We will reduce proving Theorem 3.1 to proving the following theorem.

Theorem 3.2. *We have that $\ker \varphi \cong \mathbb{Z}/2$, generated by T.*

The proof of the above theorem will constitute most of the section. First, however, we will prove Theorem 3.1, assuming that Theorem 3.2 holds.

Proof of Theorem 3.1. We have the following short exact sequence.

$$0 \to \ker \varphi \to \pi_0(\mathrm{Homeo}(S^1 \times S^2)) \xrightarrow{\varphi} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0$$

where the last map is surjective since the maps a, s and $a \circ s$ induce all of the non-trivial elements in $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. In fact, this map is split, since a and s give us a splitting $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \pi_0 \mathrm{Homeo}(S^1 \times S^2)$.

Since $\ker \varphi \cong \mathbb{Z}/2$ by Theorem 3.2, we conclude that $\pi_0 \mathrm{Homeo}(S^1 \times S^2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ as required. □

Theorem 3.2 trivially follows from the following two propositions.

Proposition 3.3. *Let $f \in \ker \varphi$ be a representative homeomorphism. Then f is isotopic to* Id *or T.*

Proposition 3.4. *The Glück twist T represents an order two element in the mapping class group of $S^1 \times S^2$. In particular, it is not isotopic to the identity.*

And we will now prove these.

3.3. **Proof of Proposition 3.3.** The proof will be split up into a number of steps. The aim is to pick a representative homeomorphism in $\ker \varphi$ and then step by step build an isotopy from it to either the identity map or the Glück twist.

First we define some notation. Let $S := \{1\} \times S^2 \subset S^1 \times S^2$, let $N \in S^2$ be the north-pole, and let $\alpha := S^1 \times \{N\}$. Let C denote a tubular neighbourhood of α, and let ∂C be the boundary torus.

Pick $f \colon S^1 \times S^2$, a homeomorphism representing an isotopy class in $\ker \varphi$. Then the steps are as follows.

(i) Isotop f to a map which fixes S pointwise.
(ii) Isotop this new map to one which fixes S and α pointwise.
(iii) Isotop this new map to one which fixes S pointwise and fixes ∂C setwise.
(iv) Isotop this new map to one which fixes S pointwise and whose restriction to ∂C is the n-fold twist map.
(v) Isotop this new map to one which fixes S pointwise and whose restriction to ∂C is either the identity, or the 1-fold twist map.
(vi) Isotop this new map to either the identity map or the Glück twist T.

We will look at steps (i), (ii), (v) and (vi) specifically.

Step (i). First, we can isotop f such that $f(S)$ does not intersect S. Now, consider the two regions of $S^1 \times S^2$ bounded by $f(S)$ and S. By the annulus theorem, both of these regions are homeomorphic to $I \times S^2$. It follows that f can be further isotoped such that $f(S) = \{-1\} \times S^2$, and then a rotational isotopy gives that we can further isotop f such that $f(S) = S$.

Now f restricts to a map $f\,|_S \colon S^2 \times S^2$ which is degree one since we assumed that $f \in \ker \varphi$. Hence, by Smale, $f\,|_S$ is isotopic to the identity, and so f is isotopic to a map which fixes S pointwise.

Step (ii). This step actually has a different interpretation, which we will explain now. Let $K \colon S^1 \hookrightarrow S^1 \times S^2$ be a knot in $S^1 \times S^2$. We have an invariant associated to K, denoted $g(K)$, called the *geometric winding number*, which is defined as the minimal possible transverse intersections of K with S. We then have the following lemma.

Lemma 3.5 (Lightbulb trick)**.** *Let K be a knot in $S^1 \times S^2$ with $g(K) = 1$. Then K is unknotted, i.e. isotopic to $\alpha \subset S^1 \times S^2$.*

Given the above lemma, the step is easy to complete. We can see that $f(\alpha)$ is a knot in $S^1 \times S^2$, and clearly $g(K) = 1$, so we can use the lemma to say that $f(\alpha)$ is isotopic to α, meaning that f can isotoped such that $f(\alpha) = \alpha$.

Step (v). Parameterise ∂C as $S^1 \times S^1$ where the first S^1 factor corresponds to the S^1-factor in $S^1 \times S^2$ and the second corresponds the meridian S^1-factor. Then

$$f\mid_{\partial C}\colon S^1 \times S^1 \to S^1 \times S^1,$$
$$(x, y) \mapsto (x, yx^n)$$

is the n-fold twist map. We can change n by an even number $2k$ by performing k belt moves, as in [REF HERE]. This defines an isotopy of f to a map whose restriction to ∂C is either the identity map (the 0-fold twist map) or the 1-fold twist map.

Step (vi). This is the final step, and the one where we use that $\pi_0 \operatorname{Homeo}(D^3, \partial)$ is trivial. Assume that f restricts to the identity ∂C. Then $S^1 \times S^2 \setminus \partial C$ consists of two 3-cells, and f cannot interchange them. Then since $\pi_0 \operatorname{Homeo}(D^3, \partial)$ is trivial, there exists an isotopy of f restricted to each of these 3-cells to the identity map that fixes the boundary throughout the isotopy. Hence, f is isotopic to the identity.

If we now assume that f restricts to the 1-fold twist map on ∂C, then $f \circ T^{-1}$ restricts to the identity map on ∂C, and hence by the previous paragraph we have that $f \circ T^{-1}$ is isotopic to the identity. Hence f is isotopic to T.

This completes the proof of Proposition 3.3. □

3.4. **Proof of Proposition 3.4.**

4. The mapping class group of lens spaces

In this section we will compute the mapping class group of lens spaces as presented by Bonahon in [Bon83]. The main result at the core of the computation is the following:

Theorem 4.1. *Up to isotopy, the lens space $L(p, q)$ contains a unique torus separating it into two solid tori.*

The importance of this theorem does not come from the existence of such a torus, as it is part of the definition of a lens space, but it concerns its uniqueness. Moreover, this result was already known before Bonahon, as it had been proved by Schubert in [Sch56]. This was done in the context of viewing lens spaces as the double branched cover of S^3 over a two-bridge knot, and considering an involution τ on said cover; he then showed that up to τ-equivariant isotopy, $L(p, q)$ contains a unique torus preserved by τ and separating $L(p, q)$ into two solid tori.

This result was largely ignored for some time, until Bonahon presented a readapted proof and used it to compute $\pi_0(\operatorname{Diff} L(p, q))$. We can also derive the classification of lens spaces from Theorem 4.1. Furthermore, it is worth noting that the mapping class group of lens space was computed independently by Hodgson and Rubinstein in 1983 [HR85].

We will now briefly mention the methods behind the proof of Theorem 4.1, but a complete discussion can be found in [Bon83, Hat01]. The idea is to consider two tori T and T' in $L(p, q)$, each separating it into two solid tori. We want to show that T and T' are isotopic, after viewing each of them in a different light.

Since T separates $L(p,q)$ into solid tori V_1 and V_2, we consider a map $i : D \to L(p,q)$, where D is a disc, such that $i(\partial D)$ is the core of V_2 , $i|\partial D : \partial D \to i(\partial D)$ is a p-sheeted cover, and $i|\int(D)$ is an embedding avoiding $i(\partial D)$. Bonahon calls $i(D)$ a generalised projective plane and notes that T is isotopic to a tubular neighbourhood of $i(\partial D)$.
Next we consider a Morse function $f : L(p,q) \to \mathbb{R}$ with one critical point for each order 0,1,2 and 3, such that T' is isotopic to a level surface between critical points of index 1 and 2. The idea is then to isotope $i(D)$ so that its singular curve $i(\partial D)$ is as simple as possible with respect to the Morse function f. Finally, prove that if $i(D)$ is in a level surface of f, then T is isotopic to T'.

4.1. **Defining $L(p,q)$.** This short subsection provides two definitions of a lens space. There are many equivalent definitions possible, but the one we will work with in this survey is actually the second definition of this subsection. Note that we are focusing on 3-dimensional lens spaces, but these definitions can be generalised to work with n-dimensional lens spaces. Throughout this section let $p \in \mathbb{N}$, $q \in \mathbb{Z}$ be such that $\gcd(p,q) = 1$.

The lens space $L(p,q)$ can be defined as the orbit space $S^3/\mathbb{Z}/p$. Here the three-sphere is seen as $S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \ ||z_1|^2 + |z_2|^2 = 1\}$ and $\mathbb{Z}/p$ acts on it by the rotation $\rho : S^3 \to S^3$ defined as $\rho(z_1,z_2) = (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi q i}{p}} z_2)$. Only the identity element fixes a point on S^3, this can be seen as a consequence of the fact that p and q are coprime. Hence the action is free. Note that when $p = 2$, ρ is the antipodal map and $L(2,1) \cong \mathbb{R}P^3$.

The next definition is the one we are going to refer to for the remainder of the section. Let $V_1 \cong V_2 \cong S^1 \times D^2$ be solid tori. Define $L(p,q) := V_1 \cup_\theta V_2$, where $\theta : \partial V_1 \to \partial V_2$ is a diffeomorphism of degree -1 defined by $\theta(u,v) = (u^r v^p, u^s v^q)$, with $qr - ps = -1$. Note that S^1 and D^2 are seen as the set of complex number with modulo equal and less than or equal to 1, respectively. From the definition it is clear that the lens space is actually determined by what happens on the boundary tori during the gluing, that is by the isotopy class of θ. Thus $\theta \in \pi_0(\mathrm{Diff}(S^1 \times S^1)) \cong SL_2(\mathbb{Z})$. With this definition in mind, θ can also be seen as the matrix $\begin{pmatrix} r & p \\ s & q \end{pmatrix}$ which maps a meridian of T $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} p \\ q \end{pmatrix}$. Note that we may always substitute the parameters of θ with r', s', q' such that $q'r' - ps' = -1$ and $q' \equiv q \mod p$.

Remark 4.2. For the rest of this discussion we will assume $p \geq 2$. Thus $L(1,0) \cong S^3$ and $L(0,1) \cong S^1 \times S^2$ will be regarded as exceptions and be excluded from consideration.

4.2. **Consequences of Theorem 4.1.** The most interesting consequences of Theorem 4.1 for us concern the mapping class group of lens spaces. Indeed, a first consequence is that any diffeomorphism of $L(p,q)$ is isotopic to a diffeomorphism preserving the torus $T = \partial V_1 = \partial V_2$.

Among such diffeomorphisms we always have the involution τ preserving both V_1 and V_2, parametrised on $V_1 \cong V_2 \cong S^1 \times D^2$ by $\tau(u, v) = (\overline{u}, \overline{v})$.

In general, there are no diffeomorphisms of $L(p, q)$ exchanging the solid tori V_1 and V_2, except when $q^2 \equiv \pm 1 \mod p$. When $q^2 \equiv +1 \mod p$ there exists an involution σ_+ of degree $+1$ which can be described by the map $(u, v) \in V_1 \leftrightarrow (u, v) \in V_2$. Similarly, when $q^2 \equiv -1 \mod p$, there exists a diffeomorphism σ_- of degree -1 and order 4, which can be expressed by $(u, v) \in V_1 \mapsto (\overline{u}, v) \in V_2$ and $(u, v) \in V_2 \mapsto (u, \overline{v}) \in V_1$. Both diffeomorphisms σ_+ and σ_- can be recovered by choosing appropriate parametrisations for V_1, V_2 (that is $r = -q$ and $r = q$, respectively), and by checking that reiterated composition with θ is well-defined. Moreover, note that τ commutes with σ_+ and σ_-, and that $\sigma_-^2 = \tau$.

Proposition 4.3. *Any diffeomorphism of $L(p, q)$ is isotopic to an element of the group generated by τ, and possibly σ_+ and σ_-.*

Observe that it is necessary to exclude $L(0, 1) \cong S^1 \times S^2$, as the Dehn twist along $\{1\} \times S^2$ is not isotopic to a composition of τ, σ_+, and σ_-.

Proof. Let φ be a diffeomorphism of $L(p, q)$, which by Theorem 4.1, can be isotoped so that $\varphi(T) = T = \partial V_1 = \partial V_2$. After possibly composing with σ_+ and σ_-, we can furthermore suppose $\varphi(V_1) = V_1$ and $\varphi(V_2) = V_2$. Let m_i be the generator of the kernel of $H_1(T) \to H_1(V_i)$, for $i = 1, 2$. We have that the m_i are distinct, and that $m_1 \cdot m_2 \neq \pm 1$, as a consequence of the fact that $p \neq 0, 1$. Moreover, $\varphi_*(m_i) = \pm m_i$ in $H_1(T)$, and thus φ_* is multiplication by ± 1 on $H_1(T)$. After possibly composing φ with τ, we can suppose φ_* is the identity on T. Conclude the proof by using meridian discs in V_1 and V_2 providing an isotopy that fixes T between φ and the identity. □

The following lemma will be of use in the computation of $\pi_0 \operatorname{Diff} L(p, q)$:

Lemma 4.4. *If $q \equiv \pm 1 \mod p$, there exists an isotopy of $L(p, q)$ exchanging V_1 and V_2. Furthermore, if $p = 2$, there exists an isotopy of $L(2, 1) \cong \mathbb{R}P^3$ coinciding with τ on $T = \partial V_1 = \partial V_2$.*

Proof. The solid torus V_1 is isotopic to $U(C_1)$, which denotes the tubular neighbourhood of $C_1 = S^1 \times \{0\}$, the core of V_1. If $q \equiv \pm 1 \mod p$, it is possible to choose a parametrisation of V_1, V_2 such that $r = \pm 1$. Now, C_1 is isotopic to $C = S^1 \times \{1\}$ in ∂V_1, which is also the curve with parametrisation $z \in S^1 \to (z^r, z^s)$ in ∂V_2. Since $r = \pm 1$, C is isotopic to C_2, the core of V_2. The latter is clearly isotopic to $U(C_2)$, and thus we have found an isotopy from V_1 to V_2.

If $p = 2$, then $L(2, 1) \cong \mathbb{R}P^3$ and C_1 is isotopic to $\mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \mathbb{R}P^3$. There exists an isotopy of $\mathbb{R}P^2$ which reverses the orientation of $\mathbb{R}P^1$ and which can be extended to $\mathbb{R}P^3$. After composing this isotopy of $\mathbb{R}P^3$ with suitable isotopies between V_1 and $U(\mathbb{R}P^1)$ we obtain the desired result. □

The lemma presented can also be rephrased in a more convenient form for our next computation. That is:

- If $q \equiv 1 \mod p$, σ_+ is isotopic to τ;
- If $q \equiv -1 \mod p$, σ_+ is isotopic to the identity;

- If $p = 2$, both σ_+ and τ are isotopic to the identity.

4.3. **Computing the mapping class group.**

Theorem 4.5. *The group* $\pi_0(\mathrm{Diff}(L(p,q))$ *for* $p \geq 2$ *is isomorphic to:*
(a) $\mathbb{Z}/2$*, generated by* τ*, if* $q^2 \not\equiv \pm 1 \mod p$*;*
(b) $\mathbb{Z}/2 \oplus \mathbb{Z}/2$*, generated by* τ *and* σ_+*, if* $q^2 \equiv 1 \mod p$ *and* $q \not\equiv \pm 1 \mod p$*;*
(c) $\mathbb{Z}/2$*, generated by* τ*, if* $q \equiv \pm 1 \mod p$ *and* $p \neq 2$*;*
(d) $\mathbb{Z}/4$*, generated by* σ_-*, if* $q^2 \equiv -1 \mod p$ *and* $p \neq 2$*;*
(e) $\mathbb{Z}/2$*, generated by* σ_-*, if* $p = 2$*.*

Proof. Let $G(p,q)$ be the abstract group with generators τ, and when $L(p,q)$ admits so, σ_+ and σ_-. The relations will be the obvious ones from the corresponding maps: $\tau^2 = \sigma_+^2 = Id$, $\sigma_-^2 = \tau$, $\tau\sigma_+ = \sigma_{+\tau}$, and only when $p = 2$, $\sigma_+\sigma_-\sigma_+ = \sigma_-^{-1}$. Therefore, the group $G(p,q)$ is isomorphic to:
(i) $\mathbb{Z}/2$ when $q^2 \not\equiv \pm 1 \mod p$;
(ii) $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ when $q^2 \equiv 1 \mod p$ and $p \neq 2$;
(iii) $\mathbb{Z}/4$ when $q^2 \equiv -1 \mod p$ and $p \neq 2$;
(iv) D_8 when $p = 2$.

We are now interested in the following composition of maps:

$$G(p,q) \xrightarrow{f} \pi_0 \mathrm{Diff}\, L(p,q) \xrightarrow{g} \mathrm{Aut}\, H_* L(p,q).$$

Note that by Proposition 4.3 f is surjective, thus we need to compute the kernel of f to determine $\pi_0 \mathrm{Diff}\, L(p,q)$. Moreover, note that on $H_1 L(p,q) \cong \mathbb{Z}/2$ the maps τ, σ_+, σ_- act by multiplication by -1, $-q$, and q, respectively. From this it follows that when $q \not\equiv \pm 1 \mod p$ the homomorphism $g \circ f$ is injective and thus $\pi_0 \mathrm{Diff}\, L(p,q) \cong G(p,q)$.

Keeping in mind these observations, we can now proceed to prove the different cases.

(a) If $q^2 \not\equiv \pm 1 \mod p$, then $q \not\equiv \pm 1 \mod p$ and, by the aforementioned statements, $\pi_0 \mathrm{Diff}\, L(p,q) \cong G(p,q) \cong \mathbb{Z}/2$. Note that in this case $G(p,q)$ is just generated by τ.

(c) Let $q \equiv 1 \mod p$ and $p \neq 2$. This implies that $q^2 \equiv 1 \mod p$ and thus $G(p,q) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Note that $\tau\sigma_+$ acts trivially on $H_1 L(p,q)$, so $\ker(g \circ f) \cong \mathbb{Z}/2$. By Lemma 4.4 σ_+ is isotopic to τ, thus $\ker(f) \cong \mathbb{Z}/2$. Hence $\pi_0 \mathrm{Diff}\, L(p,q) \cong \mathbb{Z}/2$.

Now suppose $q \equiv -1 \mod p$ and $p \neq 2$. Again, it follows that $q^2 \equiv 1 \mod p$ and $G(p,q) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. In this case σ_+ acts trivially on $H_1 L(p,q)$, so $\ker(g \circ f) \cong \mathbb{Z}/2$. By Lemma 4.4 σ_+ is isotopic to the identity, so $\ker(f) \cong \mathbb{Z}/2$ and $\pi_0 \mathrm{Diff}\, L(p,q) \cong \mathbb{Z}/2$.

(e) Suppose $p = 2$. Then in this case $q^2 \equiv 1 \equiv -1 \mod p$, and clearly $q \equiv 1 \equiv -1 \mod p$. In this case $G(p,q) \cong D_8 \cong \mathbb{Z}/4 \rtimes \mathbb{Z}/2$. Note that τ,

σ_+, σ_- act trivially on $H_1L(p,q)$, but σ_- acts non-trivially on $H_3L(p,q)$. Hence $\ker(g\circ f)\equiv \mathbb{Z}/2$. By Lemma 4.4, σ_+ and τ are both isotopic to the identity, hence $\ker(f)\cong \mathbb{Z}/2\oplus\mathbb{Z}/2$. Thus $\pi_0 \operatorname{Diff} L(p,q)\cong \mathbb{Z}/2$.

(d) Now suppose that $q^2 \equiv -1 \mod p$ and $p\neq 2$. Note that we are supposing that $q\not\equiv \pm 1 \mod p$, because if this were not the case, we would be in case (c). The action of τ and σ_- is not trivial on $H_1L(p,q)$, and we already know that σ_- acts non-trivially on $H_3L(p,q)$. Therefore $g\circ f$ is injective, and $\pi_0 \operatorname{Diff} L(p,q)\cong G(p,q)\cong \mathbb{Z}/4$.

(b) Finally let $q^2\equiv 1 \mod p$ and $p\neq 2$. Once again, note that we are assuming that $q\not\equiv\pm1 \mod p$, hence $g\circ f$ is injective. Indeed note that τ, σ_+ and $\tau\sigma_+$ act non-trivially on $H_1L(p,q)$. Then $\pi_0 \operatorname{Diff} L(p,q)\cong G(p,q)\cong \mathbb{Z}/2\oplus\mathbb{Z}/2$. □

This computation concludes the discussion about the mapping class group of lens spaces. We once again highlight the importance of Theorem 4.1, which serves as the basis for successive results. Finally, although not discussed in this section, we remark that the Smale conjecture holds for lens spaces, as proved by Waldhausen in [Wal68], and more recently by Ketover and Liokumovich in [KL23].

5. Elliptic manifolds

In this section, we will discuss elliptic manifolds. In particular, we will present their classification, discuss how to compute their isometry groups as well as their mapping class groups and discuss the generalised Smale conjecture.

Throughout this section M will denote a closed 3-manifold. We start by defining elliptic manifolds:

Definition 5.1. A closed 3-manifold M is called *elliptic*, if it is of the form $M = S^3/G$ where G is a finite subgroup of $SO(4)$ acting freely on S^3.

The definition of an elliptic manifold M is equivalent to requiring that M admits a Riemannian metric of constant curvature 1. Note that if $M = S^3/G$, then $\pi_1(M)\cong G$ and therefore every elliptic manifold has finite fundamental group. The converse follows from Thurston's geometrisation conjecture implying that elliptic manifolds are exactly the ones with finite fundamental group.

Lens spaces are exactly the elliptic manifolds with G being a cyclic group. Since we discussed lens spaces in the previous section, we will restrict here to a discussion of elliptic manifolds with non-cyclic fundamental group. We start by presenting the classification of such manifolds.

5.1. **Classification.**

We present the classification of elliptic manifolds with non-cyclic fundamental group. From the definition of elliptic manifolds, it follows that there is a one-to-one correspondence between isomorphism classes of elliptic manifolds and conjugacy classes of finite $G\subset SO(4)$ acting freely on S^3. Therefore, we need to study subgroups of $SO(4)$.

We equip S^3 with the group structure arising from viewing it as the unit sphere in the space of quaternions. Each element in S^3 acts on S^3 by left (or right) multiplication and hence defines an orientation preserving isometry of S^3 or in other words an element of $SO(4)$. Consider now the following Lie group homomorphism

$$F : S^3 \times S^3 \to SO(4),$$
$$(q_1, q_2) \mapsto (x \mapsto q_1 x q_2^{-1}).$$

F is the universal covering of $SO(4)$ and $\ker(F) = \{(1,1), (-1,-1)\} \cong \mathbb{Z}/2\mathbb{Z}$.

Let P be the 2-sphere of unit quaternions with real part 0. Then $P \subset S^3$ and since conjugating by an element in S^3 preserves the condition of having real part 0, we obtain a map

$$S^3 \to \mathrm{Isom}^+(P),$$
$$q \mapsto (c_q)|_P$$

where $c_q : S^3 \to S^3,\ x \mapsto qxq^{-1}$. Since $\mathrm{Isom}^+(P) \cong SO(3)$, we can view the above as a map

$$H : S^3 \to SO(3).$$

In fact this map is the universal covering of $SO(3)$ and $\ker(H) = \{1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$.

Let D_{2n} be the dihedral group of order $2n$. Furthermore, let T_{12}, O_{24} and I_{60} be the groups of orientation preserving symmetries of the tetrahedron, octahedron and icosahedron respectively. (The subscripts denote the order of the respecting group.) Then $D_{2n}, T_{12}, O_{24}, I_{60}$ are subgroups of $SO(3)$ and in fact (apart from cyclic ones) they are the only finite subgroups. Denote by $D^*_{4n}, T^*_{24}, O^*_{48}, I^*_{120}$ the preimages of the above groups in S^3 under the map H. These groups are called the binary dihedral group, binary tetrahedral group etc.

Let $\{1, i, j, k\}$ be the standard quaternionic basis. Then the set $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}$ forms a subgroup of S^3 called the quaternion group.

For $n \in \mathbb{N}$, let $\xi_n := \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$. Then ξ_n generates a cyclic subgroup of S^3 of order n, which we will denote by C_n.

We are now ready to present the classification of elliptic manifolds. Table 1 contains all the groups (apart from cyclic ones that correspond to lens spaces) arising as fundamental groups of elliptic manifolds. Furthermore, for each group G below, there is a unique conjugacy class in $SO(4)$ consisting of isomorphic copies of the G, which implies that for every G there is a unique elliptic manifold (up to isomorphism) with fundamental group G. (Note that this is not true in the case of lens spaces where G is cyclic.)

Table 1. Classification of elliptic manifolds

D^*_{4n}	prism
$D^*_{4n} \times C_m$	prism
"index 2"	prism
T^*_{24}	tetrahedral
$T^*_{24} \times C_n$	tetrahedral
"index 3"	tetrahedral
O^*_{48}	octahedral
$O^*_{48} \times C_n$	octahedral
I^*_{120}	icosahedral
$I^*_{120} \times C_n$	icosahedral
Q_8	quaternionic
$Q_8 \times C_n$	quaternionic

The groups "index 2" and "index 3" stand for certain index 2 subgroups of $D^*_{4n} \times C_{4m}$ with $(n, m) = 1$ and certain index 3 subgroups of $T^*_{24} \times C_{6n}$ with n odd and divisible by 3. The names "prism" etc. stand for how the corresponding manifolds are called, for example octahedral manifolds are exactly the elliptic manifolds with a subgroup of their fundamental group isomorphic to O^*_{48}. Note that all the groups in the above table are naturally subgroups of $S^3 \times S^3$, that are mapped to an isomorphic image under F and therefore can also be seen as subgroups of $SO(4)$.

5.2. **Isometries.** In this section, we will determine the isometry groups of all the elliptic manifolds occurring in the classification table from the previous section. This means that we will again omit discussing lens spaces. Since only certain lens spaces admit orientation reversing isometries, for our discussion we have $\text{Isom}(M) = \text{Isom}^+(M)$, where M is any elliptic manifold with non-cyclic fundamental group.

Let $G \subset SO(4)$ such that $M = S^3/G$ is elliptic. Given $f \in SO(4)$, we can define a unique $\bar{f} \in \text{Isom}(M)$ such that the following diagram commutes

$$\begin{array}{ccc} S^3 & \xrightarrow{f} & S^3 \\ \downarrow & & \downarrow \\ M & \xrightarrow{\bar{f}} & M, \end{array}$$

if and only if:

$$\forall x \in S^3 \; \forall g \in G \; \exists g' \in G : f(g(x)) = g'(f(x)).$$

This last condition is equivalent to $f \in \mathrm{Norm}(G) = \{h \in SO(4) : hGh^{-1} = G\}$. Hence, we obtain a map $\mathrm{Norm}(G) \to \mathrm{Isom}(M)$ which sends f to $\bar{f}$. Since S^3 is simply connected, every isometry of M can be lifted (not uniquely) to an isometry of S^3, which implies that the map $\mathrm{Norm}(G) \to \mathrm{Isom}(M)$ is surjective. Any two lifts of an isometry of M differ by a deck transformation, i.e. an element of G, so we have the short exact sequence

$$1 \to G \to \mathrm{Norm}(G) \to \mathrm{Isom}(M) \to 1.$$

We summarise the above discussion as a proposition.

Proposition 5.2. *For an elliptic manifold $M = S^3/G$ with G not cyclic, it holds that* $\mathrm{Isom}(M) \cong \mathrm{Norm}(G)/G$.

Note that the above proposition also holds for lens spaces that do not admit orientation reversing isometries. If a lens space does admit orientation reversing isometries, its isometry group is an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathrm{Norm}(G)/G$.

In table 2 we present all elliptic manifolds with non-cyclic fundamental group and their isometry groups. We also compute the group of connected components $\pi_0(\mathrm{Isom}(M))$ which we denote by $\mathcal{I}(M)$.

Table 2. Isometry groups of elliptic manifolds

G	**Isom(M)**	$\mathcal{I}$(**M**)
D^*_{4n}	$SO(3) \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$D^*_{4n} \times C_m$	$O(2) \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
"index 2"	$O(2) \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
T^*_{24}	$SO(3) \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$T^*_{24} \times C_n$	$O(2) \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
"index 3"	$O(2)$	$\mathbb{Z}/2\mathbb{Z}$
O^*_{48}	$SO(3)$	1
$O^*_{48} \times C_n$	$O(2)$	$\mathbb{Z}/2\mathbb{Z}$
I^*_{120}	$SO(3)$	1
$I^*_{120} \times C_n$	$O(2)$	$\mathbb{Z}/2\mathbb{Z}$
Q_8	$SO(3) \times S_3$	S_3
$Q_8 \times C_n$	$O(2) \times S_3$	$\mathbb{Z}/2\mathbb{Z} \times S_3$

5.3. **Mapping class groups.** Let $M = S^3/G$ be an elliptic manifold. (We should note that everything in this section also holds for lens spaces.) The mapping class group of M is $\Gamma(M) = \pi_0(\mathrm{Diff}(M))$. In the previous section, we already computed $\mathcal{I}(M) = \pi_0(\mathrm{Isom}(M))$. The following theorem (also known as the π_0-Smale conjecture) shows that these two groups are the same.

Theorem 5.3. *The inclusion* $\iota : \mathrm{Isom}(M) \to \mathrm{Diff}(M)$ *is a bijection on path components.*

Proof. See ([McC02], Theorem 3.1). □

The proof of this theorem relies on already knowing most of the groups $\Gamma(M)$, since historically these were determined earlier.

We want to present a sketch of how $\Gamma(M)$ can be computed without knowing $\mathcal{I}(M)$ for prism manifolds (i.e. the ones having a dihedral group as a subgroup of their fundamental group). See ([Asa78]) for a detailed proof.

Consider the following construction. Take an orientable S^1-bundle $\pi : N \to B$ over a Möbius band B. Then the boundary ∂N is a torus. Glue a solid torus V to N along their boundaries. The resulting manifold M turns out to be a prism manifold and in fact every prism manifold can be constructed this way.

Let α be the core curve of B. Then $\pi^{-1}(\alpha) =: K$ is a Klein bottle. One can show that K is the unique incompressible Klein bottle in M up to isotopy and that every diffeomorphism of M can be isotoped such that it fixes K. Furthermore, any diffeomorphism of K extends to one of M yielding a surjective homomorphism $\rho : \Gamma(K) \to \Gamma(M)$. It is known that $\Gamma(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and therefore computing the kernel of ρ (which will depend on the gluing used in the construction of M) determines $\Gamma(M)$.

We end this section by presenting the so called realisation theorem, which states that the mapping class group can be modelled as a group of isometries. Let $\iota : \mathrm{Isom}(M) \to \mathrm{Diff}(M)$ and $p : \mathrm{Diff}(M) \to \Gamma(M)$ be the natural inclusion and projection maps respectively. Let $\phi = p \circ \iota$ be the composition.

Theorem 5.4. *There is a subgroup* $\Theta \subset \mathrm{Isom}(M)$ *such that the map*

$$\phi : \mathrm{Isom}(M) \to \Gamma(M)$$

restricted to Θ *induces an isomorphism* $\Theta \xrightarrow{\cong} \Gamma(M)$.

Proof. Consider the short exact sequence

$$1 \to \mathrm{isom}(M) \to \mathrm{Isom}(M) \to \mathcal{I}(M) \to 1,$$

where $\mathrm{isom}(M)$ denotes the component of the identity of $\mathrm{Isom}(M)$ and $\mathcal{I}(M) = \pi_0(\mathrm{Isom}(M))$. By checking every possible case for every M in table **??** one sees that the above sequence splits. (The same holds for all lens spaces.)

Let $s : \mathcal{I}(M) \to \mathrm{Isom}(M)$ be a section of the natural map $\mathrm{Isom}(M) \to \mathcal{I}(M)$ and let $\Theta := \mathrm{Im}\, s$. Restrict $\phi : \mathrm{Isom}(M) \to \Gamma(M)$ to the subgroup Θ, i.e. consider $\phi|_\Theta : \Theta \to \Gamma(M)$. By the definition of Θ and Theorem 5.3, we obtain that $\phi|_\Theta$ is an isomorphism. □

5.4. **Smale conjecture.** Let M be an elliptic manifold. The Smale conjecture asks whether $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ is a homotopy equivalence. From theorem 5.3 we know that $\mathrm{Isom}(M)$ and $\mathrm{Diff}(M)$ have the same π_0, i.e. the same path components. Hence the Smale conjecture in the case of elliptic manifolds is equivalent to the weak Smale conjecture which asks whether the inclusion of the components of the identity $\mathrm{isom}(M) \to \mathrm{diff}(M)$ is a homotopy equivalence. This has recently been proven affirmatively, leading to the following theorem.

Theorem 5.5. *Let M be an elliptic manifold. Then* $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ *is a homotopy equivalence.*

Proof. See ([HKMR12]). □

6. Haken 3-manifolds

Throughout this section, 3-manifolds, denoted by M, will be compact, orientable, and connected.

Definition 6.1. Let M be a compact, connected, orientable 3-manifold.

1. We say that M is *irreducible* if and only if every 2-sphere in M bounds a copy of D^3 in M.
2. We say that an orientable embedded surface F in M with $\partial F \subseteq \partial M$ is *compressible* if either
   - (a) there exists a curve $k \subseteq \mathring{F}$, homotopically essential in F, and a disc $D \subseteq M$ with $\mathring{D} \subseteq \mathring{M}$ and $D \cap F = \partial D = k$; or
   - (b) there exists a ball $E \subseteq M$ with $E \cap F = \partial E$.
3. We say that M is *boundary irreducible* if and only if ∂M is incompressible.
4. We say that M is *Haken* (or sufficiently large) if it contains an incompressible surface.

Remark 6.2.

1. If $\partial M \neq \emptyset$, then M is Haken. This is because a small properly embedded boundary parallel disc is incompressible. So even D^3 is Haken, although it is not boundary irreducible.
2. A Seifert fibred space is Haken unless it has base orbifold S^3 and at most two singular fibres.

Lemma 6.3. *Let M be irreducible, boundary irreducible, and Haken. Then $M \simeq K(\pi_1(M), 1)$, with $\pi_1(M)$ infinite.*

Proof. The sphere theorem implies that $\pi_2(M) = 0$. We argue that $\pi_1(M)$ is infinite. Suppose that $\partial M = \emptyset$. Then M contains a closed incompressible surface F, which must have positive genus because M is irreducible. By the loop theorem, $\pi_1(F)$ injects into $\pi_1(M)$, and hence $\pi_1(M)$ is infinite. Now suppose that $\partial M \neq \emptyset$. If ∂M contains S^2 as a connected component, then $M \cong D^3$ by irreducibility. But ∂D^3 is compressible, and M is boundary incompressible, hence M is not D^3. It follows that every component of ∂M has positive genus. Since ∂M is incompressible, $\pi_1(\partial M) \to \pi_1(M)$ is injective, and hence $\pi_1(M)$ is infinite.

Now we consider the universal cover $\widetilde{M}$ of M. We have $\pi_1(\widetilde{M}) = 0$, and $\pi_2(\widetilde{M}) = \pi_2(M) = 0$ by the sphere theorem. Since $\pi_1(M)$ is infinite, $\widetilde{M}$ is a noncompact 3-manifold, so $H_3(\widetilde{M}) = 0$. Hence $H_i(\widetilde{M}) = 0$ for all $i > 0$. By the Hurewicz theorem, $\pi_i(\widetilde{M}) = 0$ for all $i > 0$. So $M \simeq K(\pi_1(M), 1)$ as desired. □

As a consequence of the lemma, when M is closed, there is a bijection between homotopy classes of homotopy equivalences of M and outer automorphisms of $\pi_1(M)$.

We can now state Waldhausen's main result [Wal68] on the mapping class groups of Haken 3-manifolds.

To state it we introduce the notation $\mathrm{Aut}_\partial(\pi_1(M))$, for the automorphisms of $\pi_1(M)$ that preserve the peripheral structure. Here an automorphism ψ of $\pi_1(M)$ is said to preserve the peripheral structure if for every connected component $F \subseteq \partial M$, $\psi(\pi_1(F)) \subseteq A$, for some subgroup A that is conjugate to $\pi_1(G) \subseteq \pi_1(M)$, for some connected component $G \subseteq \partial M$.

Theorem 6.4 (Waldhausen). *Let M^3 be irreducible, boundary irreducible, and Haken. Then taking the induced action on π_1 determines an isomorphism*

$$\pi_0 \operatorname{Diff}(M) \xrightarrow{\cong} \mathrm{Aut}_\partial(\pi_1(M))/\operatorname{Inn}(\pi_1(M)).$$

More generally, after Waldhausen, Hatcher and Ivanov [Hat76, Iva76] proved the following theorem. Here on both sides the boundary must be fixed pointwise.

Theorem 6.5 (Hatcher, Ivanov). *Let M^3 be irreducible, boundary irreducible, and Haken. Then the forgetful map is a homotopy equivalence*

$$\operatorname{Diff}_\partial(M) \xrightarrow{\simeq} \mathrm{hAut}_\partial(M).$$

Another interesting theorem on mapping class groups of Haken 3-manifolds is due to Johannson [Joh79].

Theorem 6.6 (Johannson). *Let M^3 be irreducible, boundary irreducible, and Haken. Suppose also that every incompressible annulus or torus in M is boundary parallel. Then $\pi_0 \operatorname{Diff}(M)$ is finite.*

This includes cases where the analogous algebraic fact is hard to see, so Johannson also proved new algebraic results on finiteness of outer automorphism of some groups, in this way.

Now we begin to outline the methods used in Waldhausen's proof. The key is Haken's notion of hierarchies.

Definition 6.7 (Hierarchies). Let M_1 be an irreducible 3-manifold. A *hierarchy* for M_1 of length n is a sequence of triples

$$(M_j, F_j \subseteq M_j, N(F_j))$$

for $j = 1, \dots, n$, where $F_j \subseteq M_j$ is an incompressible surface, and $N(F_j)$ is a tubular neighbourhood of F_j, such that $M_{j+1} = M_j \setminus \mathring{N}(F_j)$ and M_{n+1} is a union of 3-balls.

Theorem 6.8 (Existence of hierarchies)**.** *Let M_1 be an irreducible 3-manifold with $\partial M_1 \neq \emptyset$. Then there exists finite n and a hierarchy for M_1 of length n, $\{(M_j, F_j \subseteq M_j, N(F_j)) \mid j = 1, \dots, n\}$ such that $0 \neq [\partial F_j] \in H_1(\partial M_j)$ for all j.*

To apply the theorem to a closed Haken manifold, we first cut along the given incompressible surface. The condition that $0 \neq [\partial F_j]$ is important in the use of the theorem, but this will not be used in these notes.

To prove Theorem 6.8, Waldhausen defined a complexity of a handle decomposition, in terms of three nonnegative integers. He took an incompressible surface, and modified it to be a 'normal surface'. Cutting along a normal surface reduces the complexity, and hence the procedure terminates in finite time.

To sketch the proof of surjectivity in Waldhausen's theorem, we will follow Scott's exposition [Sco72], and use the following proposition, which fixes a homotopy equivalence to be a homeomorphism on surfaces in M.

Proposition 6.9. *Let M be irreducible and boundary irreducible. Let $f\colon M \to M$, with $f^{-1}(\partial M) = \partial M$, be a homotopy equivalence. Let $F \subseteq M$ be incompressible and either non-separating or if $M \setminus \mathring{N}(F)$ has two components, then neither has fundamental group $\pi_1(F)$. Then $f \sim g$ such that*

(a) $g|\colon g^{-1}(F) \to F$ is a homeomorphism; or

(b) $g(M) \subseteq \partial M$, and M is an interval bundle over F, i.e. $M \cong F \times I$ or $M \cong F \widetilde{\times} I$.

We want to use this to sketch the proof of surjectivity in Theorem 6.4. We prove the following theorem.

Theorem 6.10. *Let M be irreducible and boundary irreducible. Let $f\colon M \to M$, with $f^{-1}(\partial M) = \partial M$, be a homotopy equivalence. Then*

(a) f is properly homotopic to a homeomorphism $g\colon M \to M$; or

(b) f is properly homotopic to g with $g(M) \subseteq \partial M$, and M is an interval bundle over F, i.e. $M \cong F \times I$ or $M \cong F \widetilde{\times} I$.

The theorem almost implies surjectivity in Waldhausen's theorem. First, because M is an Eilenberg-MacLane space, every automorphism of π_1 is realised by a homotopy self-equivalence. If (a) occurs, then we are done. If (b) occurs, then we know that homotopy equivalences of surfaces are homotopic to a homeomorphism, and we can apply this and some further arguments in [Wal68] to conclude. We will sadly not discuss the further arguments needed to deal with the case of interval bundles.

Also note that it suffices to find a homeomorphism, by Hatcher's theorem that every homeomorphism between 3-manifolds is isotopic to a diffeomorphism.

Sketch proof of Theorem 6.10. Choose a hierarchy for M as in Theorem 6.8. Induct on the length of the hierarchy. Let $F \subseteq M$ be irreducible. Apply Proposition 6.9. If (b) holds in the proposition, then (b) holds in the theorem. If (a) holds in the proposition, then label one copy of M as N and think of f as a map $f\colon M \to N$. Cut N along F and cut M along $f^{-1}(F)$. We obtain a map

$$f'\colon M' \to N'$$

between the cut manifolds. Either f' satisfies the hypotheses of the theorem or $M' = N'$ is a union of copies of D^3. Also $f|_{\partial M'}$ is a homeomorphism (use the approximation for surfaces to any boundary components where this is not yet the case), By the inductive hypothesis, and the fact that the length of the hierarchy for M' is less than the length for M, either (a) or (b) hold. But $f|_{\partial M'}$ is a homeomorphism, and the homotopy is proper, so (a) must hold, therefore f' is homotopic to a homeomorphims. Glue back together to obtain a homeomorphism.

For the base case, iof the hierarchy has length one, then after cutting we have a union of 3-balls. But every homotopy equivalence of D^3 that is a homeomorphism on the boundary is homotopic to a homeomorphism by the Alexander trick. □

7. Hyperbolic manifolds

In this section, we will consider the case of hyperbolic manifolds. We will begin by discussing the basic properties of hyperbolic 3-manifolds. We will then discuss Mostow rigidity [Mos68] as well as its consequences for mapping class groups. Next we will discuss the work of Gabai [Gab97] and Gabai–Meyerhoff–Thurston [GMT03] on completing the proof that $\pi_0(\mathrm{Diff}(M)) \cong \pi_0(\mathrm{Isom}(M)) \cong \mathrm{Out}(\pi_1(M))$ for closed hyeprbolic 3-manifolds. Finally, we will discuss the resolution of the Smale conjecture for closed hyperbolic 3-manifolds due to Gabai [Gab01].

Throughout this chapter, all manifolds will be assumed to be smooth and connected. From now on, we will write $\cong_{\mathrm{Isom}}$ and $\cong$ when two manifolds are isometric or diffeomorphic respectively. We will write $\sim$ when two maps between manifolds are isotopic.

7.1. Definition and properties.

We will begin by defining what it means for a manifold to be hyperbolic.

Definition 7.1. For $n \geq 2$, an n-manifold M is *hyperbolic* if it admits a complete Riemannian metric of constant curvature -1.

Our prototypical example of a hyperbolic n-manifold is real hyperbolic space $\mathbb{H}^n$. This has multiple definitions (which are equivalent up to isometry) but, for our purposes, it will be convenient to use the *Poincaré disc model* where $\mathbb{H}^n = \mathrm{Int}\, D^n$ is taken to be the open unit ball in $\mathbb{R}^n$ with metric

$$\frac{dx_1^2 + \cdots + dx_n^2}{(1 - (x_1^2 + \cdots + x_n^2))^2}.$$

Proposition 7.2. *Every simply connected hyperbolic n-manifold is isometric to real hyperbolic space $\mathbb{H}^n$.*

An important isometry invariant of a hyperbolic manifold M is its volume:

$$\mathrm{vol}(M) := \int_M \omega_g$$

where ω_g is the volume form associated to a choice of (hyperbolic) Riemannian metric g.

If M is a hyperbolic n-manifold, then its universal cover is a simply connected hyperbolic n-manifold and so is isometric to $\mathbb{H}^n$. This implies that $M \cong \mathbb{H}^n/G$ for some $G \leq \text{Isom}(\mathbb{H}^n)$.

We will now consider the case of hyperbolic 3-manifolds M and, as usual, we will restrict to the case where M is compact. The following generalised an observation made in the previous chapter.

Proposition 7.3. *Let M be a compact hyperbolic 3-manifold. If $\partial M \neq \emptyset$ or M is non-orientable, then M is Haken.*

It therefore suffices to consider the case of closed orientable hyperbolic 3-manifolds. Since $\text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}_2(\mathbb{C})$ where $\text{PSL}_2(\mathbb{C}) := \text{SL}_2(\mathbb{C})/\pm 1$, every closed orientable hyperbolic 3-manifold is therefore of the form $\mathbb{H}^3/G$ where $G \leq \text{PSL}_2(\mathbb{C})$ is a discrete subgroup which is of finite covolume in the sense that $\text{vol}(\mathbb{H}^3/G) < \infty$ (since M is closed). Such groups G are known as *Kleinian groups*. Not all Kleinian groups correspond to manifolds since, in general, $\mathbb{H}^3/G$ will be an orbifold rather than a manifold.

Examples of hyperbolic 3-manifolds:

(1) Complements of hyperbolic knots are compact hyperbolic 3-manifolds. Since they have boundary, they are Haken.
(2) Arithmetic hyperbolic 3-manifolds. If $\mathcal{O}$ is an order in a quaternion algebra A defined over $\mathbb{Q}$ then, subject to certain conditions, one can construct a Kleinian group $G_{\mathcal{O}} \leq \text{Isom}^+(\mathbb{H}^3)$ and a corresponding closed orientable hyperbolic 3-manifold $M_{\mathcal{O}} = \mathbb{H}^3/G_{\mathcal{O}}$.
(3) Hyperbolic Dehn surgery. This is an operation which can be used construct further hyperbolic 3-manifolds from existing ones. It can be used to produce infinitely many (non-diffeomorphic) hyperbolic 3-manifolds with $\text{vol}(M) \leq V$ for some constant $V > 0$.
(4) It was shown by Jørgensen [Jr77] that there exists a compact hyperbolic 3-manifolds M which fibres over S^1.

Remark 7.4. (a) There is no analogue of hyperbolic Dehn surgery in higher dimensions. In fact, for all $n \geq 4$ and $V > 0$, there are finitely many hyperbolic n-manifolds M with $\text{vol}(M) < V$ (up to diffeomorphism). So, in some sense, there are not that many hyperbolic n-manifolds for $n \geq 4$. This is another reason why hyperbolic 3-manifolds are especially interesting.

(b) Following Example (4) above, one can also ask in what other dimensions $n \geq 4$ does there exist a hyperbolic n-manifold which fibres over S^1. Such examples do not exist for n even for elementary reasons. An example in the case $n = 5$ was found by Italiano-Martelli-Migliorini [IMM23] in 2023 but the case of $n \geq 7$ odd remains open.

Properties of closed hyperbolic 3-manifolds.

(1) They are aspherical since $\widetilde{M} \cong \mathbb{H}^3$ is contractible. This implies, for example, that $\pi_1(M) = \Gamma$ is torsion-free.
(2) They are irreducible and atoroidal. Conversely, by Thurston's hyperbolisation theorem, every closed irreducible atoroidal 3-manifold which is not Seifert fibred is hyperbolic.

(3) If M is a closed hyperbolic 3-manifold, then $\pi_1(M)$ is a hyperbolic group (in the sense of Gromov). This follows from the fact that $\pi_1(M)$ acts freely by isometries on $\mathbb{H}^3$ and so, by the Milnor-Svarc lemma, the Cayley graph of $\pi_1(M)$ is quasi-isometric to $\mathbb{H}^3$, from which the result follows. In fact, this is true even for fundamental groups of closed hyperbolic n-manifolds for all $n \geq 2$.

(4) For every closed hyperbolic 3-manifold M has a finite cover $M' \to M$ where M' fibres over the circle. This is (a special case of) the virtual fibreing conjecture, and was shown by Agol in 2013 [Ago13].

Let M be a closed hyperbolic 3-manifold. Since M is aspherical, we have that $M \simeq K(\pi_1(M), 1)$. Recall that, for a topological space X, $\mathrm{hAut}(X)$ denoted the space of homotopy automorphisms with the C^0 topology. It follows that $\pi_0(\mathrm{hAut}(X)) \cong \mathrm{hAut}(X)/\simeq$, i.e. the set of homotopy automorphisms of X considered up to homotopy.

Lemma 7.5. *Let X be a $K(G,1)$-space where G is a group. Then, for each choice of basepoint $x_0 \in X$ and identification $G = \pi_1(X, x_0)$, there is an isomorphism*

$$\phi\colon \pi_0(\mathrm{hAut}(X)) \to \mathrm{Out}(G)$$

given by sending $f \mapsto [(f_\gamma)_*]$ where $(f_\gamma)_*\colon G \to G$, $[\alpha] \mapsto [\gamma \cdot (f \circ \alpha) \cdot \gamma^{-1}]$ where $\gamma\colon [0,1] \to X$ is any choice of path from x_0 to $f(x_0)$.

Remark 7.6. The map is well defined only in the quotient $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, and the different choices of γ correspond to changing $[(f_\gamma)_*]$ by an element of $\mathrm{Inn}(G)$.

In particular, for a closed hyperbolic 3-manifold M, we have that

$$\pi_0(\mathrm{hAut}(M)) \cong \mathrm{Out}(\pi_1(M)).$$

7.2. **Mostow rigidity.** We will now discuss the following famous result, due to Mostow [Mos68].

Theorem 7.7 (Mostow rigidity)**.** *Let M, N be closed hyperbolic 3-manifolds. If $f : M \to N$ is a homotopy equivalence, then f is homotopic to an isometry.*

Remark 7.8. (a) This was generalised to not necessarily-closed finite volume hyperbolic 3-manifolds in 1971 by Prasad.

(b) This holds for closed hyperbolic n-manifolds for all $n \geq 3$ but fails for $n = 2$ where there are many the surfaces Σ_g admit infinitely many inequivalent hyperbolic metrics for any $g \geq 2$. Hyperbolic 3-manifolds therefore occupy a middle ground where the dimension $n = 3$ is large enough for Mostow rigidity but small enough that infinitely many hyperbolic 3-manifolds exist with bounded volume.

(c) The Borel conjecture can be viewed as a broad generalisation of Mostow rigidity. Let M and N be closed aspherical n-topological manifolds. Then the Borel conjecture is that, if $f : M \to N$ is a homotopy equivalence, then f is homotopic to a homeomorphism. This is true for $n = 3$ by Perelman's resolution to Thurston's geometrisation conjecture.

In particular, if two closed hyperbolic 3-manifolds are diffeomorphic, then they are isometric. That is, the choice of hyperbolic metric g on a closed hyperbolic 3-manifold M is unique. Moreover, we have:

Corollary 7.9. *Let M, N be closed hyperbolic 3-manifolds. Then:*

$$M \cong_{\mathrm{Isom}} N \Leftrightarrow M \cong N \Leftrightarrow M \simeq N \Leftrightarrow \pi_1(M) \cong \pi_1(N).$$

We will now discuss the general idea of the proof of Mostow rigidity. First define $\overline{\mathbb{H}^3} \cong D^3$ to be the closure of $\mathbb{H}^3$ and then $\partial\mathbb{H}^3 := \overline{\mathbb{H}^3} \setminus \mathbb{H}^3$ ($\cong S^2$).

Sketch of proof. Let $f : M \to N$ be a homotopy equivalence between closed hyperbolic 3-manifolds M and N. Then f induces a homotopy equivalence $F : \widetilde{M} \to \widetilde{N}$. Since $\widetilde{M} \cong \mathbb{H}^3$ and $\widetilde{N} \cong \mathbb{H}^3$, we will write this as $F : \mathbb{H}^3 \to \mathbb{H}^3$.

The proof now consists of the following three steps:

(1) Show that there exists a (continuous) extension $\overline{F} \colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ such that $\overline{F}\,|_{\partial\mathbb{H}^3} \colon \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ is a conformal diffeomorphism (i.e. an angle-preserving diffeomorphism).
(2) There is a one-to-one correspondence between isometries of $\mathbb{H}^3$ and conformal diffeomorphisms of $\partial\mathbb{H}^3$ given by

$$\mathrm{Isom}(\mathbb{H}^3) \xrightarrow{\cong} \mathrm{ConfDiffeo}(\partial\mathbb{H}^3), \quad G \mapsto \overline{G}\,|_{\partial\mathbb{H}^3}.$$

(3) Pick $G \in \mathrm{Isom}(\mathbb{H}^3)$ such that $\overline{F}\,|_{\partial\mathbb{H}^3} = \overline{G}\,|_{\partial\mathbb{H}^3}$. Then show that G induced an isometry $g : M \to N$ such that $f \simeq g$. □

We conclude this section by noting that this does not suffice to compute the mapping class group $\Gamma(M) = \pi_0(\mathrm{Diff}(M)) = \mathrm{Diff}(M)/\sim$.

It is well-known and can be proven using an elementary argument that, for M closed hyperbolic 3-manifold, the set of isometries $\mathrm{Isom}(M)$ is finite. In particular, we have that $\mathrm{Isom}(M)$ and $\pi_0(\mathrm{Isom}(M)) = \mathrm{Isom}(M)/\sim$ coincide. By combining with the results at the end of the previous section, we have the following diagram:

$$\begin{array}{ccccccc}
\overbrace{\mathrm{Isom}(M)/\sim}^{\pi_0(\mathrm{Isom}(M))} & \longrightarrow & \overbrace{\mathrm{Diff}(M)/\sim}^{\Gamma(M)} & & & & \\
\downarrow & & \downarrow & & & & \\
\mathrm{Isom}(M)/\simeq & \xrightarrow{\cong} & \mathrm{Diff}(M)/\simeq & \xrightarrow{\cong} & \underbrace{\mathrm{hAut}(M)/\simeq}_{\pi_0(\mathrm{hAut}(M))} & \xrightarrow{\cong} & \mathrm{Out}(\pi_1(M))
\end{array}$$

where the bottom arrows are all bijections. The missing ingredient is whether homotopic diffeomorpisms $f, g : M \to M$ are actually isotopic.

7.3. **The Gabai-Meyerhoff-Thurston theorem.** The aim of this section will be to discuss the following major theorem due to Gabai, Meyerhoff and N.Thurston [GMT03], building upon a previous result of Gabai [Gab97]. In the case where M is Haken, this was proved in the previous chapter.

Theorem 7.10 (Gabai-Meyerhoff-Thurston)**.** *Let M be a closed hyperbolic 3-manifold. If $f, g \in \mathrm{Diff}(M)$ and $f \simeq g$, then $f \sim g$. In particular, we have*

$$\pi_0(\mathrm{Isom}(M)) \cong \Gamma(M) \cong \pi_0(\mathrm{hAut}(M)) \cong \mathrm{Out}(\pi_1(M)).$$

Remark 7.11. (a) The main result in [GMT03] actually proved much more. Let M be a closed 3-manifold and let N be a closed hyperbolic 3-manifold. If $f : M \to N$ is a homotopy equivalence, then f is homotopic to an isometry. (In particular, M is a hyperbolic 3-manifold.)

(b) As mentioned in the previous section, $\pi_0(\mathrm{Isom}(M))$ is finite and so the above theorem shows that mapping class groups of closed hyperbolic 3-manifolds are finite. In fact, every finite group arises as the mapping class group of a closed hyperbolic 3-manifold [Koj88].

The remainder of this section will be dedicated to a discussion of the proof of this theorem. We will begin with the work of Gabai [Gab97] which proved the theorem subject to a certain technical condition which we will now discuss.

Definition 7.12. Let M be a closed hyperbolic 3-manifold and let δ be a simple closed geodesic in M. Then δ lifts to a set of hyperbolic lines $\{\delta_i\}$ in $\mathbb{H}^3$. For each $i \neq j$, define the *orthocurve* s_{ij} to be the shortest hyperbolic line segment in $\mathbb{H}^3$ between δ_i and δ_j. Define the *midplane* D_{ij} to be the hyperbolic place in $\mathbb{H}^3$ which meets s_{ij} orthogonally at its midpoint.

By considering the completion $\overline{\mathbb{H}^3}$ and corresponding boundary $\partial\mathbb{H}^3$ ($\cong S^2$), we have a curve $\lambda_{ij} = \partial D_{ij}$ which separates the pair of points $\partial\delta_i$ from $\partial\delta_j$. The set of simple closed curves $\{\lambda_{ij} \mid i \neq j\}$ in $\partial\mathbb{H}^3$ is known as the *Dirichlet insulator family* associated to the geodesic δ.

We say that a Dirichlet insulator family $\{\lambda_{ij} \mid i \neq j\}$ is *non-coalescable* if there does not exist i, j_1, j_2, j_3 such that $\lambda_{ij_1} \cup \lambda_{ij_2} \cup \lambda_{ij_3}$ separates the points in $\partial\delta_i$. See Figure 3 for an example where this condition fails.

Gabai defined a more general notion of insulator family, which is slightly more general than the notion of a Dirichlet insulator family, and proved the following.

Theorem 7.13 (Gabai)**.** *Let M be a closed hyperbolic 3-manifold. Suppose there exists a simple closed geodesic δ in M which has an associated non-coaloscable insulator family. Then, if $f, g \in \mathrm{Diff}(M)$ and $f \simeq g$, then $f \sim g$.*

Since this result is known already in the Haken case, it suffices to treat the case where M is a closed non-Haken hyperbolic 3-manifold. This assumption plays an important role in the proof.

If M contains a hyperbolic tube of radius $\log(3)/2$ (≈ 0.5) about a simple closed geodesic δ, then its associated Dirichlet insulator family is non-coalescable.

In order to prove Theorem 7.10, it remains to prove the result in the case where M is a closed hyperbolic 3-manifold such that no simple closed geodesic is contained in a hyperbolic tube of radius $\log(3)/2$. This is achieved by Gabai-Meyerhoff-Thurston as follows:

Theorem 7.14 (Gabai-Meyerhoff-Thurston)**.** *Let M be a closed hyperbolic 3-manifold such that no simple closed geodesic is contained in a hyperbolic tube of radius $\log(3)/2$. Then:*

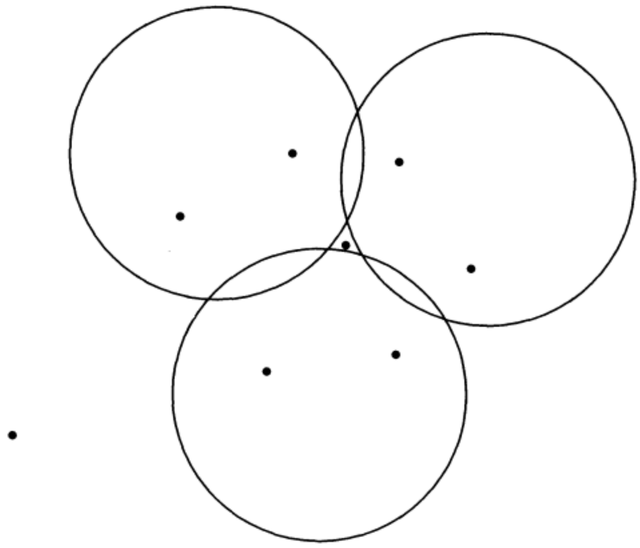


FIGURE 3. Example of a Dirichlet insulator family which is not non-coalescable: the four pairs of points are $\partial\delta_i$, $\partial\delta_{j_1}$, $\partial\delta_{j_2}$, $\partial\delta_{j_3}$, the circles are λ_{ij_1}, λ_{ij_2}, λ_{ij_3} and the diagram is drawn on a patch of the boundary sphere $\partial\mathbb{H}^3 \cong S^2$.

(a) *M is contained in one of seven families $\mathcal{R}_0, \cdots, \mathcal{R}_6$ (which can be viewed as subsets of $\mathbb{C}^2$).*
(b) *For each $M \in \mathcal{R}_i$ for $i = 1, \cdots, 6$, the shortest geodesic δ has an associated Corona insulator family $\{\kappa_{ij}\}$ which is non-coalescable.*
(c) *The set $\mathcal{R}_0$ consists of a single manifold* Vol3 *and δ in* Vol3 *has a non-coalescable insulator family (whose definition combines parts of the definition of the Corona and Dirichlet insulator families).*

It would be interesting to know whether or not any of the more recent developments in 3-manifold topology (e.g. the work of Perelman and Agol) could be used to simplify the proof of the Gabai-Meyerhoff-Thurston theorem.

7.4. **Smale conjecture.** The following was shown by Gabai [Gab01], building upon the Gabai-Meyerhoff-Thurston theorem. This resolves the Smale conjecture for closed hyperbolic 3-manifolds.

Theorem 7.15 (Gabai)**.** *Let M be a closed hyperbolic 3-manifold. Then the inclusion map*

$$\mathrm{Isom}(M) \hookrightarrow \mathrm{Diff}(M)$$

is a homotopy equivalence.

The idea of the proof is as follows. Since $\mathrm{Isom}(M)$ is finite, we have that $\mathrm{Isom}_0(M) = *$. Since $\pi_0(\mathrm{Isom}(M)) \cong \pi_0(\mathrm{Diff}(M))$ by Theorem 7.10, it therefore suffices to prove that $\mathrm{Diff}_0(M) \simeq *$ (which, in this case, is equivalent to proving the weak Smale conjecture). By Mostow rigidity, $\mathrm{Diff}_0(M)$ can be viewed equivalently as the space of hyperbolic metrics $\mathrm{Hyp}(M)$ on M. Gabai shows that this space is contractible using a generalisation of the methods used in [Gab97] (in particular, using the theory of insulators to reduce to the case of Haken 3-manifolds).

8. JSJ decompositions and mapping class groups

8.1. JSJ decompositions.

Definition 8.1. Let $a, b \in \mathbb{Z}^+$ with $\gcd(a, b) = 1$. Let $f_{a,b}\colon D^2 \to D^2$, $z \mapsto e^{2\pi i b/a} z$. A standard fibred torus is the mapping torus of $f_{a,b}$ for some a, b, i.e. $D^2 \times I/(z, 0) \sim (f_{a,b}(z), 1)$, foliated by $\{z\} \times I$ for every z.

Since $b/a \in \mathbb{Q}$, each leaf of the foliation is an embedded S^1.

Definition 8.2. M^3 is *Seifert fibred* if it is foliated by copies of S^1 such that each leaf has a neighbourhood that is a standard fibred torus.

Example 8.3. S^3 is Seifert fibred (via the Hopf map). More generally, every lens space is Seifert fibred (via its decomposition into two tori). More generally, every elliptic manifold is Seifert fibred, see ... .

Definition 8.4. M^3 is atoroidal if every incompressible torus is boundary parallel.

Theorem 8.5 (JSJ, 1979)**.** *Suppose that M^3 is oriented, irreducible and $\partial M \approx \bigsqcup^k T^2$ for some $k \geq 0$. Then*

a) There is an $m \geq 0$ and a submanifold $\bigsqcup_{i=1}^m T_i \subset M$ such that for every i, $T_i \subset M$ is an incompressible torus, and each component of M cut along $\bigsqcup_{i=1}^m T_i$ is Seifert fibred or atoroidal.

b) Any two minimal submanifolds with the above property are isotopic.

Recall that by the elliptisation theorem (Perelman) if $\pi_1(M)$ is finite, then M is elliptic (hence Seifert fibred). Recall that if M is hyperbolic, then it is atoroidal.

Theorem 8.6 (Hyperbolisation theorem (Perelman))**.** *Suppose that M^3 is oriented, irreducible and $\partial M \approx \bigsqcup^k T^2$ for some $k \geq 0$. If M is atoroidal and $\pi_1(M)$ is infinite, then it is hyperbolic with finite volume or diffeomorphic to $S^1 \times D^2$, $T^2 \times I$ or the nontrivial I-bundle over the Klein bottle.*

Note that all 3 exceptional manifolds are Seifert fibred.

Theorem 8.7 (Geometrisation theorem)**.** *Suppose that M^3 is oriented, irreducible, and $\partial M \approx \bigsqcup^k T^2$ for some $k \geq 0$.*

(a) There is an $m \geq 0$ and a submanifold $\bigsqcup_{i=1}^m T_i \subset M$ such that for every i, $T_i \subset M$ is an incompressible torus, and each component of M cut along $\bigsqcup_{i=1}^m T_i$ is Seifert fibred or hyperbolic.

(b) Any two minimal submanifolds with the above property are isotopic.

Proof. By the previous observations "each component is Seifert fibred or hyperbolic" $\Leftrightarrow$ "each component is Seifert fibred or atoroidal". □

Proposition 8.8 (...). *Suppose that M^3 is oriented, irreducible, $\partial M \approx \bigsqcup^k T^2$ for some $k \geq 0$, and M is not diffeomorphic to $S^1 \times D^2$, $T^2 \times I$ or the nontrivial I-bundle over the Klein bottle. Then M is Seifert fibred if and only if it has a geometry other than hyperbolic or Sol.*

Remark 8.9. There is also a unique minimal decomposition into geometric components, which is different from the JSJ decomposition in general (it may contain fewer tori). The Sol components may be decomposed further to get a JSJ decomposition.

8.2. **Mapping class groups.** The goal of this section is to relate the mapping class group of M to those of its JSJ components.

Remark 8.10. If M^3 has a nontrivial JSJ decomposition, then it is Haken.

Fix M^3. Suppose that M^3 is oriented, irreducible, $\partial M \approx \bigsqcup^k T^2$ for some $k \geq 0$. Fix also a JSJ decomposition $T = \bigsqcup_{i=1}^m T_i \subset M$, let $M_1, \ldots, M_n$ denote the components.

Definition 8.11. Define $\operatorname{Diff}^+(M,T) := \{f \in \operatorname{Diff}^+(M) \mid f(T) = T\}$.

Proposition 8.12. *There is a fibration*

$$\operatorname{Diff}^+(M,T) \to \operatorname{Diff}^+(M) \to (\operatorname{Emb}(T,M)/\operatorname{Diff}(T))_T$$

where $\operatorname{Emb}(T,M)/\operatorname{Diff}(T)$ *is the moduli space of submanifolds of M diffeomorphic to T and* $(\operatorname{Emb}(T,M)/\operatorname{Diff}(T))_T$ *is the path component of T.*

Proof. There is a map $t\colon \operatorname{Diff}^+(M) \to \operatorname{Emb}(T,M)/\operatorname{Diff}(T)$ given by $t(f) = f(T)$. By the uniqueness part of the JSJ theorem, $\operatorname{Im}(t) \subseteq (\operatorname{Emb}(T,M)/\operatorname{Diff}(T))_T$. By isotopy extension $\operatorname{Im}(t) \supseteq (\operatorname{Emb}(T,M)/\operatorname{Diff}(T))_T$. One must check that t is a fibration. □

Corollary 8.13. *There is a long exact sequence*

$$\cdots \to \pi_1(\operatorname{Emb}(T,M)/\operatorname{Diff}(T))_T \to \pi_0 \operatorname{Diff}^+(M,T) \to \pi_0 \operatorname{Diff}^+(M) \to 1$$

Remark 8.14. $\pi_1(\operatorname{Diff}^+(M), \operatorname{Diff}^+(M,T)) \cong \pi_1(\operatorname{Emb}(T,M)/\operatorname{Diff}(T))_T$ is called the motion group of the pair (M,T).

From now on we consider $\pi_0 \operatorname{Diff}^+(M,T)$.

Definition 8.15. We define $\operatorname{Diff}^+(M,(T_i),(M_j)) := \{f \in \operatorname{Diff}^+(M,T) \mid f(T_i) = T_i, f(M_j) = M_j$ for every $i,j\}$.

Definition 8.16. Let $G = (V,E,\alpha)$ be the decorated graph with $V = \{M_j \mid 1 \leq j \leq n\}$, $E = \{T_i \mid 1 \leq i \leq m\}$ and $\alpha(M_j) =$ [the diffeomorphism class of M_j].

Proposition 8.17. *There are exact sequences*

$$1 \to \operatorname{Diff}^+(M,(T_i),(M_j)) \to \operatorname{Diff}^+(M,T) \to \operatorname{Aut}(G)$$

$$1 \to \pi_0 \operatorname{Diff}^+(M,(T_i),(M_j)) \to \pi_0 \operatorname{Diff}^+(M,T) \to \operatorname{Aut}(G)$$

Definition 8.18. We define $\overline{\operatorname{Diff}}^+(M,(T_i),(M_j)) = \{f \in \operatorname{Diff}^+(M,(T_i),(M_j)) \mid f\big|_{T_i}\colon T_i \to T_i$ is orientation preserving for every $i\}$.

Remark 8.19. If every T_i is separating, then $\overline{\mathrm{Diff}}^+(M,(T_i),(M_j)) = \mathrm{Diff}^+(M,(T_i),(M_j))$.

Definition 8.20. Let $\overline{G} = (V, E, \alpha)$ be the 1-dimensional Δ-complex with decoration that is (the geometric realisation of) G.

Remark 8.21. There is a natural surjective map $\mathrm{Aut}(\overline{G}) \to \mathrm{Aut}(G)$, but it is not injective if G contains loops.

Proposition 8.22. *There are exact sequences*

$$1 \to \overline{\mathrm{Diff}}^+(M,(T_i),(M_j)) \to \mathrm{Diff}^+(M,T) \to \mathrm{Aut}(\overline{G})$$

$$1 \to \pi_0\overline{\mathrm{Diff}}^+(M,(T_i),(M_j)) \to \pi_0\,\mathrm{Diff}^+(M,T) \to \mathrm{Aut}(\overline{G})$$

From now on we consider $\pi_0\overline{\mathrm{Diff}}^+(M,(T_i),(M_j))$.

Definition 8.23. Define

$$\mathrm{Diff}_c^+(N) := \{f \in \mathrm{Diff}^+(N) \mid f(S) = S \text{ for every connected component } S \text{ of } \partial N\}.$$

For every connected component S of ∂N there is a natural restriction map $\mathrm{Diff}_c^+(N) \to \mathrm{Diff}^+(S)$.

Definition 8.24. Define a diagram D in the category of topological groups as follows. The objects are $\mathrm{Diff}_c^+(M_j)$ and $\mathrm{Diff}^+(T_i)$ for every j and i. There is a morphism (or two) $\mathrm{Diff}_c^+(M_j) \to \mathrm{Diff}^+(T_i)$ if T_i is a boundary component of M_j.

$\pi_0 D$ is the diagram in the category of groups obtained from D by applying π_0 to every object and morphism.

Proposition 8.25. $\overline{\mathrm{Diff}}^+(M,(T_i),(M_j)) \cong \lim D$ *and* $\pi_0\overline{\mathrm{Diff}}^+(M,(T_i),(M_j)) \cong \lim \pi_0 D$.

9. Mapping class groups of reducible 3-manifolds

9.1. **The diffeomorphism group of reducible 3-manifolds.** Let M be a reducible 3-manifolds with prime factors $P_1, \ldots, P_n$ that are different from the 3-sphere S^3 plus k $S^1 \times S^2$ summands. We introduce a space $C(M)$ modelling ways of representing M as a connected sum, possibly with some trivial S^3 summands.

Given a *finite connected graph* G whose fundamental group is a free group generated by k generators with n of the vertices being labelled as $1, \ldots, n$ and all unlabelled vertices have valence at least 3. We assign P_i to the vertex labelled i, and assign a copy of S^3, viewing as a metric object that is isometric to the standard sphere of radius 1, to each unlabelled vertex.

Each edge e corresponds to a connected-sum operation by choosing an embedding $B_e^3 \to P_i$ for each labelled vertex or an embedding $B_e^3 \to S^3$ for each unlabelled vertex connected to e and attach a product $S_e^2 \times I_e$. Everything is done isometrically by viewing $S_e^2 \times I_e$ as the product of a sphere with radius r_e and interval length l_e, and B_e^3 being the metric ball bounded by S_e^2. For unlabelled vertices, we further require that the embedding we choose is an isometry on the boundary sphere S_e^2. We perform the operations such that all products attached are disjoint in the resulting manifold.

The products $S_e^2 \times I_e$ attached are called *tubes* and parts of the 3-spheres remained in M are called *nodes*.

In the above process, we have made various choices including the radii, lengths and embeddings of 3-balls into the prime factors P_i and trivial 3-spheres. Letting these invariants vary gives rise to a space $C_G(M)$ for every such G. By allowing the following operations, we glue $C_G(M)$ into a single space $C(M)$:

- Collapse tubes between two distinct nodes.
- Collapse tubes between one node and one prime factor.

Both gives a point in $C_{G/e}(M)$ where e is the tube being collapsed. The space $C(M)$ is defined as the union of all $C_G(M)$ up to adding or collapsing degenerate tubes (tubes of length 0).

If we choose a base point $c_0 \in C(M)$, then for any $c \in C(M)$, an *exterior diffeomorphism* of M_c is given by a diffeomorphism $f_c \colon M_c \to M_{c_0}$ restricting to the identity map on $\cup_i \partial P_i$. The group of exterior diffeomorphisms is denoted by $\mathrm{Diff}_{\mathrm{ext}}(M)$. We state the theorem by Hendriks-Laudenbach [HL84], cf. [CdSR79].

Theorem 9.1 (Cesar de Sa–Rourke, Hendriks–Laudenbach)**.** *There is a principal fibre bundle from* $\mathrm{Diff}_{\mathrm{ext}}(M)$ *to* $C(M)$ *which has structure group* $\mathrm{Diff}(M, \partial)$*.*

Taking the loop suspension gives rise to the following principal fibration $\Omega C(M) \to \mathrm{Diff}(M, \partial) \to \mathrm{Diff}_{\mathrm{ext}}(M)$.

If M has one sphere boundary component S_0, then it can be constructed by taking connected sums of a 3-ball P_0 with P_i and the $S^1 \times S^2$ factors. Let D_i denote the connected sum disc in P_i, there is a (delicately defined by Hendriks–Laudenbach) subcomplex of $C_1(M)$ of $C(M)$ such that we have the following theorem:

Theorem 9.2 (Hendriks-Laudenbach [HL84])**.** *There are H-space maps:*

$$\begin{aligned} \alpha &\colon (F_k)^n \to \mathrm{Diff}(M, \partial) \\ \beta &\colon \Omega C_1(M) \to \mathrm{Diff}(M, \partial) \\ \gamma &\colon \prod_i \mathrm{Diff}(P_i, \partial P_i \cup D_i) \times \Omega O(3)^k \to \mathrm{Diff}(M, \partial). \end{aligned}$$

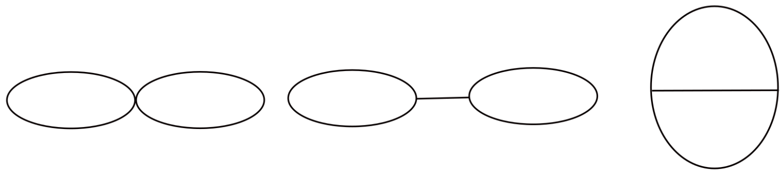
such that the map

$$\begin{aligned} h \colon (F_k)^n \times \Omega C_1(M) \times \prod_i \mathrm{Diff}(P_i, \partial P_i \cup D_i) \times \Omega O(3)^k &\to \mathrm{Diff}(M, \partial) \\ (x, y, z) &\mapsto \alpha(x) \circ \beta(y) \circ \gamma(z) \end{aligned}$$

is a homotopy equivalence.

For $(x_i) \in (F_k)^n$, the image $\alpha(x)$ corresponds to the composition of the slidings of P_i along loops represented by x_i. By the Smale conjecture, the $\Omega O(3)$ factors correspond to $\mathrm{Diff}(S^2 \times I, \partial)$.

Let B denote the closure of the complement of the $n + 2k$ connected-sum discs in P_0. In this case, the homotopy type of C_1 is described through the following proposition.



Proposition 9.3 (Hendriks-McCullough [HM87]). *The composition*

$$(F_g)^n \times \Omega C_1 \xrightarrow{\alpha\cdot\beta} \mathrm{Diff}(M,\partial) \xrightarrow{\rho} \mathrm{Emb}_e(B,M;S_0)$$

is a homotopy equivalence, where $\mathrm{Emb}_e(B,M;S_0)$ *is the space of embeddings restricting to the inclusion on* S_0 *and extendable to a diffeomorphism of* M *rel* ∂*, and* ρ *is the restriction map.*

As an example, we look at the case $n = 0$ and $k = 2$, so M is the closure of $S^1 \times S^2 \# S^1 \times S^2 \setminus B^3$. In this case it turns out that $C_1 = C$ and can be understood by listing the following three possible graphs: The proposition implies that ΩC is homotopy equivalent to $\mathrm{Emb}_e(B^3 \setminus \{\text{four points}\}, M; S^0)$.

9.2. **Finite presentation of the 3-manifold mapping class groups.** We now turn to the question about finite presentation.

Theorem 9.4 (Hatcher-McCullough [HM90]). *Let M be a compact orientable 3-manifold with a prime decomposition. If the mapping class group of each irreducible summand is finitely presented, then $\Gamma(M)$ is also finitely presented.*

Their proof, which we shall outline now, relies on a simplicial complex called the sphere complex $S(M)$.

Suppose M is reducible, we define a simplicial complex $S(M)$ whose vertices are *isotopy classes of embedded 2-spheres that don't bound 3-balls.* A collection of isotopies classes $[S_0'], \dots, [S_n']$ spans an n-simplex if and only if there is a submanifold $S_0 \cup \dots \cup S_n$ of disjoint, pairwise non-isotopic embedded 2-spheres with none of them bounding 3-balls such that S_i is isotopic to S_i' for all i.

Theorem 9.5. *The space $S(M)$ is simply-connected.*

Proposition 9.6. *The quotient $S(M)/(\Gamma(M))$ is finite.*

The proof of Theorem 9.4 is divided into the following steps.

- It suffices to prove the result for M with no 2-sphere boundary components and $M \neq S^1 \times S^2$.
- Prove by induction on the number of summands that the stabiliser of each simplex in $S(M)$ under the action of $\Gamma(M)/R(M)$ is finitely presented where $R(M)$ is the subgroup sphere twists (will be defined later).
- Prove that the stabiliser of each simplex in $S(M)$ under the action of $\Gamma(M)$ is finitely presented.
- Apply a theorem by K. Brown (stated later) on simplicial group actions.

The following proposition takes care of step 1.

Proposition 9.7. *Let M be a connected 3-manifold with finitely generated fundamental group, and suppose S is a 2-sphere boundary component of M. Let $\hat{M}$ be the manifold obtained from M by filling the S^2-boundary with a 3-ball. Then $\Gamma(M)$ is finitely presented if and only if $\Gamma(\hat{M})$ is finitely presented.*

In the rest of this section we give an outline to step 2,3 and 4.

For an embedded 2-sphere S in M, one can define a homeomorphism called the *sphere twist about* S by letting a non-contractible loop in $\pi_1 SO(3)$ based at the identity act on a product neighbourhood of S. The group of isotopy classes of sphere twists is denoted by $R(M)$. $R(M)$ is isomorphic to $(\mathbb{Z}/2)^q$ for some non-negative integer q.

Proposition 9.8. *$R(M)$ acts trivially on $S(M)$, thus the action of $\Gamma(M)$ on $S(M)$ induces an action of $\Gamma(M)/R(M)$ on $S(M)$.*

It follows that for any $\sigma \in S(M)$, the stabiliser under the action of $\Gamma(M)$ is finitely presented if and only if the stabiliser under the action of $\Gamma/R(M)$ is finitely presented. This proves that step 2 is equivalent to step 3.

Now suppose M has no 2-sphere boundary components. Let $\sigma \in S(M)$ represented by pairwise disjoint 2-spheres $S_0, \ldots, S_n$. We cut M along σ gives rise to components $M_1, \ldots, M_m$. Denote $\bar{H}_\partial(M_j)$ to be the group generated by elements of $\Gamma(M_j)/R(M_j)$ that take each component of ∂M_i to itself, and restrict to a degree 1 homeomorphism on each 2-sphere boundary component. There is a well-defined homomorphism

$$i\colon \prod \bar{H}_\partial(M_j) \to \Gamma(M)/R(M)$$

by choosing representatives restricting to the identity on each 2-sphere boundary component and glueing them together.

Proposition 9.9. *The map i is injective.*

By construction, the image of i lies in the stabiliser of σ. Conversely, take an element $[h]$ in the stabiliser of σ, by (potentially) passing to a finite index subgroup, we can assume h does not reverse sides of any S_i. Furthermore, h preserves each M_j and is isotopic to the identity on each S_i. It follows that the image of i has finite index in the stabiliser. Since the spheres S_i do not bound 3-balls, each M_j has fewer summands than M. By induction, the stabiliser of each simplex is finitely presented. This takes care of step 2.

Finally, the following theorem by K. Brown finishes the proof.

Theorem 9.10 (K. Brown, 1984)**.** *Let G be a group that acts simplicially on a simply-connected complex such that each vertex stabiliser is finitely presented, and edge stabiliser is finitely generated so that the quotient has finite 2-skeleton, then G is a finitely presented group.*

9.3. **The mapping class group of $\#_n S^1 \times S^2$.** We now focus on the manifold $M_n = \#_n S^1 \times S^2$, the connected sum of n copies of $S^1 \times S^2$. The group of sphere twists $R(M_n) \cong (\mathbb{Z}/2)^n$ is generated by the twists about the core spheres $\{*\} \times S^2$ of the n summands, and is a normal abelian subgroup of $\Gamma(M_n)$. The mapping class group $\Gamma(M_n)$ acts on the fundamental group $\pi_1(M_n)$ up to conjugation which gives rise to a homomorphism

$$\rho\colon \Gamma(M_n) \to \mathrm{Out}(\pi_1(M_n)) \cong \mathrm{Out}(F_n)$$

Laudenbach discovered the short exact sequence:

$$1 \to R(M_n) \to \Gamma(M_n) \to \mathrm{Out}(F_n) \to 1.$$

Proof sketch.

- Prove that the composition of ρ with the quotient map $\mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$ is surjective. The automorphism group $\mathrm{Aut}(F_n)$ has a countable set of standard generators:For distinct $1 \leq i, j \leq n$, elements L_{ij} and R_{ij} defined via the formulas:

$$L_{ij}(a_k) = \begin{cases} a_j a_k & \text{if } k = i \\ a_k & \text{otherwise} \end{cases} R_{ij}(a_k) = \begin{cases} a_k a_j & \text{if } k = i \\ a_k & \text{otherwise} \end{cases}$$

for $1 \leq k \leq n$, and elements

$$I_i(a_k) = \begin{cases} a_k^{-1} & \text{if } k = i \\ a_k & \text{otherwise} \end{cases}$$

for $1 \leq i \leq n$.

One way of viewing M_n is removing $2n$ disjoint 2-spheres from S^3 and identifying the boundary in pairs. We can consider M_k bounding two different 4-manifolds $\natural_n S^1 \times D^3$ and $\natural_n S^2 \times D^2$ related by surgeries. These generators can be realised by sliding one of the 2-handles (or 1-handles) corresponding to one pair of 2-spheres over another.
- Cut M_n along the core spheres to get an n-punctured S^3, denoted by M_n', whose mapping class group is generated by boundary sphere twists, and prove that any diffeomorphism in the kernel of ρ fixes $\pi_2(M_n)$ hence comes from $\Gamma(M_n')$. □

Brendle-Broaddus-Putman [BBP23] proved that this short exact sequence splits, i.e. there exists a section $\mathrm{Out}(F_n) \to \Gamma(M_n)$. Furthermore, if we choose a trivilisation $[\sigma_0]$ (up to homotopy classes) of the tangent bundle of M_n, then the section can be chosen such that the image of this section is the stabiliser of σ_0. Since $R(M_n)$ is an abelian normal subgroup, it follows that the mapping class group $\Gamma(M_n)$ is isomorphic to the semi-direct product of the twist group $R(M_n)$ and the above mentioned stabiliser. The proof is based on the notion of a crossed homomorphism.

Let G, H be groups with G acts on H on the right. A *crossed homomorphism* from G to H is a map $\lambda\colon G \to H$ such that $\lambda(g_1 g_2) = \lambda(g_1)^{g_2} \lambda(g_2)$.

Lemma 9.11. *Let $1 \to A \to G \to Q \to 1$ be a short exact sequence such that A is an abelian normal subgroup and $Q = G/A$, then the sequence splits if and only if there exists a crossed homomorphism $\lambda\colon G \to A$ that is the identity on A. Moreover, one can choose a splitting $Q \to G$ such that the image is* $\ker(\lambda)$.

To construct such a crossed homomorphism $\Gamma(M_n) \to R(M_n)$, we start by constructing a so called *derivative crossed homomorphism*

$$\mathfrak{D}\colon \Gamma(M_n) \to [M_n, \mathrm{GL}_3^+(\mathbb{R})]$$

. The upcoming construction works for general 3-manifolds as well but we focus on the case $\#_n S^1 \times S^2$.

Let $\mathrm{Fr}(TM_n)$ denote the frame bundle of M_n whose elements are orientation-preserving linear isomorphisms $\tau\colon \mathbb{R}^3 \to T_pM_n$ for $p \in M_n$. Note that $\mathrm{Fr}(TM_n)$ is a principle $\mathrm{GL}_3^+(\mathbb{R})$-bundle with $\mathrm{GL}_3^+(\mathbb{R})$ acting on the right by composition.

Recall that the set of oriented trivialisations $\mathrm{Triv}(M_n)$ consists of sections $\sigma\colon M_n \to \mathrm{Fr}(TM_n)$. The set of continuous maps $C(M_n, \mathrm{GL}_3^+(\mathbb{R}))$ inherits a group structure from $\mathrm{GL}_3^+(\mathbb{R})$ and acts simply transitively on the right on $\mathrm{Triv}(M_n)$ via $\sigma \cdot \phi = (p \to \sigma(p) \cdot \phi(p))$ for $\phi \in C(M_n, \mathrm{GL}_3^+(\mathbb{R}))$.

The diffeomorphism group $\mathrm{Diff}(M_n)$ acts on the right on $\mathrm{Triv}(M_n)$ via $\sigma^f = (Df^{-1})_* \circ \sigma \circ f$ for $f \in \mathrm{Diff}(M_n)$ where $(Df^{-1})_*$ is the induced map of the derivative Df^{-1} on $\mathrm{Fr}(TM_n)$ by composition. It also acts on the right on $C(M_n, \mathrm{GL}_3^+(\mathbb{R}))$ by composition $\phi^f = \phi \circ f$ for $f \in \mathrm{Diff}(M_n)$ and $\phi \in C(M_n, \mathrm{GL}_3^+(\mathbb{R}))$.

The above actions are compatible in the sense that for $f \in \mathrm{Diff}(M_n)$, $\phi \in C(M_n, \mathrm{GL}_3^+(\mathbb{R}))$ and $\sigma \in \mathrm{Triv}(M_n)$, we have $(\sigma \cdot \phi)^f = \sigma^f \cdot \phi^f$.

Choose a base trivialisation $\sigma_0 \in \mathrm{Triv}(M_n)$, for $f \in \mathrm{Diff}(M_n)$, there is a unique $\phi_f \in C(M_n, \mathrm{GL}_3^+(\mathbb{R}))$ such that $\sigma_0^f = \sigma_0 \cdot \phi_f$. We define the derivative crossed homomorphism

$$\mathcal{D}\colon \mathrm{Diff}(M_n) \to C(M_n, \mathrm{GL}_3^+(\mathbb{R}))$$

by $\mathcal{D}(f) = {\phi_f}^{-1}$. One can verify that by the compatibility condition between the actions, this is indeed a crossed homomorphism. It turns out that passing to homotopy classes gives rise to a derivative crossed homomorphism

$$\mathfrak{D}\colon \Gamma(M_n) \to [M_n, \mathrm{GL}_3^+(\mathbb{R})].$$

By composing with the π_1-functor, we have the following map

$$\mathfrak{T}\colon \Gamma(M_n) \to [M_n, \mathrm{GL}_3^+(\mathbb{R})] \to H^1(M_n; \mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1 M_n, \mathbb{Z}/2).$$

Proposition 9.12. *The restriction of $\mathfrak{T}$ to $R(M_n)$ is an isomorphism.*

Thus $\mathfrak{T}$ is the crossed homomorphism we need to prove the splitting of this short exact sequence:

$$1 \to R(M_n) \to \Gamma(M_n) \to \mathrm{Out}(F_n) \to 1$$

and the kernel $\mathrm{Ker}(\mathfrak{T})$ is isomorphic to $\mathrm{Out}(F_n)$.

Sketch proof of the proposition. Let S be an embedded 2-sphere in M_n, the sphere twist T_S about S is constructed by a loop $l\colon [0,1] \to SO(3)$ based at the identity which rotates S about an axis by a full twist. Let p_0 be one of the intersection points of this axis with S. For a closed embedded curve γ in M_n, we can homotope it to some γ' that intersects a neighbourhood $S \times [0,1]$ only in the form of $\{p_0\} \times [0,1]$ hence T_s fixes γ pointwise.

The loop $\mathcal{D}(T_s)(\gamma)$ in $\mathrm{GL}_3^+(\mathbb{R})$ represents $\mathfrak{T}(T_s)([\gamma])$ in $\pi_1\mathrm{GL}_3^+(\mathbb{R}) = \mathbb{Z}/2$ which counts the $\mathbb{Z}/2$-algebraic intersection number of γ with S. Thus $\mathfrak{T}(T_s)$ is the Poincare dual of $[S]$. □

10. Homological stability of mapping class groups

In this section we present a method of studying mapping class groups of 3-manifolds from a more global perspective. In particular we look at a powerful stability result for calculating the homology of mapping class groups associated to certain 3-manifolds.

Definition 10.1. A sequence of groups

$$G_0 \xrightarrow{\phi_0} G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} \dots$$

is called *homologically stable* if, for all k, the induced maps $H_k(G_n) \xrightarrow{\cong} H_k(G_{n+1})$ are isomorphisms for n sufficiently large with respect to k.

Knowing that a family of groups exhibits homological stability is an invaluable tool for examining the general behaviour of the entire family. It is most useful when used in conjuction with stable homology.

Definition 10.2. Let $\{G_n\}$ be a sequence of groups with associated inclusions. Then $G_\infty := \bigcup_i^\infty G_i$ is defined to be the limit of these groups, and we say that $H_k(G_\infty)$ is the *stable homology* of $\{G_n\}$.

Thus homological stability tells us that $H_k(G_n) \cong H_k(G_\infty)$ in a range of degrees which increases as n increases. It would be very useful to know homological stability for mapping class groups, as it would simplify homology calculations considerably.

So let us suppose that M is a connected, compact, orientable 3-manifold with boundary. We are interested in the case that M is obtained by taking the connect sum of some manifold N with a number of copies of another manifold P. In this case we will denote M by

$$N_n^P = N\#P\#P\#\cdots\#P$$

where n is the number of times we have connect summed N with P to obtain M.

Suppose further that $\partial N \neq \emptyset$, with a chosen component $\partial_0 N$ of ∂N. Fix a compact subsurface R of ∂N which contains $\partial_0 N$. We work with the mapping class groups of diffeomorphisms of M that fix R, and denote this mapping class group by

$$\Gamma_n^P(N,R) = \Gamma(N_n^P, R) = \pi_0 \operatorname{Diff}(N_n^P \text{ rel } R).$$

We can obtain N_{n+1}^P from N_n^P by taking a copy of P, removing a disc, and identifying the resulting boundary sphere with a disc in $\partial_0 N$. This gives the obvious inclusion map $N_n^P \hookrightarrow N_{n+1}^P$. Now, we can extend diffeomorphisms on N_n^P to diffeomorphisms on N_{n+1}^P by just taking them to be the identity on this new P component, which gives us a map

$$\phi_n : \Gamma_n^P(N,R) \to \Gamma_{n+1}^P(N,R).$$

Since we have a natural inclusion of the spaces $N_n^P \hookrightarrow N_{n+1}^P$, it might be tempting to conclude that the maps ϕ_n must be trivially be inclusions too. Indeed, they do define injections, but this is a subtle and non-trivial thing to prove. See Hatcher-Wahl [HW10] for details.

In summary, we have a collection of groups $\Gamma_n^P(N,R)$, and inclusions $\phi_n : \Gamma_n^P(N,R) \to \Gamma_{n+1}^P(N,R)$. This sequence exhibits homological stability.

Theorem 10.3 (Hatcher-Wahl [HW10])**.** *For any compact, connected, oriented 3-manifolds N and P and compact subsurface R of ∂N as above, the sequence $\{\Gamma_n^P(N,R), \phi_n\}$ satisfies homological stability*

$$H_k(\Gamma_n^P(N,R)) \xrightarrow{\cong} H_k(\Gamma_{n+1}^P(N,R))$$

for $n > 2k+2$.

This is a powerful theorem. Notice that we do not impose any extra conditions on P, meaning that we have stability of mapping class groups under the connect sum of any compact, connected, oriented 3-manifold. This provides us with a wealth of examples.

However, the question of calculating the stable homology of this family of mapping class groups in all degrees is still open.

11. Artin groups and 3-manifold groups

Since diffeomorphisms of 3-manifolds M do not necessarily fix a given base-point, the action of the mapping class group $\mathrm{MCG}(M)$ on the fundamental group $\pi_1(M)$ is only well-defined up to conjugation. This gives a homomorphism

$$\rho_M : \mathrm{MCG}(M) \to \mathrm{Out}(\pi_1(M)).$$

Unlike the case of compact surfaces, the homomorphism ρ_M is not always an isomorphism. However, if M is irreducible, boundary irreducible and Haken then the homomorphism ρ_M is an isomorphism onto the group

$$\mathrm{Out}_\partial(M) = \{\phi \in \mathrm{Out}(\pi_1(M)) \mid \phi \text{ preserves the peripheral structure}\}$$

since Theorem 6.4 can be applied. Hence, a characterization of the outer automorphism group of a 3-manifold group is useful to understand the mapping class group of an irreducible, boundary irreducible and Haken 3-manifold.

Artin groups provide an infinite family of 3-manifold groups. In this section we survey the classification of all the Artin groups isomorphic to the fundamental group of a 3-manifold. In particular, we will apply Theorem 6.4 for some of the 3-manifolds with fundamental group isomorphic to some Artin groups.

Definition 11.1. Let Γ be a finite graph without loops or multiple edges, such that every edge is labeled by an integer greater or equal to 2. Suppose $\mathcal{V}(\Gamma)$ is the set of vertices of Γ and let m_{ij} be the labels on the edges with endpoints a_i and a_j. The *Artin group* of type Γ is the finitely presented group

$$A(\Gamma) = \Big\langle a_1, \dots, a_n \in \mathcal{V}(\Gamma) \mid \underbrace{a_i a_j a_i \dots}_{m_{ij} \text{ times}} = \underbrace{a_j a_i a_j \dots}_{m_{ij} \text{ times}} \text{ if } a_i \text{ and } a_j \text{ are adjacent} \Big\rangle.$$

Suppose Γ is a triangle with each edge labeled with 2. The respective Artin group is isomorphic to $\mathbb{Z}^3$. In particular, the abelian free group $\mathbb{Z}^3$ is isomorphic to the fundamental group of a 3-dimensional torus.

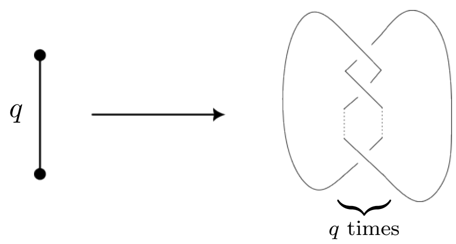


FIGURE 4. The link K_q.

If Γ consists of two vertices connected by an edge labeled by 2, then the Artin group associated is isomorphic to $\mathbb{Z}^2$. The abelian free group of rank 2 is isomorphic to the fundamental group of a 3-manifold too, which is the complement in $\mathbb{S}^3$ of a tubular neighborhood of the Hopf link.

More generally, if Γ consists of two vertices connected by an edge with an integer label $q \in \{2, 3, \dots\}$ then $A(\Gamma)$ is isomorphic to the fundamental group of the complement of a $(2, q)$ torus-link K_q in $\mathbb{S}^3$. The link K_q can be obtained from a braid group element as in Figure 4, starting from two vertical strands twisted q times in the same direction and with respective vertical endpoints reconnected by an arc. Moreover, the link K_q is a knot if and only if q is odd.

Proposition 11.2. *Let Γ_q be the labeled finite graph consisting of one edge labeled by q. The fundamental group of the complement in $\mathbb{S}^3$ of a tubular neighborhood of K_q is isomorphic to $A(\Gamma_q)$.*

Proof. Let $\nu(K_q)$ a tubular neighborhood of K_q. The Wirtinger presentation method provides the fundamental group of $\mathbb{S}^3 \setminus \nu(K_q)$ with the following presentation

$$\langle a, b \mid a^2 = b^q \rangle.$$

The map

$$\langle a, b \mid a^2 = b^q \rangle \to \langle x, y \mid \underbrace{xyx\dots}_{q \text{ times}} = \underbrace{yxy\dots}_{q \text{ times}} \rangle = A(\Gamma_q)$$

$$a \mapsto (xy)^q$$

$$b \mapsto xy$$

is a well-defined homomorphism of groups. □

The following is an easy corollary of the above proposition.

Corollary 11.3. *If q is odd, the mapping class group of $\mathbb{S}^3 \setminus \nu(K_q)$ is cyclic of order 2.*

Proof. Let G be a one-relator group generated by two elements $a, b \in G$. If G is torsion-free with non-trivial center, the outer automorphism group $\mathrm{Out}(G)$ is cyclic of order 2 provided the abelianization G^{ab} contains a copy of $\mathbb{Z}$ [GHMR00, Theorem C]. In particular, the class of automorphisms represented by $\phi : G \to G$ such that $\phi(a) = a^{-1}$ and $\phi(b) = b^{-1}$ generates $\mathrm{Out}(G)$.

The Artin groups $A(\Gamma_q)$ satisfy all the above properties if q is odd: by Lemma 6.3 the group $A(\Gamma_q)$ is a $K(\pi, 1)$ and therefore is torsion-free; the group element $a^2 = b^q$ is in the center since it commutes with both a and b; the abelianization $A(\Gamma_q)^{\mathrm{ab}}$ is isomorphic to $\mathbb{Z}$ through the homomorphism $\deg : A(\Gamma_q) \to \mathbb{Z}$ mapping every generator $a_i \in \mathcal{V}(\Gamma)$ to 1 [Mul02, Proposition 3.1].

We only need to check that ϕ preserves the peripheral structure in order to apply Theorem 6.4. The boundary $\partial(\mathbb{S}^3 \setminus K_q)$ is a torus with fundamental group generated by the elements ab^{-1} and b^q. The image of the subgroup $\langle ab^{-1}, b^q\rangle$ of $A(\Gamma)$ through the automorphism ϕ is $\langle a^{-1}b, b^q\rangle$, conjugated with $\langle ab^{-1}, b^q\rangle$ via the group element $b^{-1}a$. □

Let now Γ be a labeled tree. The Artin group $A(\Gamma)$ is the 3-manifold group of the complement of a link L_Γ in $\mathbb{S}^3$, where L_Γ is a connected sum of tori-links as in Figure 5.

More specifically, whenever two edges with labels q_1 and q_2 share a common vertex, the closure of the tubular neighborhoods of the associated tori-links K_{q_1} and K_{q_2} can be glued together on a disk. The link L_Γ can then be obtained by gluing tori-links for every pair of adjacent edges in Γ.

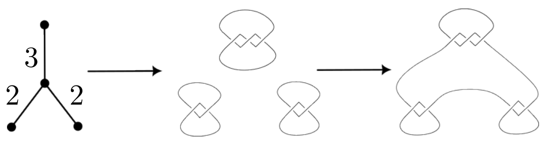


FIGURE 5. An example about how to construct L_Γ from a tree Γ.

Theorem 11.4 (Brunner [Bru92]). *Let Γ be a labeled tree. The Artin group $A(\Gamma)$ is isomorphic to the fundamental group $\pi_1(\mathbb{S}^3 \setminus \nu(L_\Gamma))$, where $\nu(L_\Gamma)$ is a tubular neighborhood of the link L_Γ.*

However, not all Artin groups are 3-manifold groups. The following theorem provides us with an obstruction.

Theorem 11.5 (Scott [Sco73]). *Every finitely generated subgroup of a 3-manifold group is finitely presented*

Sketch of the proof. Suppose the fundamental group $\pi_1(M)$ of a 3-manifold is finitely generated. Then, there exists N a compact submanifold of M such that the inclusion map $\iota : N \hookrightarrow M$ induces an isomorphism of groups $\pi_1(N) \cong \pi_1(M)$. However, every compact manifold has the homotopy type of a finite CW-complex and therefore the fundamental group $\pi_1(N)$ is finitely presented. Hence, the fundamental group $\pi_1(M)$ is also finitely presented.

More generally, if $H \leq \pi_1(M)$ is a finitely generated subgroup, then there exists a covering map $M_H \to M$ such that $\pi_1(M_H)$ is isomorphic to H. Since M_H is a 3-manifold then H is a finitely generated 3-manifold group and in particular finitely presented. □

Suppose Γ is a square graph with 4 vertices and 4 edges labeled with 2. The Artin group $A(\Gamma)$ associated is not *coherent* or, in other words, there exists a finitely generated subgroup of $A(\Gamma)$ that is not finitely presented.

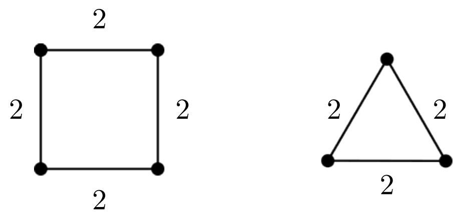


FIGURE 6. The simplicial complex Δ_Γ associated with the graph on the left of the picture is not simply connected. However, the simplicial complex Δ_Γ associated with the triangle on the right-hand-side is contractible.

Theorem 11.6 (Bestvina-Brady [BB97]). *Let Γ be a finite graph with each label equal to 2 and let Δ_Γ be the maximal simplicial complex with the graph Γ as 1-skeleton. Then, the kernel of the map $\deg : A(\Gamma) \to \mathbb{Z}$ is finitely generated if and only if Γ is connected and finitely presented if and only if Δ_Γ is simply connected.*

In particular, the simplicial complex Δ_Γ associated with the square graph Γ with all labels equal to 2 coincides with Γ, that is connected but not simply connected.

Given Theorem 11.5 as the main obstruction for an Artin group to be a 3-manifold group, Hermiller-Miller [HM99] first and Gordon then [Gor04] gave their contributions by proving the following result.

Theorem 11.7. *An Artin group $A(\Gamma)$ is the fundamental group of a connected 3-manifold if and only if Γ is either a tree or a triangle with each edge labeled 2.*

Sketch of the proof. Suppose Γ contains a circuit of length greater or equal to 4. Hermiller-Miller proved that the kernel of the degree homomorphism $\deg : A(\Gamma) \to \mathbb{Z}$ is finitely generated but not finitely presented.

Every Artin group $A(\Gamma)$ comes with a reflection group $W(\Gamma)$ by adding the relations $a_i^2 = 1$ for every $a_i \in \mathcal{V}(\Gamma)$. Moreover, the inclusion of labeled graphs induces injective homomorphisms of Artin groups [Van94]. Gordon proved that if $A(\Gamma)$ is coherent and Γ contains a triangle Ω, then the Artin group $A(\Omega)$ associated with the triangle Ω has a finite reflection group $W(\Omega)$. However, such Artin groups have been classified [SB72] and the Theorem follows after checking that all the possibilities result in a not coherent Artin group unless Γ is a tree or a triangle with each edge labeled 2. □

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Email address: `p.bader.1@research.glasgow.ac.uk`

Email address: `rachael.boyd@glasgow.ac.uk`

Email address: `g.carfora.1@research.gla.ac.uk`

Email address: `d.galvin.1@research.gla.ac.uk`

Email address: `r.giannini.1@research.gla.ac.uk`

Email address: `csaba.nagy@glasgow.ac.uk`

Email address: `john.nicholson@glasgow.ac.uk`

Email address: `w.niu.1@research.gla.ac.uk`

Email address: `isacco.nonino@glasgow.ac.uk`

Email address: `m.pencovitch.1@research.gla.ac.uk`

Email address: `mark.powell@glasgow.ac.uk`

School of Mathematics and Statistics, University of Glasgow, United Kingdom