THE CLASSIFICATION OF CLOSED ORIENTED TOPOLOGICAL 4-MANIFOLDS WITH INFINITE CYCLIC FUNDAMENTAL GROUP

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Our goal is to explain the classification of closed, connected, oriented, topological 4-manifolds with fundamental group isomorphic to $\mathbb{Z}$, up to homeomorphism.

In [FQ90], the statement and an outline of a proof is given. But they state that there is "no formal proof provided", and that they leave the proof as an "extended exercise". Later Kreck [Kre99] gave a proof in the spin case, and later still this was completed by Hambleton-Kreck-Teichner[HKT09] as a special case of their classification of 4-manifolds with geometrically 2-dimensional fundamental group. We shall explain an outline of a proof that differs from the proofs that have previously appeared, in that we apply the surgery exact sequence.

Closed 4-manifolds with fundamental group $\mathbb{Z}$ are classified, roughly speaking, by the intersection form on π_2 and by the Kirby-Siebenmann invariant. The classification has two parts. First we describe a realisation result for the invariants, then we show that any automorphism of the invariants is realised by a homeomorphism.

The intersection pairing of a closed 4-manifold M with fundamental group $\mathbb{Z}$ is defined on $\pi_2(M) \cong H_2(M;\mathbb{Z}[\mathbb{Z}])$ and denoted

$$\lambda_M \colon H_2(M;\mathbb{Z}[\mathbb{Z}]) \times H_2(M;\mathbb{Z}[\mathbb{Z}]) \to \mathbb{Z}[\mathbb{Z}].$$

It is nonsingular, Hermitian, and sesquilinear with respect to the involution on $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t,t^{-1}]$ sending $t \mapsto t^{-1}$. The Kirby-Siebenmann invariant is

$$\operatorname{ks}(M) \in H^4(M;\pi_3(\mathrm{TOP}\,/\,\mathrm{PL})) \cong H^4(M;\mathbb{Z}/2) \cong \mathbb{Z}/2.$$

1 The existence theorem

Theorem 1.1 (Existence)**.** *Let (H,λ) be a nonsingular, Hermitian, sesquilinear form over $\mathbb{Z}[\mathbb{Z}]$*

$$\lambda \colon H \times H \to \mathbb{Z}[\mathbb{Z}],$$

where $H \cong \oplus^n \mathbb{Z}[\mathbb{Z}]$ is a finitely generated free $\mathbb{Z}[\mathbb{Z}]$-module. Let $k \in \mathbb{Z}/2$. If λ is even (i.e. $\lambda(x,x) = q + \overline{q}$ for some $q \in \mathbb{Z}[\mathbb{Z}]$) assume that

$$k \equiv \operatorname{sign}(\lambda \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{R})/8 \mod 2.$$

Then there exists a closed, connected, oriented, topological 4-manifold M with $\pi_1(M) \cong \mathbb{Z}$, $\operatorname{ks}(M) = k \in \mathbb{Z}/2$, and such that there is an isomorphism $\theta \colon H \xrightarrow{\cong} H_2(M;\mathbb{Z}[\mathbb{Z}])$ inducing an isometry $\theta \colon \lambda \xrightarrow{\cong} \lambda_M$.

Proof. Start with $S^1 \times D^3$. Its boundary is $S^1 \times S^2$. Attach n 2-handles to $S^1 \times S^2$ with framing-linking matrix $\lambda(0)$. Then add clasps to the handle attaching maps by doing a finger move around an element of $\pi_1(S^1 \times S^2)$ and then clasping. Do this for each summand $\pm t^k$ with $k > 0$ on the diagonal of a matrix representing λ, and for each $\pm t^k$ summand with $k \neq 0$ above the diagonal of λ. The sign of the clasp should correspond to the $\pm$ and the element of $\pi_1(S^1 \times S^2)$ should correspond to k. With these 2-handles attaching maps we obtain a compact, oriented smooth 4-manifold W with $\pi_1(W) \cong \mathbb{Z}$ and $\lambda_W \cong \lambda$.

We want to fill in ∂W with $N \simeq S^1 \times D^3$. To get a smooth 4-manifold, we would need to add 3- and 4-handles, but this may not be possible. The manifold N will be good enough to obtain a topological 4-manifold with the right algebraic topology, however. Since $\pi_1(W) \cong \mathbb{Z}$, we have that $H_1(W;\mathbb{Z}[\mathbb{Z}]) = 0$. The long exact sequence of the pair with $\mathbb{Z}[\mathbb{Z}]$ coefficients yields:

$$H_2(W) \xrightarrow{j} H_2(W,\partial W) \to H_1(\partial W) \to 0.$$

By Poincaré duality and universal coefficients

$$H_2(W,\partial W;\mathbb{Z}[\mathbb{Z}]) \xrightarrow{PD,\cong} H^2(W;\mathbb{Z}[\mathbb{Z}]) \xrightarrow{UC,\cong} \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}]}(H_2(W;\mathbb{Z}[\mathbb{Z}]),\mathbb{Z}[\mathbb{Z}]).$$

The composition

$$UC \circ PD \circ j \colon H_2(W;\mathbb{Z}[\mathbb{Z}]) \to \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}]}(H_2(W;\mathbb{Z}[\mathbb{Z}]),\mathbb{Z}[\mathbb{Z}])$$

is the adjoint of the intersection pairing, which by assumption is nonsingular, so this composition is an isomorphism. Therefore $H_1(\partial W;\mathbb{Z}[\mathbb{Z}]) = 0$. In other words ∂W is a $\mathbb{Z}[\mathbb{Z}]$-homology $S^1 \times S^2$ i.e. the homology is $\mathbb{Z}$ in degrees 0 and 2 and is otherwise trivial. We will apply the following theorem.

Theorem 1.2. *Let M be a closed, oriented, connected 3-manifold. Suppose there exists a homomorphism $\varphi\colon \pi_1(M) \twoheadrightarrow \mathbb{Z}$ with $H_1(M;\mathbb{Z}[\mathbb{Z}]) = 0$ Then there exists a compact, oriented, connected 4-manifold N with homotopy equivalence $g\colon N \to S^1$ with $\partial N \cong M$, such that $\pi_1(M) \to \pi_1(N) \xrightarrow{g_*} \pi_1(S^1) \cong \mathbb{Z}$ agrees with φ.*

Proof of Theorem 1.2. This proof is based on that in [FQ90, Section 11.6]. First we find some 4-manifold whose boundary is M, with a map to S^1 realising φ. We use framed bordism. Every oriented 3-manifold admits a framing of its tangent bundle.

$$\Omega_3^{\mathrm{fr}}(B\mathbb{Z}) \cong \Omega_3^{\mathrm{fr}} \oplus \Omega_2^{\mathrm{fr}} \cong \mathbb{Z}/24 \oplus \mathbb{Z}/2.$$

We consider the image of (M,φ) in here. The first summand can be killed by changing the choice of framing of the tangent bundle of M. The second summand is detected by an Arf invariant. It turns out that this is determined by the order of $H_1(M;\mathbb{Z}[\mathbb{Z}])$, evaluated at $t=-1$. Since $H_1(M;\mathbb{Z}[\mathbb{Z}]) = 0$, the Arf invariant vanishes and so $(M,\varphi) = 0 \in \Omega_3^{\mathrm{fr}}(B\mathbb{Z})$. Therefore there exists a framed 4-manifold Y with framed boundary M, such that the map $M \to S^1$ associated with φ extends over Y.

Let

$$X := \mathcal{M}(M \xrightarrow{\varphi} S^1)$$

be the mapping cylinder. Then we claim that (X,M) is a Poincaré pair. To see this note that $X \simeq S^1$ so its only nontrivial homology with $\mathbb{Z}[\mathbb{Z}]$ coefficients is $H_0(X;\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Z}$. Similarly the relative homology $H_*(X,M;\mathbb{Z}[\mathbb{Z}])$ vanishes apart from $H_3(X,M;\mathbb{Z}[\mathbb{Z}]) = \mathbb{Z}$. We can also compute that the cohomology of X is $H^1(X;\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Z}$ and is otherwise zero. We have $H_4(X,M;\mathbb{Z}) \cong H_3(M;\mathbb{Z}) \cong \mathbb{Z}$. A generator is a fundamental class for X. For a complete argument one needs to show that cap product with this generator gives the correct isomorphisms.

This uses that $H_1(M;\mathbb{Z}[\mathbb{Z}]) = 0$. The pair (X,M) will be our target space. If $M = S^1 \times S^2$ then we would have $X \cong S^1 \times D^3$. We can construct a degree one normal map

$$((F,\mathrm{Id})\colon (Y,M) \to (X,M)) \in \mathcal{N}(X,M).$$

The set $\mathcal{N}(X,M)$ consists of normal bordism classes of degree one normal maps over X, where a bordism restricts to a product cobordism homeomorphic to $M \times I$ between the boundaries. Our goal is to do surgery on the interior of the domain (Y,M) to convert this into a homotopy equivalence.

Since the fundamental group $\mathbb{Z}$ is a good group, surgery theory says that this is possible, which implies that the structure set $\mathcal{S}(X,M)$ is nonempty, if and only if $\sigma^{-1}(\{0\})$ is nonempty. Here

$$\sigma\colon \mathcal{N}(X,M) \to L_4(\mathbb{Z}[\mathbb{Z}])$$

is the surgery obstruction map. Essentially it takes the intersection pairing on $H_2(Y;\mathbb{Z}[\mathbb{Z}])$ and considers it in the Witt group of nonsingular, Hermitian, even forms over $\mathbb{Z}[\mathbb{Z}]$ up to stable equivalence, where stabilisation is by hyperbolic forms

$$\left(\mathbb{Z}[\mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z}], \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

Shaneson splitting implies that

$$L_4(\mathbb{Z}[\mathbb{Z}]) \cong L_4(\mathbb{Z}) \oplus L_3(\mathbb{Z}) \cong L_4(\mathbb{Z}) \cong 8\mathbb{Z}.$$

The last isomorphism is given by taking the signature.

Theorem 1.3 (Freedman)**.** *There exists a closed, oriented, simply connected, topological 4-manifold Z with intersection form E_8 and therefore* $\operatorname{sign}(Z) = 8$.

Using Theorem 1.3 we take the connected sum of Y with copies of Z or $-Z$, to arrange that the signature becomes zero. Then the resulting normal map $Y\#^\ell Z \to X$ has trivial surgery obstruction in $L_4(\mathbb{Z})$ and therefore has is normally bordant to a homotopy equivalence

$$(F', \mathrm{Id})\colon (N, M) \to (X, M)$$

as desired. □

We return to the proof of Theorem 1.1. Apply Theorem 1.2 with $M = \partial W$. Then

$$U := W \cup_M -N$$

is a closed, connected, oriented, topological 4-manifold with $\pi_1(U) \cong \mathbb{Z}$ and $\lambda_U \cong \lambda$. For even λ, this automatically has the correct Kirby-Siebenmann invariant by Rochlin's theorem (we may have to first stabilise by copies of $S^2 \times S^2$ to get a smooth 4-manifold, to which we can apply Rochlin. But this does not change the signature nor the Kirby-Siebenmann invariant. It remains to realise both Kirby-Siebenmann invariant 0 and 1 for λ odd. We do not know whether the manifold U we have constructed has $\operatorname{ks}(U) = 0$ or $\operatorname{ks}(U) = 1$. So we need a method to alter the invariant. For this we need the following theorem.

Theorem 1.4 (Freedman)**.** *Let Σ be a $\mathbb{Z}$-homology 3-sphere. Then there exists a contractible, compact topological 4-manifold V with $\partial V = \Sigma$.*

In fact, we have already seen this in disguise, since this result was used in the construction of the E_8-manifold Z from Theorem 1.3. Attaching 2-handles to D^4 along an 8-component link with linking-framing matrix equal to the E_8 form gives a compact 4-manifold with boundary an integral homology 3-sphere. Capping off with the contractible manifold V from Theorem 1.4 yields Z.

Now, however, we need Theorem 1.4 to construct a manifold called $*\mathbb{CP}^2$ which is homotopy equivalent but not homeomorphic to $\mathbb{CP}^2$. Attach a 2-handle $D^2 \times D^2$ to D^4 along a $(+1)$-framed trefoil. The boundary is an integral homology 3-sphere, and this time capping off with V from Theorem 1.4 yields $*\mathbb{CP}^2$. It has $\operatorname{ks}(*\mathbb{CP}^2) = 1$. For comparison to obtain $\mathbb{CP}^2$ one can add a 2-handle to D^4 along a $(+1)$-framed unknot and then cap off with D^4. Moreover, in fact the construction we gave works for any knot K in place of the trefoil. Up to homeomorphism, we obtain $\mathbb{CP}^2$ if and only if $\operatorname{Arf}(K) = 0$ and $*\mathbb{CP}^2$ if and only if $\operatorname{Arf}(K) = 1$. So the Arf invariant coincides with the Kirby-Siebenmann invariant.

Now we use $*\mathbb{CP}^2$ to realise ks. Let M be a closed, oriented, connected, topological 4-manifold with $\pi_1(M) \cong \mathbb{Z}$, $\lambda_M \cong \lambda$ and λ odd. Consider

$$M \# * \mathbb{CP}^2$$

and consider $f\colon S^2 \to M \# * \mathbb{CP}^2$ representing the generator of the $\pi_2(*\mathbb{CP}^2) \cong \mathbb{Z}$ summand. Since M is odd, an application of the sphere embedding theorem implies that f is regularly homotopic to an embedding. Having an embedding of a sphere whose normal bundle has Euler number $+1$ is the same as having a $\mathbb{CP}^2$ connected summand. So we can write

$$M\# *\mathbb{CP}^2 \cong *M\#\mathbb{CP}^2$$

where this equation defines $*M$. It turns out that $M \simeq *M$ so that $\lambda_{*M} \cong \lambda_M \cong \lambda$. By additivity of the Kirby-Siebenmann invariant under connected sum we have that $\mathrm{ks}(*M) \neq \mathrm{ks}(M)$. This competes the proof that we can realise both the Kirby-Siebenmann invariants by manifolds, therefore completing the proof of Theorem 1.1. □

2 The uniqueness theorem

Now that we have understood the existence part of the proof, realising our collection of invariants, we show that these invariants also classify the manifolds.

Theorem 2.1 (Uniqueness)**.** *Let M and N be closed, connected, oriented, topological 4-manifolds with $\pi_1(M) \cong \mathbb{Z} \cong \pi_1(N)$. Suppose that* $\mathrm{ks}(M) = \mathrm{ks}(N) \in \mathbb{Z}/2$*. Let*

$$h\colon H_2(M;\mathbb{Z}[\mathbb{Z}]) \to H_2(N;\mathbb{Z}[\mathbb{Z}])$$

be an isometry of the intersection form, i.e. $h\colon \lambda_M \xrightarrow{\cong} \lambda_N$. Then there exists an orientation preserving homeomorphism $f\colon M \xrightarrow{\cong} N$ inducing h.

In particular it follows that, up to homeomorphism, there is (are) exactly one (two) 4-manifolds with $\pi_1 \cong \mathbb{Z}$ and a given intersection form λ, if λ is even (odd).

Proof. We will give a proof that explicitly uses the surgery exact sequence. Since the Whitehead group $\mathrm{Wh}(\mathbb{Z}[\mathbb{Z}]) = 0$, every homotopy equivalence is a simple homotopy equivalence, so we will not use K-theory decorations. The surgery sequence for N is:

$$\mathcal{N}(N\times I,\partial) \to L_5(\mathbb{Z}[\mathbb{Z}]) \to \mathcal{S}(N) \to \mathcal{N}(N) \to L_4(\mathbb{Z}[\mathbb{Z}]).$$

We want to compute the structure set $\mathcal{S}(N)$ and then take the quotient by the action of the self-homotopy equivalences. In the topological category, the action of $L_5(\mathbb{Z}[\mathbb{Z}])$ is defined, and the surgery sequence is exact, because $\pi_1(N) \cong \mathbb{Z}$ is a good group.

First we investigate the Wall realisation action of $L_5(\mathbb{Z}[\mathbb{Z}])$ on the structure set. We have

$$L_5(\mathbb{Z}[\mathbb{Z}]) \cong L_5(\mathbb{Z}) \oplus L_4(\mathbb{Z}) \cong L_4(\mathbb{Z}) \cong 8\mathbb{Z}.$$

The generator of this group is in the image of a degree one normal map with domain $N \times I\#_{S^1} Z \times S^1$, where Z is the E_8 manifold from Theorem 1.3. Here $\#_{S^1}$ denotes connected sum along the 1-skeleton. The surgery obstruction of this rel. boundary normal map is $8 \in 8\mathbb{Z}$. It follows that the action of $L_5(\mathbb{Z}[\mathbb{Z}])$ on $\mathcal{S}(N)$ is trivial.

Now we look at the right hand end of the surgery sequence. For some n, we have

$$\mathcal{N}(N) \cong [N, G/\mathrm{TOP}] \cong H^2(N;\mathbb{Z}/2) \oplus H^4(N;\mathbb{Z}) \cong (\mathbb{Z}/2)^n \oplus \mathbb{Z},$$

since there is a 5-equivalence

$$G/\mathrm{TOP} \to K(\mathbb{Z}/2,2) \times K(\mathbb{Z},4).$$

Recall that $L_4(\mathbb{Z}[\mathbb{Z}]) \cong L_4(\mathbb{Z}) \cong 8\mathbb{Z}$. The map from $H^4(N;\mathbb{Z}) \cong \mathbb{Z} \to 8\mathbb{Z}$ is an isomorphism, and the map $H^2(N;\mathbb{Z}/2) \to 8\mathbb{Z}$ is the zero map. We deduce that

$$\mathcal{S}(N) \cong (\mathbb{Z}/2)^n.$$

Now we start to prove the theorem. We will assume the following proposition. Its proof is nontrivial, and would be needed in order for this note to be able to claim to contain a complete proof of the uniqueness theorem.

Proposition 2.2. *Let M and N be as in Theorem 2.1. Then there is a homotopy equivalence $f\colon M \to N$ inducing $h\colon H_2(M;\mathbb{Z}[\mathbb{Z}]) \to H_2(N;\mathbb{Z}[\mathbb{Z}])$.*

This gives us elements in the structure set (M, f) with the desired properties on homology. We will show that either f is homotopic to a homeomorphism, or that there is a modified homotopy equivalence $f' \colon M \to N$ that induces the same isometry on $H_2(-;\mathbb{Z}[\mathbb{Z}])$, that is homotopic to a homeomorphism. In general we need to modify the chosen f, since its image in $\mathcal{N}(N) \cong (\mathbb{Z}/2)^n$ may be nontrivial.

Given $x \in \pi_2(N)$ with $x \cdot x \equiv 0 \mod 2$, and with $x \neq 0 \in H_2(N;\mathbb{Z}/2)$, there is a homotopy equivalence

$$\theta_x \colon N \xrightarrow{\text{pinch}} N \vee S^4 \xrightarrow{\mathrm{Id} \vee (\eta \circ \Sigma\eta)} N \vee S^2 \xrightarrow{\mathrm{Id} \vee x} N.$$

The first map takes a D^4 embedded in N, and identifies ∂D^4 to a point. The second map uses the generator of $\pi_4(S^2) \cong \mathbb{Z}/2$, and the third map sends the S^2 back into N. It turns out that θ_x induces the identity on $H_2(N;\mathbb{Z}[\mathbb{Z}])$ and that taking the composition

$$M \xrightarrow{f,\simeq} N \xrightarrow{\theta_x,\simeq} N$$

changes the image of (M, f) in $\ker(\mathcal{N}(N) \to L_4(\mathbb{Z}[\mathbb{Z}])) \cong H^2(N;\mathbb{Z}/2)$ under $\mathcal{S}(N) \to \mathcal{N}(N)$ by the dual to x in $H^2(N;\mathbb{Z}/2)$.

If $\lambda_N \cong \lambda_M$ is even, we can use this procedure to kill all classes, and thus obtain a homotopy equivalence $f' \colon M \to N$ that equals $\mathrm{Id} \colon N \to N$ in $\mathcal{S}(N)$. Here we use that $\pi_2(N) \to H_2(N;\mathbb{Z}/2)$ is onto. In other words, we have seen that

$$\mathcal{S}(N)/\,\mathrm{hAut}(N) \cong \{[N]\}.$$

By definition of the structure set, this means that there is a homeomorphism $g \colon M \to N$ such that $g = \mathrm{Id} \circ g \sim f' \colon M \to N$ are homotopic. So indeed f' is homotopic to a homeomorphism. This homeomorphism induces the same map $h \colon H_2(M;\mathbb{Z}[\mathbb{Z}]) \to H_2(N;\mathbb{Z}[\mathbb{Z}])$ as the original homotopy equivalence f from Proposition 2.2. This completes the proof of the theorem in the case that λ_N is even.

If λ_N is odd, then one class in $H^2(N;\mathbb{Z}/2)$ cannot be killed using the homotopy equivalence θ_x. But on the other hand we know from Theorem 1.1 that there exist at least two homeomorphism classes of manifolds in the homotopy class, distinguished by the Kirby-Siebenmann invariant, which we know can be realised when λ_N is odd. Therefore we have

$$\mathcal{S}(N)/\,\mathrm{hAut}(N) \cong \{[N], [*N]\}.$$

To complete the proof of the theorem, we assumed that $\mathrm{ks}(M) = \mathrm{ks}(N)$ in the hypotheses of Theorem 2.1, so we can in fact obtain a modified homotopy equivalence $f' \colon M \to N$ with trivial image in $\ker(\mathcal{N}(N) \to L_4(\mathbb{Z}[\mathbb{Z}])) \cong H^2(N;\mathbb{Z}/2)$, so that f' is homotopic to a homeomorphism. This completes the proof in the case that λ_N is odd, and therefore completes the proof of Theorem 2.1. □

Remark 2.3. Conway-Powell extended the classification of closed 4-manifolds with fundamental group $\mathbb{Z}$ to the classification of 4-manifolds with boundary provided $H_1(\partial M;\mathbb{Q}(t)) = 0$ and $\pi_1(\partial M) \to \pi_1(M) \cong \mathbb{Z}$ is surjective.

References

[FQ90] Michael H. Freedman and Frank Quinn. *Topology of 4-manifolds*, volume 39 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1990.

[HKT09] Ian Hambleton, Matthias Kreck, and Peter Teichner. Topological 4-manifolds with geometrically two-dimensional fundamental groups. *J. Topol. Anal.*, 1(2):123–151, 2009.

[Kre99] Matthias Kreck. Surgery and duality. *Ann. of Math. (2)*, 149(3):707–754, 1999.

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