TOPOLOGICAL MANIFOLDS ARE EUCLIDEAN NEIGHBOURHOOD RETRACTS

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Abstract. We show that every topological n-manifold M embeds into $\mathbb{R}^{2n+1}$ as a closed subspace and is a retract of some neighbourhood $U \subseteq \mathbb{R}^{2n+1}$.

Introduction

For smooth manifolds the following well-known result holds.

Theorem 0.1 (Whitney Embedding Theorem)**.** *Every smooth n-manifold M admits a closed smooth embedding $\iota\colon M \hookrightarrow \mathbb{R}^{2n+1}$.*

Furthermore, it can be shown that every embedded smooth manifold $M \subseteq \mathbb{R}^N$ has a *tubular neighbourhood*.

Theorem 0.2 (Tubular neighbourhood Theorem)**.** *Let $M \subseteq \mathbb{R}^N$ be an embedded smooth manifold. Then M possesses a* tubular neighbourhood*, i.e. there exists an open neighbourhood $U \subseteq \mathbb{R}^N$ of M that is diffeomorphic to a set $V \subseteq NM$ of the type*

$$V = \{(x, v) \in NM \mid |v| < \delta(x)\},$$

where $\delta\colon M \to (0, \infty)$ is continuous and NM denotes the normal bundle of M, via the map

$$\theta\colon NM \to \mathbb{R}^N,\ (x, v) \mapsto x + v.$$

Let us recall the definition of ENRs.

Definition 0.3. A topological space X is a *Euclidean Neighbourhood Retract (ENR)* if there exists a closed embedding $\iota\colon X \hookrightarrow \mathbb{R}^N$ for some $N \in \mathbb{N}$ and an open neighbourhood $U \subseteq \mathbb{R}^N$ of $\iota(X)$ such that U is a retraction of U, i.e. there exists a continuous map $r\colon U \to \iota(X)$ satisfying $r|_{\iota(X)} = \mathrm{id}_{\iota(X)}$.

It can be shown that any embedded smooth manifold $M \subseteq \mathbb{R}^N$ is a retract of every tubular neighbourhood of M, hence we have the following.

Corollary 0.4. *Every smooth manifold is an ENR.*

A detailed account is given in [Lee13, Chapter 6].

In this talk, we want to prove the following corresponding results for toplogical manifolds.

Theorem 2.1. *Let X be a second-countable locally compact Hausdorff space such that every compact subspace of X has dimension at most $n \in \mathbb{N}$. Then X admits a closed embedding $\iota\colon X \hookrightarrow \mathbb{R}^{2n+1}$.*

Theorem 3.3. *Every topological manifold is an ENR.*

This is essentially due to Hanner [Han51]. We mainly follow indications by Munkres [Mun00] and unpublished notes by Kirby and Kister [KK] adding many details.

Remark. Throughout these notes, we denote by $\mathbb{N} := \{1, 2, \ldots\}$ the set of positive integers and by $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ the set of non-negative integers.

1 Dimension Theory

Definition 1.1. Let X be a topological space and let $\mathcal{U}$ be an open covering of X. A *refinement* of $\mathcal{U}$ is an open cover $\mathcal{V}$ of X such that every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$, i.e. $V \subseteq U$.

Definition 1.2. Let X be a topological space.

(1) A collection $\mathcal{A}$ of subsets of X has *order* $m \in \mathbb{N}_0$ if m is the largest integer such that there are $m+1$ elements of $\mathcal{A}$ having a non-empty intersection.

(2) X is called *finite-dimensional* if there exists some $m \in \mathbb{N}_0$ such that every open cover of X possesses a refinement of order at most m.
The smallest such m is called the *(topological) dimension* of X, denoted by $\dim X$.

If X is a topological space and $\mathcal{A}$ is a collection of subsets of X, then $\mathcal{A}$ has order m if and only if there exists some $x \in X$ that lies in $m+1$ elements of $\mathcal{A}$ and no point of X lies in more than $m+1$ elements of $\mathcal{A}$.

Let us illuminate the notion of topological dimension with an example.

Example 1.3. Let $I := [0,1]$ denote the closed unit interval. We want to show that $\dim I = 1$. Let $\mathcal{U}$ be an open cover of I. Since I is a compact metric space, $\mathcal{U}$ has a positive Lebesgue number $\lambda > 0$, i.e. every subset of I having diameter less than λ is contained in an element of $\mathcal{U}$. For $k \in \mathbb{N}_0$, let $J_k := \left((k-1)\cdot\frac{\lambda}{4}, (k+1)\cdot\frac{\lambda}{4}\right)$. Since $\operatorname{diam} J_k = \frac{\lambda}{2} < \lambda$, we can conclude that $\mathcal{V} := \{J_k \cap I\}_{k\in\mathbb{N}_0}$ is a refinement of $\mathcal{U}$. Since $\mathcal{V}$ has order 1, this shows $\dim I \leq 1$.
In order to show that $\dim I \geq 1$, we consider the open over $\mathcal{U} := \{[0,1),(0,1]\}$. If $\dim I = 0$, $\mathcal{U}$ would have a refinement $\mathcal{V}$ of order 0. Since $\mathcal{V}$ refines $\mathcal{U}$, we get $\operatorname{card}(\mathcal{V}) \geq 2$ (note that $0 \in V_1$ and $1 \in V_2$ for some $V_1, V_2 \in \mathcal{V}$ and because $\mathcal{V}$ refines $\mathcal{U}$, we get $V_1 \subseteq [0,1)$ and $V_2 \subseteq (0,1]$ and thus $V_1 \neq V_2$). Let V be any element of $\mathcal{V}$ and let W be the union of all $V' \in \mathcal{V} \setminus V$. Then both V and W are open and $V \cup W = I$ and $V \cap W = \emptyset$, because $\mathcal{V}$ has order 0, which is a contradiction since I is connected.
Thus, $\dim I \geq 1$ and therefore $\dim I = 1$.

We can use Lebesgue numbers to show a more general result that will be needed throughout this section.

Theorem 1.4. *Let $n \in \mathbb{N}$. Every compact subspace of $\mathbb{R}^n$ has topological dimension at most n.*

Proof. Let us first divide $\mathbb{R}^n$ into unit cubes. Let

$$\mathcal{G} := \{(k,k+1)\}_{k\in\mathbb{Z}}$$
$$\mathcal{K} := \{\{k\}\}_{k\in\mathbb{Z}}.$$

If $0 \leq d \leq n$, we define $\mathcal{C}_d$ to be the set of all products

$$A_1 \times \cdots \times A_n \subseteq \mathbb{R}^n,$$

where precisely d of the sets $A_1, \ldots, A_n$ are an element of $\mathcal{G}$ and the remaining $n-d$ ones are an element of $\mathcal{K}$.
Set $\mathcal{C} := \mathcal{C}_0 \cup \cdots \cup \mathcal{C}_n$. Then for every $x \in \mathbb{R}^n$ there exists a unique $C \in \mathcal{C}$ such that $x \in C$.

Claim. Let $0 \leq d \leq n$. For every $C \in \mathcal{C}_d$, there exists an open neighbourhood $U(C)$ of C satisfying:

(1) $\operatorname{diam} U(C) \leq \frac{3}{2}$
(2) $U(C) \cap U(D) = \emptyset$ whenever $D \in \mathcal{C}_d \setminus \{C\}$.

Proof of claim. Let $x = (x_1, \ldots, x_n) \in C$. We will show that there exists a number $0 < \varepsilon(x) \leq \frac{1}{2}$ such that the open cube centered at x with radius $\varepsilon(x)$, i.e. the set

$$W_{\varepsilon(x)}(x) = (x_1 - \varepsilon(x), x_1 + \varepsilon(x)) \times \cdots \times (x_n - \varepsilon(x), x_n + \varepsilon(x)),$$

intersects no other element of $\mathcal{C}_d$. If $d = 0$, choose $\varepsilon(x) := \frac{1}{2}$. If $d > 0$, exactly d of the numbers $x_1, \dots, x_n$ are not integers. Choose $0 < \varepsilon(x) \leq \frac{1}{2}$ such that for each $1 \leq i \leq n$ that satisfies $x_i \notin \mathbb{Z}$, the interval $(x_i - \varepsilon(x), x_i + \varepsilon(x))$ contains no integer. If $y = (y_1, \dots, y_n) \in W_{\varepsilon(x)}(x)$, we have $y_i \notin \mathbb{Z}$ whenever $x_i \notin \mathbb{Z}$. Thus, either $y \in C$ or $y \in C'$ for some $C' \in \mathcal{C}_{d'}$ where $d' > d$. In conclusion, $W_{\varepsilon(x)}(x)$ intersects no other element of $\mathcal{C}_d$.
Now let $U(C)$ be the union of all $W_{\frac{\varepsilon(x)}{2}}$ where $x \in C$. Then obviously $U(C) \cap U(D) = \emptyset$ whenever $D \in \mathcal{C}_d \setminus \{C\}$. This proves (2).
If $x, y \in U(C)$, we have $x \in W_{\frac{\varepsilon(x')}{2}}(x')$ and $y \in W_{\frac{\varepsilon(y')}{2}}(y')$ for some $x', y' \in C$. By the triangle inequality

$$\|x - y\|_\infty \leq \|x - x'\|_\infty + \|x' - y'\|_\infty + \|y' - y\|_\infty \leq \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2},$$

hence establishing (1). □

Now let $\mathcal{A} := \{U(C) \mid C \in \mathcal{C}\}$. Then $\mathcal{A}$ is an open cover of $\mathbb{R}^n$ of order n by (2). Let $K \subseteq \mathbb{R}^n$ be compact and let $\mathcal{U}$ be an open cover of K. Since K is compact metric, $\mathcal{U}$ has a positive Lebesgue number $\lambda > 0$.
Consider the homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto \frac{\lambda}{3} \cdot x$. Since $\mathcal{A}$ is an open cover of order n, so is $\mathcal{A}' := \{f(U(C)) \mid C \in \mathcal{C}\}$. Since $\operatorname{diam} f(U(C)) \leq \frac{\lambda}{2} < \lambda$ for all $C \in \mathcal{C}$, we get that $\{f(U(C)) \cap K\}_{C \in \mathcal{C}}$ is an open cover of K that refines $\mathcal{U}$ and has order at most n.
Thus, $\dim K \leq n$, as desired. □

We need some more elementary properties of the topological dimension before we can proceed to manifolds.

Lemma 1.5. *Let X be a finite-dimensional topological space and let Y be a closed subspace of X. Then Y is also finite-dimensional and* $\dim Y \leq \dim X$.

Proof. Let $d := \dim X$. Let $\mathcal{U}$ be an open cover of Y. For every $U \in \mathcal{U}$ there exists some open $U' \subseteq X$ such that $U = U' \cap Y$. Let $\mathcal{A} := \{U'\}_{U \in \mathcal{U}} \cup \{X \setminus Y\}$. Then $\mathcal{A}$ is an open cover of X and thus possesses a refinement $\mathcal{B}$ of order at most d. Therefore, $\mathcal{V} := \{B \cap Y\}_{B \in \mathcal{B}}$ is an open cover of Y of order at most d that refines $\mathcal{U}$. This proves $\dim Y \leq d$. □

Theorem 1.6. *Let X be a topological space and assume $X = X_1 \cup X_2$ for some closed finite-dimensional subspaces $X_1, X_2 \subseteq X$. Then X is also finite-dimensional and*

$$\dim X = \max\{\dim X_1, \dim X_2\}.$$

Let us fix a notion for the proof of this theorem. If $\mathcal{U}$ is an open cover of X and $Y \subseteq X$ is a subspace of X, we say that $\mathcal{U}$ has *order* $m \in \mathbb{N}_0$ *in* Y if there exists some point $y \in Y$ that is contained in $m + 1$ distinct elements of $\mathcal{U}$ and no point of Y is contained in more than $m + 1$ distinct elements of $\mathcal{U}$.

Proof. By Lemma 1.5, it suffices to prove $\dim X \leq \max\{\dim X_1, \dim X_2\}$.

Claim. Let $\mathcal{U}$ be an open cover of X and let Y be a closed subspace of X such that $\dim Y \leq d < \infty$. Then $\mathcal{U}$ possesses a refinement that has order at most d in Y.

Proof of claim. Let $\mathcal{A} := \{U \cap Y\}_{U \in \mathcal{U}}$. Since $\mathcal{A}$ is an open cover of Y and $\dim Y \leq d$, there exists a refinement $\mathcal{B}$ of $\mathcal{A}$ of order at most d. For every $B \in \mathcal{B}$, there exists some open set $U_B \subseteq X$ such that $B = U_B \cap Y$. Furthermore, there exists some $A_B \in \mathcal{U}$ such that $B \subseteq A_B \cap Y$. Then, $\{U_B \cap A_B\}_{B \in \mathcal{B}} \cup \{U \setminus Y\}_{U \in \mathcal{U}}$ is an open cover of X that refines $\mathcal{U}$ and has order at most d in Y. □

Now, let $d := \max\{\dim X_1, \dim X_2\}$ and let $\mathcal{U}$ be an open cover of X. We need to show that $\mathcal{U}$ has a refinement $\mathcal{V}$ of order at most d.
Let $\mathcal{A}_1$ be a refinement of $\mathcal{U}$ of order at most d in X_1 and let $\mathcal{A}_2$ be a refinement of $\mathcal{A}_1$ of order

at most d in X_2. We can define a map $f\colon \mathcal{A}_2 \to \mathcal{A}_1$ as follows. For every $U \in \mathcal{A}_2$ choose an element $f(U) \in \mathcal{A}_1$ such that $U \subseteq f(U)$.
For all $S \in \mathcal{A}_1$, let $V(S)$ be the union of all $U \in \mathcal{A}_2$ that satisfy $f(U) = S$ and finally let $\mathcal{V} := \{V(S)\}_{S\in\mathcal{A}_1}$. Then, $\mathcal{V}$ is an open cover of X: For if $x \in X$, then $x \in U$ for some $U \in \mathcal{A}_2$ and because $U \subseteq V(f(U))$, we can deduce $x \in V(f(U))$. Furthermore, $\mathcal{V}$ refines $\mathcal{A}_1$, because $V(S) \subseteq S$ for every $S \in \mathcal{A}_1$. Since $\mathcal{A}_1$ refines $\mathcal{U}$, the cover $\mathcal{V}$ must refine $\mathcal{U}$.
Finally, we need to show that $\mathcal{V}$ has order at most d. Suppose $x \in V(S_1)\cap\cdots\cap V(S_k)$, where the sets $V(S_1), \ldots, V(S_k)$ are distinct. Thus, the sets $S_1, \ldots, S_k$ are distinct. For all $1 \le i \le k$, we can find a set $U_i \in \mathcal{A}_2$ such that $x \in U_i$ and $f(U_i) = S_i$, because $x \in V(S_i)$. Because $S_1, \ldots, S_k$ are distinct, so are $U_1, \ldots, U_k$. Thus, we have the following situation:

$$x \in U_1 \cap \cdots \cap U_k \subseteq V(S_1) \cap \cdots \cap V(S_k) \subseteq S_1 \cap \cdots \cap S_k$$

Because $X = X_1 \cup X_2$, we have $x \in X_1$ or $x \in X_2$. If $x \in X_1$, then $k \le d+1$, because $\mathcal{A}_1$ has order at most d in X_1. If $x \in X_2$, we can also conclude $k \le d+1$, because $\mathcal{A}_2$ has order at most d in X_2.
Thus, $k \le d + 1$, proving that $\mathcal{V}$ has order at most d, as desired. □

A simple induction argument then yields the following corollary.

Corollary 1.7. *Let X be a topological space and let $X_1, \ldots, X_r \subseteq X$ be closed finite-dimensional subspaces of X such that*

$$X = \bigcup_{i=1}^{r} X_i.$$

Then X is also finite-dimensional and

$$\dim X = \max\{\dim X_1, \ldots, \dim X_n\}.$$

We can now apply these results to manifolds.

Corollary 1.8. *Let M be a topological n-manifold. If $C \subseteq X$ is compact, then* $\dim C \le n$.

Proof. Since M is locally Euclidean, C can be covered by finitely many compact n-balls $B_1, \ldots, B_k \subseteq M$. By Theorem 1.4 and Lemma 1.5

$$\dim\,(B_j \cap C) \le \dim\, B_j \le n$$

(note that B_j is homeomorphic to a compact subset of $\mathbb{R}^n$) for all $1 \le j \le k$.
Since $C = \bigcup_{j=1}^{k}(B_j \cap C)$, Corollary 1.7 yields $\dim C \le n$. □

As a special case, we can note that every compact n-manifold is finite-dimensional and its topological dimension is at most n. In fact, this result can be extended to general n-manifolds. For this, we need a technical lemma.

Lemma 1.9. *Let X be a topological space and assume $X = \bigcup_{i=0}^{\infty} C_i$, where every C_i is closed, $C_0 = \emptyset$, $C_i \subseteq \mathring{C}_{i+1}$ and there exists some $d \in \mathbb{N}_0$ such that* $\dim \overline{C_{i+1} \setminus C_i} \le d$ *for all $i \in \mathbb{N}_0$. Then X is finite-dimensional and* $\dim X \le d$.

Proof. We will construct a sequence of covers $(\mathcal{V}_i)_{i\in\mathbb{N}_0}$ of X such that $\mathcal{V}_{i+1}$ refines $\mathcal{V}_i$ and $\mathcal{V}_i$ has order at most d in C_i and $\mathcal{V}_0 := \mathcal{U}$. Under these hypotheses,

$$\mathcal{V} := \{V \subseteq X \mid \exists i \in \mathbb{N}\colon V \in \mathcal{V}_i \text{ and } V \cap C_{i-1} \neq \emptyset\}$$

is a refinement of $\mathcal{U}$ of order at most d: Let $x \in X$. Then $x \in C_{i-1}$ for some $i \in \mathbb{N}$. Since $\mathcal{V}_i$ is an open cover of X, we get $x \in V$ for some $V \in \mathcal{V}_i$. But this means $V \cap C_{i-1} \neq \emptyset$ and hence $V \in \mathcal{V}$, proving that $\mathcal{V}$ is an open cover of X. Suppose now that $U_1, \ldots, U_k$ are distinct elements of $\mathcal{V}$ having nonempty intersection and let x be an element of their intersection. Then, there exists some $i_0 \in \mathbb{N}$ such that $x \in C_{i_0-1}$. For each $1 \leq j \leq k$, there exists some $i_j \in \mathbb{N}$ such that $U_j \in \mathcal{V}_{i_j}$ and $U_j \cap C_{i_j-1} \neq \emptyset$. Letting $i := \max\{i_0, i_1, \ldots, i_k\}$, we get $U_1, \ldots, U_k \in \mathcal{V}_i$ and

$$x \in \bigcap_{j=1}^{k} U_j \cap C_i.$$

Since $\mathcal{V}_i$ has order at most d in C_i, we get $k \leq d+1$, i.e. $\mathcal{V}$ has order at most d, as desired.
All that is left now is constructing the sequence $(\mathcal{V}_i)_{i\in\mathbb{N}}$. Set $\mathcal{V}_0 = \mathcal{U}$ and suppose $\mathcal{V}_1, \ldots, \mathcal{V}_i$ have already been constructed. Just as in the proof of Theorem 1.6 we can find a refinement $\mathcal{W}$ of $\mathcal{V}_n$ that has order at most d in $\overline{C_{i+1} \setminus C_i}$. Define a map $f\colon \mathcal{W} \to \mathcal{V}_i$ by choosing $f(W)$ such that $W \subseteq f(W)$ for all $W \in \mathcal{W}$. For $U \in \mathcal{V}_i$, we define $V(U)$ to be the union of all $W \in \mathcal{W}$ such that $f(W) = U$. We define $\mathcal{V}_{i+1}$ to consist of three types of set: $\mathcal{V}_{i+1}$ contains all $U \in \mathcal{V}_i$ such that $U \cap C_{i-1} \neq \emptyset$. Furthermore, $\mathcal{V}_{i+1}$ contains all $V(U)$ where $U \in \mathcal{V}_i$ such that $U \cap C_{i-1} = \emptyset$ and $U \cap C_i \neq \emptyset$. Finally, $\mathcal{V}_{i+1}$ contains all $W \in \mathcal{W}$ such that $W \cap C_i \neq \emptyset$.

Claim. $\mathcal{V}_{i+1}$ is a refinement of $\mathcal{V}_i$ that has order at most d in C_{i+1}.

Proof. Let $x \in X$. We need to show the existence of some $U \in \mathcal{V}_{i+1}$ satisfying $x \in U$.
Suppose $x \in C_{i-1}$. Since $\mathcal{V}_i$ is an open cover of X, we have $x \in U$ for some $U \in \mathcal{V}_i$. Because of $U \cap C_{i-1} \neq \emptyset$, we can conclude $U \in \mathcal{V}_{i+1}$. If $x \notin C_{n-1}$, we can find $W \in \mathcal{W}$ satisfying $x \in W$. If $W \cap C_i = \emptyset$, then $W \in \mathcal{V}_{i+1}$. Otherwise, $f(W) \subseteq W$. If $f(W) \cap C_{i-1} \neq \emptyset$, then $x \in f(W) \in \mathcal{V}_{n+1}$. If $f(W) \cap C_{i-1} = \emptyset$, then $x \in V(f(W)))$ and $V(f(W)) \in \mathcal{V}_{i+1}$, because $f(W) \cap C_{i-1} = \emptyset$ and $\emptyset \neq W \cap C_i \subseteq f(W) \cap C_i$.
In conclusion, $\mathcal{V}_{i+1}$ is an open cover of X. It is obvious that $\mathcal{V}_{i+1}$ refines $\mathcal{V}_i$.
Now let $U_1, \ldots, U_k \in \mathcal{V}_{i+1}$ be k distinct subsets of $\mathcal{V}_{i+1}$ and suppose $x \in C_{i+1}$ such that $x \in \bigcap_{j=1}^{k} U_j$. If $x \in C_{i-1}$, then necessarily $U_1, \ldots, U_k \in \mathcal{V}_i$ by the definition of $\mathcal{V}_{i+1}$ and thus $k \leq d+1$, because $\mathcal{V}_i$ has order at most d in C_i.
If $x \in C_i \setminus C_{i-1}$, then $U_1 = V(S_1), \ldots, U_k = V(S_k)$ for some distinct $S_1, \ldots, S_k \in \mathcal{V}_i$ satisfying $S_j \cap C_{i-1} = \emptyset$ and $S \cap C_i \neq \emptyset$ $(1 \leq j \leq k)$. Thus,

$$x \in \bigcap_{j=1}^{k} V(S_j) \subseteq \bigcap_{j=1}^{k} S_j,$$

implying that $k \leq d+1$, because $\mathcal{V}_i$ has order at most d in C_i.
Finally if $x \in C_{i+1} \setminus C_i$, then $U_1, \ldots, U_k \in \mathcal{W}$, hence $k \leq d+1$, because $\mathcal{W}$ has order at most d in $\overline{C_{i+1} \setminus C_i}$. In conclusion, $\mathcal{V}_{i+1}$ has order at most d in C_{i+1}. ☐

This completes the proof of Lemma 1.9. ☐

If X is a second-countable locally compact Hausdorff space, then we can decompose X as in the statement of Lemma 1.9.

Lemma 1.10. *Every second-countable locally compact Hausdorff space X can be* exhausted by compact subsets*, i.e. there exist compact subsets $(C_i)_{i\in\mathbb{N}}$ such that $C_i \subseteq \mathring{C}_{i+1}$ and $X = \bigcup_{i=1}^{\infty} C_i$.*

Proof. Let $\mathcal{B}$ be a countable basis of the topology of X and let

$$\mathcal{B}' := \{V \in \mathcal{B} \mid \overline{V} \text{ is compact}\}.$$

Since X is locally compact, $\mathcal{B}'$ is again a basis of X. Let us now write $\mathcal{B}' = \{V_i\}_{i\in\mathbb{N}}$. Let $C_1 := \overline{V_1}$. Assume now, that compact subsets $C_1, \ldots, C_k$ satisfying $V_j \subseteq C_j$ and $C_{j-1} \subseteq \mathring{C}_j$ for all $1 \leq j \leq k$ (where $C_0 := \emptyset$) have already been constructed. Because C_k is compact, there exists some $m_k \leq k+1$ satisfying $C_k \subseteq \bigcup_{j=1}^{m_k} V_j$. Letting $C_{k+1} := \bigcup_{j=1}^{m_k} \overline{V_j}$, we see that C_{k+1} is compact and $C_k \subseteq \mathring{C}_{k+1}$ as well as $V_{k+1} \subseteq C_{k+1}$. Thus $(C_i)_{i\in\mathbb{N}}$ is an exhaustion of X by compact subsets. ☐

Now, we can finally prove that all topological manifolds are finite-dimensional.

Theorem 1.11. *Let M be a topological n-manifold. Then M is finite-dimensional and* $\dim M \leq n$.

Proof. Since M is a second-countable locally compact Hausdorff space, M can be exhausted by compact subsets $(C_i)_{i\in\mathbb{N}}$. Each C_i is closed and furthermore each $\overline{C_{i+1} \setminus C_i}$ is compact since $\overline{C_{i+1} \setminus C_i} \subseteq C_{i+1}$. Thus, $\dim \overline{C_{i+1} \setminus C_i} \leq n$ by Corollary 1.8. Lemma 1.9 now yields $\dim M \leq n$. □

2 The embedding theorem

We want to make use of the fact that manifolds are finite-dimensional. The aim of this section is the proof of the following statement.

Theorem 2.1. *Let X be a second-countable locally compact Hausdorff space such that every compact subspace of X has dimension at most $n \in \mathbb{N}$. Then X admits a closed embedding $\iota \colon X \hookrightarrow \mathbb{R}^{2n+1}$.*

Since every n-manifold M is a second-countable locally compact Hausdorff space such that $\dim C \leq n$ for all compact $C \subseteq M$, we can thus conclude that M admits a closed embedding $M \hookrightarrow \mathbb{R}^{2n+1}$.

If X is a topological space, we denote by $C(X, \mathbb{R}^N)$ the set of all continuous maps $X \to \mathbb{R}^N$. We shall equip $\mathbb{R}^N$ with the metric

$$\delta(x, y) := \min\{1, \|x - y\|_\infty\},$$

where $x, y \in \mathbb{R}^N$. Then δ induces the same topology on $\mathbb{R}^N$ as $\|\cdot\|_\infty$ and $(\mathbb{R}^N, \delta)$ is a complete metric space. We equip $C(X, \mathbb{R}^N)$ with the metric

$$\rho(f, g) := \sup_{x\in X} \delta(f(x), g(x)),$$

where $f, g \in C(X, \mathbb{R}^N)$. Since (R^N, δ) is complete, so is $(C(X, \mathbb{R}^N), \rho)$.

Our proof of Theorem 2.1 is based on [Mun00, p. 315, Exercise 6].

Definition 2.2. Let X be a topological space and let $f \in C(X, \mathbb{R}^N)$. We write $f(x) \xrightarrow{x\to\infty} \infty$, if for all $R > 0$ there exists some compact subset $C \subseteq X$ such that $\|f(x)\|_\infty > R$ for all $x \in X \setminus C$.

Remark. Note that $f(x) \xrightarrow{x\to\infty} \infty$ whenever X is compact.

Lemma 2.3. *Let X be a topological space and let $f, g \in C(X, \mathbb{R}^N)$ such that $\rho(f, g) < 1$ and $f(x) \xrightarrow{x\to\infty} \infty$. Then also $g(x) \xrightarrow{x\to\infty} \infty$.*

Proof. Let $R > 0$. There exists some compact subset $C \subseteq X$ such that $\|f(x)\|_\infty > R + 1$ whenever $x \in X \setminus C$. The triangle inequality yields

$$\|f(x)\|_\infty \leq \|g(x)\|_\infty + \|f(x) - g(x)\|_\infty < \|g(x)\|_\infty + 1$$

and hence $\|g(x)\|_\infty > R$ whenever $x \in X \setminus C$. This proves $g(x) \xrightarrow{x\to\infty} \infty$. □

Lemma 2.4. *Let $f \in C(X, \mathbb{R}^N)$ such that $f(x) \xrightarrow{x\to\infty} \infty$. Then f is* proper, *i.e. $f^{-1}(K)$ is compact whenever $K \subseteq \mathbb{R}^N$ is compact. If f is injective as well, then f is also a closed embedding.*

Proof. Let $K \subseteq \mathbb{R}^N$ be compact. Thus, $K \subseteq [-R,R]^N$ for some $R > 0$. We can find a compact subset $C \subseteq X$ such that $\|f(x)\|_\infty > R$ whenever $x \in X \setminus C$. Therefore, $f^{-1}(K) \subseteq f^{-1}\left([-R,R]^N\right) \subseteq C$. This shows that $f^{-1}(K)$ is compact as a closed subset of the compact space C. Therefore, f is proper.
Since f is proper and $\mathbb{R}^N$ is locally compact Hausdorff, f must also be closed. Thus, if f is injective, then it will be a closed embedding. □

Suppose X is a second-countable locally compact Hausdorff space. We can choose a metric d on X that induces the topology of X (see [Bre97, Chapter I, Theorem 12.12]). For every $f \in C(X, \mathbb{R}^N)$ and $C \subseteq X$ compact, we let

$$\Delta(f, C) := \sup_{z \in f(C)} \operatorname{diam} f^{-1}(\{z\}).$$

Lemma 2.5. *Given $\varepsilon > 0$ and $C \subseteq X$ compact, we let*

$$U_\varepsilon(C) := \{f \in C(X, \mathbb{R}^N) \mid \Delta(f, C) < \varepsilon\}.$$

Then $U_\varepsilon(C)$ is open in $C(X, \mathbb{R}^N)$.

Proof. Let $f \in U_\varepsilon(C)$ and let $b > 0$ such that $\Delta(f, C) < b < \varepsilon$. Furthermore, let

$$A := \{(x, y) \in C \times C \mid d(x, y) \geq b\}.$$

Since A is closed in the compact space $C \times C$, A is also compact. The continuous map

$$X \times X \to \mathbb{R}, (x, y) \mapsto \delta(f(x), f(y))$$

is strictly positive on A and thus $r := \frac{1}{2} \cdot \min_{(x,y) \in A} \delta(f(x), f(y))$ satisfies $r > 0$. We will show that $B_\rho(f, r) \subseteq U_\varepsilon(C)$: Let $g \in B_\rho(f, r)$, i.e. $\rho(f, g) < r$. If $(x, y) \in A$, then $\delta(f(x), f(y)) \geq 2r$. Since $\delta(f(x), g(x)) < r$ and $\delta(f(y), g(y)) < r$, we get $g(x) \neq g(y)$. Thus, by contraposition, if $g(x) = g(y)$ for some $x, y \in C$, then $(x, y) \notin A$ and thus $d(x, y) < b$.
This shows $\Delta(g, C) \leq b < \varepsilon$. □

We recall the notion of affine independence.

Definition 2.6. A set of points $S \subseteq \mathbb{R}^N$ is *affinely independent* if for all distinct $p_0, \ldots, p_k \in S$ and $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$, the equations

$$\sum_{i=0}^{k} \alpha_i \cdot p_i = 0 \text{ and } \sum_{i=0}^{k} \alpha_i = 0$$

imply that $\alpha_0 = \cdots = \alpha_k = 0$.

Geometrically speaking, if $S \subseteq \mathbb{R}^N$ is affinely independent and $\operatorname{card}(S) = k$, then the points of S uniquely determine a k-plane in $\mathbb{R}^N$.

Lemma 2.7. *Let $x_1, \ldots, x_n \in \mathbb{R}^N$ be distinct points and let $r > 0$. Then, there exist distinct points $y_1, \ldots, y_n \subseteq \mathbb{R}^N$ such that:*

(1) $\|x_i - y_i\|_\infty < r$ for all $1 \leq i \leq n$.
(2) $\{y_1, \ldots, y_n\}$ is in general position, *i.e. every subset $S \subseteq \{y_1, \ldots, y_n\}$ such that* $\operatorname{card}(S) \leq N + 1$ *is affinely independent.*

Proof. We construct the points $y_1, \ldots, y_n$ inductively. Let $y_1 := x_1$. Now, suppose $y_1, \ldots, y_k$ have already been constructed and are in general position as well as $\|x_i - y_i\|_\infty < r$ for all $1 \leq i \leq k$. Consider the union P of all the affine subspaces that are generated by subsets $A \subseteq \{y_1, \ldots, y_k\}$ such that $\operatorname{card}(A) \leq N$. Since every l-plane in $\mathbb{R}^N$ is closed and has empty interior whenever $l < N$, we can deduce $\mathring{P} = \emptyset$, because $\mathbb{R}^N$ is a Baire space as a complete metric space (see [Bre97, Chapter I, Theorem 17.1]). Choose any $y_{k+1} \in \mathbb{R}^N \setminus P$ satisfying $\|x_{k+1} - y_{k+1}\|_\infty < r$. This process yields the sought points $y_1, \dots, y_n$. □

Another fact from point-set topology that we need are partitions of unity. We shall only state the result here and omit the proof.

Theorem 2.8. *Let X be a paracompact space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. Then there exists a* partition of unity $\{\phi_i\}_{i\in I}$ subordinate to $\mathcal{U}$, *i.e.*

(1) Each $\phi_i \colon X \to [0,1]$ is a continuous map.
(2) supp $\phi_i \subseteq U_i$ *for all $i \in I$.*
(3) $\{\text{supp } \phi_i\}_{i\in I}$ *is locally finite, i.e. each point $x \in X$ has a neighbourhood that intersects only finitely many of the* $\{\text{supp } \phi_i\}_{i\in I}$.
(4) $\sum_{i\in I} \phi_i(x) = 1$ *for all $x \in X$.*

For a proof see [Mun00, Theorem 41.7]. Recall that second-countable locally compact Hausdorff spaces are paracompact.

Lemma 2.9. *Suppose, X is a second-countable locally compact Hausdorff space such that every compact subspace of X has topological dimension at most $n \in \mathbb{N}$. If $\emptyset \neq C \subseteq X$ is compact, then $U_\varepsilon(C)$ is dense in $C(X, \mathbb{R}^N)$ for every $\varepsilon > 0$.*

Proof. Choose a metric d on X and let $f \in C(X, \mathbb{R}^{2n+1})$ and let $1 > r > 0$. We need to find a $g \in U_\varepsilon(C)$ satisfying $\rho(f, g) \leq r$. Since C is compact, we can cover C by finitely many open (open in C) sets $U_1, \dots, U_m \subseteq C$ such that

(1) diam $U_i < \frac{\varepsilon}{2}$ for all $1 \leq i \leq m$,
(2) diam$f(U_i) \leq \frac{r}{2}$ for all $1 \leq i \leq m$,
(3) $\{U_1, \cdot, U_m\}$ has order at most n.

Let $\{\phi_1, \dots, \phi_m\}$ be a partition of unity subordinate to $\{U_1, \dots, U_m\}$. For each $1 \leq i \leq m$ choose a point $x_i \in U_i$. Then choose $z_1, \dots, z_m \in \mathbb{R}^{2n+1}$ such that $\|f(x_i) - z_i\|_\infty < \frac{r}{2}$ and $\{z_1, \dots, z_m\}$ is in general position (Lemma 2.7). Finally, let

$$\tilde{g} \colon C \to \mathbb{R}^{2n+1},\ x \mapsto \sum_{i=1}^{m} \phi_i(x) \cdot z_i.$$

Claim. $\|\tilde{g}(x) - f(x)\|_\infty < r$ for all $x \in C$.

Proof of claim. For all $x \in C$, we have

$$\tilde{g}(x) - f(x) = \sum_{i=1}^{m} \phi_i(x) \cdot (z_i - f(x_i)) + \sum_{i=1}^{m} \phi_i(x) \cdot (f(x_i) - f(x)),$$

where we have used $\sum_{i=1}^{m} \phi_i(x) = 1$. We have $\|z_i - f(x_i)\|_\infty < \frac{r}{2}$ for all $1 \leq i \leq m$. Also if $\phi_i(x) \neq 0$, then $x \in U_i$ and since diam $f(U_i) < \frac{r}{2}$, we can conclude $\|f(x_i) - f(x)\|_\infty < \frac{r}{2}$. Thus,

$$\|\tilde{g}(x) - f(x)\|_\infty < \sum_{i=1}^{m} \phi_i(x) \cdot \frac{r}{2} + \sum_{i=1}^{m} \phi_i(x) \cdot \frac{r}{2} = r.$$

□

Claim. If $x, y \in C$ satisfy $\tilde{g}(x) = \tilde{g}(y)$, then $d(x, y) < \frac{\varepsilon}{2}$.

Proof of claim. We will prove that $\tilde{g}(x) = \tilde{g}(y)$ implies $x, y \in U_i$ for some $1 \leq i \leq m$. Since diam $U_i < \frac{\varepsilon}{2}$, the claim follows.
$\tilde{g}(x) = \tilde{g}(y)$ implies $\sum_{i=1}^{m} (\phi_i(x) - \phi_i(y)) \cdot z_i = 0$. Because the cover $\{U_1, \dots, U_m\}$ has order at most n, at most $n + 1$ of the numbers $\phi_1(x), \dots, \phi_m(x)$ and at most $n + 1$ of the numbers $\phi_1(y), \dots, \phi_m(y)$ are non zero. Letting

$$S := \{z_i \mid 1 \leq i \leq m \text{ and } \phi_i(x) - \phi_i(y) \neq 0\},$$

we can deduce $\text{card}(S) \leq 2n+2$. Note that $\sum_{i=1}^{m}(\phi_i(x) - \phi_i(y)) = 0$ and since $\{z_1, \ldots, z_m\} \subseteq \mathbb{R}^{2n+1}$ are in general position and $\text{card}(S) \leq 2n+1+1$, we can conclude $\phi_i(x) - \phi_i(y) = 0$ for all $1 \leq i \leq m$. Since $\phi_i(x) > 0$ for some $1 \leq i \leq m$, we get $\phi_i(x) = \phi_i(y) > 0$ and thus $x, y \in U_i$. □

In conclusion,

$$h\colon C \to [-r, r]^{2n+1},\ x \mapsto f(x) - \tilde{g}(x)$$

is a well-defined continuous map. As a locally compact Hausdorff space, X is also normal. Thus, we can apply the Tietze extension theorem (see [Mun00, Theorem 35.1]): h can be extended to a continuous map $H : X \to [-r, r]^{2n+1}$. Letting

$$g\colon X \to \mathbb{R}^{2n+1}, x \mapsto f(x) - H(x),$$

we have $g|_C = \tilde{g}$ and thus $\Delta(g, C) \leq \frac{\varepsilon}{2} < \varepsilon$ and $\rho(f, g) \leq r$. □

Let X be as in Theorem 2.1 or Lemma 2.9 and choose a metric d on X. Since $(C(X, \mathbb{R}^{2n+1}, \rho)$ is a Baire space, every intersection of countably many open dense subsets of $C(X, \mathbb{R}^{2n+1})$ is again dense in $C(X, \mathbb{R}^{2n+1})$. Consider an exhaustion of X by compact subsets $(C_k)_{k\in\mathbb{N}}$ (Lemma 1.10). Then the set $\bigcap_{k=1}^{\infty} U_{1/k}(C_k)$ is dense in $C(X, \mathbb{R}^{2n+1})$.

Lemma 2.10. *Every $f \in \bigcap_{k=1}^{\infty} U_{1/k}(C_k)$ is injective.*

Proof. Let $x, y \in X$ such that $f(x) = f(y)$. There exists some $k_0 \in \mathbb{N}$ such that $x, y \in C_k$ whenever $k \geq k_0$. Because $f \in U_{1/k}(C_k)$, we get $d(x, y) \leq \frac{1}{k}$ for all $k \geq k_0$. Hence, $d(x, y) = 0$ and therefore $x = y$. □

Lemma 2.11. *If X is a second-countable locally compact Hausdorff space, then there exists a map $f \in C(X, \mathbb{R}^N)$ such that $f(x) \xrightarrow{x\to\infty} \infty$.*

Proof. It suffices to consider the case $N = 1$. Let $\{U_k\}_{k\in\mathbb{N}}$ be cover of X by open sets such that $\overline{U_k}$ is compact for each $k \in \mathbb{N}$. Since X is second-countable locally compact Hausdorff, X is paracompact and we can find a partition of unity $\{\phi_k\}_{k\in\mathbb{N}}$ subordinate to $\{U_k\}_{k\in\mathbb{N}}$. Letting

$$f\colon X \to \mathbb{R},\ x \mapsto \sum_{k=1}^{\infty} k \cdot \phi_k(x),$$

we see that $f(x) \xrightarrow{x\to\infty} \infty$. □

We can now proceed to the proof of Theorem 2.1.

Proof. Begin with a continuous map $f\colon X \to \mathbb{R}^{2n+1}$ such that $f(x) \xrightarrow{x\to\infty} \infty$ from Lemma 2.11. Consider an exhaustion of X by compact subsets $(C_k)_{k\in\mathbb{N}}$ (Lemma 1.10). Since $\bigcap_{k=1}^{\infty} U_{1/k}(C_k)$ is dense in $C(X, \mathbb{R}^{2n+1})$, we can find $\iota \in \bigcap_{k=1}^{\infty} U_{1/k}(C_k)$ such that $\rho(f, \iota) < 1$. Then ι is injective by Lemma 2.10 and $\iota(x) \xrightarrow{x\to\infty} \infty$ by Lemma 2.3. Then, $\iota\colon M \hookrightarrow \mathbb{R}^{2n+1}$ is a closed embedding by Lemma 2.4, as desired. □

3 ANRs and ENRs

Definition 3.1. A topological space X is called an *Absolute Neighbourhood Retract (ANR)* if for every paracompact space P and every continuous map $f\colon A \to X$, where $A \subseteq P$ is closed, there exists an extension $\overline{f}\colon W \to X$ of f where W is an open neighbourhood of A.

Why are we interested in ANRs? In this section, we want to prove the following.

Theorem 3.2. *Every topological manifold is an ANR.*

Why do we want to prove this? Here is the reason.

Theorem 3.3. *Every topological manifold is an ENR.*

Proof. Let M be a topological n-manifold and let $\iota\colon M \hookrightarrow \mathbb{R}^{2n+1}$ be a closed embedding. Because M is an ANR by Theorem 3.2, so is $\iota(M)$. Since $\iota(M) \subseteq \mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n+1}$ is paracompact, the map $f\colon \iota(M) \to \iota(M)$, $x \mapsto x$ can be extended to a map $r\colon U \to \iota(M)$ where U is an open neighbourhood of $\iota(M)$. This r is a retraction. □

We will prove Theorem 3.2 by a series of lemmas and we will follow [KK].

Lemma 3.4. *Every open subset of an ANR is again an ANR.*

Proof. Let X be an ANR and let $U \subseteq X$ be open. Let $f\colon A \to U$ be continuous where $A \subseteq P$ is closed, P is paracompact. Letting $\tilde{f} := i \circ f$, where $i\colon U \hookrightarrow X$ is the standard embedding, $\tilde{f}$ can be extended to a map $\overline{f}\colon W \to X$, where W is an open neighbourhood of A. Then, $\overline{f}|_{\overline{f}^{-1}(U)}$ is the sought extension. □

Lemma 3.5. *Let X be paracompact and assume further* dim $X \leq n$. *If $\mathcal{U}$ is an open cover of X, there exist $n+1$ collections of open subsets $\mathcal{V}_0, \ldots, \mathcal{V}_n$ such that $\mathcal{V} := \bigcup_{k=0}^{n} \mathcal{V}_k$ is a locally finite refinement of $\mathcal{U}$.*

Proof. Since dim $X \leq n$, we can assume that $\mathcal{U} = \{U_i\}_{i\in I}$ has order at most n. Let $\{\phi_i\}$ be a partition of unity subordinate to $\mathcal{U}$. For each $i \in I$, we let

$$V_i := \{x \in X \mid \forall j \in I \setminus \{i\}\colon \phi_i(x) > \phi_j(x)\}.$$

Then $V_i \subseteq \text{supp } \phi_i \subseteq U_i$ and $V_i \cap V_j = \emptyset$ whenever $i \neq j$. Let $\mathcal{V}_0 := \{V_i\}_{i\in I}$.
Now let $0 \leq k \leq n$ and let $i_0, \ldots, i_k \in I$ be distinct indices. Let

$$V_{i_0,\ldots,i_k} := \{x \in X \mid \phi_i(x) > \phi_j(x) \text{ whenever } i \in \{i_0, \ldots, i_k\} \text{ and } j \notin \{i_0, \ldots, i_k\}\}$$

Note that $V_{i_0,\ldots,i_k} \cap V_{j_0,\ldots,j_k} = \emptyset$ whenever $\{i_0, \ldots, i_k\} \neq \{j_0, \ldots, j_k\}$, because for $i \in \{i_0, \ldots, i_k\} \setminus \{j_0, \ldots, j_k\}$ and $j \in \{j_0, \ldots, j_k\} \setminus \{i_0, \ldots, i_k\}$ we have $\phi_i(x) > \phi_j(x)$ for all $x \in V_{i_0,\ldots,i_k}$ and $\phi_j(y) > \phi_i(y)$ for all $y \in V_{j_0,\ldots,j_k}$.
Define $\mathcal{V}_k$ be the set of all such $V_{i_0,\ldots,i_k}$ and let $\mathcal{V} := \bigcup_{k=0}^{n} \mathcal{V}_k$.
We need to show that $\mathcal{V}$ covers X. Let $x \in X$ and let $J := \{i \in I \mid \phi_i(x) > 0\}$. Then, card$(J) \leq n+1$ since $\mathcal{U}$ has order at most n. Writing $J = \{j_0, \ldots, j_k\}$, we get $x \in V_{j_0,\ldots,j_k}$. Obviously, $\mathcal{V}$ is a refinement of $\mathcal{U}$. It only remains to show that $\mathcal{V}$ is locally finite. Let $x \in X$. There exists a neighbourhood N of x that intersects only finitely many of the $\{\text{supp } \phi_i\}_{i\in I}$. Let $J := \{i \in I \mid \text{supp } \phi_i \cap N \neq \emptyset\}$. Then card$(J) < \infty$. Assume $V_{j_0,\ldots,j_k} \in \mathcal{V}$ intersects N. Let $y \in N \cap V_{j_0,\ldots,j_k}$. Then $\phi_{j_l}(y) > 0$ for all $1 \leq l \leq k$. Thus $\{j_0, \ldots, j_k\} \subseteq J$. But there are only finitely many subsets of J and hence only finitely many elements of $\mathcal{V}$ intersect N. □

Lemma 3.6. *Let X be paracompact space and let $\mathcal{U} := \{U_i\}_{i\in I}$ be an open cover of X. Then there exista a locally finite cover $\mathcal{V} = \{V_i\}_{i\in I}$ of X satisfying $\overline{V_i} \subseteq U_i$ for all $i \in I$.*

For a proof see [Mun00, Lemma 41.6].

The following lemma is needed for local-to-global results.

Lemma 3.7. *Let X be a paracompact space and suppose* dim $X \leq n$. *Let $\mathcal{U}$ be an open cover of X satisfying the following.*

(1) If $V \subseteq X$ is open and $V \subseteq U$ for some $U \in \mathcal{U}$, then $V \in \mathcal{U}$.
(2) If $\mathcal{V} \subseteq \mathcal{U}$ and $V_1 \cap V_2 = \emptyset$ for any $V_1, V_2 \in \mathcal{V}$ such that $V_1 \neq V_2$, then $\bigcup_{V\in\mathcal{V}} V \in \mathcal{U}$.
(3) If $U_1, U_2 \in \mathcal{U}$ and $V_1, V_2 \subseteq X$ are open and $\overline{V}_1 \subseteq U_1$, $\overline{V}_2 \subseteq U_2$, then $V_1 \cup V_2 \in \mathcal{U}$.

Then $X \in \mathcal{U}$.

Proof. Let $\mathcal{V} := \bigcup_{k=0}^{n} \mathcal{V}_k$ as in Lemma 3.5. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, we have $\mathcal{V} \subseteq \mathcal{U}$ by (1). Thus, by (2), we have $V_k := \bigcup_{V \in \mathcal{V}_k} V \in \mathcal{U}$ for all $0 \leq k \leq n$. Then $\{V_k\}_{0 \leq k \leq n}$ is an open cover of X by $n+1$ elements of $\mathcal{U}$. By Lemma 3.6, there exists an open cover $\{W_k\}_{0 \leq k \leq n}$ of X satisfying $\overline{W}_k \subseteq V_k$ for all $0 \leq k \leq n$. Then $W_0 \cup W_1 \in \mathcal{U}$ by (3). By using (1), we see that $\{W_0 \cup W_1, \dots, W_n\}$ is an open cover of X by n elements of $\mathcal{U}$. Repeat this process with $\{W_0 \cup W_1, \dots, W_n\}$ instead of $\{W_k\}_{0 \leq k \leq n}$ to get a covering of X by $n-1$ elements of $\mathcal{U}$ and so on until X is covered by one element of $\mathcal{U}$, which eventually yields $X \in \mathcal{U}$. □

Before we proceed to the proof of Theorem 3.2, we should notice that the closed unit interval I is an ANR as a consequence of the Tietze extension theorem and hence so is I^n for any $n \in \mathbb{N}_0$. We now come to the proof of Theorem 3.2.

Proof of Theorem 3.2. Let M be a topological n-manifold and let $\mathcal{U}$ be the collection of all open subsets of M which are ANRs. Then, $\mathcal{U}$ is an open cover of M since every point $p \in M$ lies in a neighbourhood that is homeomorphic to an open subset of I^n and is thus an ANR by Lemma 3.4.

We will be done, once we show that $\mathcal{U}$ satisfies the conditions (1) - (3) in Lemma 3.7. Condition (1) ia met since every open subset of an ANR is again an ANR.

For condition (2) consider a subset $\mathcal{V} := \{V_i\}_{i \in I}$ of $\mathcal{U}$ that consists of disjoint sets and let $f\colon A \to V := \bigcup_{i \in I} V_i$ be a continuous map where A is a closed subset of a paracompact space P. Each V_i is clopen in V since the $\{V_i\}_{i \in I}$ are disjoint. Thus, $A_i := f^{-1}(V_i)$ is closed in P for each $i \in I$ as is $\bigcup_{i \in J} A_i$ for all $J \subseteq I$.

If we can find a disjoint collection of open sets $\{W_i\}_{i \in I}$ of P such that $A_i \subseteq W_i$ for all $i \in I$, we will be done: Then we can extend $f|_{A_i}\colon A_i \to V_i$ to $\overline{f}_i\colon W_i' \to V_i$, where W_i' is open in P (because V_i is an ANR) and define $W := \bigcup_{i \in I}(W_i \cap W_i')$ as well as $\overline{f}\colon W \to V$ by $\overline{f}|_{W_i \cap W_i'} := \overline{f}_i|_{W_i \cap W_i'}$.

Claim. If P is paracompact and $\{A_i\}_{i \in I}$ is a disjoint collection of closed sets such that $\bigcup_{i \in J} A_i$ is closed for any $J \subseteq I$, then there exists a disjoint collection $\{W_i\}_{i \in I}$ of open sets such that $A_i \subseteq W_i$ for all $i \in I$.

Proof of the claim. P is normal as a paracompact space und thus we can find open sets $\{Y_i\}_{i \in I}$ such that $A_i \subseteq Y_i$ and $\overline{Y}_i \cap \bigcup_{j \in I \setminus \{i\}} A_i = \emptyset$. Let $A := \bigcup_{i \in I} A_i$. Then $\{Y_i\}_{i \in I} \cup \{P \setminus A\}$ is an open cover of P. By Lemma 3.6, there is a locally finite open cover $\{Z_i\}_{i \in I} \cup \{Z\}$ such that $\overline{Z}_i \subseteq Y_i$ for all $i \in I$ and $\overline{Z} \subseteq P \setminus A$. By local finiteness, the equality $\bigcup_{j \in J} \overline{Z}_j = \overline{\bigcup_{j \in J} Z_j}$ holds for any $J \subseteq I$, hence $\bigcup_{j \in J} \overline{Z}_j$ is closed. Thus, $W_i := Z_i \setminus \bigcup_{j \in I \setminus \{i\}} \overline{Z}_j$ is an open set such that $A_i \subseteq W_i$ and the $\{W_i\}_{i \in I}$ are disjoint. □

This proves condition (2). All that is left is proving condition (3). Let $U_1, U_2 \in \mathcal{U}$ and let $V_1, V_2 \subseteq M$ be open such that $\overline{V}_1 \subseteq U_1$ and $\overline{V}_2 \subseteq U_2$. We need to show that $V_1 \cup V_2 \in \mathcal{U}$, i.e. $V_1 \cup V_2$ is an ANR.

Let $f\colon A \to V_1 \cup V_2$ be continuous, where A is a closed subset of a paracompact space P. Let $B_0 := f^{-1}(\overline{V}_1 \cap \overline{V}_2)$, $B_1 := f^{-1}(\overline{V}_1)$, $B_2 := f^{-1}(\overline{V}_2)$. Then, B_0, B_1 and B_2 are closed subsets of P. Let $A_0 := f^{-1}(U_1 \cup U_2)$. Then A_0 is open in A, hence there exists some open subset $X_0 \subseteq P$ such that $A_0 = X_0 \cap P$. Because P is normal as a paracompact space, we can find an open subset $Y_0 \subseteq P$ such that $B_0 \subseteq Y_0 \subseteq \overline{Y}_0 \subseteq X_0$.

Since $f(\overline{Y}_0 \cap A) \subseteq U_1 \cup U_2$ and $U_1 \cap U_2$ is an ANR, we can extend $f|_{\overline{Y}_0} \cap A$ to a map $\overline{f}_0\colon Z_0 \to U_1 \cap U_2$ where Z_0 is an open neighbourhood of $\overline{Y}_0 \cap A$. Use normality again to find an open set $W_0 \subseteq P$ such that $B_0 \subseteq W_0 \subseteq \overline{W}_0 \subseteq Y_0 \cap Z_0$.

Thus $\overline{f}_0$ is defined on $\overline{W}_0$ and extends $f|_{\overline{W}_0 \cap A}$. For $i \in \{1, 2\}$, let $f_i\colon B_i \cup \overline{W}_0 \to U_i$ be defined by $f_i(x) := f(x)$ for all $x \in B_i$ and $f(x) := \overline{f}_0(x)$ for all $x \in \overline{W}_0$. We can extend f_i to $\overline{f}_i\colon Z_i \to U_i$ where Z_i is an open neighbourhood of $B_i \cup \overline{W}_0$, because U_i is an ANR.

Since

$$(B_1 \setminus W_0) \cap (B_2 \setminus W_0) = (B_1 \cap B_2) \setminus W_0 = B_0 \setminus W_0 = \emptyset,$$

and both $B_1 \setminus W_0$ and $B_2 \setminus W_0$ are closed, we can once again use normality to find disjoint open sets $W_1, W_2 \subseteq P$ such that $B_i \setminus W_0 \subseteq W_i \subseteq Z_i$ for each $i \in \{1, 2\}$.
Finally, let $\overline{f}\colon W_0 \cup W_1 \cup W_2 \to U_1 \cup U_2$ be defined by $f|_{W_i} := \overline{f}_i|_{W_i}$ where $i \in \{0, 1, 2\}$. By letting $W := \overline{f}^{-1}(V_1 \cup V_2)$, we can conclude that $\overline{f}|_W$ is an extension of f to an open neighbourhood W of A. This proves (3) and therefore, M is an ANR. □

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