MFO LOW DIMENSIONAL TOPOLOGY || DEVELOPMENTS IN 4-MANIFOLDS

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This is the extended abstract of my talk in Oberwolfach on Monday 17th February 2020. I was asked to give an overview of the current state of the art in 4-manifold topology, and to describe some recent advances.

We will consider some of the major open problems on 4-manifolds, discuss what classification results are known, and then talk about recent interest in studying symmetries of 4-manifolds.

Fundamental problems. I want to start by drawing the following stark contrast in our knowledge. The following five statements are true and known in the topological category with locally flat embeddings for every n, but are unknown and open in the smooth category for $n = 4$.

(1) The Poincaré conjecture that $M^n \simeq S^n$ implies $M \cong S^n$.
(2) The Schoenflies problem, that for every embedding $f\colon S^{n-1} \hookrightarrow S^n$, $f(S^{n-1})$ is unknotted.
(3) The unknot problem, that for every embedding $K\colon S^{n-2} \hookrightarrow S^n$ with $S^n \setminus K(S^{n-2}) \simeq S^1$ is unknotted.
(4) Let $F\colon D^n \xrightarrow{\cong} D^n$ be an equivalence with $F|_{\partial D^n} = \mathrm{Id}$. Then F is isotopic to Id.
(5) If $M^n \simeq T^n$ then $M \cong T^n$.

Let me briefly mention the status of the statements for other n in the smooth and PL categories. They are also relatively well understood. The statements all hold for $n \leq 3$ in the smooth and PL categories. For smooth manifolds, statements (1), (4), and (5) are generally false in dimensions $n \geq 5$, for example due to the existence of exotic spheres. Statements (1) and (4) do hold in the PL category for $n \geq 5$, but (5) does not. Statements (2) and (3) also hold in the smooth and PL categories for $n \geq 5$.

What do we know about 4-manifolds? Thanks to the work of Freedman and Quinn, surgery theory allows one to classify 4-manifolds for certain fundamental groups. I will briefly describe the known classifications. These exhibit cases where the topology of 4-manifolds corresponds closely to the algebra of intersection forms. In each case, by a classification I mean that there is a collection of algebraic-topological invariants of a 4-manifold in the relevant class, and these invariants coincide if and only if the associated 4-manifolds are homeomorphic. The intersection form on the middle homology and the Kirby-Siebenmann invariant always appear.

(a) Freedman and Quinn classified closed, simply connected 4-manifolds [FQ90, Chapter 10].
(b) Freedman and Quinn classified closed, orientable 4-manifolds with fundamental group $\mathbb{Z}$ [FQ90, Chapter 10].
(c) Wang classified closed, nonorientable 4-manifolds with fundamental group $\mathbb{Z}$ [Wan95].

(d) Hambleton and Kreck classified closed 4-manifolds with finite cyclic fundamental groups [HK88].
(e) Hambleton, Kreck and Teichner classified closed nonorientable 4-manifolds with fundamental group $\mathbb{Z}/2$ [HKT94].
(f) Hambleton, Kreck and Teichner classified closed orientable 4-manifolds with fundamental group $\mathbb{Z} \ltimes \mathbb{Z}[1/2]$ [HKT09].
(g) Freedman and Quinn classified closed aspherical 4-manifolds with good fundamental groups for which the high dimensional Borel conjecture is known [FQ90].
(h) Brookman, Davis, and Kahn classified 4-manifolds homotopy equivalent to the connected sum of two copies of projective space $\mathbb{RP}^4\#\mathbb{RP}^4$ [BDK07].
(i) Boyer classified simply connected compact 4-manifolds with a fixed 3-manifold as the boundary [Boy86].
(j) There are no known classifications for 4-manifolds with boundary that have nontrivial fundamental group. The only groups that admit a surjection to $\mathbb{Z}/2$ for which nonorientable 4-manifolds are classified are $\mathbb{Z}$ and $\mathbb{Z}/2$.

In joint work with A. Conway and D. Crowley, we are working on a classification of compact 4-manifolds with fundamental group $\mathbb{Z}$ and nonempty boundary. There is a corresponding and equally interesting discussion of results on embedded surfaces on 4-manifolds, but I omitted it for space reasons.

All of these classifications rely on Freedman's disc embedding theorem. In work with Ray and Teichner, I have filled a gap in the proof from the book of Freedman and Quinn relating to the existence of transverse spheres for the embedded discs that come out of the disc embedding theorem. These transverse spheres are vital in applications to surgery. With Behrens, Kalmár, Kim and Ray, I am also an editor of a new book, written in a collaborative project with 20 authors, that gives a new and complete proof of the disc embedding theorem.

One of the big themes of 4-manifold topology is the ubiquity of exoticness, for example 4-manifolds that are homeomorphic but not diffeomorphic, diffeomorphisms that are topologically but not smoothly isotopic, and locally flat embeddings that are not smoothable. There is a wealth of fascinating complication in smooth 4-manifolds, which makes any kind of classification scheme for smooth structures hard to approach.

Stable classification. One way to obtain classification results for 4-manifolds in terms of algebraic topology is to consider the stable classification.

Two closed 4-manifolds M and N are said to be *stably diffeomorphic* if there are natural numbers m and n such that $M\#^m S^2 \times S^2$ and $N\#^n S^2 \times S^2$ are diffeomorphic.

Kreck reduced this question to bordism: for example two spin 4-manifolds with fundamental group π are stably diffeomorphic if and only if they represent, for some choices of spin structure and maps to π, bordant elements of $\Omega_4^{Spin}(B\pi)$. This enables one to reduce the question to algebraic topological invariants again. Hambleton-Kreck-Teichner [HKT09] studied the case of 2-dimensional groups. With Kasprowski, Land, and Teichner [KLPT17], I studied the case of 4-manifolds with 3-dimensional groups. In a forthcoming paper with Kasprowski and Teichner, we study spin 4-manifolds with abelian fundamental groups.

Another paper with Kasprowski and Teichner [KPT18] studied the analogous question where one stabilises with copies of $\mathbb{CP}^2$ instead of $S^2 \times S^2$.

Diffeomorphisms of 4-manifolds. A lot of recent activity has been on symmetries of 4-manifolds. Despite our lack of knowledge in the smooth category, with regards to the five questions I started with, one can still obtain information on the symmetries and families of such symmetries. The following is a useful guiding question.

Question. For a fixed 4-manifold M, what are the homotopy types of $\mathrm{Homeo}_\partial(M)$ and $\mathrm{Diff}_\partial(M)$?

These are the spaces, in fact topological groups, of homeomorphisms and diffeomorphisms of M that restrict to the identity on the boundary, equipped with the compact-open and Whitney topologies respectively.

There are two reasons to study the homotopy type. First, it can be easier than trying to study these purely as groups. Second, the homotopy type contains information on the classification of fibre bundles with fibre M.

In dimension 2, every connected component of the space of diffeomorphisms of a compact surface is contractible, so one is left with studying the mapping class group. In dimension 3, there is a strong understanding of diffeomorphism spaces, thanks in particular to Thurston, Hatcher, and Perelman. In dimension 4, our knowledge is somewhat limited, but there has been some exciting progress recently.

Theorem (Watanabe, 2018)**.** *$\pi_k(\mathrm{Diff}_\partial(D^4)) \neq 0$ for $k = 1, 4, 8$.*

Watanabe's theorem [Wat18] implies in particular that the 4-dimensional generalised Smale conjecture does not hold, that is $\mathrm{Diff}(S^4)$ is not homotopy equivalent to $O(5)$. Note that it was already known, and is not too hard to see, that $\mathrm{Diff}(S^n) \simeq \mathrm{Diff}_\partial(D^n) \times O(n+1)$.

The idea of Watanbe's proof is to construct bundles $D^4 \to E \to S^{k+1}$, using a 4-dimensional version of the Goussarov-Habiro clasper surgery. He then evaluates Kontsevich configuration space integrals on these bundles, using parametrised Morse theory. In 3-dimensions that analogous integrals give invariants of diffeomorphism classes of 3-manifolds, whereas in dimension 4 these are invariants of *families*.

Here are some more selected results on diffeomorphism and homeomorphism spaces of 4-manifolds. I find it striking that there are interesting results of a similar flavour coming from such different techniques. We already mentioned that Watanabe used configuration space integral characteristic classes.

Let Diff_0 denote the subset of diffeomorphisms that are proper homotopy equivalent to the identity.

Theorem (Gabai, 2017)**.** *$\pi_0(\mathrm{Diff}_0(S^2 \times D^2)/\,\mathrm{Diff}_0(D^4)) = 0$.*

Gabai's proof [Gab17] uses intricate geometric constructions to show that homotopy implies isotopy for certain embedded 2-spheres in 4-manifolds.

Theorem (Budney-Gabai, 2019)**.** *$\pi_0(\mathrm{Diff}_\partial(S^1 \times D^3)/\,\mathrm{Diff}_\partial(D^4)) \neq 0$.*

Budney-Gabai [BG19] use the Goodwillie-Klein-Weiss embedding calculus.

Theorem (Baraglia-Konno, 2019)**.** *$\pi_1(\mathrm{Homeo}(K3)/\!/\,\mathrm{Diff}(K3)) \neq 0$.*

Here $\mathrm{Homeo}(K3)/\!/\,\mathrm{Diff}(K3)$ denotes the homotopy quotient, which is homotopy equivalent to the homotopy fibre of the forgetful map $\mathrm{BDiff}(K3) \to \mathrm{BDiff}(K3)$. Baraglia and Konno [BK19] use family Seiberg-Witten theory.

Theorem (Galatius-Randal-Williams, 2014). *The limit homology*

$$\mathrm{colim}_{n\to\infty} H_k(\mathrm{BDiff}_{\frac{1}{2}\partial}(\natural^n S^2 \times S^2 \setminus \mathring{D}^4); \mathbb{Q})$$

is generated by a collection of well-understood characteristic classes, the generalised Miller-Morita-Mumford κ*-classes.*

In particular, this homology is computed. Galatius and Randal-Williams [GRW14] apply a notion of parametrised surgery to the Galatius-Madsen-Tillman-Weiss theorem on the homotopy type of the cobordism category. It is unknown whether, for a fixed k, there is an n such that the homology equals the limit homology.

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