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Brunnian Exotic Surface Links in the 4-Ball

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Abstract. This paper investigates exotic phenomena exhibited by links of disconnected surfaces with boundary properly embedded in the 4-ball. Our main results provide two different constructions of exotic pairs of surface links, which are Brunnian, meaning that all their proper sublinks are trivial. Furthermore, we modify these core constructions to vary the number of components in the exotic links, the genera of the components, and the number of components that must be removed before the surfaces become unlinked.

1. Introduction

We construct exotic pairs of surface links in the 4-ball where every sublink is smoothly trivial. A *surface link* is a smooth, oriented, properly embedded, 2-dimensional submanifold $\Sigma \subseteq B^4$ whose connected components Σ_i each have exactly one boundary component that lies in S^3. When Σ is connected, it is called a *surface knot*. If each component Σ_i is a disk, we say that Σ is a *disk link* (or *disk knot,* in the connected case). We will say that two surface links Σ and Σ' with $\partial\Sigma = \partial\Sigma'$ form an *exotic pair* if they are topologically ambiently isotopic rel. boundary, but there is no ambient diffeomorphism of W carrying Σ to Σ'.

A fundamental open question in the study of surfaces in 4-manifolds asks whether there exists an orientable, exotic pair of surfaces in S^4. Exotic pairs of orientable surfaces have been constructed in many other 4-manifolds X with $b_2(X) > 0$; see, for example, the foundational paper by Fintushel and Stern [FS97]. In a related vein, S^4 is known to contain exotic nonorientable surfaces [FKV88]. However, these results rely on the topology of the ambient 4-manifold or nonorientability of the surfaces in an essential way, such as to support the use of gauge-theoretic invariants.

Juhász, Miller, and Zemke [JMZ21] and Hayden [Hay20] produced exotic, orientable, connected surface knots. This paper studies surface links and exhibits increasingly subtle forms of exotic behavior. Specifically, we uncover exotic behavior among a particularly delicate class of surface links that are analogous to Brunnian links in S^3 [Bru92].

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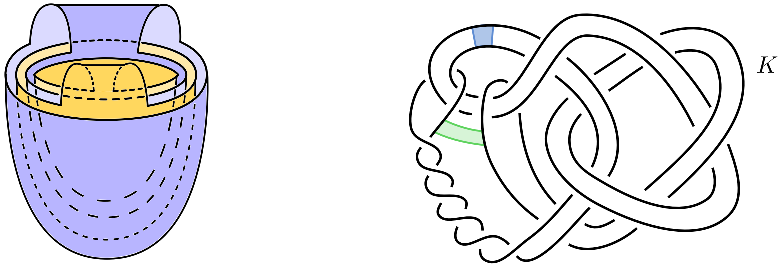

(a) (b)

Figure 1 (a) The Bing double of a disk consists of two disjoint disks. (b) The knot K, together with two bands, leading to the different slice disks D_1 and D_2, whose Bing doubles appear in Theorem 1.2.

DEFINITION.

(a) A surface link $\Sigma \subseteq B^4$ is *unlinked* if $\partial\Sigma$ is an unlink in S^3 and Σ is smoothly isotopic to a Seifert surface for the unlink in S^3.

(b) A surface link $\Sigma \subseteq B^4$ is *Brunnian* if removing any component from Σ yields an unlinked surface link.

Our notion of Brunnian differs from the usual convention in 3-dimensions, where a Brunnian link is required to be nonsplit. According to our definition, an unlink is Brunnian.

THEOREM 1.1. *Let $n \geq 2$ be an integer.*

(i) *There exists an exotic pair of n-component Brunnian disk links in B^4.*

(ii) *There exists an infinite family of pairwise exotic n-component Brunnian surface links in B^4 that each consists of a single genus-one surface and $n-1$ disks.*

We prove Theorem 1.1 by induction on n, with $n = 2$ the base case. Our constructions utilize the *Bing doubling* operation on a disk knot, which we define precisely in Section 2. For now, it suffices to know that the Bing doubling operation replaces one disk component of a surface link with two disjoint disks in a neighborhood of the original, as illustrated in Figure 1(a).

THEOREM 1.2. *The knot K in Figure 1(b) bounds two different slice disks D_1 and D_2 in B^4 (pictured in Figure 6). Their Bing doubles* $\mathrm{BD}(D_1)$ *and* $\mathrm{BD}(D_2)$ *form an exotic pair of 2-component Brunnian disk links in B^4.*

We briefly outline the strategy that distinguishes the diffeomorphism classes of the disk links in Theorem 1.2. The disk links $\mathrm{BD}(D_1)$ and $\mathrm{BD}(D_2)$ have the same boundary in S^3, a link L. We identify a knot γ in the exterior of L such that any diffeomorphism of pairs sending $(B^4, \mathrm{BD}(D_1))$ to $(B^4, \mathrm{BD}(D_2))$ must preserve the isotopy class of γ. By showing that γ is slice in the complement $B^4 \setminus \mathrm{BD}(D_2)$

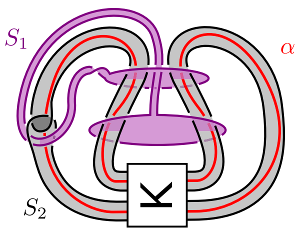

Figure 2 The surface link $\Sigma = S_1 \sqcup S_2$. At the bottom of the figure, two bands in S_2 are knotted through K. The diagram of K should be taken to have writhe zero.

but not in $B^4 \smallsetminus \mathrm{BD}(D_1)$, we deduce that the disk link exteriors are not diffeomorphic.

Next, we briefly sketch our proof of the $n = 2$ case of Theorem 1.1(ii). Our construction of an infinite family of pairwise exotic 2-component Brunnian surface links relies on *rim surgery,* a technique first introduced in [FS97]. It takes as input an essential curve α on a surface Σ and a knot J and outputs a new surface $\Sigma(\alpha; J)$ (see Section 4.3).

THEOREM 1.3. *Let K be a strongly quasipositive topologically slice knot, and let Σ denote the surface link in Figure 2. For each integer $m \geq 0$, let $J_m \subseteq S^3$ be a knot whose mod 2 Alexander polynomial $\Delta_{J_m}(t) \in \mathbb{F}_2[t^{\pm 1}]$ has m irreducible factors (counted with multiplicity). The 2-component Brunnian surface links $\{\Sigma(\alpha; J_m)\}_{m=0}^{\infty}$, each consisting of a disk and a genus one surface, are pairwise exotic.*

The surface links of Theorem 1.3 are constructed explicitly in Section 4. A key ingredient in the construction is a choice of curve $\alpha \subseteq \Sigma$ such that:

(i) α bounds a locally flat disk with interior in the complement of Σ,
(ii) α bounds a smooth disk with interior in the complement of each component Σ_i, but
(iii) the interior of every smooth disk with boundary α intersects Σ.

This ensures that rim surgery along α preserves both the topological isotopy type of the link and the smooth isotopy type of each component, while potentially changing the smooth isotopy type of the link itself. We note that for the last item to hold, α must be essential, so the construction cannot work for disk links.

To obstruct smooth equivalence of each pair, we use the link Floer cobordism maps induced by each surface link, following the methods of Juhász, Miller, and Zemke [JMZ21], and use link Floer cobordism maps. First, they showed that the cobordism map induced by a quasipositive surface is nontrivial, and then they applied a theorem of Juhász and Zemke [JZ23] to show that rim surgery can potentially change the smooth isotopy type of a quasipositive surface. Although the surface Σ in Theorem 1.3 is not quasipositive, we may compose it with another link cobordism and obtain a quasipositive surface. The composition laws for cobordism maps guarantee that the map associated with Σ is nonvanishing.

To prove Theorem 1.1 and transition from the $n = 2$ cases in Theorems 1.2 and 1.3 to $n \geq 3$, we use a technique inspired by work of Cha and Kim [CK08] on covering links in S^3; this is done in Section 6. We show that given an exotic pair Σ and Σ' of Brunnian 2-component surface links constructed as in Theorem 1.2 or Theorem 1.3, iteratively Bing doubling one disk component of each of Σ and Σ' yields an exotic pair of n-component surface links. We prove this inductively by investigating the 2-fold covers of B^4 branched along a trivial disk component in each of these iterated Bing doubles.

1.1. Increasing the Genera

We modify the pairs of surface links from Theorem 1.3 to increase the genera of the surface components in the 2-component surface link case.

THEOREM 1.4. *Fix a pair of integers $r \geq 0$ and $s \geq 1$. There exists an infinite family of Brunnian 2-component surface links in B^4 consisting of a genus r surface and a genus s surface, any two of which form an exotic pair.*

In particular, let $\Sigma_0^{r,s}$ be the surface link constructed in Figure 21. For each $n \geq 0$, let $J_n \subseteq S^3$ be a knot whose mod 2 Alexander polynomial has n irreducible factors (counted with multiplicity). Then the 2-component Brunnian surface links $\{\Sigma_0^{r,s}(\alpha; J_n)\}$, each consisting of a genus r surface and a genus s surface, are pairwise exotic.

The primary constructive technique requires banding the surface from Theorem 1.3 with another simple Brunnian surface link of controlled genus. An analogous application of the link Floer argument from Theorem 1.3 obstructs a diffeomorphism of pairs.

1.2. (n, k)Brunnian Disk Links

We study a natural generalization of Brunnian links.

DEFINITION 1.5. An n-component surface link is (n, k)*-Brunnian* if every sublink of fewer than k components is an unlink, but every sublink of at least k components is nontrivial.

The notion of (n, k)-Brunnian was introduced by Debrunner [Deb61] for links in S^3. Debrunner's original definition requires all sublinks of at least k components to be nonsplit; in our adaptation to dimension four, we ask only for these sublinks to be nontrivial. By combining his construction of (n, k)-Brunnian links with our construction of exotic disk links, we prove the following.

THEOREM 1.6. *For any integers n and k with $n \geq 2$ and $1 \leq k \leq n$, there exists a pair of (n, k)-Brunnian disk links in B^4 forming an exotic pair.*

Heuristically, the index k measures the extent to which the link components are entangled. At one extreme are (n, n)-Brunnian links: these are the nontrivial n-component Brunnian links. The $(n, 1)$-Brunnian links lie at the other extreme: all of their nonempty sublinks are nontrivial. Thus Theorem 1.6 encapsulates a wide spectrum of exotic behavior.

The extreme cases of Theorem 1.6 follow immediately from the above: for $k = n$, take the disk links from Theorem 1.1; for $k = 1$, consider a split union of n copies of D_1 and the same of D_2, then apply [Hay20]. The intermediate cases require a new construction.

1.3. Exotic Closed Surfaces

Finally, in Section 8, we show that exotic surface links in B^4 can be used to construct exotic links of closed surfaces in larger 4-manifolds. This is illustrated using the disk links $\mathrm{BD}(D_1)$ and $\mathrm{BD}(D_2)$ from Theorem 1.2. We first consider the 4-manifold X obtained from B^4 by attaching 0-framed 2-handles along the link in S^3 bounding the iterated Bing doubles $\mathrm{BD}^k(D_1)$ and $\mathrm{BD}^k(D_2)$. Capping off these disk links with the cores of the 2-handles yields a pair of exotic sphere links in X. When $k = 1$, we then show that X further embeds into a closed 4-manifold in which these sphere links remain exotic. In each case, we distinguish the exotic sphere links by considering the effect of surgering the ambient 4-manifold along these spheres.

1.4. Open Problems

Here are some natural further questions that remain open.

(1) Construct exotic pairs/families of n-component Brunnian surface links where the components have arbitrary genera.
(2) Construct examples of infinitely many Brunnian disk links that are pairwise exotic.
(3) Does there exist a distinct pair of surface links (or knots) in B^4 with equivalent Bing doubles? This can be interpreted in both categories, or asked purely for exotic pairs.

The results of this article could be viewed as evidence towards a negative answer to (3).

1.5. Organization

In Section 2, we recall the technique of Bing doubling disks. Section 3 constructs the exotic pair of Brunnian 2-component disk links promised in Theorem 1.2. In Section 4, we recall the necessary Heegaard Floer theory and construct the pairwise exotic Brunnian surface links of Theorem 1.3. Section 5 extends this construction to produce an infinite family of 2-component Brunnian links with arbitrary genera. Covering surfaces are introduced in Section 6; this section also contains the proof of Theorem 1.1. Section 7 constructs exotic pairs of (n, k)-Brunnian disk links. Finally, Section 8 produces our examples of exotic closed

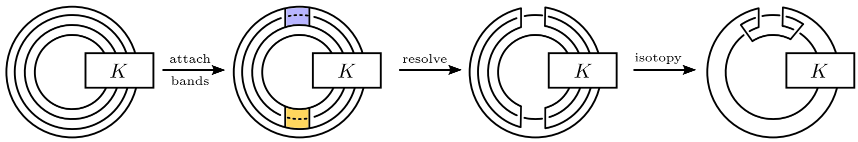

Figure 3 Constructing BD(D), the Bing double of a slice disk D, from four copies of D and two bands.

surfaces. The appendices detail the computer-assisted calculations used to prove Theorems 1.1 and 1.2.

1.6. Conventions

We provide some of our conventions.

- J, K, and L denote knots or links in S^3, whereas Σ, S, and D denote surfaces in B^4, with D always a disk.
- We work in the category DIFF of smooth manifolds with smooth embeddings and smooth isotopies, and the category TOP of topological manifolds with locally flat embeddings and topological ambient isotopies. We specify whether an isotopy or surface is in DIFF or TOP.
- Given a smooth or locally flat proper submanifold $Y \subseteq X$, νY will denote an open tubular neighborhood of Y. In the smooth case, the existence of tubular neighborhoods is standard differential topology. In the locally flat case, existence follows from [FQ90, Theorem 9.3] when X is a 4-manifold. A closed tubular neighborhood is denoted by $\overline{\nu} Y$.

2. Background on Bing Doubling

In this section, we review Bing doubling, the primary tool used throughout to produce surface links with arbitrarily many components.

We obtain the *Bing double* of a slice disk D in B^4 with $\partial D = K$ as follows. First, define a model Bing double of the core disk $D^2 \times \{0\} \subseteq D^2 \times D^2$ from four parallel copies of the disk, joined with two bands, as depicted in Figure 3. The result is two disjoint slice disks in $D^2 \times D^2 \cong B^4$ for the Bing double of $S^1 \times \{0\} \subseteq S^1 \times D^2 \subseteq \partial(D^2 \times D^2) \cong S^3$. Now, for a slice disk D in B^4, choose an orientation-preserving diffeomorphism of a tubular neighborhood νD with $D^2 \times D^2$ such that D maps to $D^2 \times \{0\}$ via an orientation-preserving diffeomorphism. Embed the model Bing double in B^4 using the inverse of this identification to yield the pair of disks BD(D) $\subseteq \nu D$. Their boundary is the Bing double BD(K) $\subseteq \nu K$ of the knot K. Since any two orientation-preserving diffeomorphisms of $D^2 \times D^2$ are isotopic (not necessarily rel. boundary), the isotopy class of BD(D) is independent of the choice of identification $\nu D \cong D^2 \times D^2$.

Going a step further, we define Bing doubling as an operation on ordered surface links. Let $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_k$ be a surface link with $\Sigma_1 \cong D^2$. To perform the Bing doubling operation and obtain a new surface link BD(Σ), replace

Σ_1 with $\mathrm{BD}(\Sigma_1)$ and label the two new disks $\mathrm{BD}(\Sigma)_1$ and $\mathrm{BD}(\Sigma)_2$; then label $\mathrm{BD}(\Sigma)_i := \Sigma_{i-1}$ for $3 \le i \le k+1$. This defines $\mathrm{BD}(\Sigma)$ as an ordered surface link obtained from Σ via Bing doubling. For an ordered surface link Σ, we will also write $\mathrm{BD}(\Sigma)$, where the doubling operation is always performed on the first component.

Viewing BD as an operator on ordered surface links, we may perform an iterated Bing doubling. We write $\mathrm{BD}^k(\Sigma)$ for the surface link obtained after Bing doubling *the first component* k times. (The ordering of the two components of $\mathrm{BD}(\Sigma_1)$, assigned arbitrarily at each stage, does not affect the construction.) This operation is integral to the proof of Theorem 1.1.

We prove some foundational lemmas about Bing doubling surfaces.

LEMMA 2.1. *For* CAT, *either* DIFF *or* TOP, *let* $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_n$ *and* $\Sigma' = \Sigma'_1 \sqcup \cdots \sqcup \Sigma'_n$ *be surface links in* B^4 *with* $\partial\Sigma = \partial\Sigma' \subseteq S^3$. *Suppose that* Σ *and* Σ' *are* CAT *ambiently isotopic rel. boundary as ordered surface links. Assume that* $\Sigma_1 \cong D^2 \cong \Sigma'_1$ *and that* $\partial(\mathrm{BD}(\Sigma_1)) = \partial(\mathrm{BD}(\Sigma'_1))$. *Then the surface links* $\mathrm{BD}(\Sigma) = \mathrm{BD}(\Sigma_1) \sqcup \Sigma_2 \sqcup \cdots \sqcup \Sigma_n$ *and* $\mathrm{BD}(\Sigma') = \mathrm{BD}(\Sigma'_1) \sqcup \Sigma'_2 \sqcup \cdots \sqcup \Sigma'_n$, *obtained by Bing doubling the first component of each surface link, are also* CAT *ambiently isotopic rel. boundary.*

Proof. Let $F_t \colon B^4 \to B^4$, $t \in [0,1]$, be a CAT ambient isotopy with $F_t|_{S^3} = \mathrm{Id}_{S^3}$ for all t, such that $F_0 = \mathrm{Id}_{B^4}$ and $F_1(\Sigma) = \Sigma'$. Let $G \colon D^2 \times D^2 \xrightarrow{\cong} \overline{\nu}\Sigma_1$ and $G' \colon D^2 \times D^2 \xrightarrow{\cong} \overline{\nu}\Sigma'_1$ be two identifications used for the definitions of $\mathrm{BD}(\Sigma_1)$ and $\mathrm{BD}(\Sigma'_1)$ respectively.

Every homeomorphism of D^2 is smoothable, and every orientation preserving rel. boundary diffeomorphism of D^2 is isotopic rel. boundary to the identity by Smale's theorem [Sma59]. Therefore, by the isotopy extension theorem [EK71], we arrange that the isotopy between Σ and Σ' sends Σ_1 to Σ'_1 respecting fixed choices of parametrizations $D^2 \to \Sigma_1$ and $D^2 \to \Sigma'_1$. In other words, extending F_t with a further ambient isotopy supported in a neighborhood of Σ_1, we arrange that the composition

$$D^2 \times \{0\} \xrightarrow{G} \Sigma_1 \xrightarrow{F_1} \Sigma'_1 \xrightarrow{(G')^{-1}} D^2 \times \{0\}$$

is the identity. Similarly, by uniqueness of normal bundles for 2-dimensional submanifolds of 4-manifolds (which holds for CAT = TOP by [FQ90, Chapter 9]), we may assume that F_1 sends a given parametrization of $\overline{\nu}\Sigma_1$ as $D^2 \times D^2$ to a given such parametrization of $\overline{\nu}\Sigma'_1$. The disks $\mathrm{BD}(\Sigma_1) \subseteq \overline{\nu}\Sigma_1$ are therefore sent to the disks $\mathrm{BD}(\Sigma'_1) \subseteq \overline{\nu}\Sigma'_1$ by the new F_1. □

LEMMA 2.2. *Let* $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_n \subseteq B^4$ *be a Brunnian surface link with* $\Sigma_1 \cong D^2$. *Then* $\mathrm{BD}(\Sigma) := \mathrm{BD}(\Sigma_1) \sqcup \Sigma_2 \sqcup \cdots \sqcup \Sigma_n$ *is Brunnian.*

Proof. Removing Σ_i for $i \ge 2$ from $\mathrm{BD}(\Sigma)$ yields a surface link smoothly isotopic to an unlinked surface with one component Bing doubled by Lemma 2.1

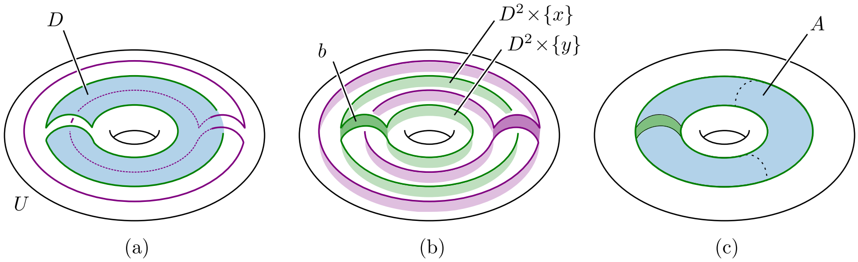

Figure 4 Throughout, we see $\partial\Sigma_1$ (resp. $\partial\Sigma_2$) as the green (resp. purple) curves inside $U \approx (\partial D^2) \times D^2 \subset S^3$. Left: The disk $D \subset U$ with $\partial D \approx \partial\Sigma_1$. Middle: Both Σ_1 and Σ_2 are formed from two disk fibers of $D^2 \times (\partial D^2)$ and a band inside U. The band b joins $D^2 \times \{x\}$ and $D^2 \times \{y\}$ to form Σ_1'. Right: The band b and the disk D together form an annulus A inside the solid torus U.

with CAT = DIFF. This is also unlinked. On the other hand, removing one component of $\mathrm{BD}(\Sigma_1)$ renders the second one an unknotted disk, split from $\Sigma_2 \sqcup \cdots \sqcup \Sigma_n$. Since the latter surface link is also unlinked by the Brunnian hypothesis, $\mathrm{BD}(\Sigma)$ is indeed Brunnian, as desired. $\square$

LEMMA 2.3. *Let $\Sigma = \Sigma_1 \cup \Sigma_2$ denote the Bing double of the core disk $D^2 \times \{0\}$ in $D^2 \times D^2$, and let Σ_1' be a properly embedded disk in $D^2 \times D^2$ obtained as the push-in of a standard Seifert disk $D \subset S^3$ for $\partial\Sigma_1$, depicted in Figure 4(a). The surface links $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\Sigma' = \Sigma_1' \cup \Sigma_2$ are smoothly isotopic rel. boundary in $D^2 \times D^2$.*

Proof. We will show that the 2-sphere $\Sigma_1 \cup D$ bounds a 3-ball Δ in $D^2 \times D^2$ whose interior is disjoint from Σ_2. This implies that, in the complement of Σ_2, the disk Σ_1 is isotopic (rel. boundary) to the push-in of the disk D, as desired.

Let U denote the solid torus $(\partial D^2) \times D^2 \subset S^3$ containing $\partial\Sigma$. Note that each circle fiber $(\partial D^2) \times \{pt\}$ bounds a disk $D^2 \times \{pt\}$ inside $D^2 \times D^2$, and that distinct circle fibers bound disjoint disks. Each of Σ_1 and Σ_2 is formed from a pair of such disks by joining them via a band inside the solid torus U. In particular, suppose that Σ_1 is formed from two disk fibers $D^2 \times \{x\}$ and $D^2 \times \{y\}$ via a band b, as in Figure 4(b). Also note that D and Σ_2 meet along a single ribbon intersection.

Together the band $b \subset \Sigma_1$ and the Seifert disk D form an annulus A cobounded by the circle fibers $\partial D^2 \times \{x\}$ and $\partial D^2 \times \{y\}$. We see the annulus A in Figure 4(c). Moreover, we observe that all of A decomposes as a union of circle fibers. Taking the union of the disjoint disks in $D^2 \times D^2$ bounded by these circle fibers yields a 3-ball Δ with boundary

$$\partial\Delta = (D^2 \times \{x\}) \cup A \cup (D^2 \times \{y\}),$$

as depicted in Figure 5. Finally, by rewriting A as $D \cup b$ and noting that

$$\Sigma_1 = (D^2 \times \{x\}) \cup b \cup (D^2 \times \{y\}),$$

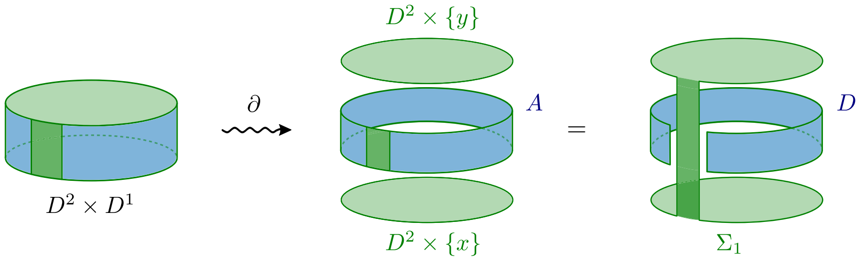

Figure 5 Left: The 3-ball Δ is realized as $D^2 \times D^1$. Middle: The boundary of the 3-ball, $\partial\Delta$, is decomposed as $(D^2 \times \{x\}) \cup A \cup (D^2 \times \{y\})$. Right: By realizing A as $b \cup D$ and the band b as part of Σ_1, we deduce that $\partial\Delta$ is exactly $D \cup \Sigma_1$.

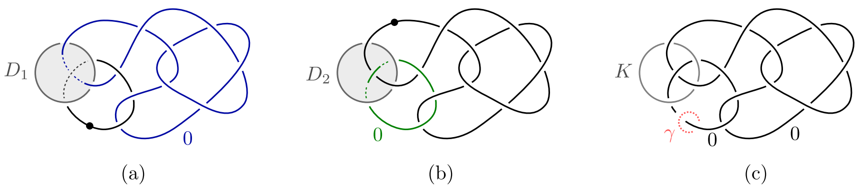

Figure 6 Parts (a) and (b) depict disks D_1 and D_2 in nonstandard handle diagrams of B^4. Part (c) depicts the slice knot $K = \partial D_1 = \partial D_2$ and a distinguished curve $\gamma \subseteq S^3 \setminus K$, drawn in a nonstandard surgery description of S^3.

we conclude that $\partial\Delta = \Sigma_1 \cup D$, as desired; see the right side of Figure 5. Moreover, we see that Δ meets Σ_2 only along the ribbon intersection between Σ_2 and D. □

3. An Exotic Pair of Disk Links

We begin with the pair of disks D_1 and D_2 in B^4 depicted in Figure 6; these are obtained using the construction from [Hay20, Section 2] (as applied to the link $L14n_{40949}$), but are distinct from the examples used in that paper. The figure shows two handle diagrams for B^4. To see this, note that by erasing the gray disk from either picture, the remaining dotted and 0-labeled components form a Hopf link, corresponding to a cancelling 1- and 2- handle pair. These disks are disjoint from the 1-handle curves, and all intersections between the disks and the 2-handles' attaching regions occur in the disks' interiors. Therefore, after pushing the disks' interiors into the 0-handle, the disks are indeed embedded in B^4.

By construction, these disks are bounded by the same knot K in S^3, which we redraw in the standard diagram for S^3 in Figure 1(b). We show how to achieve this in Figure 7.

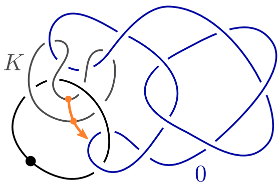

Figure 7 Redrawing K in the standard diagram of S^3.

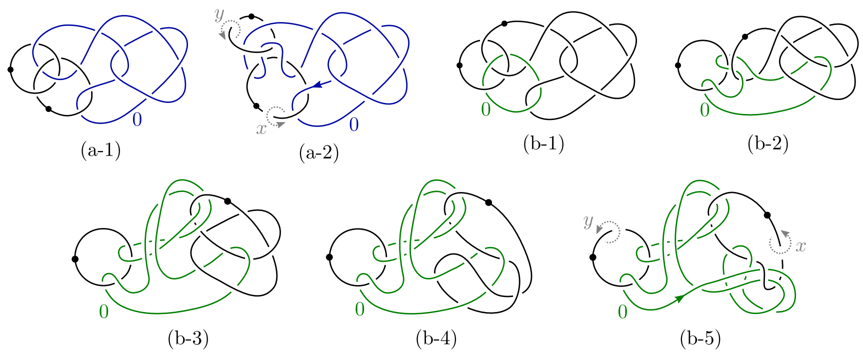

Figure 8 Redrawing the disk exteriors to compute their fundamental groups.

PROPOSITION 3.1. *The disks D_1 and D_2 are topologically ambiently isotopic rel. boundary.*

Proof. The disks are smoothly embedded and are bounded by the same knot in S^3, so it suffices to show that these disks' exteriors have $\pi_1 \cong \mathbb{Z}$ [CP21, Theorem 1.2]. Handle diagrams for these disk exteriors are drawn in parts (a-1) and (b-1) of Figure 8; the remaining parts of Figure 8 manipulate and decorate these diagrams to simplify the π_1 calculation. By tracing the 2-handle curves from parts (a-2) and (b-5) starting at the labeled arrows, we obtain the following presentations:

$$\pi_1(B^4 \setminus D_1) = \langle x, y \mid x^{-1}xy^{-1}x^{-1}y = 1\rangle = \langle x, y \mid y^{-1}x^{-1}y = 1\rangle \cong \mathbb{Z},$$
$$\pi_1(B^4 \setminus D_2) = \langle x, y \mid xx^{-1}x^{-1}xyx^{-1}y^{-1} = 1\rangle = \langle x, y \mid yx^{-1}y^{-1} = 1\rangle \cong \mathbb{Z}.$$

It follows that the disks are topologically ambiently isotopic rel. boundary. □

Now take the Bing doubles of the disks D_1 and D_2. Note that each of $\mathrm{BD}(D_1)$ and $\mathrm{BD}(D_2)$ are bounded by the 2-component link $\mathrm{BD}(K)$.

COROLLARY 3.2. *The disk links* $\mathrm{BD}(D_1)$ *and* $\mathrm{BD}(D_2)$ *are topologically ambiently isotopic rel. boundary.*

Proof. This follows by combining Proposition 3.1 and Lemma 2.1 with CAT = TOP. □

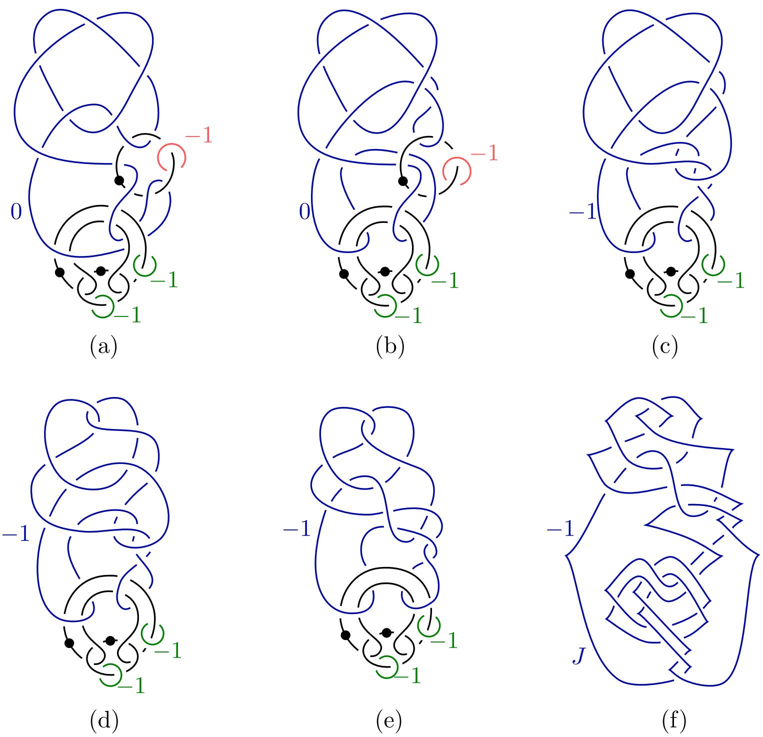

Figure 9 Attaching three 2-handles to the exterior of $\mathrm{BD}(D_1)$ and simplifying. All steps except (b) to (c) and (e) to (f) are isotopies. From (b) to (c), we cancel the red 2-handle with a 1-handle, introducing a full twist into all blue strands passing through the 1-handle. Diagrams (e) and (f) are related as in Figure 10.

By Lemma 2.2, $\mathrm{BD}(D_1)$ and $\mathrm{BD}(D_2)$ are Brunnian. To prove Theorem 1.2, it therefore remains to establish the following proposition.

PROPOSITION 3.3. *The disk links* $\mathrm{BD}(D_1)$ *and* $\mathrm{BD}(D_2)$ *are not smoothly equivalent*.

The proof of the proposition uses the following technical lemma.

LEMMA 3.4. *The knot complement* $S^3 \smallsetminus K$ *has a hyperbolic structure with trivial isometry group*.

Proof. This is verified using SnapPy [C+] and Sage [19]; see Appendix A for additional documentation regarding this calculation. □

Proof of Proposition 3.3. To distinguish $\mathrm{BD}(D_1)$ and $\mathrm{BD}(D_2)$, we will examine the simple closed curve $\gamma \subseteq S^3 \smallsetminus K$ shown in Figure 6(c). This curve is redrawn in the exteriors of $\mathrm{BD}(D_1)$ and $\mathrm{BD}(D_2)$ in Figure 11. The curve γ bounds a smooth

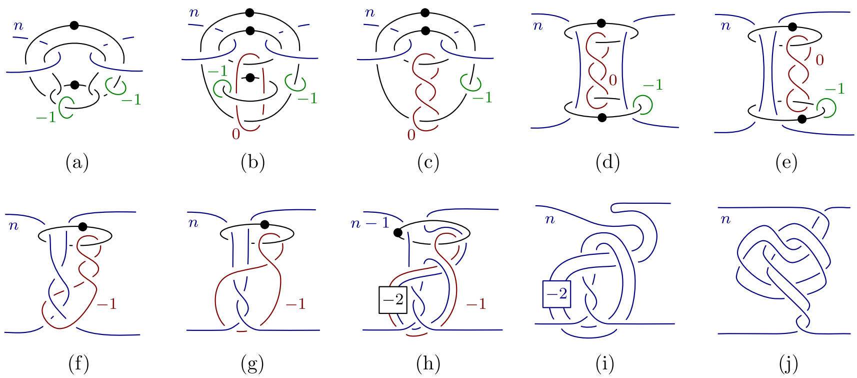

Figure 10 The process of simplifying (e) to (f) in Figure 9. From (a) to (b), we add a cancelling handle pair. From (b) to (c), we remove a cancelling pair. From (c) to (d) to (e) is simply isotopy of the diagram. From (e) to (f), we remove a cancelling pair and then isotope to obtain (g). From (g) to (h), we do one handle slice, and then from (h) to (i) we remove a cancelling pair and further isotope to obtain (j).

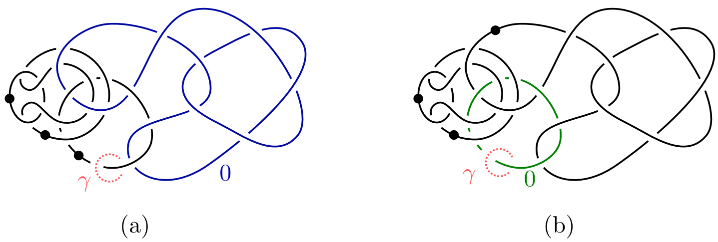

Figure 11 Handle diagrams for the exteriors of the disk links $\mathrm{BD}(D_1)$ and $\mathrm{BD}(D_2)$ in parts (a) and (b), respectively.

disk in the exterior of $\mathrm{BD}(D_2)$ since it bounds an obvious disk in the diagram from Figure 11(b) that only intersects the 2-handle attaching circle (and not the 1-handle curve).

On the other hand, we claim that γ does not bound a smoothly embedded disk in the exterior of $\mathrm{BD}(D_1)$. To prove this, we begin by eliminating the 1-handles in the exterior of $BD(D_1)$ by attaching three additional -1-framed 2-handles along meridians to the dotted 1-handle curves as shown in Figure 9, yielding a larger 4-manifold W. Note that, if γ were to bound a smoothly embedded disk in the exterior of $\mathrm{BD}(D_1)$, then this disk could be capped off with the core of the -1-framed 2-handle attached along γ to produce a smoothly embedded 2-sphere of square -1 in W. We will show that no such 2-sphere can exist. After simplifying the handle diagram (Figures 9–10), we see that the resulting 4-manifold W is obtained from B^4 by attaching a single -1-framed 2-handle along a knot $J \subseteq S^3$.

In Figure 9(f), the knot J has been drawn as a Legendrian knot with Thurston–Bennequin number $tb = 0$. (The Thurston–Bennequin number is computed from a Legendrian knot diagram as the diagram's writhe minus the number of right cusps.) Since the 2-handle is attached with framing $-1 = tb - 1$, W admits a Stein structure [Eli90]. By [LM98], it follows that W cannot contain any smoothly embedded 2-sphere of square -1. It follows that γ cannot bound a smoothly embedded disk in the exterior of $\mathrm{BD}(D_1)$.

This proves that $\mathrm{BD}(D_1)$ and $\mathrm{BD}(D_2)$ are not smoothly equivalent rel. boundary. To obtain the stronger conclusion stated in the theorem, we will show that any supposed diffeomorphism $F\colon (B^4, \mathrm{BD}(D_1)) \to (B^4, \mathrm{BD}(D_2))$ must fix γ up to isotopy. Observe that, along the boundary, F restricts to a self-diffeomorphism of $(S^3, \mathrm{BD}(K))$. It follows that it further restricts to a self-diffeomorphism of $S^3 \setminus \nu\, \mathrm{BD}(K)$. By Lemma 3.4, K has a hyperbolic complement (with trivial isometry group, to be used in what follows). A JSJ decomposition of $S^3 \setminus \nu\, \mathrm{BD}(K)$ is therefore given by

$$S^3 \setminus \nu\, \mathrm{BD}(K) = (S^3 \setminus \nu K) \cup (S^1 \times D^2 \setminus \mathrm{BD}(S^1 \times \{0\})).$$

The latter piece is diffeomorphic to the complement of the Borromean rings. By uniqueness of JSJ decompositions of 3-manifolds [JS79; Joh79] (see also Theorem 6.6), up to isotopy, any diffeomorphism of pairs must restrict on $S^3 \setminus \nu K \subseteq S^3 \setminus \nu\, \mathrm{BD}(K)$ to a self-diffeomorphism of $S^3 \setminus \nu K$. We need to show that every diffeomorphism of $S^3 \setminus \nu K$ is isotopic to the identity to conclude that γ is fixed up to isotopy. By Mostow rigidity [Mos68], every homotopy equivalence of a hyperbolic 3-manifold is homotopic to an isometry. Waldhausen's [Wal68] work, combined with Hatcher's proof of the Smale conjecture [Hat83], shows that for compact, orientable 3-manifolds $N \neq B^3$ with nonempty boundary, homotopic diffeomorphisms are isotopic. Combining these two statements, it follows that every diffeomorphism of a hyperbolic knot complement is isotopic to an isometry. Since the isometry group of $S^3 \setminus \nu K$ is trivial, this completes the proof of the claim. □

4. Infinitely Many Exotic Surface Links

In this section, we construct the 2-component surface links promised in Theorem 1.3. In Section 4.1, we begin with some background on Heegaard Floer cobordism maps, which provides our diffeomorphism obstructions. In Section 4.3, we review the rim surgery construction. Section 4.4 constructs infinitely many pairwise exotic Brunnian surface links.

4.1. *Heegaard Floer Cobordism Maps*

In this section, the obstruction to smooth isotopy comes from maps on link Floer homology. Link Floer homology was introduced by Ozsváth and Szabó [OS04; OS08], whereas these cobordism maps were later defined by Juhász [Juh16]. We refer the reader to, for example, [JMZ21] for a more detailed discussion; for our purposes, the following brief description suffices.

DEFINITION 4.1. Let l be a link in a closed, oriented, connected 3-manifold M. A *multi-pointed link* ℓ is the link l with two basepoints, labeled w and z, chosen per component of l.

Let $\mathrm{DLink}^\times$ be the groupoid of multi-pointed links in closed, oriented, connected 3-manifolds, where the morphisms are smooth multi-pointed isotopies, considered up to isotopy of isotopies. Let $\mathrm{Vect}^\times$ be the groupoid of $\mathbb{F}_2$-vector spaces with linear isomorphisms. Link Heegaard Floer homology gives rise to a functor

$$\widehat{HFL}\colon \mathrm{DLink}^\times \to \mathrm{Vect}^\times .$$

Given the input of a multi-pointed link ℓ in a 3-manifold M, we call $\widehat{HFL}(M,\ell)$ the *link Floer homology* of ℓ.

DEFINITION 4.2. Let M_0 and M_1 be closed, oriented, connected 3-manifolds. A *decorated cobordism* between multi-pointed links $\ell_0 \subseteq M_0$ and $\ell_1 \subseteq M_1$ consists of a compact, oriented surface Σ properly embedded in a compact, connected, oriented 4-manifold W with:

(1) $\partial W = M_0 \sqcup -M_1$;
(2) $\partial \Sigma = l_0 \sqcup -l_1$; and
(3) a collection of arcs $\mathcal{A}$ properly embedded in Σ such that:
- $\mathcal{A}$ does not meet any of the w or z basepoints of ℓ_i;
- each component of $\ell_i - \{w \text{ and } z \text{ basepoints}\}$ meets exactly one endpoint of $\mathcal{A}$;
- the components of $\Sigma \smallsetminus \mathcal{A}$ can be sorted into two subsurfaces Σ_w and Σ_z of Σ so that all w basepoints are in Σ_w and all z basepoints are in Σ_z.

We may view a properly embedded surface Σ in a 4-manifold W with connected boundary as a cobordism from $(\emptyset,\emptyset)$ to the boundary (M,l). In this setting, we can decorate Σ after choosing w and z basepoints on each component of l in an essentially canonical way: take Σ_w to consist of small bigons including each w point and $\Sigma_z = \Sigma \smallsetminus \Sigma_w$. We call this a *trivial decoration of* Σ. If no other decoration is specified, then we always assume a surface is trivially decorated.

Juhász [Juh16] showed that decorated link cobordisms give rise to an extension of the functor $\widehat{HFL}$. Let DLink denote the category of multi-pointed links (M,ℓ), where the morphisms are smooth ambient isotopy classes of decorated cobordisms $[(W,\Sigma,\mathcal{A})]$. Let Vect denote the category of $\mathbb{F}_2$-vector spaces with linear transformations as morphisms. Since we can consider a multi-pointed isotopy as a decorated cobordism, $\mathrm{DLink}^\times \subseteq \mathrm{DLink}$ and $\mathrm{Vect}^\times \subseteq \mathrm{Vect}$ are subcategories that contain all the objects but fewer morphisms. Juhász constructed a functor

$$\widehat{HFL}\colon \mathrm{DLink} \to \mathrm{Vect}$$

with the same value $\widehat{HFL}(M,\ell)$ on multi-pointed links as the functor $\widehat{HFL}:$ $\mathrm{DLink}^\times \to \mathrm{Vect}^\times$ introduced above; this explains why we use the same notation for both functors.

Next, we make explicit the above formalism and recall a key computation from [JMZ21].

REMARK 4.3.

(1) Given a decorated link cobordism $(W, \Sigma, \mathcal{A})$ from (M_0, ℓ_0) to (M_1, ℓ_1), there is an induced map

$$F_{W,\Sigma}\colon \widehat{HFL}(M_0, \ell_0) \to \widehat{HFL}(M_1, \ell_1).$$

We will omit the arcs $\mathcal{A}$ from the notation for cobordism maps. This map is well-defined up to smooth ambient isotopy of Σ rel. boundary. Given a surface Σ whose boundary is a multi-pointed link ℓ in a 3-manifold M, we may view Σ as having trivial decoration; moreover, Σ naturally induces a map

$$F_{W,\Sigma}\colon \widehat{HFL}(\emptyset) \to \widehat{HFL}(M, \ell).$$

The vector space $\widehat{HFL}(\emptyset)$ is a copy of $\mathbb{F}_2$, so $F_{W,\Sigma}$ is determined by the image of the single generator.

(2) According to [Juh16, Theorem 11.3], link cobordism maps behave well under composition. That is, if a decorated cobordism (W, Σ) from (M_0, ℓ_0) to (M_1, ℓ_1) can be split as the composition of decorated cobordisms (W_0, Σ_0) from (M_0, ℓ_0) to (M', ℓ'), followed by (W_1, Σ_1) from (M', ℓ') to (M_1, ℓ_1), then

$$F_{W,\Sigma} = F_{W_1,\Sigma_1} \circ F_{W_0,\Sigma_0}.$$

(3) By [JMZ21, Corollary 8.4], if $\Sigma \subseteq B^4$ is obtained by pushing the interior of a quasipositive Seifert surface for a knot in S^3 into B^4, then $F_{B^4,\Sigma}$ is nonvanishing. (In fact, $\Omega(\Sigma) = 0$; see Section 4.2.)

The composition law of Remark 4.3 (2) provides a means of decomposing a complicated cobordism into simpler ones. Remark 4.3 (3) is useful because in practice link cobordism maps are difficult to compute; this gives a large family of examples where we at least know the induced map is nontrivial. For example, the genus one Seifert surface for any iterated positive untwisted Whitehead double of a positive knot K (i.e. $\mathrm{Wh}_+(\mathrm{Wh}_+(\cdots(\mathrm{Wh}_+(K))\cdots)))$ is quasipositive [Lee01].

4.2. The **Ω***-Invariant*

The link cobordism map $F_{W,\Sigma}$ is an invariant of Σ only up to isotopy rel. boundary. To obstruct two surfaces from being smoothly equivalent, we use the invariant Ω from [JMZ21, Section 6], defined for surfaces embedded in B^4.

Given an oriented, properly embedded surface Σ in B^4 with positive genus and connected boundary, there exists an invariant $\Omega(\Sigma) \in \mathbb{Z}^{\geq 0} \cup \{-\infty\}$ with the property that if (B^4, Σ) is diffeomorphic to (B^4, Σ'), then $\Omega(\Sigma) = \Omega(\Sigma')$. The invariant $\Omega(\Sigma)$ is defined to be $-\infty$ if and only if $F_{B^4,\Sigma}$ vanishes.

In [JMZ21], the surface Σ is assumed to have connected boundary; to avoid re-writing that material for the disconnected boundary case, we continue to use Ω in the connected boundary setting. The details of the construction of Ω are beyond the scope of this paper, so we provide a heuristic description of $\Omega(\Sigma)$ when Σ has genus g.

- We can view $F_{B^4,\Sigma}(1)$ as an element of $\widehat{HFL}(S^3, \ell) \otimes \mathbb{F}_2[\mathbb{Z}^{2g}]$.

- For each element a of $\mathbb{F}_2[\mathbb{Z}^{2g}]$, $\Omega(a)$ denotes the number of irreducible factors of a, counted with multiplicity (here we use the fact that $\mathbb{F}_2[\mathbb{Z}^{2g}]$ is a UFD).
- Now let $\Omega(\Sigma) = \max\{\Omega(a) \mid F_{B^4,\Sigma}(1) = a \cdot y \text{ for some } y\}$.

Note that this is a description and not the definition, which requires deformed knot Floer homology; see Remark 4.8 for a longer discussion and [JMZ21] for even more details.

4.3. Rim Surgery

We briefly recall the technique *rim surgery.* This well-known method for producing potentially exotic pairs of surfaces in 4-manifolds was invented by Fintushel and Stern [FS97] and generalizes the twist-spin construction of Zeeman [Zee65].

DEFINITION 4.4. Let X be a 4-manifold and let $\Sigma \subseteq X$ be a smoothly embedded surface. Let $\alpha \subseteq \Sigma$ be a simple closed curve with $w_1^\Sigma(\alpha) = 0$, and $J \subseteq S^3$ be a knot. (Here, w_1^Σ is the first Steifel–Whitney class of Σ, so the condition $w_1^\Sigma(\alpha) = 0$ requires that α admits a neighborhood homeomorphic to an annulus.)

Choose a framing η of the 2-subbundle of the normal bundle of α that is normal to Σ. Now we can identify $\nu\alpha$ with $B^3 \times S^1$; choose this identification so that $\alpha = \{0\} \times S^1$, $\Sigma \cap \nu\alpha = I \times S^1$ for a fixed vertical arc $I \subseteq B^3$, and η restricts to the same pair of vectors in $T_0 B^3$ for each $\{0\} \times \theta$. We construct the *rim surgered surface* $\Sigma(\alpha; J) \subseteq X$ as follows: the surface agrees with Σ outside of $\overline{\nu}\alpha$. Note that $(\overline{\nu}\alpha, \Sigma \cap \overline{\nu}\alpha) = (B^3 \times S^1, I \times S^1)$, where I is an unknotted arc in B^3. To obtain $\Sigma(\alpha; J)$, replace each (B^3, I) with $(B^3, \mathring{J})$, that is, the tangle obtained from (S^3, J) by deleting a small ball from S^3 centered at a point on J.

As written, $\Sigma(\alpha; J)$ depends on our choice of framing η. There are an integers' worth of such framings that induce a fixed orientation on the subbundle. Choosing a different η twists $B^3 \times \theta$ an integer number of times about a fixed axis with boundary $\partial \mathring{J}$ as θ goes from 0 to 2π (see Figure 12, second row vs. third row).

One setting in which it is easy to specify a framing η is in the case that α bounds a framed locally flat disk Δ in the complement of Σ. Here, the disk Δ is *framed* if a section of the normal bundle of α that lies in $T\Sigma$ extends to a nonvanishing section over all of Δ. We can then specify that, for each $x \in \alpha$, the first coordinate of $\eta(x)$ points into Δ. This ensures that Δ intersects $(\nu\alpha)$ in an annulus of the form (arc) $\times S^1$. Here, we say that replacing $\Sigma \cap \overline{\nu}\alpha$ with $\mathring{J} \times S^1$ yields $\Sigma(\alpha; J, 0)$. We call this *0-twisted rim surgery* (relative to Δ). If we twist the copy of $\mathring{J}$ a total of n times, as above, we instead obtain a surface that we call $\Sigma(\alpha; J, n)$. We call this *n-twisted rim surgery.* When Δ has been implicitly specified, we write $\Sigma(\alpha; J, n)$ as a well-defined surface without reference to Δ. In Figure 12, we illustrate a neighborhood of α intersecting Σ, $\Sigma(\alpha; T, 0)$, and $\Sigma(\alpha; T, 1)$ for T the right-handed trefoil.

REMARK 4.5. Let Σ be an unknotted sphere in S^4 obtained by doubling (B^4, D), where D is a boundary-parallel disk in B^4. Let $\alpha = \partial D$, so α is a curve in Σ

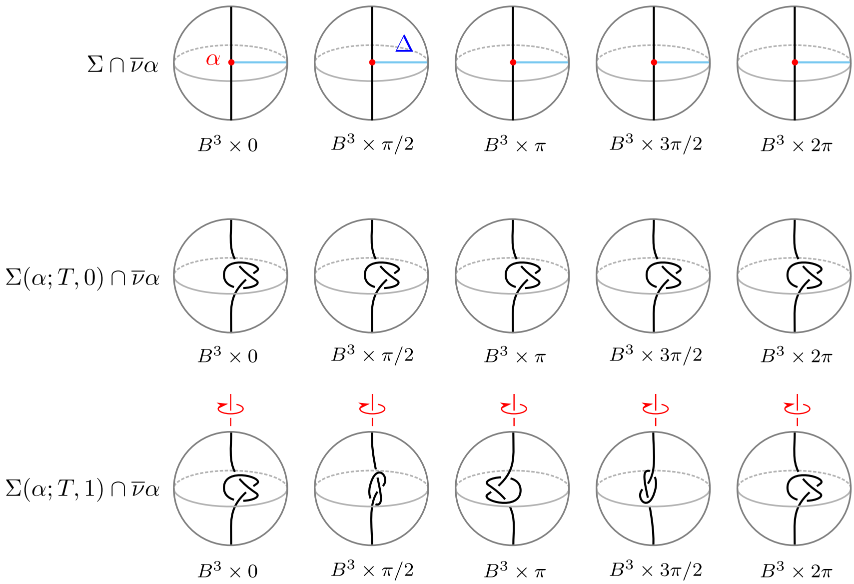

Figure 12 Top row: We draw a neighborhood $\overline{\nu}\alpha$ of α, a simple closed curve in Σ with annular neighborhood. In blue, we draw the portion of a framed disk Δ bounded by α that intersects $\overline{\nu}\alpha$. We draw $\overline{\nu}\alpha$ as $B^3 \times S^1$ with α as $0 \times S^1$. The surface Σ intersects $\overline{\nu}\alpha$ in (vertical arc) $\times S^1$ as illustrated. The parameterization of $\overline{\nu}\alpha$ is chosen so that Δ intersects it in (arc from 0 to ∂B^3) $\times S^1$, as illustrated. Middle row: Replacing $\Sigma \cap \overline{\nu}\alpha = I \times S^1$ with $\mathring{T} \times S^1$ yields $\Sigma(\alpha; T, 0)$. Bottom row: Replacing the $I \times \theta$ s of $\Sigma \cap \overline{\nu}\alpha$ with copies of $\mathring{T}$ that rotate once about a vertical axis as θ runs from 0 to 2π yields $\Sigma(\alpha; T, 1)$.

that bounds a framed disk (a parallel copy of D) into the complement of Σ. Then $\Sigma(\alpha; J, n)$ is the n-twist spin of the knot J as constructed by Zeeman [Zee65]. Zeeman also showed that the 1-twist spin of any knot J is an unknotted $S^2 \subseteq S^4$.

Next, we give a criterion for rim surgery to preserve the isotopy class.

LEMMA 4.6. ([JMZ21, *Corollary 2.7*]) *If α bounds a* CAT*-embedded framed disk D whose interior lies in $B^4 \setminus \Sigma$, then $\Sigma(\alpha; J, 1)$ and Σ are* CAT *ambiently isotopic rel. boundary.*

Proof. This proof is based on Zeeman's [Zee65] work on twist-spun knots. Since D is framed, there is a thickening of D to $D \times I$ meeting Σ in $(\partial D) \times I$ (i.e. we can take parallel copies of D that are disjoint from D and have boundary on Σ). Let D' and D'' be two parallel copies of D, pushed to have boundary curves on opposite sides of α in Σ, and compress Σ along both D' and D'' to obtain a surface Σ'. Thus, Σ' has a 2-sphere component U bounding $D \times I$, and U is CAT-unknotted in the complement of the rest of Σ'.

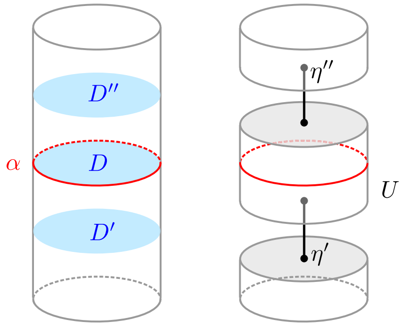

Figure 13 Left: A neighborhood of a framed disk D with boundary α on Σ. Since D is framed, we may find two parallel copies D' and D'' of D on opposite sides of D (in Σ) with boundary on Σ. Right: Compressing Σ along D' and D'' yields a disconnected surface Σ' with an unknotted 2-sphere component U. Surgering Σ' along the arcs η', η'' (with framing chosen to yield an surface orientable in the pictured neighborhood of D) yields Σ again.

The surface Σ can be obtained from Σ' by surgery along two framed arcs η' and η'' (undoing the compressions along D' and D'', respectively), as in Figure 13. Additionally, the curve α exists in both Σ and Σ', so rim surgery can be performed on both surfaces along α. By the same logic, we see that $\Sigma(\alpha; J, 1)$ can be obtained from $\Sigma'(\alpha; J, 1)$ by surgery along η' and η''. Note that one component of $\Sigma'(\alpha; J, 1)$ is the rim-surgered 2-sphere $U(\alpha; J, 1)$. As in Remark 4.5, Zeeman [Zee65] showed that the 1-twist spin $U(\alpha; J, 1)$ is CAT-unknotted for any J, so $\Sigma'(\alpha; J, 1)$ is CAT-isotopic to Σ'. Since η' and η'' are arcs each meeting νU in only one interval, we may arrange for this isotopy to take η' and η'' to η' and η'', respectively; this uses the fact that $\nu U \cong S^2 \times D^2$ is simply connected and that every pair of embedded arcs in a 4-manifold which are homotopic to one another rel. endpoints are isotopic to one another rel. endpoints. Therefore, $\Sigma(\alpha; J, 1)$ is CAT-isotopic to the result of surgering Σ' along η' and η'', which we already noted is isotopic to Σ. Thus $\Sigma(\alpha; J, 1)$ is CAT-isotopic to Σ, as desired. □

On the other hand, rim surgery does often change the isotopy type of a surface when working outside the setting of Lemma 4.6. One potential obstruction to smooth isotopy comes from the link cobordism maps of Section 4.1: if surfaces Σ_1 and Σ_2 in X are isotopic rel. boundary, then $F_{X,\Sigma_1} = F_{X,\Sigma_2}$ (where we choose w, z basepoints of $\partial\Sigma_1 = \partial\Sigma_2$ to agree).

LEMMA 4.7 ([JZ23, Theorem 1.1] (see also [JMZ21, Theorem 5.1])). *Let Σ be a smooth surface properly embedded in B^4. Let α be an essential, nonseparating simple closed curve on Σ and J be a knot. Then*

$$F_{B^4,\Sigma(\alpha;J)}(1) = \Delta_J(y_\alpha) F_{B^4,\Sigma}(1) \in \widehat{\mathrm{HFL}}(S^3, \partial\Sigma).$$

In particular, if $F_{B^4,\Sigma}$ is nonvanishing and J has nontrivial Alexander polynomial, then Σ and $\Sigma(\alpha; J)$ are not smoothly isotopic rel. boundary. More generally, if $F_{B^4,\Sigma}$ is nonvanishing and if J_1 and J_2 are knots with Alexander polynomials that are distinct up to multiplication with a monomial $\pm t^k$, then the surfaces $\Sigma(\alpha; J_1)$ and $\Sigma(\alpha; J_2)$ are not smoothly isotopic rel. boundary.

For $\partial\Sigma$ connected and a nonnegative integer k equal to the number of irreducible factors of the mod 2 Alexander polynomial $\Delta_J(t) \in \mathbb{F}_2[t^{\pm 1}]$ of J (counted with multiplicity), we conclude $\Omega(\Sigma(\alpha; J)) = \Omega(\Sigma) + k$.

REMARK 4.8. To parse Lemma 4.7, we will discuss the totally twisted link Floer cobordism maps of [JZ23] and [JMZ21].

Given an n-tuple $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ of closed 2-forms on a manifold X, we obtain an action of $C_2^{\mathrm{sm}}(X; \mathbb{Z})$ (the group of smooth 2-chains on X) on $\mathbb{R}^n$ defined by

$$h \cdot (a_1, \dots, a_n) = (a_1 + \langle \omega_1, h\rangle, \dots, a_n + \langle \omega_n, h\rangle).$$

Note that for h in the image of $\partial_3 \colon C_3^{\mathrm{sm}}(X; \mathbb{Z}) \to C_2^{\mathrm{sm}}(X; \mathbb{Z})$, this is the trivial action, since

$$\langle \omega_i, h\rangle = \langle \omega_i, \partial_3 g\rangle = \langle d\omega_i, g\rangle = \langle 0, g\rangle = 0$$

if $h = \partial_3 g$ is a boundary. Thus, there is in fact an action of coker(∂_3) on $\mathbb{R}^n$ induced by $\boldsymbol{\omega}$.

We want to extend the action of $C_2^{\mathrm{sm}}(X; \mathbb{Z})$ on $\mathbb{R}^n$ to an action of $\mathbb{F}_2[C_2^{\mathrm{sm}}(X; \mathbb{Z})]$ on $\mathbb{F}_2[\mathbb{R}^n]$, and then consider the induced action of $\mathbb{F}_2[\mathrm{coker}(\partial_3)]$ on $\mathbb{F}_2[\mathbb{R}^n]$. Let $\{y_1, \dots, y_n\}$ be the standard basis of $\mathbb{R}^n$, and write the group structure of $\mathbb{R}^n$ multiplicatively. Thus $y_1^{a_1} \cdots y_n^{a_n}$ denotes the element of $\mathbb{F}_2[\mathbb{R}^n]$ corresponding to $(a_1, \dots, a_n) \in \mathbb{R}^n$. Similarly, write e^h to indicate the element of $\mathbb{F}_2[\mathrm{coker}(\partial_3)]$ corresponding to $h \in \mathrm{coker}(\partial_3)$. Then we can extend the action above to the promised action on $\mathbb{F}_2[\mathbb{R}^n]$ via

$$e^h \cdot y_1^{a_1} \cdots y_n^{a_n} = y_1^{a_1 + \langle \omega_1, h\rangle} \cdots y_n^{a_n + \langle \omega_n, h\rangle}.$$

We write $\mathbb{F}_2[\mathbb{R}^n]_{\boldsymbol{\omega}}$ to indicate $\mathbb{F}_2[\mathbb{R}^n]$ considered as an $\mathbb{F}_2[\mathrm{coker}(\partial_3)]$-module equipped with the action determined by $\boldsymbol{\omega}$.

Juhász and Zemke [JZ23] give a way of twisting the sutured Floer homology of a sutured manifold (M, γ) (e.g. the complement of $\partial\Sigma$ in S^3) via an n-tuple $\boldsymbol{\omega}$ of closed 2-forms on M. This is relevant to our setting because the sutured Floer homology of 0-surgery on a link in S^3 is tautologically equal to the hat version of link Floer homology of that link, so this might be viewed as another perspective on link Floer homology.

Let (M, γ) be a balanced sutured 3-manifold, and let (S, α, β) be an admissible sutured Heegaard decomposition for (M, γ) in the sense of [Juh06]. The twisted sutured Floer chain complex $\underline{CF}(S, \alpha, \beta, \mathfrak{s}; \mathbb{F}_2[\mathrm{coker}(\partial_3)])$ is generated by $\mathbf{x} \otimes e^h$ across all $h \in \mathrm{coker}(\partial_3)$ and $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $s_z(\mathbf{x}) = \mathfrak{s}$. Here, $\mathfrak{s}$ is a relative spin^c structure on M, and $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are the totally real tori in the complex manifold $\mathrm{Sym}^{\mathrm{genus}(S)}(S)$ (of real dimension $2 \cdot \mathrm{genus}(S)$), which is the genus($S$)-fold symmetric product of S with itself used in the construction of (every flavor of) Heegaard–Floer homology.

The differential on the twisted sutured Floer complex is given by

$$\partial(x \otimes e^h) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot \mathbf{y} \otimes e^{\widetilde{D}(\phi)} \cdot e^h,$$

where $\widetilde{D}(\phi) \in \operatorname{coker}(\partial_3)$ is obtained from the domain of ϕ by smoothly capping off with compressing disks for α, β [JZ23]. Let

$$\underline{CF}(S, \alpha, \beta; \mathbb{F}_2[\operatorname{coker}(\partial_3)]) := \bigoplus_{\mathfrak{s} \in \operatorname{Spin}^c(M)} \underline{CF}(S, \alpha, \beta, \mathfrak{s}; \mathbb{F}_2[\operatorname{coker}(\partial_3)]).$$

Now we set

$$CF(S, \alpha, \beta, \mathfrak{s}; \mathbb{F}_2[\mathbb{R}^n]_\omega) := \underline{CF}(S, \alpha, \beta, \mathfrak{s}; \mathbb{F}_2[\operatorname{coker}(\partial_3)]) \bigotimes_{\mathbb{F}_2[\operatorname{coker}(\partial_3)]} \mathbb{F}_2[\mathbb{R}^n]_\omega.$$

This is the *sutured Floer chain complex perturbed by* $\boldsymbol{\omega}$. Analogous to above, we also define

$$CF(S, \alpha, \beta; \mathbb{F}_2[\mathbb{R}^n]_\omega) := \bigoplus_{\mathfrak{s} \in \operatorname{Spin}^c(M)} CF(S, \alpha, \beta, \mathfrak{s}; \mathbb{F}_2[\mathbb{R}^n]_\omega).$$

Choosing a different admissible Heegaard diagram for (M, γ) preserves the chain complex $\underline{CF}(S, \alpha, \beta; \mathbb{F}_2[\operatorname{coker}(\partial_3)])$ up to chain homotopy equivalence and the action of $\mathbb{F}_2[\operatorname{coker}(\partial_3)]$. (Baldwin and Sivek [BS16] call this a *projective transitive system.*) See Section 6 of [JZ23] to understand transition maps. We denote the homology of this chain complex by

$$\underline{SHF}(M, \gamma; \mathbb{F}_2[\operatorname{coker}(\partial_3)]),$$

and we write

$$SHF(M, \gamma; \mathbb{F}_2[\mathbb{R}^n]_\omega)$$

for the homology of $CF(S, \alpha, \beta; \mathbb{F}_2[\mathbb{R}^n]_\omega)$.

Back in the 4-dimensional world, given a suitably decorated sutured cobordism W from (M, γ) to (M', γ'), Juhász and Zemke [JZ23] construct a twisted map

$$\underline{F}_W : \underline{SHF}(M, \gamma; \mathbb{F}_2[\operatorname{coker}(\partial_3))]) \to \underline{SHF}(M', \gamma'; \mathbb{F}_2[\operatorname{coker}(\partial_3')]).$$

This procedure involves decomposing W into elementary pieces and defining a map associated with each piece; we refer the interested reader to Section 7 of [JZ23]. Finally, we define

$$F_{W;\boldsymbol{\omega}} : SHF(M, \gamma; \mathbb{F}_2[\mathbb{R}^n]_\omega) \to SHF(M', \gamma'; \mathbb{F}_2[\mathbb{R}^n]_\omega)$$

to be given by

$$F_{W;\boldsymbol{\omega}} = \underline{F}_W \otimes 1_{\mathbb{F}_2[\mathbb{R}^n]_\omega}.$$

This means that if $\underline{F}_W(1)$ has a term of the form $[x] \cdot e^h$, then $F_{W;\omega}(1)$ has a corresponding term $[x] \cdot y_1^{\langle \omega_1, h \rangle} \cdots y_n^{\langle \omega_n, h \rangle}$. Thus, $F_{W;\boldsymbol{\omega}}(1)$ is a polynomial in $y_1, \dots, y_n$, albeit with real exponents.

In the setting of our paper, let $\boldsymbol{\omega}$ be an n-tuple (with n the sum of the genera of the connected components of Σ) of closed 2-forms on $X := B^4 \setminus \nu\Sigma$, so $F_{B^4,\Sigma;\boldsymbol{\omega}} = F_{X;\boldsymbol{\omega}}$. Then $F_{B^4,\Sigma;\boldsymbol{\omega}}(1)$ is a polynomial in $y_1, \dots, y_n$ (although again note that we mean with real-valued exponents and coefficients in $\widehat{HFL}(S^3, \partial\Sigma)$). Lemma 4.7 says that

$$F_{B^4,\Sigma(\alpha;J);\boldsymbol{\omega}}(1) = \Delta_J(y_1^{\langle\omega_1,[T_\alpha]\rangle} \cdots y_n^{\langle\omega_n,[T_\alpha]\rangle}) \cdot F_{B^4,\Sigma;\boldsymbol{\omega}}(1),$$

where Δ_J is the mod 2 Alexander polynomial of J and T_α is the rim 2-torus $\partial\nu\Sigma|_\alpha$ centered about α.

Generally, it does not make sense to count factors of polynomials with real-valued exponents. However, [JMZ21, Proposition 4.3] shows that there exists a monomial $m \in \mathbb{F}_2[\mathbb{R}^n]$ such that $m \cdot F_{B^4,\Sigma;\boldsymbol{\omega}}(1) \in \widehat{HFL}(S^3, \partial\Sigma) \otimes \mathbb{F}_2[\mathbb{Z}^n]$. Then $\Omega_{\boldsymbol{\omega}}(\Sigma)$ is defined to be the largest number of irreducible factors of any element of $\mathbb{F}_2[\mathbb{Z}^n]$ dividing $m \cdot F_{B^4,\Sigma;\boldsymbol{\omega}}(1)$, so in this language it is clear that $\Omega_{\boldsymbol{\omega}}(\Sigma(\alpha; J)) = \Omega_{\boldsymbol{\omega}}(\Sigma) + k$, where k is the number of irreducible factors of $\Delta_J(t)$ as a polynomial in $\mathbb{F}_2[t^{\pm}]$. Finally, [JMZ21, Section 6.2] shows that $\Omega_{\boldsymbol{\omega}}$ is independent of the choice of $\boldsymbol{\omega}$ and set $\Omega := \Omega_{\boldsymbol{\omega}}$. See [JZ23] and [JMZ21] for more details.

4.4. Construction of Exotic Surface Links

We now construct an infinite family of exotic 2-component Brunnian links. Each link will have a disk component and a genus one component. Fix K to be some strongly quasipositive topologically slice knot, for example, the positive untwisted Whitehead double of the right-handed trefoil. This is topologically slice by [FQ90, Theorem 11.7B] (as the Alexander polynomial of any untwisted Whitehead double is trivial) and strongly quasipositive by [Lee01].

Let $\Sigma = S_1 \sqcup S_2$ be the surface link depicted in Figure 2. We draw a disk and a genus one surface immersed in S^3 with ribbon intersections. Pushing the interiors of this surface slightly into B^4 yields disjoint surfaces. Moreover, each knot ∂S_i is an unknot, S_1 is an unknotted disk, and S_2 is an unknotted genus one surface.

Proposition 4.9. *Let α be the curve on S_2 pictured in Figure 14. Then α bounds framed disks Δ and Δ' into the complement of S_2, with Δ smooth and Δ' locally flat, and with Δ' disjoint from S_1. Moreover, Δ and Δ' can be taken to agree near their common boundary α.*

Proof. In the left two frames of Figure 14, we illustrate that α bounds a framed locally flat disk Δ' whose interior is disjoint from both S_2 and S_1. The construction of the disk is explained in detail in the caption of Figure 14. This disk is obtained from the two schematically pictured locally flat slice disks for the topologically slice knot K by gluing them along the arc where their boundaries coincide.

Viewed as a knot in S^3 on the ribbon surface S_2, α is an unknot and the framing induced by S_2 on α is the 0-framing. We conclude that α bounds a framed smooth disk Δ into B^4 that is disjoint from S_2 in its interior. Moreover, as drawn, Δ' induces the 0-framing on α as a knot in S^3, so we can arrange for Δ and Δ' to agree near their boundaries. □

Now that we have specified framed disks (that agree near their boundaries) bounded by α, we can consider n-twisted rim surgery on α.

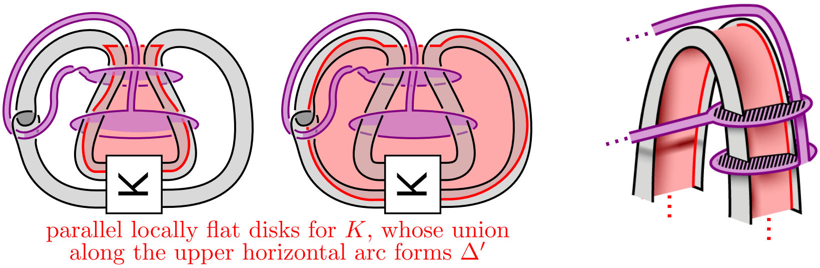

Figure 14 Left two panels: Because K is topologically slice, α bounds a topological locally flat disk Δ' whose interior is disjoint from both S_2 and S_1 (and which is normal to S_2 near its boundary). The disk is obtained by gluing the two indicated locally flat disks together along the two planar surfaces indicated, which are themselves glued along an arc. The visible portion of Δ' (projected to S^3) intersects S_1 in two ribbon arcs. Right panel: A close-up of a portion of the first two figures. In the projection to S^3, the indicated portion of Δ' intersects S_1 in two ribbon intersections. The projections of S_1, S_2 also have ribbon intersections, and the projection of S_1 has a ribbon self-intersection. In B^4, the shaded regions of S_1 lie further toward the center of B^4 so that the interiors of all surface are disjoint.

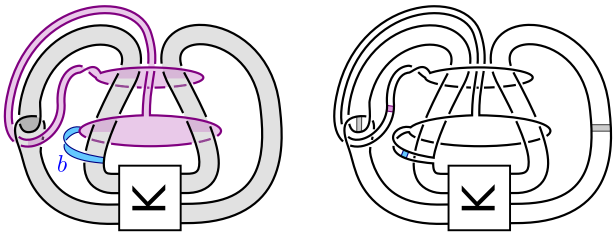

Figure 15 Left: We form a connected surface S from S_2 and S_1 via attaching the band b pictured. Right: We draw the knot ∂S with four bands contained in S representing index-1 critical points of the radial height function on B^4 restricted to S. Surgering ∂S along these bands yields an unlink; disks bounded by this unlink give the minima of S.

Corollary 4.10. *For any knot J, $S_2(\alpha; J, 1)$ is smoothly isotopic rel. boundary to S_2 and $\Sigma(\alpha; J, 1)$ is topologically isotopic rel. boundary to Σ.*

Proof. This follows immediately from Proposition 4.9 and Lemma 4.6. □

To apply Lemma 4.7 and obstruct smooth isotopy rel. boundary of the surface links, we must show that the map $F_{B^4,L} : \widehat{HFK}(\emptyset) \to \widehat{HFK}(S^3, \partial\Sigma)$ is nonvanishing. We achieve this by obtaining a strongly quasipositive surface from Σ, even though Σ itself is *not* strongly quasipositive (since S_2 is not).

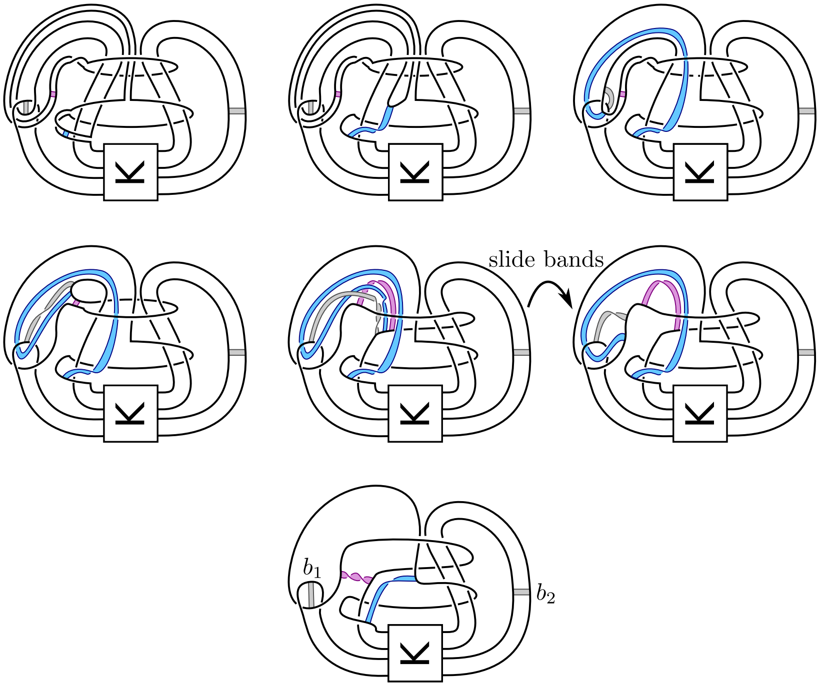

Figure 16 Left to right, top to bottom: We isotope the diagram for S obtained in Figure 15. Between the fifth and sixth frame, we slide two bands over another band. Isotopy and slides of bands describe isotopy of S. In the final diagram, we see that $\partial S = \mathrm{Wh}_+(\mathrm{Wh}_+(K))$ and note that the bands b_1 and b_2 lie on the standard genus one surface for $\mathrm{Wh}_+(\mathrm{Wh}_+(K))$.

PROPOSITION 4.11. *Let S be the connected surface obtained from Σ by attaching S_2 and S_1 via the band depicted in the left of Figure 15 (with interior pushed slightly into B^4). S is smoothly isotopic to the standard genus one Seifert surface for* $\mathrm{Wh}_+(\mathrm{Wh}_+(K))$ (*with interior pushed slightly into B^4*).

Proof. In Figure 16, we isotope S to see $\partial S = \mathrm{Wh}_+(\mathrm{Wh}_+(K))$ (see also Figure 17). The surface S is embedded in B^4 with four index-1 and three index-0 points with respect to the standard radial height function on B^4. The index-1 points are flattened and drawn as bands attached to ∂S; surgering ∂S along these bands yields a 3-component unlink. Moreover, two of these bands, labeled b_1 and b_2, lie on the standard genus one Seifert surface for $\mathrm{Wh}_+(\mathrm{Wh}_+(K))$. After surgering ∂S along these bands, we obtain an unknot with two trivial bands attached, as shown in Figure 18. We conclude that the remaining two bands (i.e. index-1 points of S) can be removed via isotopy of S. Thus, S is isotopic to the standard genus one Seifert surface for $\mathrm{Wh}_+(\mathrm{Wh}_+(K))$. □

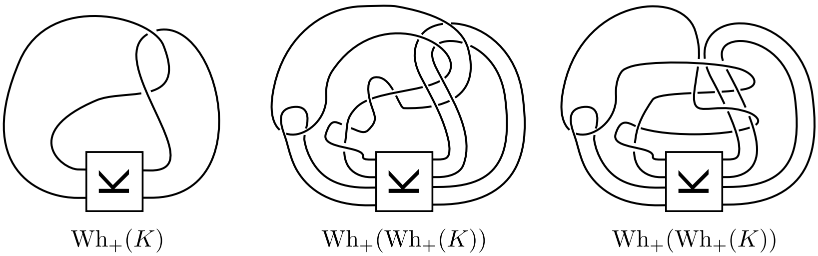

Figure 17 Left: The knot $\mathrm{Wh}_+(K)$. Note that this diagram has writhe 2. Middle: The knot $\mathrm{Wh}_+(\mathrm{Wh}_+(K))$. Right: We isotope the middle drawing of $\mathrm{Wh}_+(\mathrm{Wh}_+(K))$ to agree with ∂S as in the last panel of Figure 16.

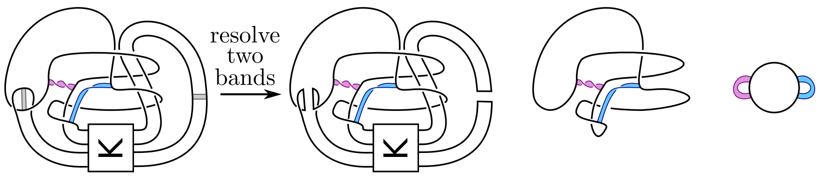

Figure 18 Starting with the last diagram of Figure 16, we surger ∂S along the two bands lying on the standard surface for $\mathrm{Wh}_+(\mathrm{Wh}_+(K))$. We then isotope the result to obtain the diagram on the right: two trivial bands attached to an unknot. We conclude that the remaining two bands are cancellable, and therefore S is isotopic to the standard genus one surface for $\mathrm{Wh}_+(\mathrm{Wh}_+(K))$.

COROLLARY 4.12. *The map* $F_{B^4,\Sigma}\colon \widehat{\mathrm{HFK}}(\emptyset) \to \widehat{\mathrm{HFK}}(S^3, \partial\Sigma)$ *is nonvanishing.*

Proof. Let B denote the link cobordism described by the band b, as shown on the left of Figure 15. This B is a cobordism from $\partial\Sigma$ to $\mathrm{Wh}_+(\mathrm{Wh}_+(K))$. By Proposition 4.11, $B \circ \Sigma$ is isotopic to the standard genus one Seifert surface S for $\mathrm{Wh}_+(\mathrm{Wh}_+(K))$. By Remark 4.3 (2), we have

$$F_{B^4,S} = F_{S^3\times I,B} \circ F_{B^4,\Sigma}.$$

Since K is a strongly quasipositive knot, S is strongly quasipositive surface [Lee01]. By Remark 4.3 (3), $F_{B^4,S}$ is nonvanishing. Thus, $F_{B^4,\Sigma}$ is also nonvanishing. □

THEOREM 1.3. *Let K be a strongly quasipositive topologically slice knot, and let Σ_0 denote the surface link of Figure 2. For each integer $n \geq 1$, let $J_n \subseteq S^3$ be a knot whose Alexander polynomial has n irreducible factors* (*counted with multiplicity*). *Let $\Sigma_n := \Sigma_0(\alpha; J_n, 1)$, where α is the curve on S_2 illustrated in Figure 14. Then each Σ_n is Brunnian. Additionally, for $n \neq m$, we find that Σ_n and Σ_m form an exotic pair.*

The positive untwisted Whitehead double of the right-handed trefoil is one possible choice for K. The knot J_n may, for example, be a connect sum of n trefoils.

REMARK 4.13. In Theorem 1.3, if we ask only that for $n \neq m$, J_n and J_m have inequivalent mod 2 Alexander polynomials, up to multiplication by a unit in $\mathbb{F}_2[t^{\pm 1}]$, rather than that their mod 2 Alexander polynomials have different numbers of irreducible factors, then we may still conclude that Σ_n and Σ_m are not smoothly isotopic rel. boundary by applying Lemma 4.7 and Corollary 4.12. But, for us at least (and also for [JMZ21]), the invariant Ω is needed to obstruct diffeomorphism of pairs.

Proof of Theorem 1.3. Note that Σ_0 is Brunnian by construction. It follows from Corollary 4.10 that each Σ_n is Brunnian and that for all n, Σ_n is topologically isotopic rel. boundary to Σ_0.

Assume $n < m$. Let S denote the genus one Seifert surface for $\mathrm{Wh}_+(\mathrm{Wh}_+(K))$ with interior pushed slightly into B^4. By Proposition 4.11, there is a band b so that $\Sigma_n \cup b$ is isotopic rel. boundary to $S(\alpha; J_n, 1)$. Applying Remark 4.3 (3) and Lemma 4.7, we deduce that $\Omega(S(\alpha; J_n, 1)) = n$. Suppose there is a diffeomorphism $f : (B^4, \Sigma_n) \to (B^4, \Sigma_m)$. We cannot assume that f restricts to the identity on ∂B^4, so this does not yield an equivalence from $S(\alpha; J_n, 1)$ to $S(\alpha; J_m, 1)$. Instead, we obtain an equivalence from $(B^4, S(\alpha; J_n, 1))$ to $(B^4, f(b) \cup \Sigma_m)$, where $f(b)$ is some other band attached to $\partial \Sigma_m$. Note that $f(b) \cup \Sigma_m$ is obtained from $f(b) \cup \Sigma_0$ by 1-twist rim surgery along α using J_m. Applying Lemma 4.7, we have either

$$\Omega(f(b) \cup \Sigma_m) = -\infty \quad \text{if } F_{B^4, f(b) \cup \Sigma_0} \text{ vanishes, or}$$
$$\Omega(f(b) \cup \Sigma_m) \geq m \quad \text{if } F_{B^4, f(b) \cup \Sigma_0} \text{ does not vanish.}$$

In both cases, $\Omega(f(b) \cup \Sigma_m) \neq \Omega(S(\alpha; J_n, 1)) = n$, yielding a contradiction. We conclude that there is no diffeomorphism from (B^4, Σ_n) to (B^4, Σ_m). ☐

5. Surfaces of Higher Genus

Next, we extend the construction of Section 4.4 to produce higher genus surfaces, proving Theorem 1.4. From now on, let $\Sigma_0 = S_1 \cup S_2$ be as in Theorem 1.3 (i.e. the surface link in Figure 2, with K a topologically slice, strongly quasipositive knot). Recall that S_1 has genus zero and S_2 has genus one. In shorthand, we write that Σ_0 has genus $(0, 1)$.

To use Σ_0 to produce exotic surfaces of higher genus, first we describe a certain high genus Brunnian link. Fix a positive integer g and let $R^g = R_1^g \sqcup R_2^g$ be the surface link in B^4 illustrated in Figure 19. We also draw a band b_R^g between the two components of R^g. The link R^g is Brunnian of genus $(0, g)$.

PROPOSITION 5.1. *The surface $\widetilde{R}^g$ obtained by gluing R_1^g and R_2^g together along the band b_R^g, as in Figure 19, is isotopic to a quasipositive Seifert surface (pushed slightly into B^4).*

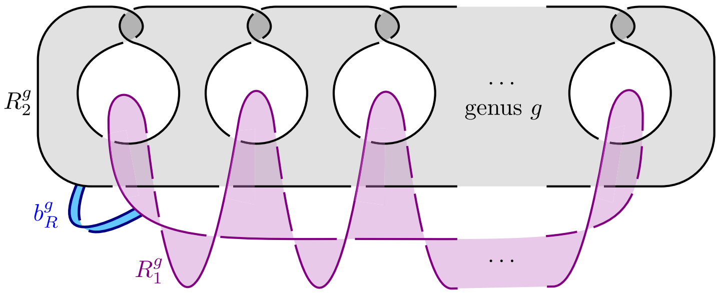

Figure 19 A 2-component surface link $R^g = R_1^g \sqcup R_2^g$ and a band b_R^g between the components of R^g. Note that R_1^g is an unknotted disk and R_2^g is an unknotted genus g surface. Viewed as a ribbon surface in S^3, there are g arcs of intersection between the pictured ribbon surfaces.

Proof. In Figure 20, we illustrate an isotopy from $\widetilde{R}^g$ to a quasipositive Seifert surface (slightly pushed into B^4) for a connected sum of g right-handed trefoil knots. □

We use R^g to increase the genus of Σ_0. For any integers r, s with $r \geq 0$ and $s \geq 1$, let $\Sigma_0^{r,s}$ be obtained as follows (see Figure 21 (top)).

1. Start with a copy of Σ_0.
2. If $r > 0$, then band a copy of R^r to Σ_0. The bands are indicated in Figure 21 (top); note that the disk component of R^r is banded to S_2, whereas the positive genus component of R^r is banded to S_1.
3. If $s > 1$, then band a copy of R^{s-1} to Σ_0. The bands are again indicated in Figure 21 (top); note that the disk component of R^{s-1} is banded to S_1, whereas the positive genus component of R^{s-1} is banded to S_2.
4. Call the result genus (r, s) surface $\Sigma_0^{r,s}$.

In Figure 21 (bottom), we illustrate a certain band b between the two components of $\Sigma_0^{r,s}$ and a curve α (the same as in Theorem 1.3) that lies on $\Sigma_0^{r,s}$. Just as in Proposition 4.9, we see that α bounds a smooth framed disk into the complement of the component on which it lies, and a locally flat framed disk into the complement of $\Sigma_0^{r,s}$. The following proposition illustrates why we chose the specific bands in Figure 21 (top).

PROPOSITION 5.2. *The surface obtained by gluing the components of $\Sigma_0^{r,s}$ together along b is isotopic to a quasipositive Seifert surface (pushed slightly into B^4).*

Proof. In Figure 22, we isotope the result of gluing $\Sigma_0^{r,s}$ along b to obtain a copy of $\Sigma_0 \cup b$ trivially boundary-summed with $R^r \cup b_R^r$ and $R^{s-1} \cup b_R^{s-1}$. (If $r = 0$ or $s = 1$, then ignore the corresponding R.) From Propositions 4.11 and 5.1, we find that this is a quasipositive Seifert surface (pushed into B^4) for $\mathrm{Wh}_+(\mathrm{Wh}_+(K))\#(\#_{r+s-1}(\text{right-handed trefoils}))$. □

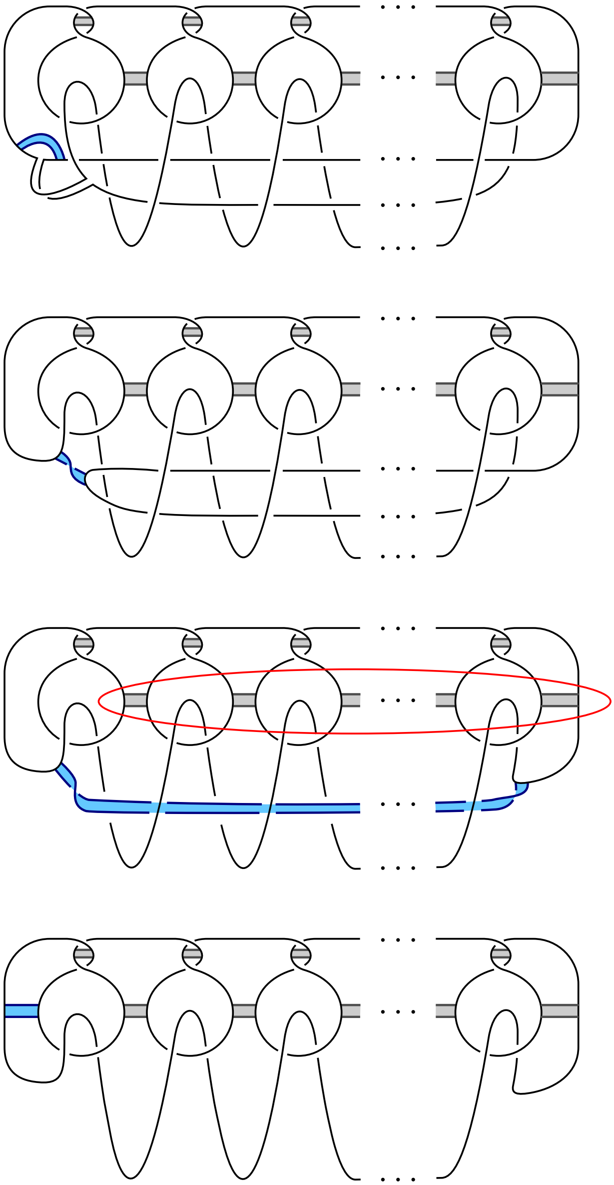

Figure 20 We draw the boundary of $\widetilde{R}^g$ with bands indicating index-1 critical points into B^4. From top to bottom, we isotope $\widetilde{R}^g$. To get from the third to the fourth picture, we also slide the bottommost band over all of the circled bands. In the end, we obtain a positive genus g Seifert surface (slightly pushed into B^4) for a connected sum of g right-handed trefoil knots.

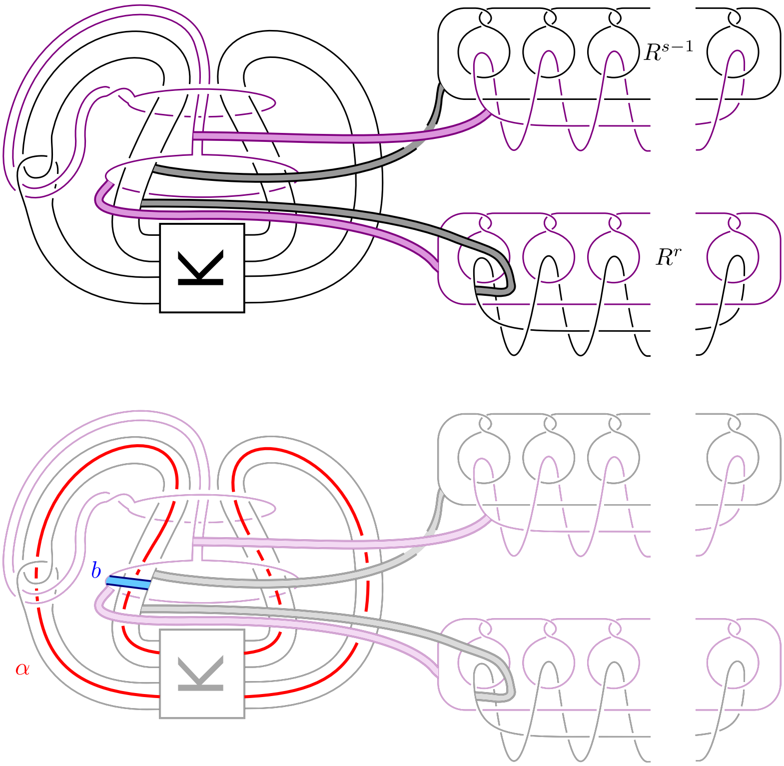

Figure 21 Top: The 2-component surface link $\Sigma_0^{r,s}$ (here we draw as if $r > 0$ and $s > 1$). Bottom: A band b between the components of $\Sigma_0^{r,s}$ and a curve α lying on $\Sigma_0^{r,s}$.

Theorem 1.4 follows from the next statement.

THEOREM 5.3. *Fix integers r, s with $r \geq 0$ and $s \geq 1$. For each $n \in \mathbb{N}$, let J_n be a knot in S^3 whose Alexander polynomial has n irreducible factors counted with multiplicity. Let $\Sigma_n^{r,s} = \Sigma_0^{r,s}(\alpha; J_n, 1)$. Then, for all $m > n \geq 0$, we have that*:

- *$\Sigma_n^{r,s}$ is a Brunnian genus (r, s) link,*
- *$\Sigma_n^{r,s}$ and $\Sigma_m^{r,s}$ are topologically isotopic rel. boundary, and*
- *$(B^4, \Sigma_n^{r,s})$ is not diffeomorphic to $(B^4, \Sigma_m^{r,s})$.*

REMARK 5.4. As in Remark 4.13, if we only have that $\Delta_{J_n} \neq \Delta_{J_m}$ up to a unit in $\mathbb{F}_2[t^{\pm 1}]$, then we may still deduce that $(B^4, \Sigma_n^{r,s})$ and $(B^4, \Sigma_m^{r,s})$ are not smoothly isotopic rel. boundary.

Proof. The curve α bounds a smooth disk into the complement of the component of $\Sigma_0^{r,s}$ on which it lies. Additionally, $\Sigma_0^{r,s}$ is Brunnian: the two components are trivial. We need to see this is true after the rim surgery. The rim surgery only

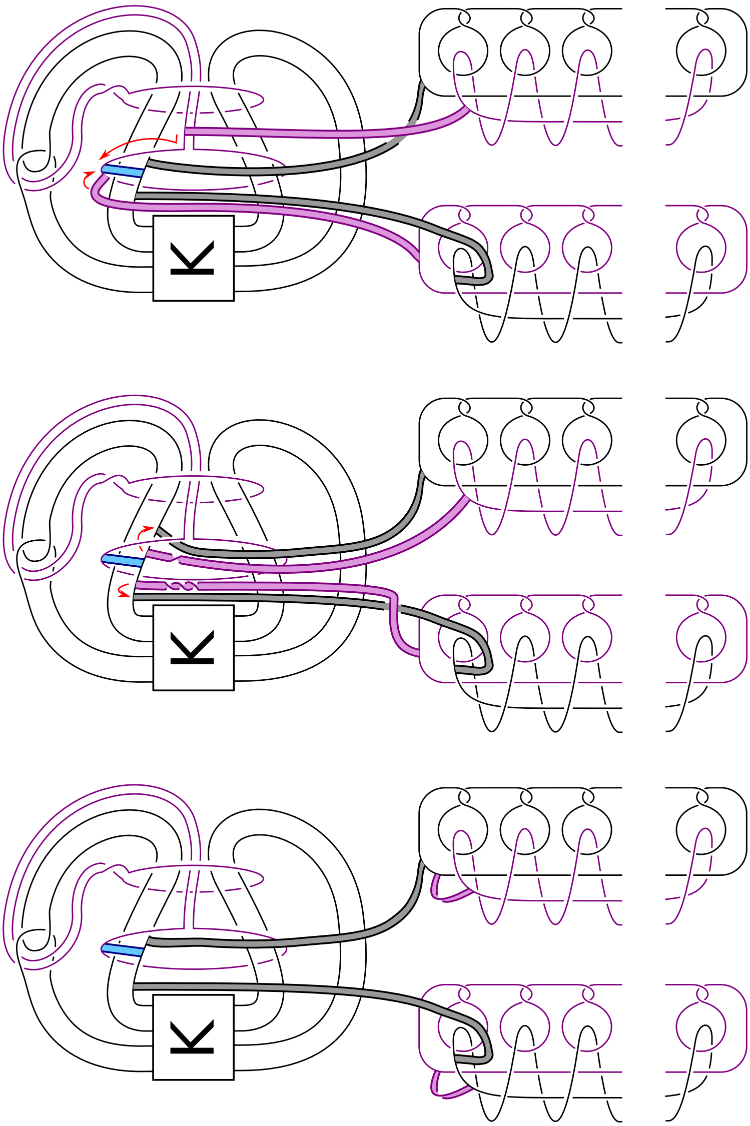

Figure 22 We glue the components of $\Sigma_0^{r,s}$ together along the band b. From top to bottom, we perform isotopy via band slides. We draw arrows to preemptively indicate band slides. Note at the end the resemblance to Figures 16 and 20.

changes one of the components, the one containing α, so we only have to check that the component is still trivial. Since α bounds a smooth disc in the complement of that component, Lemma 4.6 shows that the component is still trivial.

Moreover, since α bounds a locally flat disk into the complement of $\Sigma_0^{r,s}$, it follows also from Lemma 4.6 that $\Sigma_n^{r,s}$ is topologically isotopic rel. boundary to $\Sigma_0^{r,s}$ for any n.

By Proposition 5.2, $\Sigma_0^{r,s} \cup b$ is isotopic to a strongly quasipositive Seifert surface pushed into B^4. It follows from Remark 4.3 (3) and Lemma 4.7 that $\Omega(\Sigma_n^{r,s} \cup b) = n$. As in Theorem 1.3, we note that if $\Sigma_n^{r,s}$ and $\Sigma_m^{r,s}$ are smoothly equivalent, then $\Sigma_n^{r,s} \cup b$ is smoothly equivalent to $\Sigma_m^{r,s} \cup b'$ for some band b', so $\Omega(\Sigma_m^{r,s} \cup b') = n$. But $(\Sigma_m^{r,s} \cup b')$ is obtained from 1-twist rim surgery using knot J_m on $(\Sigma_0^{r,s} \cup b')$, so by Lemma 4.7 we have

$$\Omega(\Sigma_m^{r,s} \cup b') = -\infty \text{or} \Omega(\Sigma_m^{r,s} \cup b') \geq m.$$

This yields a contradiction, so we conclude that $(B^4, \Sigma_n^{r,s})$ is not diffeomorphic to $(B^4, \Sigma_m^{r,s})$. □

6. Surfaces with More than Two Components

In this section, we construct Brunnian exotic surface pairs with arbitrarily many components. To prove Theorem 1.1, we take a pair of surface links from either Theorem 1.2 or Theorem 1.3, and apply iterated Bing doubling to the first component, yielding exotic pairs.

This proof uses covering space methods and was inspired by work of Cha and Kim [CK08] on concordance of 1-links obtained from Bing doubling. We extend their covering link methods to the 4-dimensional setting. The main obstacle is to show that there is no diffeomorphism of pairs relating Bing doubled surfaces, provided we start with a pair of surface links that admit no diffeomorphism of pairs.

Section 6.1 introduces the notion of covering surfaces and investigates covering surfaces of Bing doubles. Section 6.2 recalls some of the theory of JSJ decompositions of 3-manifolds and applies it to Bing doubles and their covering links. Section 6.3 uses this to show that, in a fairly general setting, Bing doubling a pair of exotic surface links yields another pair of exotic surface links.

6.1. Coverings and Bing Doubles

LEMMA 6.1. *Let $\Sigma = \Sigma_0 \sqcup \Sigma_1 \sqcup \cdots \sqcup \Sigma_n$ be a surface link in B^4, and suppose that Σ_0 is an unknotted disk. Fix $k \geq 1$. Suppose that $H_1(\Sigma \smallsetminus \Sigma_0; \mathbb{Z}) \to H_1(B^4 \smallsetminus \Sigma_0; \mathbb{Z}) \cong \mathbb{Z}$ has image in $k\mathbb{Z}$. Let $p\colon B^4 \to B^4$ be a k-fold branched covering map with branching set Σ_0. Then $p^{-1}(\Sigma_1 \sqcup \cdots \sqcup \Sigma_n)$ is a kn-component surface link.*

Proof. The homological condition guarantees that each component of $\Sigma_1 \sqcup \cdots \sqcup \Sigma_n$ lifts to a k component surface link. □

DEFINITION 6.2. Let Σ be a surface link. A *covering surface link* of Σ is a surface link obtained from Σ by a finite sequence of the following two operations.

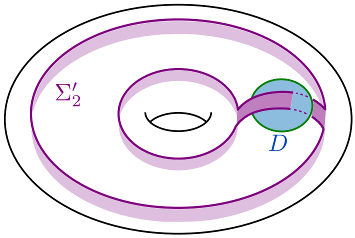

Figure 23 The solid torus $U = \partial N \cap S^3$. The disk Σ_1' is the push-in of the Seifert disk $D \subseteq U$ shown in the figure. The disk Σ_2' is formed from two push-offs of Σ_1 (represented by the light purple collars, which extend to disks out of view) joined by a band in U.

(1) Taking the pre-image of Σ under a branched covering map p, as in Lemma 6.1, with branching set an unknotted disk, and forgetting the branching set.
(2) Passing to a sublink.

The following lemma is key to our use of covering surfaces in arguments by contradiction. It states that there is a covering surface of a Bing double isotopic to the original surface link. Recall that $\mathrm{BD}(\Sigma)$ is defined to be $\mathrm{BD}(\Sigma_1) \sqcup \Sigma_2 \sqcup \cdots \sqcup \Sigma_n$. The "first" component of $\mathrm{BD}(\Sigma)$, denoted $BD(\Sigma)_1$, is one of the two components of $\mathrm{BD}(\Sigma_1)$. By symmetry of the Bing doubling operation, these two components can be arbitrarily ordered.

LEMMA 6.3. *Let $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_n$ be an n-component surface link, where $n \geq 2$. If Σ_1 is a trivial disk, then Σ can be realized as a covering surface of $BD(\Sigma)$.*

Proof. We begin by setting some notation. Fix a neighborhood $N \cong D^2 \times D^2$ of Σ_1 where Σ_1 is identified with $D^2 \times \{0\}$. Denote $\mathrm{BD}(\Sigma)$ by Σ'; its components are labeled so that Σ_1' and Σ_2' form $\mathrm{BD}(\Sigma_1)$ and so that $\Sigma_i' = \Sigma_{i-1}$ for $i \geq 3$. Note that Σ_1' and Σ_2' lie inside N, whereas Σ_i' lies in $B^4 \setminus N$ for $i \geq 3$. Let U denote the solid torus $(\partial D^2) \times D^2 = (\partial N) \cap S^3$; this solid torus is identified with $\nu(\partial \Sigma_1)$ in S^3 and contains $\partial \Sigma_1'$ and $\partial \Sigma_2'$.

The disk Σ_2' consists of two push-offs of Σ_1 joined by a band in U. By Lemma 2.3, we may isotope Σ_1' so that it is the push-in of a Seifert disk D for the unknot $\partial \Sigma_1'$ as in Figure 23. Note that D and Σ_2' meet in a single ribbon intersection that occurs in U, as shown.

Since Σ_1' is an unknotted disk in B^4, the double cover of B^4 branched along Σ_1' is B^4. However, to lift the remaining surface link components, we describe the branched cover more precisely. As shown in the proof of Lemma 2.3, the disks Σ_1' and D cobound a 3-ball Δ lying in N. Let Q denote the modified 4-ball (with corners) obtained by cutting B^4 open along Δ. The double branched cover of Σ_1' is obtained by gluing together two copies of this space, Q and $Q^\star$, as depicted schematically in Figure 24, along the part of the boundary $\Delta_- \cup \Delta_+$ obtained

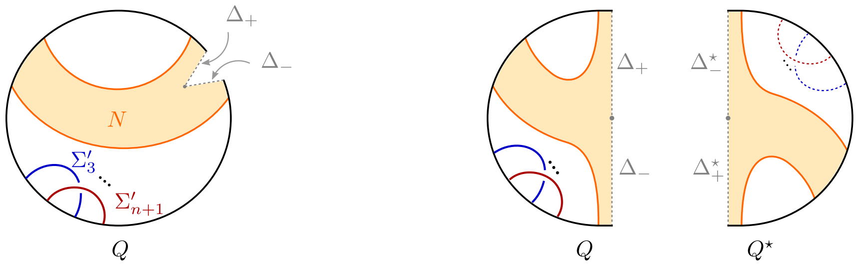

Figure 24 A schematic illustrating how to produce Q and the double cover of B^4 branched over Σ_1'. Left: Obtaining Q by slicing open B^4 along the 3-ball Δ; the push-offs $\Delta_\pm$ of Δ lie in the boundary of Q. Right: Gluing two copies Q and $Q^\star$ to produce the double cover of B^4 branched over Σ_1'. The lifts of the surface components $\Sigma_3', \ldots, \Sigma_{n+1}'$ in $Q^\star$ are dashed because we discard them when constructing the covering surface link.

from cutting along Δ. Fix a diffeomorphism

$$Q \bigcup_{\Delta_- \cup \Delta_+} Q^\star \cong B^4.$$

Note that the 3-ball Δ is disjoint from the surface link components Σ_i' for $i \geq 3$. The inclusion $Q \to Q \bigcup_{\Delta_- \cup \Delta_+} Q^\star$ therefore induces an embedding of $\bigcup_{i \geq 3} \Sigma_i'$ into B^4.

Since we are only aiming to produce a covering surface for Σ' and not reconstruct the entire lift, we discard the copies of the components Σ_i' with $i \geq 3$ induced by the embedding $Q^\star \to Q \bigcup_{\Delta_- \cup \Delta_+} Q^\star$, keeping only those in Q (this is the only meaningful distinction between Q and $Q^\star$).

We still have to understand the lifts of the remaining component Σ_2'. To this end, observe that cutting along Δ separates Σ_2' into two copies of the original disk Σ_1. The copies of these disks in Q are glued to the copies in $Q^\star$ along the arcs where they meet the 3-ball Δ; see the first row of Figure 25. In $Q^\star$, where we have discarded the lifts of Σ_i' for $i \geq 3$, we may isotope these copies of the trivial disk Σ_1 back into the boundary of $Q^\star$ while preserving the arcs where they are glued to the corresponding disks from Q; this is illustrated in the second row of Figure 25. We may then pull the portions of the disks lying in $Q^\star$ back through the gluing region so that the lifts of Σ_2' lie entirely in Q, as in the final row of Figure 25. After this isotopy, these disks become parallel copies of Σ_1. Discarding one of these copies of Σ_1 and identifying $Q \cup Q^\star$ with B^4 as before, we see that the resulting covering link is smoothly equivalent to Σ, as desired. □

Our newfound understanding of coverings of Bing doubles allows us to study other knots and surfaces in the complement of Σ.

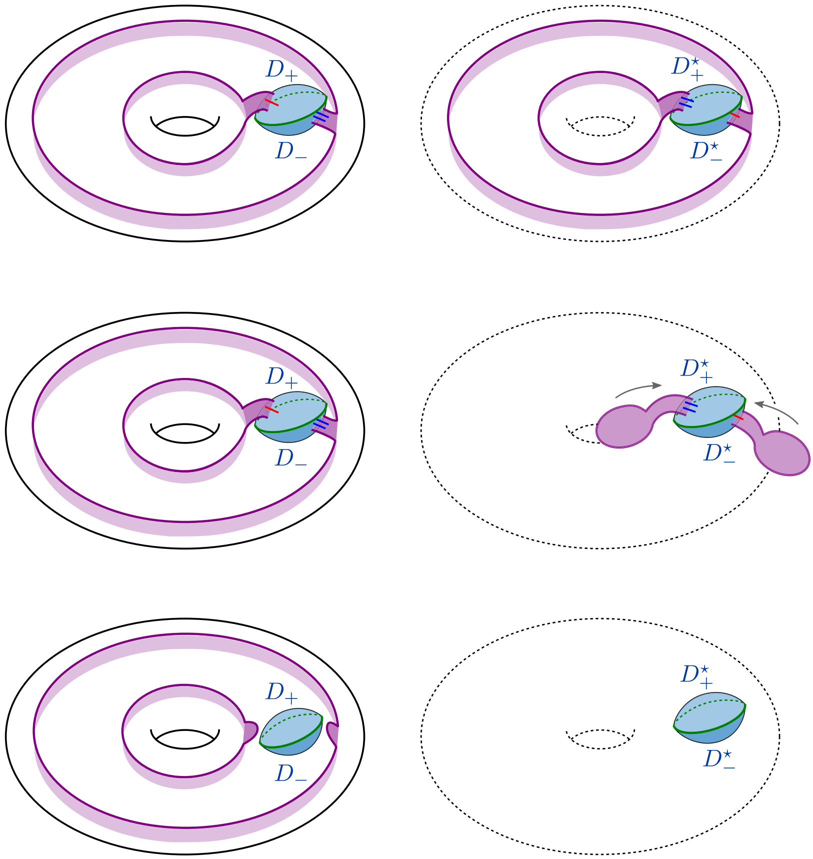

Figure 25 The left and right sides depict $\partial Q \cap U$ and $\partial Q^\star \cap U^\star$, respectively. (Note that ∂Q also contains the 3-ball $\Delta_+ \cup \Delta_-$, which is out of view because it lies on the other side of the 2-sphere $D_+ \cup D_-$; an analogous statement holds in $\partial Q^\star$.) The torus is de-emphasized in $\partial Q^\star \cap U^\star$ because the lifts of $\Sigma_3', \ldots, \Sigma_{n+1}'$ in $Q^\star$ have been deleted. Top row: We see the lifts of Σ_2' in ∂Q and $\partial Q^\star$. Middle row: After deleting the lifts of the components $\Sigma_3', \ldots, \Sigma_{n+1}'$ from $Q^\star$, the copies of Σ_1 in $Q^\star$ can be isotoped into the boundary. This isotopy requires that the copies of Σ_1 in $Q^\star$ temporarily exit the neighborhood $N^\star$ before they can be pulled into the boundary, so they are shown intersecting the dotted torus (which is the corner of $\partial N^\star$). Bottom row: After pulling the portions of Σ_1 in $Q^\star$ through the gluing region, the lifts of Σ_2' lie entirely in Q.

COROLLARY 6.4. *Let Σ be a surface link with first component a trivial disk. Let J be a knot in $S^3 \setminus \nu(\partial\Sigma)$. Suppose that J bounds a smooth disk D into the complement of* $\mathrm{BD}(\Sigma)$. *Then J bounds a smooth disk into the complement of Σ.*

Proof. Let $\Sigma_D := \mathrm{BD}(\Sigma) \cup D$. As usual, call the components of $\mathrm{BD}(\Sigma)$ by the names $\Sigma_1, \dots, \Sigma_{n+1}$, where Σ_1, Σ_2 are the components obtained by taking the Bing double of the first component of Σ. Consider the preimage of $\Sigma_D \smallsetminus \Sigma_1$ in the 2-fold cyclic cover of B^4 branched along Σ_1, as in Lemma 6.3.

For $i = 2, \dots, n+1$, let Σ_i^1 and Σ_i^2 denote the two lifts of Σ_i, and let D^1 and D^2 denote the two lifts of D. By Lemma 6.3, we know that (after perhaps exchanging the names of Σ_i^1 and Σ_i^2 for some values of i) the surface link $\widetilde{\Sigma} := \bigsqcup_{i=2}^{n+1} \Sigma_i^1$ is equivalent to Σ. Moreover, since $\partial\Sigma_1$ is an unknot split from J, the link $\partial\widetilde{\Sigma} \cup \partial D^1$ is equivalent to $\partial\Sigma \cup J$ (perhaps after switching the roles of D^1 and D^2). Then D^1 is a slice disk for J into the complement of Σ, as desired. □

6.2. *JSJ Trees*

We recall some 3-manifold theory that will be used in the forthcoming arguments.

Every compact, irreducible 3-manifold X with toroidal boundary admits a JSJ decomposition into finitely many codimension zero submanifolds, each of which are again compact, irreducible 3-manifolds with toroidal boundary. Each of these submanifolds is either Seifert fibered or has the property that every incompressible torus is boundary parallel [JS79; Joh79]. A JSJ decomposition is obtained by cutting the 3-manifold X along incompressible, nonboundary-parallel tori. A minimal collection of these tori are called *JSJ tori.* The closures of the connected components of the complement of the JSJ tori are called *JSJ pieces.* The JSJ tori are unique up to isotopy and are preserved (although possibly permuted) by any diffeomorphism from X to itself.

The JSJ decomposition of a 3-manifold naturally yields a graph with a vertex corresponding to each JSJ piece and an edge between two vertices if and only if the corresponding pieces share a JSJ torus. When the 3-manifold is the complement of a link in S^3, which is the only case considered here, this graph is a forest of trees [Bud06, Section 4]. The number of connected components of the graph is given by the number of split components of the link.

The next lemma is entirely 3-dimensional, but we state it in terms of surface links in B^4 in order to use our established Bing doubling terminology. Note that the exterior of the Bing double pattern in a solid torus is diffeomorphic to the exterior of the Borromean rings (see Figure 26), so this 3-manifold will appear frequently in our JSJ decomposition arguments.

LEMMA 6.5. *Let Σ be a surface link in B^4. Let T be the tree associated with the JSJ decomposition of $X := S^3 \smallsetminus \nu(\partial\Sigma)$. Let $Y \subseteq X$ be the JSJ piece that contains $\partial\Sigma_1$, and let v_Y be the vertex of T corresponding to Y. Assume that $\partial\Sigma_1$ is not a split unknotted component of $\partial\Sigma$. The JSJ tree of $X_{\mathrm{BD}} := S^3 \smallsetminus \nu(\partial\,\mathrm{BD}(\Sigma))$ is obtained from T by first adding a new vertex v_E corresponding to a JSJ piece diffeomorphic to the exterior E of the Borromean rings, and then adding an edge connecting v_E and v_Y.*

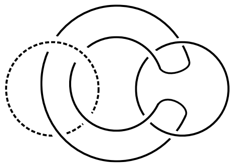

Figure 26 The exterior of the Borromean rings in S^3 can be identified with the exterior of the Bing double of the unknot in the solid torus $S^3 \smallsetminus \nu(U)$, where U denotes the dashed component.

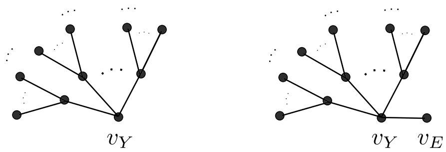

Figure 27 JSJ trees for $S^3 \smallsetminus \nu(\partial\Sigma)$ and $S^3 \smallsetminus \nu(\partial\,\mathrm{BD}(\Sigma))$.

Proof. Let $K := \partial\Sigma_1$ and let F be the torus $\partial\overline{\nu}K$. Note that $X_{\mathrm{BD}} \cong X \bigcup_F E$, where the zero-framed longitude of one Borromean component is glued to the meridian of K on F, and the meridian of the Borromean component becomes the longitude of F. We claim that F is a JSJ torus of X_{BD}. Since K is not a split unknotted component, the longitude of F is nontrivial in $\pi_1(X)$. In X_{BD}, the exterior of the satellite link, the longitude of F is therefore again nontrivial. The meridian of F is nontrivial in X. The surface F is also incompressible in E. Therefore F is a JSJ torus of X_{BD}, as desired. To obtain the JSJ tree for X_{BD}, we start with T, the JSJ tree for X. Then, as shown in Figure 27, we add a new vertex corresponding to E and an edge connecting it to v_Y, since Y is precisely the component that shares a JSJ torus with E. This gives the asserted JSJ tree. □

By repeatedly applying Lemma 6.5, we see that iterated Bing doubling adds a line to the JSJ tree of the link exterior with as many edges as applications of BD. This feature of the tree allows us to restrict the possible automorphisms of the underlying 3-manifold, using the following result, which is a direct consequence of the uniqueness of JSJ decompositions.

THEOREM 6.6. *A diffeomorphism of a link exterior to itself induces an automorphism of the associated JSJ tree. Moreover, each vertex, corresponding to a JSJ piece Y say, must be sent to another vertex corresponding to a JSJ piece Y' such that $Y \cong Y'$.* □

The next lemma refers to the *3-keychain link:* the three-component link consisting of an unknot and two copies of its meridian (see Figure 28). The exterior of this link is diffeomorphic to the cartesian product of a circle S^1 with a connected planar surface with three boundary components (i.e. a pair of pants).

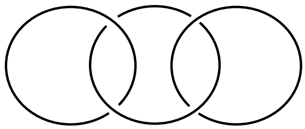

Figure 28 The 3-keychain link.

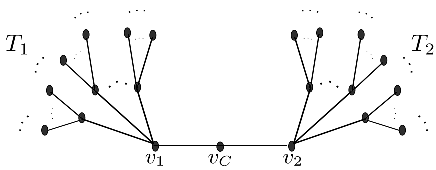

Figure 29 JSJ tree for the exterior of the covering link $\widetilde{L}$ from a JSJ tree for the exterior of L.

LEMMA 6.7. *Let L be an n-component link in S^3 with first component L_1 unknotted. Let $\widetilde{L}$ be the covering link obtained by taking the 2-fold branched cover with branching set the first component of* $\mathrm{BD}(L)$, *forgetting the branching set, and forgetting one of the lifts of* $(\mathrm{BD}(L))_2$ (*i.e. the other component of* $\mathrm{BD}(L_1)$). *The JSJ tree of the exterior of $\widetilde{L}$ is obtained as follows. Let T_1 and T_2 be two copies of the JSJ tree for L, and let v_1 and v_2 be the vertices corresponding to the JSJ piece containing the boundary of a tubular neighborhood of L_1, in each copy. Take $T_1 \sqcup T_2$, add a new vertex v_C, and add two new edges, one joining v_C to v_1 and one joining v_C to v_1. The JSJ piece corresponding to v_C is the 3-keychain link exterior.*

Proof. Observe that $\widetilde{L}$ is isotopic to the result of taking two split copies of L in disjoint copies of B^3 and performing a trivial band sum on the two copies of L_1. This follows from the argument in the proof of Lemma 6.3, and restricting to S^3; see Figure 25. Therefore, the exterior of $\widetilde{L}$ has JSJ decomposition consisting of two copies of the JSJ decomposition for L joined by a central 3-keychain link exterior C. It is a standard fact that, in the JSJ decomposition of exteriors of connected sums, keychain link exteriors appear in between the JSJ pieces of the summands (see e.g. [Sch53]). The conclusion for the resulting JSJ tree is immediate. □

6.3. *Bing Doubling Exotic Surface Link Pairs*

We now arrive at the main goal of this section. By combining our understanding of JSJ decompositions with our work on covering surfaces, as in Section 6.1, we show that if two surfaces are smoothly inequivalent, then under some fairly weak assumptions, the results of Bing doubling one disk component of each are also smoothly inequivalent.

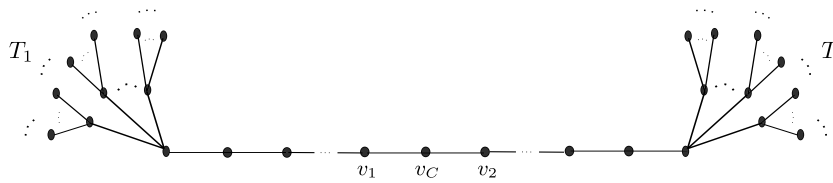

Figure 30 JSJ tree for the exterior of a covering link of an iterated Bing double.

Combining the work in Sections 6.1 and 6.2 yields the following proposition, which shows that the disk links from Section 3 are not just nonequivalent, but in fact, any of their iterated Bing doubles have nondiffeomorphic complements.

PROPOSITION 6.8. *Let* $\Sigma_1 = \mathrm{BD}(D_1)$ *and* $\Sigma_2 = \mathrm{BD}(D_2)$ *be the disk links from Proposition 3.3. For any* $n \geq 0$, $\mathrm{BD}^n(\Sigma_1)$ *and* $\mathrm{BD}^n(\Sigma_2)$ *have nondiffeomorphic complements.*

Proof. Recall that Σ_1 and Σ_2 are each a link of two disks in B^4 whose components are individually unknotted disks, $\partial\Sigma_1 = \partial\Sigma_2$, and Σ_1 and Σ_2 are topologically isotopic rel. boundary.

In Proposition 3.3, we showed that Σ_1 and Σ_2 are not smoothly equivalent by studying a knot γ in $S^3 \setminus \nu(\partial\Sigma_i)$, which is preserved up to isotopy by any automorphism of $(S^3, \partial\Sigma_i)$. The knot γ bounds a smooth disk into the complement of Σ_2 but not into Σ_1, implying that there is no diffeomorphism between (B^4, Σ_1) and (B^4, Σ_2).

Since γ bounds a smooth disk in the complement of Σ_2, γ bounds a smooth disk (e.g. the same disk) into the complement of $\mathrm{BD}^n(\Sigma_2)$ for any $n \geq 0$. Since γ does not bound a smooth disk into the complement of Σ_1, it follows from Corollary 6.4 that γ does not bound a smooth disk into the complement of $\mathrm{BD}^n(\Sigma_1)$ for any $n \geq 0$. Therefore, if there is a diffeomorphism from the complement of $\mathrm{BD}^n(\Sigma_1)$ to the complement of $\mathrm{BD}^n(\Sigma_2)$, it must not preserve the isotopy class of γ. We claim this is not possible.

In Figure 31, we show a JSJ decomposition for $S^3 \setminus \nu(\mathrm{BD}^n(\Sigma_i))$. Recall that $\partial\Sigma_i$ is a Bing double of a hyperbolic knot K (cf. Lemma 3.4), so $S^3 \setminus \nu(\partial\Sigma_i)$ has a JSJ decomposition of one essential torus cutting the manifold into two pieces: the complement of K and a Borromean ring complement. Then $S^3 \setminus \nu(\mathrm{BD}^n(\Sigma_i))$ admits a JSJ decomposition with $n+2$ pieces in a line, as illustrated in Figure 31, with essential tori $T_1, \ldots, T_{n+1}$ in between. The first piece (bounded by T_1) is the complement of K, and every piece thereafter is a Borromean ring complement.

The boundary M^3 of $B^4 \setminus \nu(\mathrm{BD}^n(\Sigma_i))$ is obtained from $S^3 \setminus \nu(\partial\,\mathrm{BD}^n(\Sigma_i))$ by performing 0-framed Dehn filling along every torus boundary component. Note that this Dehn filling on the Borromean rings exterior yields the 3-torus T^3. In this closed 3-manifold M^3, T_{n+1} bounds a copy of $(T^2 \setminus \mathring{D}^2) \times S^1$, T_1 bounds a

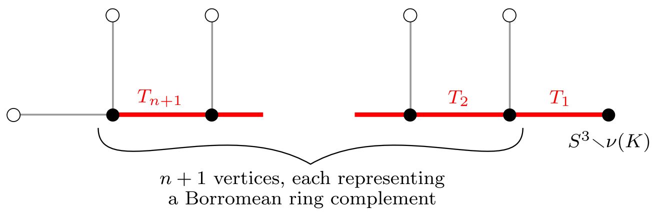

Figure 31 A JSJ decomposition of $S^3 \smallsetminus \nu(\partial L_i)$ as in Proposition 6.8. The bold red edges represent essential tori. The black vertices are the components of the JSJ decomposition. The white vertices represent torus boundary components; grey edges between white and black vertices show to which JSJ component the corresponding boundary belongs.

copy of $S^3 \smallsetminus \nu(K)$, and $T_i \sqcup T_{i+1}$ bounds

$$T^3 \smallsetminus \nu(\text{two independent primitive curves})$$

for $i = 1, \dots, n$. This is again a JSJ decomposition for M^3. Since the complement of K is not homeomorphic to any other piece in this JSJ decomposition, we conclude that any automorphism of M^3 preserves T_1 up to isotopy and induces an automorphism of $S^3 \smallsetminus \nu(K)$. Since $S^3 \smallsetminus \nu(K)$ has trivial automorphism group (Lemma 3.4), we conclude that γ is preserved up to isotopy by any automorphism of M^3. Therefore, there is no diffeomorphism from the complement of $\mathrm{BD}^n(\Sigma_1)$ to the complement of $\mathrm{BD}^n(\Sigma_2)$, as desired. □

Proposition 6.8 allows us to immediately conclude that for every $n \geq 0$, $\mathrm{BD}^n(\Sigma_1)$ and $\mathrm{BD}^n(\Sigma_2)$ is an exotic pair of $(n+1)$-component Brunnian links. Next, we prove our main technical result on covering surfaces; it has the advantage that it is applicable to a more general family of surface links, for example, all of the families in Section 4 and those in Theorem 5.3.

PROPOSITION 6.9. *Let $\Sigma = \Sigma_1 \sqcup \Sigma_2$ and $\Sigma' = \Sigma'_1 \sqcup \Sigma'_2$ be 2-component surface links in B^4 with Σ_1 and Σ'_1 trivial disks. Suppose that $\partial\Sigma = \partial\Sigma' \subseteq S^3$, and let X be the link exterior in S^3. Suppose that either:*

(i) *X has no Borromean rings exterior and no 3-keychain link exterior in its JSJ decomposition; or*
(ii) *X has no 3-keychain link exterior and exactly one Borromean rings exterior E in its JSJ decomposition, but two of the three boundary components of E are ∂X.*

Then, if the iterated Bing doubles $\mathrm{BD}^n(\Sigma)$ and $\mathrm{BD}^n(\Sigma')$ are smoothly equivalent (Bing doubling the first component at each iteration), then so are Σ and Σ'.

Proof. Let $S := \mathrm{BD}^n(\Sigma)$ and $S' := \mathrm{BD}^n(\Sigma')$. The first components S_1 and S'_1 are both unknotted disks since they arose from Bing doubling. Consider the 2-fold branched coverings $p\colon B^4 \to B^4$ and $p'\colon B^4 \to B^4$ with branching sets the first components S_1 and S_1' of each surface link, respectively. Since first components are unknotted, the branched covering spaces are indeed again B^4.

By Lemma 6.3, $\mathrm{BD}^{n-1}(\Sigma)$ is a covering surface link of $S = \mathrm{BD}^n(\Sigma)$. Moreover, $\mathrm{BD}^{n-1}(\Sigma)$ is a sublink of the inverse image $p^{-1}(S \setminus S_1)$. Analogously, $\mathrm{BD}^{n-1}(\Sigma')$ is a sublink of the inverse image $p^{-1}(S' \setminus S_1')$.

Let $F\colon (B^4, S) \to (B^4, S')$ be a diffeomorphism of pairs. We claim that it must send S_1 to either S_1' or S_2'.

To prove the claim, we consider the JSJ decomposition and tree for $S^3 \setminus \nu(\partial S)$. There is a unique univalent vertex in the JSJ tree for $S^3 \setminus \nu(\partial S)$ that corresponds to a copy of the exterior E of the Borromean rings. This vertex must be sent to the analogous vertex in the JSJ tree for $S^3 \setminus \nu(\partial S')$ by Theorem 6.6. The two boundary components of $S^3 \setminus \nu(\partial S')$ that lie in this copy of E are S_1' and S_2'. It follows that $F(S_1)$ is S_1' or S_2'. After potentially relabeling, we can and shall assume, without loss of generality, that the diffeomorphism F in fact sends S_1 to S_1'.

Since the diffeomorphism $F\colon (B^4, S) \to (B^4, S')$ preserves the branching set (i.e. $F(S_1) = S_1'$), we may lift F to a diffeomorphism of pairs

$$\widetilde{F}\colon (B^4, p^{-1}(\mathrm{BD}^n(\Sigma))) \to (B^4, p^{-1}(\mathrm{BD}^n(\Sigma'))).$$

In the inverse image, $p^{-1}(\mathrm{BD}^n(\Sigma))$, we see two copies of each component of $\mathrm{BD}^{n-1}(\Sigma)$. Restrict the diffeomorphism $\widetilde{F}$ to

$$\mathrm{BD}^{n-1}(\Sigma) \subseteq p^{-1}(S \setminus S_1) \subseteq p^{-1}(S) = p^{-1}(\mathrm{BD}^n(\Sigma)).$$

The image is a sublink of $p^{-1}(S' \setminus S_1')$. We claim that the image is a copy of $\mathrm{BD}^{n-1}(\Sigma')$. Assuming this claim, we are done: by induction, we see that Σ and Σ' are smoothly equivalent, as desired.

To prove the claim, we again argue by considering the effect of the diffeomorphism restricted to the link exteriors in S^3

$$\widetilde{F}\,|\colon S^3 \setminus \nu(\partial(p^{-1}(S \setminus S_1))) \to S^3 \setminus \nu(\partial(p^{-1}(S' \setminus S_1'))).$$

Since the links on the boundary coincide, so do their covering links. We will restrict how the diffeomorphism can permute components using the relative rigidity of 3-manifold diffeomorphisms, in particular by studying the JSJ decomposition and applying Theorem 6.6 again.

Lemma 6.7 provides a description of the JSJ tree of the covering link obtained by forgetting the branching set and one of the lifts of ∂S_2 and $\partial S_2'$. As before, we choose a lift of S_2, and then the lift of S_2' is determined by the diffeomorphism F. The automorphisms of the JSJ tree of this covering link are given by composing automorphisms of the two copies of the JSJ tree for the exterior of ∂S with either:

(i) the identity, or
(ii) a flip map that switches the two subtrees that are copies of the JSJ tree for the exterior of ∂S.

To see this, we consider the structure of the JSJ trees. The JSJ tree $T(n-1)$ of the exterior of $\partial\,\mathrm{BD}^{n-1}(\Sigma)$ is obtained by iterating the procedure pictured in Figure 27 and consists of the JSJ tree for X with a tail of $(n-1)$ Borromean exteriors attached to the vertex corresponding to the JSJ piece containing $\partial\Sigma_1$. The JSJ tree for the covering link is pictured in Figure 30 and is obtained from two copies of $T(n-1)$ by adding one more vertex corresponding to the 3-keychain link exterior, with two edges connecting this new vertex to the ends of both tails, as described in Lemma 6.7 and shown in Figure 29. By our assumptions on the JSJ decomposition of X, the Borromean and 3-keychain link exteriors cannot be replicated anywhere in the tree but at its center, and the tails must get either fixed or swapped with each other by $\widetilde{F}$. The link components, which have boundaries of tubular neighborhoods that lie in JSJ pieces corresponding to one half of the tree, form a copy of $\mathrm{BD}^{n-1}(\Sigma)$ in the domain and form a copy of $\mathrm{BD}^{n-1}(\Sigma')$ in the codomain. Determining how $\widetilde{F}$ acts on connected components of our surface links is enough to control how $\widetilde{F}$ acts on their boundary components. Therefore, we have completed the proof of the claim that the image $\widetilde{F}(\mathrm{BD}^{n-1}(\Sigma))$ is a copy of $\mathrm{BD}^{n-1}(\Sigma')$. This proves the inductive step and therefore completes the proof of the proposition. □

Finally, we prove the following, which is a more specific version of the statement of Theorem 1.1 in Section 1. The version stated in Section 1 immediately follows.

THEOREM 1.1. *Let Σ and Σ' be an exotic pair of Brunnian 2-component surface links constructed as in Theorem 1.2 or Theorem 1.3, with orderings such that the first components are disks. For all $n \geq 3$, the n-component surface links $\mathrm{BD}^{n-2}(\Sigma)$ and $\mathrm{BD}^{n-2}(\Sigma')$ are an exotic pair of Brunnian surface links.*

The disk links in Theorem 1.2 arose by taking two slice disks D_1 and D_2 in B^4 for a fixed knot K in S^3, and taking the Bing doubles $\Sigma := \mathrm{BD}(D_1)$ and $\Sigma' := \mathrm{BD}(D_2)$. This is an exotic pair of Brunnian disk links. Theorem 1.1 states that iterated Bing doubling both of these links yields an exotic pair with any specified number of components.

The surface disk links in Theorem 1.3 were constructed by taking a particular 2-component surface link Σ consisting of a disk and a genus one surface, and applying 1-twisted rim surgery (using a knot J) along a curve α that bounds a locally flat disk in the exterior of Σ but does not bound any smoothly embedded disk in the exterior. By varying J, we obtain exotic surface links. Theorem 1.1 shows that applying Bing doubling to the disk component of Σ produces n-component surface links such that the same rim surgeries give rise to an infinite family of pairwise exotic surface links.

Proof of Theorem 1.1. By Lemma 2.2, the surface links $\mathrm{BD}^{n-2}(\Sigma)$ and $\mathrm{BD}^{n-2}(\Sigma')$ are still Brunnian. Moreover, Lemma 2.1 (with CAT = TOP) ensures they are still topologically isotopic (we may assume the doubling is done such that the boundaries are the same link in S^3).

If the assumptions of Proposition 6.9 hold, then $\mathrm{BD}^{n-2}(\Sigma)$ and $\mathrm{BD}^{n-2}(\Sigma')$ being smoothly equivalent implies that Σ and Σ' are too, which is a contradiction. Therefore we need to check that the hypotheses of the proposition hold. For the disk links of Theorem 1.2, the JSJ decomposition of the exterior of $\partial\Sigma = \partial\Sigma'$ consists of a Borromean exterior and the exterior of a hyperbolic knot

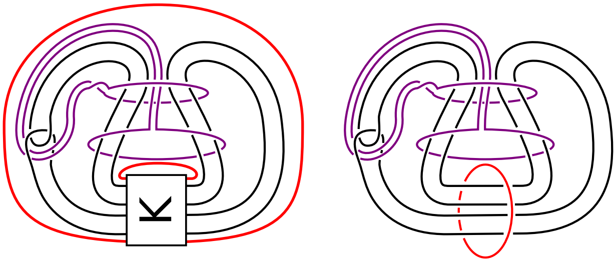

Figure 32 Left: An essential torus in the exterior of $\partial\Sigma$, where Σ is the surface link from Theorem 1.3. This torus splits the exterior of $\partial\Sigma$ into the exterior of K glued to the exterior of a hyperbolic 3-component link L. Right: The link L.

(Lemma 3.4), and so Proposition (6.9)(ii) is satisfied, and therefore the proposition applies.

For the surface links of Theorem 1.3, we note that their boundary is a satellite of the knot K, as shown on the left of Figure 32. In this case, the JSJ decomposition of $\partial S = \partial S'$ consists of the exterior of the hyperbolic (verified by SnapPy [C+]) 3-component link L illustrated on the right of Figure 32 along with a JSJ decomposition of the exterior of K in S^3. Since L has pairwise zero linking numbers, the exterior of L is not the exterior of a 3-keychain link. Moreover, as verified by SnapPy [C+], the exterior of L has symmetry group $\mathbb{Z}/2$, so it is also not a Borromean rings exterior. Taking K to be (for example) the positive untwisted Whitehead double of the trefoil, whose exterior admits a JSJ decomposition consisting of a trefoil exterior and a Whitehead link exterior, we find that Proposition (6.9)(i) is satisfied and the proposition applies. This completes the proof; see Appendix B for further documentation of this use of SnapPy. □

7. Links of Other Brunnian Types

In this section, we apply our techniques to construct exotic surfaces of any Brunnian type, in the language of Debrunner [Deb61].

Definition 7.1. Let Σ be an n-component surface link in B^4. Fix an integer k with $1 \le k \le n$. The surface link Σ is (n,k)-*Brunnian* if every sublink of Σ with fewer than k components is an unlink of surfaces, whereas every sublink of Σ with at least k components is nontrivial.

We adapt the Bing doubling construction of Brunnian links (in the usual sense) to produce (n,k)-Brunnian links.

Procedure 7.2. *Constructing an* (n,k)*-Brunnian surface link.* Let D be a disk in B^4 whose boundary is a nontrivial knot in S^3. To simplify the exposition, we let n_k denote $\binom{n}{k}$. Let $\Sigma_{n,k}(D)$ be the n-component surface link constructed as follows (see also Figures 33 and 34).

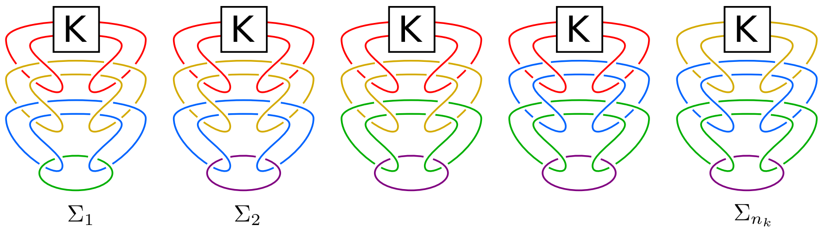

Figure 33 To build $\Sigma_{n,k}(D)$, we start with a split union of n_k copies $\Sigma_1, \ldots, \Sigma_{n_k}$ of $\mathrm{BD}^{k-1}(D)$. Here we draw the boundaries of $\Sigma_1 \sqcup \cdots \sqcup \Sigma_{n_k}$, where $K = \partial D$. In addition, we color every component of each Σ_i with a color in $\mathcal{C}$, such that (1) no two components of Σ_i may be the same color, and (2) for $i \neq j$, the links Σ_i and Σ_j may not use the same subset of colors in $\mathcal{C}$. In this figure, $(n, k) = (5, 4)$.

(1) For $i = 1, 2, \ldots, n_k$, let Σ_i be a copy of $\mathrm{BD}^{k-1}(D)$. Arrange $\Sigma_1, \ldots, \Sigma_{n_k}$ from left to right, as n_k split copies of $\mathrm{BD}^{k-1}(D)$.
(2) Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ indicate a set of n distinct colors. For each i, color every component of Σ_i a color in $\mathcal{C}$ so that no two components of Σ_i are the same color and so that for $i \neq j$, the surface links Σ_i and Σ_j are not colored with the same size-k subset of $\mathcal{C}$. See Figure 33 for an example.
(3) Let $B_1, \ldots, B_{n_k}$ be disjoint 4-balls containing $\Sigma_1, \ldots, \Sigma_{n_k}$, respectively. We now band all components of $\Sigma_1 \sqcup \cdots \sqcup \Sigma_{n_k}$ of each single color together to form an n-component link of disks, such that the bands are chosen to be trivial near their ends in the B_is and disjoint from other B_js.

Said differently: for each i and j: when Σ_i includes a color-C_j component and there is a color-C_j component in Σ_s for some $s > i$, let $i_j = \min\{s > i \mid \Sigma_s \text{ has a color-}C_j \text{ component}\}$ and band the color-C_j components in Σ_i and Σ_{i_j} together by a band that is trivial in B_i and B_{i_j} and does not intersect any other B_l. See Figure 34.

When restricted to S^3, this construction differs from Debrunner's [Deb61]; in particular, we elect to use Bing doubles to construct our generalized Brunnian links. Notice that if $n = k$, $n_n = 1$, hence $\Sigma_{n,n}(D) = \mathrm{BD}^{n-1}(D)$. When $k = 1$, $\Sigma_{n,1}(D)$ is a split union of n copies of D.

PROPOSITION 7.3. *The surface link* $\Sigma := \Sigma_{n,k}(D)$ *from Procedure 7.2 is* (n, k)*-Brunnian.*

Proof. The proof of Proposition 7.3 naturally divides into two claims.

CLAIM 1. *Let* Σ' *be a* $(k-1)$*-component sublink of* Σ*. Then* Σ' *is an unlink of disks.*

Proof of Claim 1. By construction, each component of Σ is colored by a different element of a size-n set $\mathcal{C}$. Suppose that Σ' is obtained from Σ by deleting the

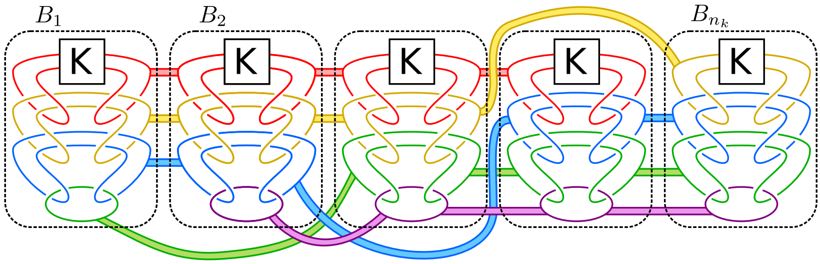

Figure 34 Starting from Figure 33, to complete our construction of $\Sigma_{n,k}(D)$, we band together all C_i-colored components of $\Sigma_1 \sqcup \cdots \sqcup \Sigma_{n_k}$ for each $C_i \in \mathcal{C}$. The bands are taken to be trivial in the 4-balls $B_1, \ldots, B_{n_k}$ but otherwise may be chosen freely so long as they are disjoint. The result is an n-component link of disks that we call $\Sigma_{n,k}(D)$.

components of colors $\mathcal{C}_r \subseteq \mathcal{C}$, where $|\mathcal{C}_r| = n - k + 1$. Since $\mathcal{C}$ has n colors and each Σ_i has k distinct colors, we conclude that Σ' is obtained from $\Sigma = \bigcup_{\text{bands}} \Sigma_i$ by deleting a nonzero number of components from each Σ_i.

Note that

$$\Sigma' = \bigcup_{\text{along bands}}^{i \geq 1} \Sigma_i \setminus (\text{components of colors } \mathcal{C}_r).$$

In Figure 35 (left), we depict the ball neighborhood B_1 of Σ_1 (the leftmost copy of $\mathrm{BD}^{k-1}(D)$ in the construction of Σ). The ball B_1 meets the bands in Σ', which are attached to Σ_1, trivially: all bands extend from Σ_1 towards the right. The surface $B_1 \cap \Sigma'$ is obtained from $B_1 \cap \Sigma_1$ by deleting at least one component of Σ_1 (and any attached bands). Thus, in B_1, we may isotope $B_1 \cap \Sigma'$ rel. $(\partial \overline{B_1}) \cap \mathring{B}^4$ to obtain trivial disks, as indicated in Figure 35. (Note that although we only draw the boundary of Σ' in Figure 35, once any component of $\Sigma' \cap B_1$ is deleted, the remaining components are determined up to smooth isotopy rel. boundary as trivial disks in B^4.) We conclude

$$\Sigma' \approx \bigcup_{\text{along bands}}^{i \geq 2} \Sigma_i \setminus (\text{components of colors } \mathcal{C}_r),$$

where we use "$\approx$" to denote smooth isotopy throughout.

Now proceed by induction, assuming

$$\Sigma' \approx \left(\bigcup_{\text{along bands}}^{i \geq s} \Sigma_i \setminus (\text{components of colors } \mathcal{C}_r) \right) \cup (\text{an unlink of disks})$$

for some $s > 1$ (up to smooth isotopy). In Figure 35 (right), we depict the ball neighborhood B_s of Σ_s. The ball B_s meets bands in Σ' trivially attached to Σ_s, with all bands extending from Σ_s toward the right. Note that some components of Σ_s may not have any bands extending to the right—in the construction of Σ, this means that such a component is the last of its color. The surface $\Sigma' \cap B_s$ is obtained from $\Sigma_s \cup$ (bands extending to the right) by deleting at least one component (and any attached bands). Then, in B_s, we may isotope $B_s \cap \Sigma'$ rel. $(\partial \overline{B}_s) \cap \mathring{B}^4$ to obtain trivial disks as indicated in Figure 35. Note that some disks might be split in the interior of B_s. We conclude that

$$\Sigma' \approx \Bigg(\bigcup_{\text{along bands}}^{i \geq s+1} \Sigma_i \setminus (\text{components of colors } \mathcal{C}_r) \Bigg) \cup (\text{an unlink of disks}).$$

Inductively, we obtain

$$\begin{aligned} \Sigma' &\approx (\Sigma_{n_k} \setminus (\text{components of colors } \mathcal{C}_r)) \cup (\text{an unlink of disks}) \\ &\approx \text{an unlink of disks} \end{aligned}$$

since Σ_{n_k} is Brunnian and includes at least one color in $\mathcal{C}_r$. This completes the proof of the claim.

CLAIM 2. *Let Σ'' be a k-component sublink of Σ. Then Σ'' is nontrivial and, in fact, Σ'' is smoothly isotopic to* $\mathrm{BD}^{k-1}(D)$.

Proof of Claim 2. First, note that ∂D is a nontrivial knot, so $\partial\, \mathrm{BD}^{k-1}(D)$ is not an unlink, and $\mathrm{BD}^{k-1}(D)$ is not an unlinked disk link.

The proof of Claim 2 is similar to that of Claim 1. Suppose that Σ'' is obtained from Σ by deleting the components of colors $\mathcal{C}_r \subseteq \mathcal{C}$, where $|\mathcal{C}_r| = n - k$. Since $\mathcal{C}$ has n colors and each Σ_i has k distinct colors, we conclude that Σ'' is obtained from $\Sigma = \bigcup_{\text{bands}} \Sigma_i$ by deleting a nonzero number of components from each Σ_i for all $i \neq m$, where m is the unique integer such that Σ_m is colored with all of the colors in $\mathcal{C} - \mathcal{C}_r$. Again, we write

$$\Sigma'' = \bigcup_{\text{along bands}}^{i \geq 1} \Sigma_i \setminus (\text{components of colors } \mathcal{C}_r).$$

Applying the proof of the previous claim applied to Σ_1 and working from left to right,

$$\Sigma'' \approx \Bigg(\bigcup_{\text{along bands}}^{i \geq m} \Sigma_i \setminus (\text{components of colors } \mathcal{C}_r) \Bigg) \cup (\text{an unlink of disks}).$$

By horizontally mirroring the argument and applying it to Σ_{n_k} and working from right to left, we find

$$\begin{aligned} \Sigma'' &\approx (\Sigma_m \setminus (\text{components of colors } \mathcal{C}_r)) \cup (\text{an unlink of disks}) \\ &\approx \Sigma_m \cup \text{an unlink of disks} \\ &\approx \Sigma_m \approx \mathrm{BD}^{k-1}(D), \end{aligned}$$

since Σ'' and Σ_m each have k components. Since $\mathrm{BD}^{k-1}(D)$ is a nontrivial disk link, this completes the proof of the claim.

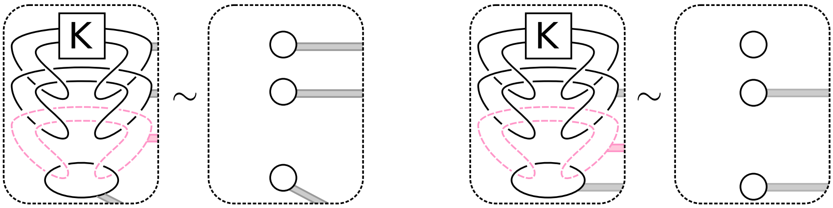

Figure 35 Left: The ball B_1 intersecting Σ'. If we delete at least one component, for example, the one highlighted, then we obtain a trivial link of disks with trivial bands extending out of B_1. Right: The ball B_s intersecting Σ' after applying an inductive hypothesis in the proof of Proposition 7.3. If we delete at least one component, for example, the one highlighted, then we obtain a trivial link of disks with trivial bands extending from some components out of B_s.

This completes the proof of the proposition. □

THEOREM 1.6. *For any integers n and k with $n \geq 2$ and $1 \leq k \leq n$, there exists a pair of (n, k)-Brunnian disk links in B^4 forming an exotic pair.*

Proof. Let D_1 and D_2 be the exotic pair of disks of Proposition 3.1. Recall that D_1 and D_2 are topologically isotopic rel. boundary. We construct $\Sigma := \Sigma_{n,k}(D_1)$ and $\Sigma' := \Sigma_{n,k}(D_2)$ as in Procedure 7.2, such that all choices of colors and bands agree in S^3, and $\partial\Sigma = \partial\Sigma'$. By Proposition 7.3, Σ and Σ' are (n, k)-Brunnian.

The disks D_1 and D_2 are topologically isotopic rel. boundary; applying Lemma 2.1, so are $\mathrm{BD}^{k-1}(D_1)$ and $\mathrm{BD}^{k-1}(D_2)$. Since Σ and Σ' are obtained by banding together copies of $\mathrm{BD}^{k-1}(D_1)$ and $\mathrm{BD}^{k-1}(D_2)$ with the same choices of bands in S^3, we conclude that Σ and Σ' are topologically isotopic rel. boundary.

Suppose that $f : (B^4, \Sigma) \to (B^4, \Sigma')$ is a diffeomorphism, and let Σ_k be a k-component sublink of Σ. Thus, $f(\Sigma_k)$ is a k-component sublink of Σ'. By the proof of Claim 2 within Proposition 7.3, we know that Σ_k is smoothly isotopic to $\mathrm{BD}^{k-1}(D_1)$ and $f(\Sigma_k)$ is smoothly isotopic to $\mathrm{BD}^{k-1}(D_2)$. This contradicts Theorem 1.1; we conclude there is no diffeomorphism from (B^4, Σ) to (B^4, Σ'). □

8. Closed Surfaces

We show that in some cases our exotic Brunnian surface links may be promoted to exotic links of closed surfaces in 4-manifolds with positive second Betti number.

THEOREM 8.1. *Let Σ_1 and Σ_2 be the 2-component exotic disk links in B^4 from Proposition 3.3. Fix a nonnegative integer n. Let X^4 be the 4-manifold obtained from B^4 by attaching a 0-framed 2-handle along every boundary component of* $\mathrm{BD}^n(\Sigma_i)$. *Let $S_i \subseteq X$ be the $(n+1)$-component link of 2-spheres obtained from*

$\mathrm{BD}^n(\Sigma_i)$ *by gluing a core of each 2-handle to the corresponding component of* $\mathrm{BD}^n(\Sigma_i)$. *Then* S_1 *and* S_2 *are topologically isotopic in* X, *but* (X, S_1) *is not diffeomorphic to* (X, S_2).

Proof. Since S_1 and S_2 agree outside $\mathrm{BD}^n(\Sigma_1)$ and $\mathrm{BD}^n(\Sigma_2)$, which are topologically isotopic rel. boundary, it is immediate that S_1 and S_2 are topologically isotopic.

Let Y_i^4 be the 4-manifold obtained from X by surgering each component of S_i (i.e. for each component F of Σ_i, delete $\nu(F) \cong S^2 \times \mathring{D}^2$ and reglue $B^3 \times S^1$ to obtain Y_i). We claim that $Y_i \cong B^4 \smallsetminus \nu(\mathrm{BD}^n(\Sigma_i))$. This can be seen via Kirby calculus [GS99, Section 5.4]: attaching a 0-framed 2-handle along an unknot and then surgering along a 2-sphere consisting of a trivial disk in B^4 glued to a core of that 2-handle has the effect in a Kirby diagram of first adding a 0-framed circle (the 2-handle) and then changing the 0 to a dot. The latter represents the result of carving out a trivial disk from B^4 with the specified boundary. To see that the 0-dot exchange on an unknot realizes the desired surgery, note that surgery on $S^2 \times \{0\} \subseteq S^2 \times D^2$ yields $S^2 \times S^1 \times [0, 1] \bigcup_{S^1 \times S^1 \times \{1\}} B^3 \times S^1 \cong B^3 \times S^1$.

If (X, S_1) were diffeomorphic to (X, S_2), then Y_1 would be diffeomorphic to Y_2. But Proposition 6.8 says precisely that Y_1 is not diffeomorphic to Y_2, so we conclude that (X, S_1) and (X, S_2) are not diffeomorphic. □

In the 2-component case we can obtain the analogous result to Theorem 8.1 in a closed 4-manifold Z.

THEOREM 8.2. *The 2-component disk links* $\Sigma_1 = \mathrm{BD}(D_1)$ *and* $\Sigma_2 = \mathrm{BD}(D_2)$ *from Proposition 3.3 smoothly embed into 2-component sphere links* S_1 *and* S_2, *respectively, in a closed 4-manifold* Z *such that*

(i) S_1 *and* S_2 *are topologically isotopic in* Z, *and*

(ii) (Z, S_1) *is not diffeomorphic to* (Z, S_2).

Proof. Much as in the proof of Theorem 8.1, we begin by attaching 0-framed 2-handles to B^4 along the two unknotted components of the link $\mathrm{BD}(K)$, where K is the underlying slice knot from the proof of Proposition 3.3. Denote this 4-manifold by X. As in the proof of Theorem 8.1, the disk links $\mathrm{BD}(D_1)$ and $\mathrm{BD}(D_2)$ give rise to a pair of exotic sphere links S_1 and S_2 in X such that surgering S_i turns X into the disk link exterior $B^4 \smallsetminus \nu(\mathrm{BD}(D_i))$.

Our strategy will be to embed X into a closed 4-manifold Z by constructing a well-chosen 4-dimensional "cap" C with $\partial C = \partial X$ (but oriented so that $\partial C = -\partial X$). The induced sphere links S_1 and S_2 will remain topologically isotopic in the larger 4-manifold Z. To show that S_1 and S_2 also remain smoothly distinct, we will distinguish the 4-manifolds $B^4 \smallsetminus \nu(\mathrm{BD}(D_1)) \cup C$ and $B^4 \smallsetminus \nu(\mathrm{BD}(D_2)) \cup C$ obtained by surgering Z along S_1 and S_2, respectively. The cap C will be constructed with this goal in mind.

To this end, recall from the proof of Proposition 3.3 that we can attach three 2-handles to $B^4 \smallsetminus \nu(\mathrm{BD}(D_1))$ to produce a 4-manifold W that admits a Stein structure; see Figure 9. For later use, we note that one of these is a -1-framed

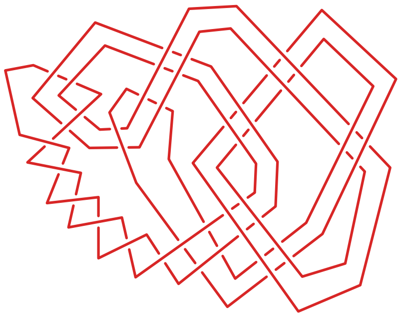

Figure 36 The diagram of K input into SnapPy.

2-handle attached along the curve γ from Figure 11. By [LM97], we can then further embed the Stein domain W into a closed, minimal Kähler surface, which we denote by Q; here *minimality* implies that Q contains no smoothly embedded 2-spheres of self-intersection -1. We define the desired cap C to be the exterior of $B^4 \smallsetminus \nu(\mathrm{BD}(D_1))$ in Q, and let $Z = X \cup C$ as discussed before.

It remains to distinguish the 4-manifolds obtained from Z by surgering S_1 and S_2. The resulting 4-manifolds are $B^4 \smallsetminus \nu(\mathrm{BD}(D_1)) \cup C$ (which is the minimal Kähler surface Q) and $B^4 \smallsetminus \nu(\mathrm{BD}(D_2)) \cup C$. To distinguish them, we first recall that the curve γ bounds a smoothly embedded disk in the complement of $\mathrm{BD}(D_2)$. After gluing C to $B^4 \smallsetminus \nu(\mathrm{BD}(D_2))$, this disk can be glued to the core of the 2-handle attached along γ to yield a smoothly embedded 2-sphere of self-intersection -1 in $B^4 \smallsetminus \nu(\mathrm{BD}(D_2)) \cup C$. It follows that surgering Z along S_1 and S_2 yields distinct 4-manifolds, so the pairs (Z, S_1) and (Z, S_2) are not diffeomorphic. □

A. The Isometry Group of K

We began by drawing K in SnapPy's link editor, as depicted in Figure 36, and extracted a Dowker–Thistlethwaite code for this knot projection:

$$\mathrm{DT}{:}[(-70, 20, -84, -82, -80, 42, -32, 56, -66, 2, -44, 18, 36, -52, 40, 76, -16, -64, 28, -60, -78, 68, -22, -12, -30, 62, -26, -74, 14, 50, -38, 54, 72, 46, -4, -24, 34, -58, 48, -10, -8, -6)].$$

Running SnapPy inside Sage, we then verified that K is a hyperbolic knot whose complement has a trivial isometry group.

```
sage: import snappy
sage: K = snappy.Manifold('DT:[(-70,20,-84,-82,-80,
      42,-32,56,-66,2,-44,18,36,-52,40,76,-16,-64,28,
      -60,-78,68,-22,-12,-30,62,-26,-74,14, 50,-38,54,
      72,46,-4,-24,34,-58,48,-10,-8,-6)]')
sage: K.solution_type()
```

```
'all tetrahedra positively oriented'
sage: K.verify_hyperbolicity()
(True, [0.69085717467? + 0.50830991237?*I,
-0.09695795674? + 0.91647294852?*I,
0.97074783390? + 0.28150095915?*I, 1.26374636309?
+ 0.55883319058?*I,
0.13653313161? + 0.57669418470?*I, 0.18676207068?
+ 0.42212622171?*I,
1.71203281718? + 0.96878529641?*I, 0.16077780284?
+ 2.5757092416?*I,
-0.35397677376? + 0.58988805254?*I, -0.03195090969?
+ 1.66462568439?*I,
-0.7299955531? + 1.9313492842?*I, 0.16909543612?
+ 0.10116074904?*I,
-1.1477637844? + 1.2694777149?*I, 0.23479520751?
+ 0.83739365476?*I,
-0.6690405348? + 0.68508313707?*I, 1.6959300535?
+ 2.5175042223?*I,
-0.9638799118? + 2.3142283467?*I, 1.11640893998?
+ 0.55523026497?*I,
0.7891261692? + 1.31673271484?*I, 0.98323205817?
+ 1.45469490101?*I,
-0.56189716658? + 0.44115682612?*I, -0.4883656525?
+ 2.1993281961?*I,
0.19274256226? + 0.71697136107?*I])
sage: R = K.canonical_retriangulation(verified = True)
sage: len(R.isomorphisms_to(R))
     #This gives the size of the isometry group.
1
```

The size of the isometry group is 1, so the identity is the unique isometry of $S^3 \setminus K$.

B. The Isometry Group of L

As in Appendix A, we began by drawing L in SnapPy's link editor, as depicted in Figure 37, and extracted a Dowker–Thistlethwaite code for this knot projection:

$$\begin{aligned}\text{DT:}[&(16, 34, -64, 54, 40, 68, -44, -60, 32, 50, 42, -66, -38, -56, 70, -48),\\ &(30, -58, 8, -24, -12, 20, 52, -2, -62, -14, -36, -26, 6, 46, 18),\\ &(-4, 28, 10, -22)].\end{aligned}$$

Running SnapPy inside Sage, we then verified that L is a hyperbolic link whose complement has an isometry group $\mathbb{Z}/2\mathbb{Z}$.

```
sage: import snappy
sage: L = snappy.Manifold('DT:[(16,34,-64,54,40,68,
```

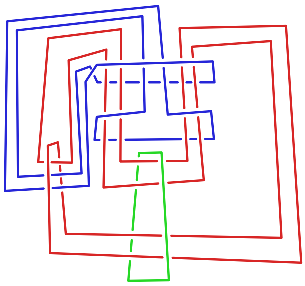

Figure 37 The diagram of L input into SnapPy.

```
-44,-60,32,50,42,-66,-38,-56,70,-48), (30,-58,8,-24,-
12,20,52,-2,-62,-14,-36,-26,6,46,18),
(-4,28,10,-22)]')
sage: L.solution_type()
'all tetrahedra positively oriented'
sage: L.verify_hyperbolicity()
(True, [0.63324006006? + 1.10322026006?*I,
0.10993343042? + 0.47600086029?*I,
0.16413577877? + 0.79246676123?*I, 0.66490644413?
+ 1.23999412266?*I,
0.38972085196? + 0.84475722622?*I, -0.02052870871?
+ 0.71226998786?*I,
-1.09458686743? + 1.17956342451?*I, 1.08258208868?
+ 0.80923399264?*I,
0.07133786888? + 0.68448305451?*I, 0.63642037823?
+ 0.39539813066?*I,
0.31403980641? + 1.10759837099?*I, 0.40489568556?
+ 0.46123907137?*I,
0.16243243753? + 0.76871625189?*I, 0.38972085196?
+ 0.84475722622?*I,
0.66413540183? + 0.62635898843?*I, 0.50355806795?
+ 0.72289062001?*I,
0.50203604063? + 0.63579926026?*I, 1.13580695569?
+ 1.15030349850?*I,
0.07522244611? + 0.83274531566?*I, -0.09975044831?
+ 0.76471739458?*I,
-0.38903974680? + 0.93113075790?*I, 0.20490074235?
+ 1.28331759380?*I,
1.63503890307? + 1.75198642689?*I, 0.25525121024?
+ 0.99618331677?*I,
0.38848526399? + 0.87511804068?*I, 0.81713558269?
+ 0.50449818986?*I,
```

```
0.33037527178? + 0.55709566053?*I, 0.6350389031?
+ 1.75198642689?*I,
0.33037527178? + 0.55709566053?*I])
sage: R = L.canonical_retriangulation(verified = True)
sage: len(R.isomorphisms_to(R))
     #This gives the size of the isometry group.
2
```

We conclude that L is hyperbolic and not the Borromean rings.

ACKNOWLEDGMENTS. This work is the product of a research group formed under the auspices of the American Institute for Mathematics (AIM) in their virtual Research Community on 4-dimensional topology. We are grateful to AIM, and especially to the program organizers Miriam Kuzbary, MM, Juanita Pinzón-Caicedo, and Hannah Schwartz. We thank Slava Krushkal for providing valuable feedback on an earlier draft of this paper. SK thanks Jen Hom and John Etnyre for helpful conversations. AK and MP are grateful to the Max Planck Institute for Mathematics in Bonn, where they were visitors while this paper was written. We used KLO [SKK21], developed by Frank Swenton, for initial exploration.

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EPILOGUE

Peculiar surfaces live in B^4*,*
delicately tangled sorts:
drab if untethered,
but exotic together.
Bing doubling yields them in scores!