The geometry of finitely generated kleinian groups*

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To my father on his 70th birthday

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Introduction

Until recently little of a general nature was known about finitely generated kleinian groups, indeed it could well have been questioned whether any general theory was possible. Poincaré, in his fundamental paper of 1883, attempted to treat these groups in a manner analogous to his treatment of fuchsian groups: He recognized that they can be extended from acting on the complex plane to acting on upper half space (hyperbolic 3-space) and therefore that there are fundamental polyhedra which play the role of the fundamental polygons for fuchsian groups. At this point, however, his general analysis ended. Even though a few additional general facts were discovered in the interim, the state of knowledge was essentially unchanged until the appearance in 1964 of Ahlfors' finiteness theorem and soon afterwards by Bers' inequalities giving actual estimates for Ahlfors' theorem. Their beautiful results were proved by deep analytic methods enabling them to construct from the group certain spaces of differentials on the quotient surfaces; the depth of these results is merely confirmed in our present study. Kra has woven their results into an elegant cohomology theory.

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In contrast to the analytic approach of Ahlfors and Bers, Maskit has pioneered in the study of kleinian groups by purely geometric methods in the complex plane. His work complements the insight provided by the analytic approach and yields some very deep and fundamental knowledge concerning certain special classes of groups. In particular Maskit develops the concept that the class of "nice" groups, that is the class for which generalizations of the classical results for fuchsian groups can be fruitfully sought, is the class of constructible groups. These are the groups that arise from his far-reaching generalizations of the Klein combination theorems. Although we have taken a different path in this paper, Maskit's work has been a great influence. Indeed, there are many points of contact between his work and ours although the precise relation remains unclear.

The program of this paper is to carry forward Poincaré's original approach towards the goal of providing a general theory of kleinian groups. Following Poincaré, we view a (finitely generated) group G as giving rise to a 3-manifold $\mathfrak{M}(G)$ with boundary. A large number of problems concerning G can be reformulated as topological questions about $\mathfrak{M}(G)$ and a major share of our work consists of a topological analysis of $\mathfrak{M}(G)$. This is possible at the present time only because understanding of the topology of 3-manifolds has reached a rather mature stage. We will use most directly some striking recent results of Waldhausen; how much we are indebted to his work will be clear to the reader.

The topological analysis of $\mathfrak{M}(G)$ does not directly involve the limit set $\Lambda(G)$ of G. To pass from information about $\mathfrak{M}(G)$ to information about $\Lambda(G)$ we consistently use Gehring's extension theorem which says that a quasiconformal map of upper half space can be extended to be quasiconformal on the bounding plane. This is applied by lifting a piecewise linear map between two (for instance) compact manifolds $\mathfrak{M}(G)$ and $\mathfrak{M}(H)$ with smooth triangulations. Like Mostow (see below) we find this theorem plays a very fundamental, yet incompletely understood, role in determining the possible deformations of G.

The analysis in this paper is concerned with groups without elliptic elements (groups without torsion). We plan to supplement this material at a later date with a discussion of the more general case; in some cases the extension is immediate, in others somewhat different techniques are required. In any case, with this restriction the fundamental group of $\mathfrak{M}(G)$ is isomorphic to G.

This paper is divided into two main parts. The first is directed toward a topological analysis of Ahlfors' theorem and Bers' inequality (Chapters

3–7). In order to carry out a complete topological analysis we find it necessary to make certain assumptions (§§ 6.1, 12.3) on the 3-manifolds involved. This is because of the lack of information about non-compact manifolds although very recently G. P. Scott made an important breakthrough in this direction (see Chapter 13). In particular his results imply that the assumptions are satisfied if G is not a free product. It seems reasonable to believe this is true in general. In any case the 3-manifolds involved can be rather well described in their relation to compact manifolds (Theorem 6.4). Our work includes new proofs of some of Maskit's theorems and in particular provides the 3-dimensional analogue to his study of B-groups (Chapters 3 and 5). With the assumptions we are also able to recover the topological part of Ahlfors' theorem and a sharpened form of Bers' inequality (Chapter 7).

The second part of this paper is directed toward analyzing that class $\mathcal{C}$ of groups which have a finite-sided fundamental polyhedron in hyperbolic 3-space. This assumption is a natural generalization of the usual assumption in Lie group theory that the coset space $\mathcal{G}/\Gamma$ have finite volume for a discrete subgroup Γ of $\mathcal{G}$ (see Lemma 4.7). But more important is the fact that the groups in $\mathcal{C}$ are exactly those groups for which $\mathfrak{M}(G)$ can be compactified in a natural way (Proposition 4.2). There are two principal results concerning the groups in $\mathcal{C}$. One (Theorem 8.1) implies that if $\psi: G \rightarrow H$, $G \in \mathcal{C}$, is an isomorphism induced by a quasiconformal homeomorphism of regular sets $f: \Omega(G) \rightarrow \Omega(H)$, then f has a quasiconformal extension to the extended plane. A consequence of this might be regarded as the analogue for kleinian groups of Mostow's rigidity theorem (see Theorem 8.3 for its most general expression), namely: If f above is conformal, then G and H are conjugate groups.

We also prove (Theorem 10.1) that a $G \in \mathcal{C}$ is "strongly stable". In other words every small homomorphism $\varphi: G \rightarrow \mathrm{SL}(2, \mathrm{C})/\pm 1$ which sends parabolic elements to parabolic elements is in fact an isomorphism induced by a quasiconformal map of the extended plane with small dilatation. This result is just what is required to fit the groups in $\mathcal{C}$ into Bers' deformation theory for kleinian groups (§ 10.6: This general theory includes the classical Teichmüller space theory). Together with his results one now knows that the deformation space $T(G)$ based on a fixed $G \in \mathcal{C}$ is a manifold with a natural complex analytic structure induced from the matrices, or more precisely from the deformation variety of G. Perhaps surprisingly, the dimension of $T(G)$ depends only on the topological type of the boundary of the 3-manifold $\mathfrak{M}(G)$ associated with G, not on the "internal structure" of the manifold itself. On the other hand this fact can be regarded as being consistent with Mostow's rigidity theorem, which can be interpreted to say in particular that $T(G)$, for a discrete subgroup G of $\mathrm{SL}(2, \mathrm{C})/\pm 1$ of finite volume, is a point (Corollary 9.5).

In Chapter 11, we try to indicate the relation of constructive methods to the theory we have presented. This is done by displaying certain forms of the Klein combination theorems which have general applicability in constructing more complicated groups from simpler ones, and then showing that $\mathcal{C}$ is preserved under these operations.

It is a pleasure for me to acknowledge the advice and inspiration I received in discussions with L. Bers, B. Maskit, and C. Papakyriakopoulos.

1. Topological preliminaries

1.1. Throughout our study, and in particular in this chapter, we will be concerned with oriented 3-manifolds $\mathfrak{M}$ (usually with boundary $\partial\mathfrak{M}$) which have a C^∞ differential structure. Later on, the manifolds will actually have a natural conformal structure and the C^∞ structure will be taken from that. Neither $\mathfrak{M}$ nor $\partial\mathfrak{M}$ will necessarily be compact. Of course 3-manifolds can be triangulated but for analytic purposes we will need not just a topological triangulation but a "smooth" triangulation. It is known that $\mathfrak{M}$ has a C^∞ triangulation and in fact a C^∞ triangulation of $\partial\mathfrak{M}$ can be extended to a C^∞ triangulation of $\mathfrak{M}$ (for details and definitions we refer to Munkres' book [44]). Smoothly imbedded simplices are required for the following reason.

Definition. A homeomorphism $f\colon \mathfrak{M}_1 \to \mathfrak{M}_2$ between two C^∞ manifolds is *quasiconformal* if there exist coordinate coverings (u_i, h_i) of $\mathfrak{M}_1$ and (u'_j, h'_j) of $\mathfrak{M}_2$ and a constant K, $1 \leqq K < \infty$, such that the composed mappings $h'_j \circ f \circ h_i^{-1}$, whenever they are defined, are K-quasiconformal in the euclidean sense.

Of course for the study of quasiconformal maps of manifolds, this definition is not satisfactory because K is not invariantly defined. However, for our purposes, as illustrated by the following result, this is sufficient.

LEMMA 1.1. *A piecewise linear* (PL) *homeomorphism between two compact C^∞ manifolds with C^∞ triangulation is quasiconformal.*

We omit the proof since it is a direct consequence of the basic properties of quasiconformal mappings (see [16, 41]).

1.2. Notation and conventions. $\mathfrak{M}^0$ denotes the interior of $\mathfrak{M}$ and $\mathfrak{M}^-$ the closure of $\mathfrak{M}$ in some larger manifold. If $\eta \subset \mathfrak{M}$ is a submanifold we will make frequent use of the notation $\partial_0\eta$ for the relative boundary of $\eta \cap \mathfrak{M}^0$ in $\mathfrak{M}^0$.

Unless otherwise specified our work will be carried out in the PL category but using only C^∞ triangulations. When, because of the nature of the problem, a shift to the C^∞ category is necessary, it will be explicitly described.

If S is a compact surface imbedded in $\mathfrak{M}$, S will always be assumed to (i) be non-singular, and (ii) satisfy $\partial S = \partial\mathfrak{M} \cap S$.

A *regular neighborhood* $N(S)$ of S is a compact submanifold of $\mathfrak{M}$ such that there exists a homeomorphism (PL) $f\colon N(S) \to S \times [0, 1]$ for which $f(S) = S \times \{1/2\}$. By sufficiently subdividing the triangulation of $\mathfrak{M}$ near S, $N(S)$ can be made arbitrarily close to S.

A *non-trivial* loop in a surface S is a loop which is not contractible in S to a point.

$H_1(\mathfrak{M})$ denotes the first integral homology group.

1.3. We begin with two lemmas that relate the first homology group $H_1(\partial\mathfrak{M})$ of $\partial\mathfrak{M}$ to $H_1(\mathfrak{M})$.

LEMMA 1.2 (cf. [27]). *If g is the total genus of $\partial\mathfrak{M}$ (= sum of genera of the components of $\partial\mathfrak{M}$) then $g \leqq$* rank $H_1(\mathfrak{M})$.

LEMMA 1.3. *Assume $\mathfrak{M}$ is compact and the inclusion map $H_1(\partial\mathfrak{M}) \to H_1(\mathfrak{M})$ is surjective. If g is the total genus of $\partial\mathfrak{M}$ then* rank $H_1(\mathfrak{M}) = g$.

Proof. Observe that the first Betti numbers satisfy $\beta_1(\mathfrak{M}) \leqq \beta_1(\partial\mathfrak{M}) = 2g$ and the first relative Betti number satisfies $\beta_1(\mathfrak{M}, \partial\mathfrak{M}) = m - 1$ where m is the number of components of $\partial\mathfrak{M}$. Indeed, choose $(m - 1)$ 1-chains connecting the m components of $\partial\mathfrak{M}$. Given any relative 1-cycle γ_0 in $\mathfrak{M}$ there is a 1-chain on $\partial\mathfrak{M}$ and a linear combination γ_c of the connecting chains so that when these are added to γ_0 we obtain a 1-cycle γ. Since γ is homologous to a cycle on $\partial\mathfrak{M}$ we see that γ_0 is homologous, mod $\partial\mathfrak{M}$, to γ_c. Now the double $\hat{\mathfrak{M}}$ of $\mathfrak{M}$ is compact so the Euler characteristics satisfy $2\chi(\mathfrak{M}) - \chi(\partial\mathfrak{M}) = \chi(\hat{\mathfrak{M}}) = 0$. Using Lefschetz duality and the facts $\beta_3(\mathfrak{M}) = 0$, $\beta_0(\mathfrak{M}) = 1$, $\beta_0(\partial\mathfrak{M}) = \beta_2(\partial\mathfrak{M}) = m$, Lemma 1.3 follows from

$$2(-\beta_3 + \beta_2 - \beta_1 + \beta_0)(\mathfrak{M}) = (\beta_2 - \beta_1 + \beta_0)(\partial\mathfrak{M}) .$$

1.4. The contemporary theory of 3-manifolds is based on Dehn's lemma, the loop theorem, and the sphere theorem which we will use in the following special forms.

LEMMA 1.4 (Papakyriakopoulos [45], [46]). *If γ is a non-trivial loop in $\partial\mathfrak{M}$ which is contractible to a point in $\mathfrak{M}$, and if $\gamma \subset N \subset \partial\mathfrak{M}$ is a neighborhood of γ, there is a simple loop $\gamma_0 \subset N$, non-trivial in $\partial\mathfrak{M}$, which bounds a disk D in $\mathfrak{M}$. If γ is a simple loop, D can be chosen so that $\partial D = \gamma$.*

LEMMA 1.5 (Papakyriakopoulos [45], Whitehead [52]). *If $\pi_2(\mathfrak{M}^0) \neq 0$,* *then* $\mathfrak{M}^0$ *contains a sphere which does not bound a ball; if* $\pi_2(\mathfrak{M}^0) = 0$ *and* $\pi_1(\mathfrak{M}^0)$ *is infinite, then* $\pi_3(\mathfrak{M}^0) = 0$.

The following result will be frequently used to cut up our manifolds.

LEMMA 1.6 (Waldhausen [49]). THE CYLINDER THEOREM. *Suppose* γ_1, γ_2 *are mutually disjoint loops on* $\partial\mathfrak{M}$ *neither of which is homotopic to a point in* $\mathfrak{M}$. *Assume* γ_1 *is freely homotopic in* $\mathfrak{M}$ *to* γ_2 *and* $\gamma_1 \subset N_1$, $\gamma_2 \subset N_2$ *are disjoint neighborhoods. Then there exist simple loops* $\alpha_1 \subset N_1$, $\alpha_2 \subset N_2$, *neither of which is contractible in* $\mathfrak{M}$, *which are the boundary components of a cylinder in* $\mathfrak{M}$. *If* γ_1 *and* γ_2 *are simple loops, we can take* $\alpha_1 = \gamma_1$ *and* $\alpha_2 = \gamma_2$.

The corresponding fact for surfaces is in [28].

1.5. The following test for compactness will also turn out to be useful.

LEMMA 1.7. *Suppose* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *have no boundary and satisfy* $\pi_2(\mathfrak{M}_i) = \pi_3(\mathfrak{M}_i) = 0$, $i = 1, 2$. *Assume* $\mathfrak{M}_1$ *is compact and* $\pi_1(\mathfrak{M}_1)$ *is isomorphic to* $\pi_1(\mathfrak{M}_2)$. *Then the integral homology group* $H_k(\mathfrak{M}_1)$ *is isomorphic to* $H_k(\mathfrak{M}_2)$, $k = 0, 1, 2, 3$. *In particular* $\mathfrak{M}_2$ *is also compact.*

Proof. As always $\mathfrak{M}_1$, $\mathfrak{M}_2$ are oriented 3-manifolds. There exists a continuous map $f: \mathfrak{M}_1 \to \mathfrak{M}_2$ which induces the given isomorphism $\pi_1(\mathfrak{M}_1) \to \pi_1(\mathfrak{M}_2)$ ([19, p. 198]). It follows from the Whitehead theorem ([19, p. 167]) that f induces an isomorphism between the integral homology groups. The group $H_3(\mathfrak{M}_2)$ tells that $\mathfrak{M}_2$ is compact.

1.6. LEMMA 1.8. (van Kampen [39].) *In the notation of Lemma* 1.4, *if* $\mathfrak{M}$-D *has two components* $\mathfrak{M}_1$, $\mathfrak{M}_2$ *then* $\pi_1(\mathfrak{M}) = \pi_1(\mathfrak{M}_1) * \pi_1(\mathfrak{M}_2)$ (*free product*); *if there is only one component, then* $\pi_1(\mathfrak{M}) = \pi_1(\mathfrak{M} - D) * \mathbf{Z}$.

LEMMA 1.9. (Grushko [39].) *If* $G = G_1 * G_2$ *then the ranks satisfy* $r(G) = r(G_1) + r(G_2)$.

1.7. We will be in the fortunate position that the universal covering spaces of the manifolds $\mathfrak{M}^0$ under consideration in Chapter 2 and beyond will turn out to be euclidean space R^3. A simple, but for our purposes extremely important, consequence of this is that *every imbedded* 2-*sphere in* $\mathfrak{M}$ *bounds a ball in* $\mathfrak{M}$. (A manifold with this property is called *irreducible.*) This follows from Alexander's theorem which asserts that the closure of each complementary component of a polyhedral 2-sphere in S^3 is a 3-cell. Because our manifolds have this property, the fact that the Poincaré conjecture is unproved in no way affects our work.

Of course in addition it is true that the homotopy groups $\pi_2(\mathfrak{M}) = \pi_3(\mathfrak{M}) = 0$.

For the rest of this section, the universal covering space of $\mathfrak{M}^0$ is assumed to be $\mathbf{R}^3$.

LEMMA 1.10 (Waldhausen [50]). *Suppose* $\mathfrak{M}$ *is compact with* $\partial\mathfrak{M} \neq \emptyset$. *If* $\pi_1(\mathfrak{M})$ *is a non-trivial free product then there exists a disk* D *in* $\mathfrak{M}$, $\partial D \subset \partial\mathfrak{M}$, *which induces some splitting of* $\pi_1(\mathfrak{M})$ *into a non-trivial free product.*

Definition. A compact surface $S \subset \mathfrak{M}$ with $\partial S = \partial\mathfrak{M} \cap S$ is *incompressible* if ker $(\pi_1(S) \to \pi_1(\mathfrak{M})) = 0$.

LEMMA 1.11. *The following statements are equivalent* (π: $\mathbf{R}^3 \to \mathfrak{M}^0$ *is the natural projection*).

(i) S *is incompressible in* $\mathfrak{M}$.

(ii) *Each component of* $\pi^{-1}(S^0)$ *in* $\mathbf{R}^3$ *is simply connected.*

(iii) *No non-trivial simple loop in* S *bounds a disk in* $\mathfrak{M} - S$.

Proof. These results seem pretty standard. The implications (ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) are trivial. If a component S^* of $\pi^{-1}(S^0)$ is not simply connected, by the general van Kampen theorem [39] applied to S^* and the two components of $\mathbf{R}^3 - S^*$, there exists a non-trivial loop $\alpha^* \subset S^*$ which is contractible in one of the components of $\mathbf{R}^3 - S^*$. By Lemma 1.4 there is a non-trivial simple loop $\alpha_1^* \subset S^*$ which bounds a disk D_1^* with $D_1^{*0} \subset \mathbf{R}^3 - S^*$.

Now D_1^* intersects only a finite number of components of $\pi^{-1}(S^0)$. We can assume all of these intersections consist of simple loops in D_1^*. From this finite set of simple loops choose an innermost one α_2^*; α_2^* bounds a disk $D_2^* \subset D_1^*$. But α_2^* also appears as a simple loop in a component S_1^* of $\pi^{-1}(S^0)$. The projection α_2 of α_2^* is a non-trivial loop in S which violates (i) and, via Lemma 1.4, (iii) as well.

In the following lemma, H^+ denotes the closed upper half space in $\mathbf{R}^3$.

LEMMA 1.12. *Let* P *be a simply connected, complete, polyhedral surface in* $\mathbf{R}^3$. *Then* $\mathbf{R}^3 - P$ *has two components and the closure of each is topologically equivalent to* H^+.

Proof. We begin by recalling that Alexander's theorem implies that if σ is a polyhedral 2-sphere in $\mathbf{R}^3$, $\mathbf{R}^3 - \sigma$ has two components and the closure B of the bounded component is a 3-cell. It is also true that if $D \subset B$ is a disk with $D \cap \sigma = \partial D$ then the closure of each component of $B - D$ is a 3-cell.

That $\mathbf{R}^3 - P$ has two components follows from elementary homology considerations.

Fix a compact set K in $\mathbf{R}^3$ such that $K \cap P$ is connected. We will show that there exists a (polyhedral) sphere σ such that

(i) K lies in the bounded component of $\mathbf{R}^3 - \sigma$,

(ii) $\sigma \cap P$ is a simple loop.

Start with a sphere σ_1 with property (i). We may assume that $\sigma_1 \cap P$ consists of a finite number of simple loops and along each of these, σ_1 crosses P. We will successively reduce the number of components of $\sigma_1 \cap P$, preserving property (i), until we obtain a sphere which also satisfies (ii).

Suppose α is a component of $\sigma_1 \cap P$ and D is the closure of the bounded component of $P - \alpha$; D is a disk. Assume $D \cap K = \emptyset$. If necessary, by replacing α by an innermost loop in the set $\sigma_1 \cap D$, we can also assume that $\sigma_1 \cap D^0 = \emptyset$. Now $\sigma_1 - \alpha$ has two components whose closures D_1, D_2 are disks. Each of $D \cup D_1$, $D \cup D_2$ are spheres and exactly one of them, say $D \cup D_1$, is such that K is contained in the bounded component of $\mathbf{R}^3 - D \cup D_1$. By moving $D \cup D_1$ slightly away from P near D, we obtain a new sphere σ_2 which satisfies (i) but such that $\sigma_2 \cap P$ has one fewer component than $\sigma_1 \cap P$. Repeating this process at most a finite number of times we end up with a sphere σ_m which satisfies property (i) and has the additional quality that if α is a component of $\sigma_m \cap P$, then the bounded component of $P - \alpha$ contains $P \cap K$.

If $\sigma_m \cap P$ is connected we are done. Otherwise, let α be the innermost component of $\sigma_m \cap P$, i.e., the component for which $D^0 \cap \sigma_m = \emptyset$. Find the component β of $\sigma_m \cap P$ which is closest to α; then $\alpha \cup \beta$ is the boundary of a closed annulus $A \subset P$ with $A^0 \cap \sigma_m = \emptyset$. $\alpha \cup \beta$ is also the boundary of a closed annulus A' in σ_m.

Let σ_{m+1} denote the sphere that is obtained from σ_m by replacing A' by A. We claim that the bounded component B_{m+1} of $\mathbf{R}^3 - \sigma_{m+1}$ contains K. To see this first observe that A^0 lies in the unbounded component of $\mathbf{R}^3 - \sigma_m$. This is true because the bounded component D^0 of $P - \alpha$ lies in the bounded component B_m of $\mathbf{R}^3 - \sigma_m$ since it intersects K. Secondly, the torus $A \cup A'$ bounds a solid torus T. T^0 lies in the unbounded component of $\mathbf{R}^3 - \sigma_m$ because A^0 does. Consequently $B_{m+1} = (B_m \cup T)^0$ contains K. Moving σ_{m+1} slightly away from P near A we obtain a sphere σ_{m+2} such that $\sigma_{m+2} \cap P$ has two fewer components than $\sigma_m \cap P$. Repeating this process at most a finite number of times we obtain a sphere $\sigma = \sigma_n$ which has both properties (i) and (ii).

With this preliminary step, Lemma 1.12 is easily proved. We can find inductively a sequence of spheres $\{\sigma_n\}$ such that (i) the closure B_n of the bounded component of $\mathbf{R}^3 - \sigma_n$ contains σ_{n-1} in its interior; (ii) $\sigma_n \cap P$ is connected; and (iii) $\lim \sigma_n = \infty$. Let H be the closure of one of the two components of $\mathbf{R} - P$. Each $B_n \cap H$ is a cell and so is $(B_{n+1} - B_n)^- \cap H$ (its boundary is a sphere).

In $\mathbf{R}^3$ choose a sequence $\{\sigma_n^+\}$ of standard spheres with the properties (i), (ii), (iii) with respect to the plane ∂H^+. This will be our model. Let B_n^+ denote the closure of the bounded component of $\mathbf{R}^3 - \sigma_n^+$. Working step by step, we can set up homeomorphisms, $B_1 \cap H \to B_1^+ \cap H^+$, $(B_{n+1} - B_n)^- \cap H \to (B_{n+1}^+ - B_n^+)^- \cap H^+$, which agree on the common boundary as we pass from the n^{th} to the $(n+1)^{\text{st}}$ stage. In this way we can construct a homeomorphism of

$$H = (B_1 \cap H) \bigcup_1^\infty [(B_{n+1} - B_n)^- \cap H] \longrightarrow H^+$$
$$= (B_1^+ \cap H^+) \bigcup_1^\infty [(B_{n+1}^+ - B_n^+)^- \cap H^+] .$$

COROLLARY 1.13. *Assume S is incompressible in $\mathfrak{M}$ and $\mathfrak{M} - S$ has two components $\mathfrak{M}_1$, $\mathfrak{M}_2$. Then if $\gamma_1 \subset \mathfrak{M}_1$ is freely homotopic to $\gamma_2 \subset \mathfrak{M}_2$, there is a loop in S which is freely homotopic in $\mathfrak{M}_i$ to γ_i, $i = 1, 2$.*

Proof. Fix a component $\mathfrak{M}_1^*$ of $\pi^{-1}(\mathfrak{M}_1^0)$ where $\pi: \mathbf{R}^3 \to \mathfrak{M}^0$ is the natural projection. Lemma 1.12 implies that $\mathfrak{M}_1^*$ is a topological ball since each component of $\pi^{-1}(S^0)$ is simply connected (Lemma 1.11). The lift of γ_1 from a point $x \in \mathfrak{M}_1^*$ terminates at $T(x)$ for some cover transformation T. Although $T(\mathfrak{M}_1^*) = \mathfrak{M}_1^*$, T may not preserve any component of $\partial_0\mathfrak{M}_1^*$. But the hypothesis on γ_2 implies that T also preserves a component of $\pi^{-1}(\mathfrak{M}_2^0)$. This forces T to preserve at least one component S^* of $\partial_0\mathfrak{M}_1^*$. It follows that γ_1 is freely homotopic to a loop in S^0. The same reasoning holds for γ_2.

1.8. In this section we shall gather together some useful results concerning pasting together mappings of 3-manifolds, both in the PL and C^∞ categories.

LEMMA 1.14. *Let S be a closed, orientable, triangulated (resp. C^∞) surface and f, g two orientation preserving* PL *homeomorphisms (resp. diffeomorphisms) of S onto itself which are homotopic. Consider the 3-manifold $S \times [0, 1]$ triangulated so as to extend the given triangulation of $S \cong S \times \{0\} \cong S \times \{1\}$ (resp. with the product C^∞ structure). Interpret f, g as acting as follows: $f: S \times \{0\} \to S \times \{0\}$, $g: S \times \{1\} \to S \times \{1\}$. There exists a* PL *homeomorphism (resp. diffeomorphism) $F: S \times I \to S \times I$ which restricts to f and g on $S \times \{0\}$ and $S \times \{1\}$ respectively.*

Proof. Case 1: f and g are PL maps. There exists an isotopy H connecting f and g [12] which can be interpreted as a level-preserving homeomorphism $S \times I \to S \times I$ which restricts to f and g on the boundary. A theorem of Bing [9, Theorem 9] implies that H can be approximated arbitrarily closely (with respect to any given distance function) by a PL homeomorphism F, with the additional property that F is equal to H and hence to f and g on the boundary components $S \times \{0\}$, $S \times \{1\}$ of $S \times [0, 1]$.

Case 2: f and g are diffeomorphisms. The work of Earle and Eells [10] shows that two orientation preserving diffeomorphisms of a compact, closed surface onto itself, which are homotopic, in fact lie in the same path component of the group of diffeomorphisms of the surface onto itself. According to Munkres [43, p. 523] it is known that when this occurs for a manifold there is actually a differentiable isotopy h between the two diffeomorphisms. In other words there exists a diffeomorphism $F(x, t): S \times I \to S \times I$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ which is defined by $(x, t) \to (h(x, t), t)$.

COROLLARY 1.15. *Let* $\mathfrak{M}$ *be a compact, oriented, triangulated* (*resp.* C^∞) 3-*manifold and* $f: \partial\mathfrak{M} \to \partial\mathfrak{M}$ *a* PL *homeomorphism* (*resp. diffeomorphism*) *which is homotopic to the identity on each component of* $\partial\mathfrak{M}$. *Then there exists a* PL *homeomorphism* (*resp. diffeomorphism*) $F: \mathfrak{M} \to \mathfrak{M}$ *with* $F|\partial\mathfrak{M} = f$.

Proof. Assume first $\partial\mathfrak{M}$ is connected. If f is PL, take a closed neighborhood N of $\partial\mathfrak{M}$ which is PL equivalent to $\partial\mathfrak{M} \times [0, 1]$. Apply Lemma 1.14 in N to the maps f and identity on $\partial\mathfrak{M}$ and $\partial N - \partial\mathfrak{M}$ respectively. Extend the resulting map to $\mathfrak{M} - N$ by setting it equal to the identity.

In the smooth case apply the collaring theorem [44, Theorem 5.9] to obtain a compact, C^∞ submanifold $N \subset \mathfrak{M}$ with $\partial\mathfrak{M} \subset \partial N$ and a diffeomorphism $h: N \to \partial\mathfrak{M} \times I$. Because of this and Lemma 1.14 there exists a diffeomorphism $F_1: N \to N$ such that $F_1|\partial\mathfrak{M} = f$ and f restricted to $R = \partial N - \partial\mathfrak{M}$ is the identity. Extend F_1 to all $\mathfrak{M}$ by setting $F_1 =$ identity in $\mathfrak{M} - N$. Then F_1 is a diffeomorphism except possible near R. Using Munkres' technique [42], F_1 can then be smoothed in a neighborhood of R.

The case in which $\partial\mathfrak{M}$ is not connected is handled in the same way.

The proof used for the following lemma can obviously be applied to prove a much more general result, but this special case is the only one we will encounter here.

LEMMA 1.16. *Suppose f and g are orientation preserving* PL *homeomorphisms* (*resp. diffeomorphisms*) *of the cylinder* $A = \{z \in \mathrm{C}: 1/2 \leqq |z| \leqq 1\}$ *onto itself which preserve the boundary components. Consider the manifold* $A \times I$ *with a triangulation which restricts to that of* $A \cong A \times \{0\} \cong A \times \{1\}$ (*resp. product* C^∞ *structure*). *Interpret the maps f and g as acting on* $A \times \{0\}$, $A \times \{1\}$ *respectively. Then there exists a* PL *homeomorphism* (*resp. diffeomorphism*) $F: A \times I \to A \times I$ *such that* $F|A \times \{0\} = f$, $F|A \times \{1\} = g$.

Proof. Case 1: f and g are PL maps. Since f and g are homotopic (in fact homotopic to the identity), hence isotopic, in A there exists a level-preserving homeomorphism $H: A \times I \to A \times I$ which restricts to f and g on

$A \times \{0\}$ and $A \times \{1\}$. However, H is not necessarily PL on the remainder of $\partial(A \times I)$, namely on the two annuli $A_1 = \{z: |z| = 1/2\} \times I$ and $A_2 = \{z: |z| = 1\} \times I$. On A_1, for example, H is a PL map of each component of ∂A_1 onto itself. From [12] we deduce that $H|A_1$ is isotopic to a PL homeomorphism of A_1 onto itself via an isotopy which keeps the components of ∂A_1 pointwise fixed. Using this isotopy on A_1 and a corresponding one on A_2 we can adjust H so as to obtain a homeomorphism H_1 of $A \times I \to A \times I$ which is PL on $\partial(A \times I)$ and also restricts to f and g. Now Bing's theorem [9] is applicable and there is a PL homeomorphism $F: A \times I \to A \times I$ with $F|\partial(A \times I) = H_1|\partial(A \times I)$.

Case 2: f and g are diffeomorphisms. The proof is a repetition of the corresponding case of Lemma 1.14 but this time we refer to Earle and Schatz [11] to find that f and g are in the same path component of the space of diffeomorphisms of A onto itself. Again [44, p. 523] asserts the existence of a differentiable isotopy joining f and g. This in turn can be interpreted as a diffeomorphism of $A \times I$ which preserves each shell $A \times \{t\}$, $0 \leqq t \leqq 1$.

1.9. We shall also have occasion to use the following two results. The first is required in shifting between the PL and C^∞ categories. The second is a smoothing theorem.

LEMMA 1.17. *Suppose S is a finite union of mutually disjoint compact, C^∞ submanifolds of $\mathfrak{M}$ such that $\partial S \cap \partial\mathfrak{M} = S \cap \partial\mathfrak{M}$ and, if $S \cap \partial\mathfrak{M} \neq \emptyset$, S is transverse to $\partial\mathfrak{M}$ along $S \cap \partial\mathfrak{M}$. Then there exists a C^∞ triangulation of $\mathfrak{M}$ which restricts to a triangulation of S.*

Proof. For simplicity assume S is connected. The hypothesis implies that the double $D(S)$ of S is a C^∞ submanifold of the double $D(\mathfrak{M})$ of $\mathfrak{M}$. As submanifolds of $D(\mathfrak{M})$, $D(S)$ and $\partial\mathfrak{M}$ intersect transversely and have no boundaries. As a consequence of the bicollaring theorem [40, Corollary 3.6] there is a C^∞ triangulation of a closed neighborhood in $D(\mathfrak{M})$ of each of the surfaces $\partial\mathfrak{M}$, $D(S)$ which restricts to a triangulation of $\partial\mathfrak{M}$, $D(S)$ respectively. By [44, 10.11] there is a C^∞ triangulation of a closed neighborhood of $\partial\mathfrak{M} \cup D(S)$ in $D(\mathfrak{M})$ which restricts to a triangulation of *both* $\partial\mathfrak{M}$ and $D(S)$. Now apply [44, 10.7] which asserts that this induced triangulation of $\partial\mathfrak{M} \cup D(S)$ can be extended to a C^∞ triangulation of $D(\mathfrak{M})$. This in turn restricts to a triangulation of $\mathfrak{M}$.

LEMMA 1.18. *Suppose $f: \mathfrak{M}_1 \to \mathfrak{M}_2$ is a homeomorphism, d a metric on $\mathfrak{M}_2$, and δ a positive continuous function on $\mathfrak{M}_1$. There exists a diffeomorphism $g: \mathfrak{M}_1 \to \mathfrak{M}_2$ such that $d(f(p), g(p)) < \delta(p)$ for all $p \in \mathfrak{M}_1$. In addition* *suppose* A_0 *and* A *are closed subsets of* $\mathfrak{M}_1$ *such that* $A_0 \subset A^0$ *and* A *has a* C^∞ *triangulation which restricts to a triangulation of* A_0. *Then if* f *is already a diffeomorphism in a neighborhood of* A, g *can be chosen so that* $g = f$ *in a neighborhood of* A_0.

Proof. The first statement is due to Munkres [43]. The second is too, after some preliminary observations. By a C^∞ triangulation of A we mean a C^∞ imbedding of a euclidean complex into $\mathfrak{M}_1$ whose image is A. By subdividing the triangulation if necessary, we can find a closed set A_1 such that $A_0 \subset A_1^0 \subset A_1 \subset A^0$ and the triangulation of A also restricts to a triangulation of A_1. There is also a C^∞ triangulation of $f(A)$ with respect to which $f: A \to f(A)$ is a PL map. By [44, 10.7] the restriction of these triangulation to A_1, $f(A_1)$ can in turn be extended to C^∞ triangulations of $\mathfrak{M}_1$, $\mathfrak{M}_2$ respectively. Now the homeomorphism $f: \mathfrak{M}_1 \to \mathfrak{M}_2$ is PL on $A_1 \to f(A_1)$. Bing's theorem [9, Theorem 9] allows us to approximate f arbitrarily closely by a PL homeomorphism $f_1: \mathfrak{M}_1 \to \mathfrak{M}_2$ with the property that $f_1 = f$ on A_1. Apply Munkres' smoothing process [43] to f_1. This process consists of smoothing f_1 in small neighborhoods of the 2-, 1-, and 0-simplices of the triangulation of $\mathfrak{M}_1$. In particular, in a neighborhood of A_0, $f_1 = f$ can be left undisturbed.

Remark. The PL version of Lemma 1.18 is also true. It is due to Bing and we have used it in the course of the above proof.

2. Basic properties of kleinian groups

2.1. We will consider the group **M** of Möbius transformations $\{z \mapsto (az + b)/(cz + d), ad - bc = 1\}$ usually viewed as acting on the Riemann sphere, which will be denoted by $\partial\mathfrak{B}$. Other representations of **M** are the simple Lie groups SL(2, C)/$\pm$ 1 with the obvious identification and SO(1, 3)/$\pm$ 1 with identification via stereographic projection and homogeneous coordinates. There is a standard classification that $T \in \mathbf{M}$ is loxodromic, parabolic (unipotent), or elliptic according to whether T is conjugate in **M** to $ke^{i\Theta}z$, $k \neq 1$ (T is hyperbolic if $\Theta = 0$), to $z + 1$, or to $e^{i\Theta}z$, $\Theta \neq 0 \pmod{2\pi}$, respectively. These expressions allow us to see the geometry of each transformation T.

As Poincaré [47] first observed, each Möbius transformation acting, say, on the 2-sphere $\partial\mathfrak{B}$ has a unique extension to a conformal automorphism of the 3-ball $\mathfrak{B}$. In this way **M** extends from $\partial\mathfrak{B}$ to the group of orientation preserving isometries of hyperbolic 3-space $\mathfrak{B}$. In different language **M** extends to a certain closed subgroup of SO(1, 4) which has $\mathfrak{B}$ as its associated symmetric space.

The point $p \in \mathfrak{B}^-$ is a *limit point* of a subgroup G of **M** if there is an infinite sequence $\{A_n\}$ of distinct elements of G and a point $q \in \mathcal{B}^-$ such that $\lim A_n(q) = p$. Let $\Lambda(G)$ denote the set of limit points which lie on $\partial\mathcal{B}$. It turns out (cf. [22]) that $\Lambda(G)$ is a closed set on $\partial\mathcal{B}$. The complementary open set $\Omega(G)$ on $\partial\mathcal{B}$ is called the *regular set*.

As a matrix subgroup of $\mathrm{SL}(2, \mathbf{C})/\pm 1$, G is discrete if and only if it acts *discontinuously* on the open ball $\mathcal{B}$. This means no $p \in \mathcal{B}$ is a limit point. If G is discrete its limit set with respect to the closed ball $\mathcal{B}^-$ is precisely $\Lambda(G)$. If in addition $\Omega(G) \neq \emptyset$, G is traditionally called a *discontinuous* group, but it is also true that the requirement $\Omega(G) \neq \emptyset$ in itself implies discreteness. A *kleinian group* is a discontinuous group G which has more than two limit points. In particular G is not finite, cyclic, or free abelian of rank 2. The limit set $\Lambda(G)$ of a kleinian group is a perfect, nowhere dense subset of $\partial\mathcal{B}$ and the components of $\Omega(G)$ are either simply or infinitely connected (provided G is finitely generated).

2.2. Throughout this paper we will assume that the kleinian group G is *finitely generated* and has *no torsion* (i.e., contains no elliptic elements). Then G acts freely on $\mathcal{B} \cup \Omega(G)$. $\Omega(G)/G = \bigcup S_i$ is a union of Riemann surfaces S_i none of which is conformally a sphere, nor a once or twice punctured sphere, nor a torus. We will assign to the 3-manifold

$$\mathfrak{M}(G) = \mathcal{B} \cup \Omega(G)/G$$

the orientation, C^∞ structure, and conformal structure induced from $\mathcal{B} \cup \Omega(G)$ by the natural projection

$$\pi\colon \mathcal{B} \cup \Omega(G) \longrightarrow \mathfrak{M}(G) .$$

Of course $\partial\mathfrak{M}(G) = \bigcup S_i$ and, since there is no torsion, $\pi_1(\mathfrak{M}) \cong G$.

As we have already discussed in § 1.7, $\mathfrak{M}(G)$ has the fortunate property that every imbedded 2-sphere bounds a ball.

We will use the following terminology. A loop $\gamma \subset \mathfrak{M}(G)$ *determines the transformation* $T \in G$ if for some $x^* \in \{\pi^{-1}(x)\}$, where x is the initial point of γ, the lift of γ from x^* terminates at $T(x^*)$. Any two transformations determined by γ are conjugate in G. Conversely if T is determined by γ and T_1 is conjugate to T in G then T_1 is also determined by γ. Thus we can also speak of the *conjugacy class of transformations* in G *determined* by γ.

In a different direction, if γ_1 is freely homotopic to γ then γ and γ_1 determine the same conjugacy class of transformations in G. Conversely two loops γ, γ_1 which determine the same conjugacy class are freely homotopic.

It should be emphasized that the configuration $\pi\colon \mathcal{B} \cup \Omega(G) \rightarrow \mathfrak{M}(G)$ may have two characteristics which are unusual from a topological point of view. The first is that a component S of $\partial\mathfrak{M}$ may have a *puncture* p. By this we mean that p is an isolated ideal boundary component of S which has a neighborhood conformally equivalent to a once punctured disk. It follows from [29], for example, that if γ is a simple loop in S retractible to p then transformation in G determined by γ is parabolic. The converse of this statement is false as we will discover later. The geometry of parabolic transformations will be discussed in § 2.5 below.

The second possibility we want to mention is that a component S^* of $\pi^{-1}(S)$ is not necessarily the universal covering surface of S because S^* need not be simply connected. It is simply a regular planar covering.

2.3. *Example* 1. $\Lambda(G) = \{z\colon |z| = 1\}$. Then G is a *fuchsian group* of the first kind and $\Omega(G)$ has exactly two components which correspond under the map $z \to \bar{z}^{-1}$. $\Omega(G)/G$ is the union of two finitely punctured compact surfaces S_1, S_2 which are anti-conformally equivalent. In a simple geometric manner $\mathfrak{M}(G)$ may be realized as $S_1 \times [0, 1]$.

Example 2. If a component Ω_0 of $\Omega(G)$ is invariant under G then G is called a *function group*; if in addition Ω_0 is simply connected, G is called a *B-group*. For these groups the inclusion $\pi_1(S) \to \pi_1(\mathfrak{M}(G))$, $S = \Omega_0/G$, is an isomorphism and the structure of $\mathfrak{M}$ is considerably simplified as we shall discover. It is also true that the topology of $\Omega(G)$ and of $\partial\mathfrak{M}$ can be expressed directly in terms of Ω_0 and subsurfaces of S. For B-groups this situation was completely worked out by Maskit [35], and several years ago I found a direct proof of Ahlfors' finiteness theorem as applied to the invariant region for function groups (unpublished). The simplest type of function group not a B-group is the classical Schottky group. This is generated by transformations which pair $2g$ circles bounding a $2g$-connected region (or more generally analytic Jordan curves), sending the exterior of one onto the interior of its partner. In this case $\Omega(G)$ is connected and infinitely connected, $\Omega(G)/G$ is a compact surface of genus g, and $\mathfrak{M}(G)$ is a handlebody of genus g.

Example 3. G is a *degenerate group* if $\Omega(G)$ is connected and simply connected. These peculiar groups exist and represent every conformal type of surface (arising from a kleinian group) save the triply punctured sphere. Degenerate groups were found by Bers [7] using the boundary corresponding to his imbedding of Teichmüller space in a space of quadratic differentials, and by Maskit in some unpublished work, using geometric methods on $\partial\mathfrak{B}$. It is an open question whether $\mathfrak{M}(G) \cong (\Omega(G)/G) \times [0, 1]$.

2.4. AHLFORS' FINITENESS THEOREM [2]. *If G is a kleinian group with*

$N < \infty$ generators, then $\partial\mathfrak{M}(G)$ is a finite union of surfaces, each of which is a finitely punctured compact surface.

Note that this says in particular that no components of $\partial\mathfrak{M}$ are disks. This is false if $N = \infty$ (Accola [1]). The fact that every ideal boundary component of a component of $\partial\mathfrak{M}$ is conformally a puncture in itself, generally, is highly non-trivial. For example no geometic proof of this fact is known for degenerate groups.

If one did not insist on proving that every ideal boundary component on $\partial\mathfrak{M}(G)$ is a puncture, then the work here would be independent of Ahlfors' theorem. In particular this is the case in Chapters 5–7 (cf. Lemma 3.1). However, we will not be so dogmatic and will always refer to ideal boundary components as punctures.

Bers found a very important refinement of Ahlfors' theorem as follows.

BERS' ESTIMATE [6]. *If $\partial\mathfrak{M}(G) = \bigcup S_i$ is the decomposition of $\partial\mathfrak{M}$ into its components S_i and if S_i has genus g_i and b_i punctures then*

$$\sum (2g_i + b_i - 2) \leqq 2(N - 1) .$$

2.5. Suppose p is the fixed point of a parabolic transformation in the kleinian group G. The subgroup

$$M_p = \{T \in G \colon T(p) = p\}$$

is the *maximal parabolic subgroup* corresponding to p. It is known that M_p contains only parabolic transformations and furthermore that M_p is either infinite cyclic or free abelian of rank two.

Denote by $\sigma_p(r)$ the sphere with radius r which is internally tangent to $\partial\mathfrak{B}$ at p and let $\beta_p(r) \subset \mathfrak{B}$ be the open ball bounded by $\sigma_p(r)$. Both σ_p and β_p are invariant under all $T \in M_p$.

For all sufficiently small r, $\beta_p(r)$ has the property that $T(\beta_p(r)) \cap \beta_p(r) = \varnothing$ for all $T \in G$, $T \notin M_p$. When M_p is free abelian of rank two, Fatou's proof of this [13, p. 159] is complete. On the other hand, as A. Beardon has pointed out, his proof for cyclic M_p has a serious gap. However, Beardon [5] among others has given a direct, elementary proof which is valid in all cases.

Case 1. M_p is free abelian of rank two. In this case, for each small r, $(\sigma_p(r) - \{p\})/M_p$ is a torus in $\mathfrak{M}(G)$. It will be called a *canonical cusp torus.* This torus bounds $\beta_p(r)/M_p \cong S^1 \times \{z \in \mathbf{C} \colon 0 < |z| < 1\}$ (S^1 is the 1-sphere). The *closure* of this will be referred to as a *canonical solid cusp torus.* Each solid cusp torus has finite hyperbolic volume.

Suppose $\mathcal{T}_1$, $\mathcal{T}_2$ are two mutually disjoint, canonical solid cusp tori in $\mathfrak{M}(G)$ (or in different manifolds $\mathfrak{M}(G)$ and $\mathfrak{M}(H)$). There is a natural C^∞

quasiconformal map $f: \mathcal{T}_1 \to \mathcal{T}_2$. Fix components $\mathcal{T}_i^*$ of $\pi^{-1}(\mathcal{T}_i)$ and represent $\mathcal{T}_i^*$ as closed upper half space $\mathcal{H} = \{(x_1, x_2, x_3): x_3 \leqq 0\}$, $i = 1, 2$. There is a level-preserving affine map $f^*: \mathcal{H} \to \mathcal{H}$ which induces an isomorphism of the group of cover transformations of $\mathcal{T}_1^*$ onto that of $\mathcal{T}_2^*$. Its projection is f.

Case 2. M_p is infinite cyclic. In this case for each small r, $(\sigma_p(r) - \{p\})/M_p$ will be called a *doubly infinite cylinder associated* with p. It is naturally imbedded in $\mathfrak{M}(G)$ and bounds there a region $\beta_p(r)/M_p$ homeomorphic to $\{z \in \mathbf{C}: 0 < |z| < 1\} \times (0, 1)$.

In each case as we vary r, for small r we obtain a nested family of the above objects. Since there are at most a countable number of conjugacy classes of maximal parabolic subgroups of G, it is possible to choose the associated canonical cusp tori and doubly infinite cylinders so as to be mutually disjoint in $\mathfrak{M}(G)$. Note too, they are C^∞ submanifolds of $\mathfrak{M}(G)$.

2.6. In the case of cyclic M_p, in general nothing is known regarding its effect on $\mathfrak{M}(G)$ beyond the existence of the doubly infinite cylinders which can be constructed from spheres. However, in a certain situation, which we will now examine, one can say much more. This special case will be very important in the remainder of our work.

A closed submanifold $\mathcal{T}$ of $\mathfrak{M}(G)$ is a *tube* if $\mathcal{T} \cong \{z \in \mathbf{C}: 0 < |z| \leqq 1\} \times [0, 1]$.

Definition 2.1. Suppose p_1 and p_2 are distinct punctures on $\partial\mathfrak{M}(G)$. p_1 and p_2 are said to be *paired* if there is a (closed) cylinder C and a tube $\mathcal{T}$ in $\mathfrak{M}(G)$ such that

(i) $C \cap \partial\mathfrak{M} = \partial C \cap \partial\mathfrak{M}$ consists of two simple loops, one retractible in $\partial\mathfrak{M}$ to p_1, the other retractible in $\partial\mathfrak{M}$ to p_2.

(ii) $\partial_0(\mathcal{T} \cap \mathfrak{M}^0) = C^0$ (∂_0 denotes the relative boundary in $\mathfrak{M}^0$).

(iii) $(\mathcal{T} \cap \partial\mathfrak{M})^0$ is the union of a neighborhood of p_1 and of p_2, each one conformally equivalent to a once punctured disk.

We will refer to p_1 and p_2 as *paired* by the *pairing cylinder* C or the *pairing tube* $\mathcal{T}$.

Assuming p_1 and p_2 are paired we will construct a family of *canonical pairing cylinders* and *tubes*. To do this consider the situation in $\Omega(G) \subset \partial\mathcal{B}$.

Let α_1, α_2 denote the two components of $C \cap \partial\mathfrak{M}$ where C is a pairing cylinder. Fix a lift α_1^* of α_1 in $\Omega(G)$. Then α_1^* determines a parabolic transformation $A \in G$. There is also a lift α_2^* of α_2 which determines A. Let γ_i^* denote the component of $\pi^{-1}(\alpha_i)$ which contains α_i^*, $i = 1, 2$. Then γ_1^*, γ_2^* are both open Jordan arcs in $\Omega(G)$ invariant under A. When the fixed point p of A is added, γ_1^*, γ_2^* become Jordan curves "tangent" to each other at p. The maximal subgroup M_p is infinite cyclic and generated by A.

The Jordan curves $\gamma_1^* \cup \{p\}$, $\gamma_2^* \cup \{p\}$ bound mutually disjoint topological disks D_1^*, $D_2^* \subset \Omega(G)$ with $A(D_i^*) = D_i^*$; $\pi(D_i^*)$ is a neighborhood of the puncture p_i. It is known [22] that there exist circles $C_1(r)$, $C_2(r)$ of radius r which lie on $\partial\mathcal{B}$ such that for all sufficiently small r,

(i) $C_1(r)$, $C_2(r)$ are tangent at p ,
(ii) $C_i(r) \subset D_i^* \cup \{p\}$, $i = 1, 2$,
(iii) $A(C_i(r)) = C_i(r)$, $i = 1, 2$.

The existence of the circles is usually formulated in the case that A is of the form $z \to z + 1$ and C_1, C_2 are the horizontal lines in the plane $\{y = \pm N\}$ for large $N > 0$.

Let $\sigma_1'(r)$, $\sigma_2'(r)$ denote the spheres orthogonal to $\partial\mathcal{B}$ along $C_1(r)$, $C_2(r)$ respectively and set $\sigma_i(r) = \sigma_i'(r) \cap \mathcal{B}$. Let $X_i(r)$ denote the component of $\mathcal{B} - \sigma_i(r)$ which is adjacent to a subset of D_i^*, $i = 1, 2$. Determine the radius $r_1 = r_1(r)$ of $\beta_p(r_1)$ (cf. § 2.5) as follows. Let p^* be the diametrically opposite point to p. Under a conformal map f of $\mathcal{B}^-$ to closed upper half space with $f(p) = \infty$, $f(p^*) = 0$, the three regions $f(X_1(r))$, $f(X_2(r))$, $f(\beta_p(r))$ are to be equidistant from 0. Then let $\beta_p^*(r)$ denote the relative closure in $\mathcal{B} \cup \Omega(G)$ of $\beta_p(r_1) \cup X_1(r) \cup X_2(r)$.

LEMMA 2.2. *For all sufficiently small r, $\pi(\beta_p^*(r))$ is a tube in $\mathfrak{M}(G)$ pairing p_1 and p_2. If q_1 and q_2 are also paired punctures on $\partial\mathfrak{M}(G)$ (or on a different $\partial\mathfrak{M}(G_1)$) then for all sufficiently small r there is a function $r_2 = r_2(r)$ such that $\pi(\beta_p^*(r))$ is conformally equivalent to $\pi(\beta_q^*(r_2))$ in such a way that p_1 corresponds to q_1 and $r_2 \to 0$ as $r_1 \to 0$.*

Proof. For sufficiently small r, $\beta_p^*(r)$ has the property that $T(\beta_p^*(r)) \cap \beta_p^*(r) = \varnothing$ for all $T \in G$, $T \notin M_p$ while $T \in M_p$ preserves $\beta_p^*(r)$. If $\mathcal{T}$ is a tube pairing p_1 and p_2, take r small enough so that $\pi(\beta_p^*(r)) \subset \mathcal{T}$. The second statement follows from the fact that any two parabolic transformations are conjugate in **M** and from our normalized choice of $r_1(r)$.

The pairing tubes $\pi(\beta_p^*(r))$ are called *canonical pairing tubes* and the closures of the relative boundaries $\partial_0(\pi(\beta_p^*(r)) \cap \mathfrak{M}^0)$ are called *canonical pairing cylinders*.

Our work leads to the following algebraic characterization.

LEMMA 2.3. *Two distinct punctures on $\partial\mathfrak{M}(G)$ are paired if and only if they determine the same conjugacy class of parabolic transformations in G.*

Proof. The necessity is trivial. If the two punctures determine the same conjugacy class, the construction above can be applied directly to obtain a canonical cylinder which pairs the punctures.

2.7. Although Lemma 2.2 contains the basic result we need on "compactifying" manifolds, there are certain additional technical complications that must be ironed out. These have to do with the fact that if $\mathcal{T}$ is a canonical pairing tube, $\mathfrak{M} - \mathcal{T}^0$ is not a C^∞ submanifold of $\mathfrak{M}$ because there are "corners" on $\mathcal{T}$.

In order to get an explicit picture of what a canonical pairing tube $\mathcal{T}$ looks like, it is convenient to replace $\mathcal{B}$ by upper half space $\mathcal{H} = \{(x_1, x_2, x_3) \in \mathbf{R}^3: x_3 > 0\}$ and assume A is the translation $(x_1, x_2, x_3) \rightarrow (x_1 + 1, x_2, x_3)$ (notation as in § 2.6). The intersection of $\beta_\infty^*(r)$ with the plane $\{x_1 = 0\}$ is the complement R in the closed half plane $\{x_3 \geqq 0\}$ of some open rectangle as in Fig. 1.

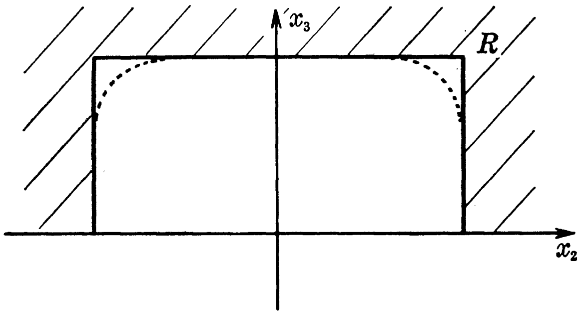

FIG. 1.

$\mathcal{T}$ is obtained by identifying the faces $R \times \{0\}$, $R \times \{1\}$ of $R \times I$ under A.

By rounding the corners of R as indicated in Fig. 1, $(\partial_0\mathcal{T})^-$ becomes a C^∞ submanifold of $\mathfrak{M}(G)$, transverse to $\partial\mathfrak{M}(G)$. This smoothing can be done in a canonical way for all canonical pairing tubes so that both assertions of Lemma 2.2 continue to hold. We will speak of these smoothed tubes $\mathcal{T}$ as *smoothed canonical pairing tubes*. The closure of the relative boundaries $(\partial_0\mathcal{T})^-$ will be called *smoothed canonical pairing cylinders.*

Suppose $\mathcal{T}$ and $\mathcal{T}_1$ are smoothed canonical pairing tubes such that $\mathcal{T} \subset \mathcal{T}_1^0$. We claim

LEMMA 2.4. *$\mathcal{T}_1 - \mathcal{T}^0$ is diffeomorphic to* $\{z \in \mathrm{C}: 1/2 \leqq |z| \leqq 1\} \times I$.

Proof. The intersection of $\beta_\infty^*(r_1) - \beta_\infty^*(r)^0$ with the plane $\{x_1 = 0\}$ is the closed region R in Fig. 2. There is a conformal map h of R^0 onto a rectangle R_1^0, the four arcs of ∂R corresponding to the four sides of R_1. The boundary properties of conformal mappings and the reflection principle show that h is C^∞ in R. Hence the map $(z, t) \rightarrow (h(z), t)$ of $R \times I \rightarrow R_1 \times I$ is a diffeomorphism. Lemma 2.4 now follows.

In reference to Lemma 1.17 we obtain the following.

LEMMA 2.5. *If S is a canonical cusp torus or a smoothed canonical* *pairing cylinder in* $\mathfrak{M}(G)$, *there is a* C^∞ *triangulation of* $\mathfrak{M}(G)$ *with respect to which* S *is a polyhedral surface.*

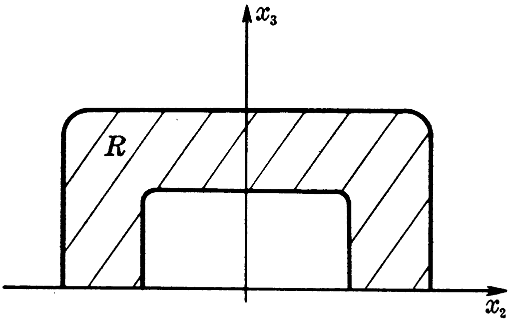

FIG. 2.

2.8. Our approach to the limit set $\Lambda(G)$ of a kleinian group G will always be by means of the following theorem. It plays a very fundamental role in passing from the group action on $\mathfrak{B}$ to the group action on $\partial\mathfrak{B}$.

GEHRING'S EXTENSION THEOREM [16]. *A quasiconformal homeomorphism* $\mathfrak{B} \to \mathfrak{B}$ *can be extended to a quasiconformal homeomorphism* $\mathfrak{B}^- \to \mathfrak{B}^-$.

3. Quasi-fuchsian groups

3.1. We will begin with a lemma which has considerable application here and in later chapters.

LEMMA 3.1. *Suppose* C *is a closed cylinder in* $\mathfrak{M} = \mathfrak{M}(G)$ *such that*

(i) $C \cap \partial\mathfrak{M} = \partial C$ *consists of two simple loops in* $\partial\mathfrak{M}$ *not homotopic to a point in* $\mathfrak{M}$ *nor freely homotopic on* $\partial\mathfrak{M}$, *and*

(ii) *for some base point* $0 \in \mathfrak{M}^0 - C^0$ *the inclusion* $\pi_1(\mathfrak{M}^0 - C^0; 0) \to \pi_1(\mathfrak{M}^0; 0)$ *is an isomorphism.*

Then there are two punctures p_1, p_2 *on* $\partial\mathfrak{M}$, *uniquely determined by* C, *which are paired by* C. *Furthermore, no puncture* p_3 *on* $\partial\mathfrak{M}$, $p_3 \neq p_1\ p_2$, *is paired with either* p_1 *or* p_2.

Proof. Hypothesis (ii) implies C separates $\mathfrak{M}$, for otherwise there would be a loop in $\mathfrak{M}$ with non-zero intersection number with C. One component of $\mathfrak{M} - C$ contains 0; denote the closure of the other component by $\mathcal{T}$.

Hypothesis (i) implies that C is incompressible in $\mathfrak{M}$ and Corollary 1.13 then says that the inclusion $\pi_1(C) \to \pi_1(\mathcal{T})$ is an isomorphism. Let R be one of the components of $\mathcal{T} \cap \partial\mathfrak{M}$. We claim that the inclusion $\pi_1(R) \to \pi_1(\mathcal{T})$ is an isomorphism, in particular that $\pi_1(R)$ is infinite cyclic and because of (i), $\mathcal{T} \cap \partial\mathfrak{M}$ has two components.

If this is not the case, there is a non-trivial loop $\gamma \subset R^0$ in the kernel of this inclusion. By Lemma 1.4 there is a non-trivial simple loop $\gamma_0 \subset R^0$ which bounds a disk D in $\mathcal{T}$ with $D \cap R = \partial D = \gamma_0$. D separates $\mathcal{T}$ and moreover, if R_0 is the component of $R - \gamma_0$ that does not border on a component of $C \cap \partial\mathfrak{M}$, every loop in R_0 is freely homotopic to a loop on $C^0 = \partial_0\mathcal{T}$ and hence to a loop on D. In other words every loop in R_0 is trivial in $\mathcal{T}$ and consequently in $\mathfrak{M}$. We conclude that every component of $\pi^{-1}(R_0)$ in $\Omega(G)$ is homeomorphic to R_0. This is possible only if R_0 is a disk.

Thus each of the two components R_1, R_2 of $\mathcal{T} \cap \partial\mathfrak{M}$ has an infinite cyclic fundamental group. Either by applying Ahlfors' finiteness theorem or by simple direct analysis we conclude that each of these components is a once punctured disk. That is, C is a cylinder pairing two punctures p_1, p_2 on $\partial\mathfrak{M}$.

Finally, a puncture $p_3 \neq p_1$, p_2 cannot also be paired with p_1. The easiest way of seeing this is as follows. Fix components R_i^* of $\pi^{-1}(R_i)$, $i = 1, 2$, which are "tangent" at the fixed point p of the parabolic transformation $T \in G$. Each of $R_i^* \cup \{p\}$ is a topological disk invariant under T (we may even assume $\partial(R_i^* \cup \{p\})$ is a circle). If p_3 were paired with p_1 also there would be a third topological disk $R_3^* \cup \{p\}$, with $R_3^* \cap R_1^* = R_3^* \cap R_2^* = \emptyset$, also invariant under T. The simple geometric properties of T rule out such a configuration.

Remark. It is possible that two punctures on the same component of $\partial\mathfrak{M}$ are paired.

3.2. *Definition.* The (finitely generated, torsion free) kleinian group G is *quasi-fuchsian* if two of the components of $\Omega(G)$ are each invariant under G.

In this chapter, we shall completely describe the topological structure of $\mathfrak{M}(G)$ for quasi-fuchsian groups G.

LEMMA 3.2. *For a quasi-fuchsian group G, exactly two components, say Ω_1, Ω_2, of $\Omega(G)$ are invariant under G and each of these is simply connected. Each of $S_1 = \Omega_1/G$ and $S_2 = \Omega_2/G$ is a finitely punctured compact surface.*

Proof. The first statement is proved in Accola [1] and also follows from the topological considerations of Chapter 1. The second is a consequence of Ahlfors' finiteness theorem, or more simply, an elementary analysis of each of the at most finite number of ideal boundary components of S_1 and S_2.

We remark that two punctures on the same component S_1 or S_2 cannot be paired. Two curves in S_1, for example, which are freely homotopic in $\mathfrak{M}$ are freely homotopic in S_1 since the inclusion $\pi_1(S_1) \to \pi_1(\mathfrak{M})$ is an isomorphism. However, the totality of punctures on S_1 is paired with the totality of punctures on S_2 as we shall see in the next lemma.

LEMMA 3.3. *For a quasi-fuchsian group G with invariant components Ω_1, Ω_2 of $\Omega(G)$, $\mathfrak{M}(G)$ is diffeomorphic to* $(\Omega_1/G) \times [0, 1]$.

3.3. *Proof. Case* 1: $S_1 = \Omega_1/G$ is compact. This case was proved by Waldhausen [51, Lemma 5.1]. Because we will have to use his construction in the general case, it is appropriate to outline his proof here. Take a system of simple loops α_i, $1 \leqq i \leqq 2g$, on S_1, $g =$ genus S_1, such that α_i intersects α_{i+1} exactly once (transversely) while $\alpha_i \cap \alpha_j = \varnothing$, $j \neq i$, $i+1$. Applying Lemma 1.6, there exists a corresponding set $\{\alpha'_i\}$ of simple loops on $S_2 = \Omega_2/G$ such that (i) each pair (α_i, α'_i) bounds a cylinder C_i in $\mathfrak{M}$, and (ii) $C_i \cap C_j$ is a simple arc from $\alpha_i \cap \alpha_{i+1}$ to $\alpha'_i \cap \alpha'_{i+1}$ while $C_i \cap C_j = \varnothing$ for $j \neq i$, $i+1$.

Let $N(C)$ be a regular neighborhood of $C = \bigcup C_i$ (§ 1.2). Then $\partial_0 N(C) = \partial N(C) \cap \mathfrak{M}^0$ is an open cylinder in $\mathfrak{M}$ and $(\partial_0 N(C))^- \cap S_1$ is a simple loop that bounds the region $D_1 = [S_1 - S_1 \cap N(C)]^-$ which is a disk. But also $D_2 = [S_2 - S_2 \cap N(C)]^-$ is a disk in S_2. For the simple loop ∂D_2 bounds the disk $\partial_0 N(C) \cup D_1$ in $\mathfrak{M}$. Since the inclusion map $\pi_1(S_2) \to \pi_1(\mathfrak{M})$ is injective, ∂D_2 is homotopic to a point in S_2 as well as in $\mathfrak{M}$. This implies ∂D_2 bounds a disk, necessarily D_2, in S_2.

Set $M = [\mathfrak{M} - N(C)]^-$. We have just seen that ∂M is a 2-sphere, namely the union of the cylinder $\partial_0 N(C)$ and two caps D_1, D_2. Consequently (cf. § 1.7) M is a ball and is homeomorphic to $D_1 \times [0, 1]$ in such a way that $(\partial_0 N(C))^-$ corresponds to $\partial D_1 \times [0, 1]$. Adjoining $N(C)$ to M we obtain Lemma 3.3 in this case.

3.4. *Case* 2: Now suppose S_1 has punctures. Following Waldhausen's proof take the simple loops $\{\alpha_i\}$ on S_1, $1 \leqq i \leqq 2g$, $g =$ genus S_1, and construct the corresponding cylinders C_i exactly as above. For a thin regular neighborhood $N(C)$ about $C = \bigcup C_i$, each region $R_i = [S_i - S_i \cap N(C)]^-$, $i = 1, 2$ is again planar since all the handles on S_1 and consequently S_2 have been accounted for in the construction of the C_i. But in this case, R_1 is not simply connected since it contains all the punctures of S_1 (this is why $N(C)$ is taken to be thin; it is thin enough that all punctures of $S_1 \cup S_2$ lie in $R_1 \cup R_2$).

At this point we make the important observation that each component M^* of $\pi^{-1}(M^0)$, where $M = \mathfrak{M} - N(C)$, is a topological ball in $\mathfrak{B}$. This follows from Lemma 1.12 because each component of $\pi^{-1}(\partial_0 N(C))$ is simply connected by Lemma 1.11. Furthermore, each of $M^{*-} \cap \Omega_i$, $i = 1, 2$, is connected and simply connected. Thus if $T \in G$ preserves $M^{*-} \cap \Omega_1$, then T also preserves $M^{*-} \cap \Omega_2$.

The next step is to construct mutually disjoint canonical pairing tubes in M between the punctures in R_1 and those in R_2. This construction proceeds as follows. Choose a puncture p in R_1 and a small simple loop γ_1 and R_1 contractible to p. Because Ω_2, as well as Ω_1, is invariant under G, and in view of the observation above, γ_1 is freely homotopic in M to a loop γ_2 in R_2. By Lemma 1.6, γ_2 can be taken as a simple loop which together with γ_1 bounds a cylinder C in M. But since the inclusion $\pi_1(S_1) \to \pi_1(\mathfrak{M})$ is an isomorphism, the hypotheses of Lemma 3.1 are satisfied. Consequently C pairs p with a uniquely determined puncture on R_2.

Continuing in this manner to obtain disjoint pairing tubes, and then using Lemma 2.2, we obtain mutually disjoint canonical pairing tube $\mathcal{T}_i$, pairing in M the totality of punctures in R_1 with the totality of punctures in R_2.

3.5. Now (changing notation) let M denote the closure in $\mathfrak{M}$ of $\mathfrak{M} - N(C) - \bigcup \mathcal{T}_i$ and set $R_1 = M \cap S_1$, $R_2 = M \cap S_2$. M has the properties (i) $(\partial_0 M)^-$ is the union of cylinders $((\partial_0 N(C))^-$ and $(\partial_0(\mathcal{T}_i)^-)$ and $\partial M - (\partial_0 M)^-$ is the union of the homeomorphic planar regions R_1 and R_2, (ii) each component of $\pi^{-1}(M^0)$ is a topological ball and each component of $\pi^{-1}(R_i)$ is simply connected, (iii) the inclusion $\pi_1(R_i) \to \pi_1(M)$ is an isomorphism, $i = 1, 2$. (Recall $\partial_0 M$ is the relative boundary of $M \cap \mathfrak{M}^0$ in $\mathfrak{M}^0$.) To complete the proof it suffices to show that

(*) $\qquad M \cong R_1 \times [0, 1]$ in such a way that $(\partial_0 M)^- \cong \partial R_1 \times [0, 1]$.

It will then follow that $\mathfrak{M}(G) \cong S_1 \times [0, 1]$.

The proof of (*) proceeds by induction on the number of components of $\partial_0 M$ as M ranges over the set of submanifolds of $\mathfrak{M}(G)$ which satisfy properties (i)–(iii) above. The case that $\partial_0 M$ has one component has been settled in § 3.3. Assuming that (*) is true for those M such that $\partial_0 M$ has n components, consider an M with $\partial_0 M$ having $n + 1$ components.

Fix two (cylindrical) components C_1, C_2 of $(\partial_0 M)^-$. Choose three simple, closed arcs α_1, α_2, α_3 in ∂M as follows: (a) α_1 in C_1 from a point p_1 on $C_1 \cap R_2$ to a point q_1 on $C_1 \cap R_1$; (b) α_2 in R_1 from q_1 to a point q_2 on $C_2 \cap R_1$; (c) α_3 in C_2 from q_2 to a point p_2 on $C_2 \cap R_2$. We will show below that p_2 can be connected to p_1 by a simple arc in R_2 such that the resulting simple loop γ is homotopic to a point in M. Note that γ is non-trivial in ∂M since $C_1 - \gamma \cap C_1$, for example, is connected.

Assuming that γ can be constructed as specified, by Lemma 1.4 the simple loop γ is the boundary of a closed disk D with $D^0 \subset M^0$. Note that $M - D$ is connected. Let $N(D)$ be a regular neighborhood of D and denote by M_1 the closure in M of $M - N(D)$. M_1 has the required topological properties (i), (ii), (iii) and $(\partial_0 M_1)^-$ is the union of n cylinders, so by the induction hypothesis, (*) is true for M_1. Using our construction, we see that (*) is true for M as well.

3.6. Let α be the simple arc from p_1 to p_2, constructed above, that consists of a segment α_1 in C_1, a segment α_2 in R_1, and a segment α_3 in C_2. Fix a component M^* of $\pi^{-1}(M)$ and a component α^* of $\pi^{-1}(\alpha)$ in ∂M^*. Let p_1^*, p_2^* denote the initial and terminal point of the simple arc α^* and choose a simple arc β^* in R_2^* $(=$ the component of $\pi^{-1}(R_2)$ that lies in $\partial M^*)$ from p_2^* to p_1^*.

We claim that β^* can be suitably chosen so that $\beta = \pi(\beta^*)$ is a simple arc in R_2. Since M^* is a ball and $\gamma^* = \alpha^* \cup \beta^*$ a simple loop in ∂M^*, this will complete the proof of Lemma 3.3.

Because the proof uses standard techniques in surface topology (cf. [26]) we shall only provide an outline. Fix a set Γ of simple loops which are free generators of $\pi_1(R_2^0 \cup \{p_2\}; p_2)$. We may assume $\beta = (\prod \beta_i)\cdot\beta_0$ where each β_i, $i \geqq 1$, is an element of Γ and β_0 is a simple arc in R_2 from p_2 to p_1. Let γ_1^* be a lift $\neq \gamma^*$ of $\gamma = \pi(\gamma^*)$; γ_1^* is a simple loop.

By our construction γ_1^* can intersect γ^* only in the segment β^*. Now β^* divides R_2^* into two components. If γ_1^* meets both of these (open) components, there is a relatively compact region K_1^* in R_2^* such that ∂K_1^* consists of segments of β^* and of γ_1^*. At most a finite number of lifts of γ intersect K_1^*. Let K_2^* be a component of $K_1^* - K_1^* \cap \{\pi^{-}(\gamma)\}$ and set $K_2 = \pi(K_2^*)$. Then K_2 is a relatively compact region in $R_2^0 \cup \{p_2\}$ bounded by some combination of elements of the set $\{\Gamma, \beta_0\}$. This is impossible since Γ is a set of free generators.

We conclude that no lift $\neq \gamma^*$ of γ crosses γ^*. By analyzing the order in which the β_i, $i \geqq 0$, comprising β appear about the point p_2, it follows that β must be homotopic to a simple arc in R_2.

3.7. PROPOSITION 3.4. *Given a quasi-fuchsian group G there exists a fuchsian group H and a quasiconformal diffeomorphim $f\colon \mathfrak{M}(H) \to \mathfrak{M}(G)$.*

Proof. Lemma 3.3 says that if $S_1 = \Omega_1/G$ then $\mathfrak{M}(G) \cong S_1 \times [0, 1]$. Let H be a fuchsian group such that Δ/H is quasiconformally equivalent to S_1, where Δ is the open unit disk. Then also $\mathfrak{M}(H) \cong S_1 \times [0, 1]$ (all homeomorphisms are orientation preserving).

Case 1. S_1 is compact. Starting with a homeomorphism $\mathfrak{M}(H) \to \mathfrak{M}(G)$, Lemma 1.18 gives a diffeomorphism $f\colon \mathfrak{M}(H) \to \mathfrak{M}(G)$. In this case f is automatically quasiconformal.

Case 2. For simplicity of notation assume S_1 has only one puncture. Thus there are two punctures on $\partial\mathfrak{M}(H)$ and these are paired. Fix four nested and smoothed canonical pairing tubes $\mathcal{T}_i$ in $\mathfrak{M}(H)$ such that $\mathcal{T}_{i+1} \subset \mathcal{T}_i^0$, $1 \leqq i \leqq 3$. Take $\mathcal{T}_1$ sufficiently thin so that there exists a conformal map χ which maps each $\mathcal{T}_i$ onto a smoothed canonical pairing tube $\mathcal{T}'_i$ in $\mathfrak{M}(G)$. In addition χ can be chosen to satisfy $\mathcal{T}_i \cap (\Delta/H) \to \mathcal{T}'_i \cap S_1$. By Lemma 1.17 there exists a C^∞ triangulation of $\mathfrak{M}(H)$ which restricts to a triangulation of each $\mathcal{T}_i$. Using χ the triangulation of $\mathcal{T}_1$ determines a triangulation of $\mathcal{T}'_1$ and then all $\mathcal{T}'_i$. Applying [44, § 10.7] the restriction of this particular triangulation to $\mathcal{T}'_2$ can in turn be extended to a C^∞ triangulation of $\mathfrak{M}(G)$. $\chi: \mathcal{T}_2 \to \mathcal{T}'_2$ is PL. From now on however we will restrict χ to $\mathcal{T}_3$.

The proof of Lemma 3.3 shows that there exists a PL homeomorphism $f: \mathfrak{M}(H) - \mathcal{T}_2^0 \to \mathfrak{M}(G) - \mathcal{T}_2'^0$ which sends $(\partial_0\mathcal{T}_2)^- \to (\partial_0\mathcal{T}'_2)^-$ and $(\partial_0\mathcal{T}_2)^- \cap (\Delta/H) \to (\partial_0\mathcal{T}'_2)^- \cap S_1$. Now apply the PL version of Lemma 1.16 to interpolate in $\mathcal{T}_2 - \mathcal{T}_3^0$ between the maps f on the cylinder $(\partial_0\mathcal{T}_2)^-$ and χ on $(\partial_0\mathcal{T}_3)^-$. This gives us a PL homeomorphism $g: \mathfrak{M}(H) \to \mathfrak{M}(G)$ with $g = \chi$ in $\mathcal{T}_3$.

Of course g is also a quasiconformal map. To get a quasiconformal map which is a diffeomorphism, apply Lemma 1.18 to obtain h such that $h = \chi$ in $\mathcal{T}_4$.

COROLLARY 3.5 (Maskit [36]). *If G is a quasi-fuchsian group, there is a fuchsian group H and a quasiconformal map of $\partial\mathcal{B}$ onto itself which induces an isomorphism $G \to H$.*

Proof. Lift the quasiconformal map given by Proposition 3.4 to $\mathcal{B} \cup \Omega(H)$ and apply Gehring's theorem.

4. Fundamental polyhedra

4.1. Fix a point $0 \in \mathcal{B}$. To each T in a kleinian group G, $T \neq$ id., corresponds the hyperbolic plane

$$H(T) = H(T; 0) = \{p \in \mathcal{B}: d(0, p) = d(p, T(0)) = d(0, T^{-1}(p))\}$$

where d is the hyperbolic distance in $\mathcal{B}$. We note that $H(T)^- \cap H(T^{-1})^-$ is empty if T is hyperbolic, is a point p if T is parabolic with fixed point p, but otherwise is an arc of a circle orthogonal to $\partial\mathcal{B}$, or a point on $\partial\mathcal{B}$.

The convex polyhedron

$$\mathcal{P}_0 = \{p \in \mathcal{B}: d(0, p) \leqq d(p, T(0)) \text{ for all } T \in G\} ,$$

which is relatively closed in $\mathcal{B}$ is a "fundamental region" for G in $\mathcal{B}$ and will be called a *Poincaré fundamental polyhedron* for G with *center* at 0. Its faces are arranged in distinct pairs (f, f') where $Tf' = f$ for some $T \in G$ and $f' \subset H(T^{-1})$, $f \in H(T)$. In fact $\mathcal{P}_0$ may be equally well defined as the intersection

$$\mathcal{P}_0 = \bigcap_{T \in G} \mathcal{H}(T)$$

where $\mathcal{H}(T)$ is the closed half space in $\mathcal{B}$ determined by the plane $H(T)$ and containing 0.

We will always work with $\mathcal{P}$, the euclidean closure of $\mathcal{P}_0$ in $\mathcal{B}^-$. $\mathcal{P} \cap \Omega(G)$ is a fundamental set for G in $\Omega(G)$. $\mathfrak{M}(G)$ is the manifold resulting from identification of the opposite faces of $\mathcal{P} \cap (\mathcal{B} \cup \Omega(G))$. This will be considered in much greater detail in Chapter 9.

4.2. Among the properties of $\mathcal{P}$ that we need, the following is not too familiar.

LEMMA 4.1. *Assume that the polyhedron $\mathcal{P}$ for any choice of center has a finite number of faces. Then the center* 0 *can be chosen so that*

(i) *no two parabolic fixed points in $\mathcal{P}$ are equivalent under G.*

If $p \in \mathcal{P}$ is a parabolic fixed point and M_p the maximal parabolic subgroup at p,

(ii) *if M_p is cyclic then p lies on exactly two faces of $\mathcal{P}$ which are tangent at p and paired by a generator of M_p,*

(iii) *otherwise p lies on exactly four faces of $\mathcal{P}$, the opposite faces of each pair are tangent at p and are paired by a generator of M_p,*

(iv) *$p \notin H(T)^-$ for all $T \in G$, $T \notin M_p$.*

Proof. We will outline the proof as follows. If T is loxodromic or parabolic but not in M_p and the parabolic fixed point p lies in $H(T)^-$, then there is a uniquely determined hyperbolic plane $H_1(T; p)$ orthogonal to $H(T)$ with $0,\ T(0) \in H_1(T; p)$. The equation of $H_1(T; p)$ depends on p and T but not on 0. In fact if $T = (az + b)/(cz + d)$ acts in upper half space and $p = \infty$ then $H_1(T; p) = \{(x, y, \xi),\ z = x + iy\colon |c|^2\xi + |cz + d|^2 = 1,\ \xi > 0\}$ where $c \neq 0$ since T cannot share a fixed point with a parabolic transformation in M_p. Observe that *any* point $0_1 \in H_1(T; p)$ has the property that p lies on the plane $H(T; 0_1)^-$ since $T\colon 0_1 = (z_0, \xi_0) \to (T(z_0), \xi_0)$ (see [13, p. 159]).

There are countably many transformations in G and countably many planes $H_1(T; p)$. The orbit under G of these planes yields the planes for the conjugacy class of p. Do this for each of the countable number (actually we will show below there are a finite number) of conjugacy classes of parabolic fixed points to obtain in the end a countable system of planes in $\mathcal{B}$. As long as we choose the point 0 so that it does not lie on any of these planes, the corresponding polyhedron $\mathcal{P}$ will have the desired properties.

4.3. The following result is the bridge between statements about the group G and statements about the manifold $\mathfrak{M}(G)$. Recall that $\partial_0\mathfrak{M}_0$ denotes the relative boundary of $\mathfrak{M}_0 \cap \mathfrak{M}(G)^0$ in $\mathfrak{M}(G)^0$ for a submanifold $\mathfrak{M}_0$.

PROPOSITION 4.2. *G has a finite sided Poincaré fundamental polyhedron* $\mathfrak{Q}$ *if and only if there is a compact submanifold* $\mathfrak{M}_0$ *of* $\mathfrak{M}(G)$ *such that each component of* $(\partial_0\mathfrak{M}_0)^-$ *is either a canonical cusp torus or a canonical cylinder pairing two punctures on* $\partial\mathfrak{M}(G)$ (cf. §§ 2.5, 2.6). *If one Poincaré polyhedron is finite sided, they all are.*

In particular each component of $\mathfrak{M} - \mathfrak{M}_0$ *is homeomorphic either to* $\{0 < |z| < 1\} \times [0, 1]$ *or to* $\{0 < |z| < 1\} \times S^1$.

Proof. Assume first that $\mathfrak{M}(G)$ has the structure described but $\mathfrak{Q}$ has an infinite number of faces. Then there is a component η of $\mathfrak{M} - \mathfrak{M}_0$ which contains the projection of infinitely many faces of $\mathfrak{Q}$. Because any given compact set in $\mathfrak{B} \cup \Omega(G)$ is covered by a finite number of images of $\mathfrak{Q}$ under G, the projections of only a finite number of faces of $\mathfrak{Q}$ intersect $\partial_0\eta$. Consequently there is a component R_1 of $\eta - \eta \cap \pi(\partial_0\mathfrak{Q})$ which contains infinitely many of the projected faces in $\pi(\partial_0\mathfrak{Q})$ in its boundary ($\partial_0\mathfrak{Q}$ is the relative boundary of $\mathfrak{Q} \cap \mathfrak{B}$ in $\mathfrak{B}$).

Set $R_1^* = \pi^{-1}(R_1) \cap \mathfrak{Q}$. R_1^* is contained in a component η^* of $\{\pi^{-1}(\eta)\}$ and therefore uniquely determines a maximal parabolic subgroup M_p of G whose common fixed point p lies in the closure of R_1^* in $\mathfrak{B}^-$. In fact if 0 is the center of $\mathfrak{Q}$, then R_1^* is contained in the cusp

$$C_p(0) = \{x \in \mathfrak{B}: d(0, x) \leqq d\big(x, T(0)\big) \text{ for all } T \in M_p\} .$$

If $\{f_n\}$ is an infinite sequence of faces of $\mathfrak{Q}$ contained in $\partial_0 R_1^*$ then $\lim f_n = p$. An infinite number of the opposite faces $\{f_n'\}$ cannot also bound R_1^* since $T(\eta^*) \cap \eta^* = \emptyset$ for all $T \notin M_p$. Consequently there is another component $R_2 \neq R_1$ of $\eta - \eta \cap \pi(\partial_0\mathfrak{Q})$ whose lift R_2^* to $\mathfrak{Q}$ contains (as we may assume) the opposite faces $\{f_n'\}$ in its boundary. Like R_1^*, R_2^* determines a maximal parabolic subgroup M_q, $q \neq p$.

Let $\{S_n\}$ denote the necessarily distinct transformations in G which pair f_n and f_n': $S_n(f_n') = f_n$. Then M_q is conjugate in G to M_p and in fact $M_q = S_n^{-1}M_pS_n$ since a conjugating transformation is determined by any arc in $\pi(\mathfrak{Q}^0)$ which connects the opposite sides of any $\pi(f_n)$.

Moreover, the regions R_1^* and $S_1(R_2^*)$ which are adjacent along f_1 must lie in $C_1 = C_p(0) \cup C_p\big(S_1(0)\big)$. C_1 in turn is contained in a finite number of images of $C_p(0)$ under M_p. This implies $T(C_1) \cap C_1 \neq \emptyset$ for at most a finite number of elements of M_p. But each $S_nS_1^{-1}$ is in M_p and $S_nS_1^{-1}(C_1) \cap C_1 \neq \emptyset$. We conclude that at most a finite number of $S_nS_1^{-1}$ can be distinct, which is impossible.

The converse is more classical. Suppose $\mathfrak{Q}$ has a finite number of sides. Only parabolic fixed points can lie in $\mathfrak{Q}$, for the non-euclidean line joining two fixed points of $T \in G$ is preserved by T. In particular each limit point of G in $\mathfrak{Q}$ lies on the boundary of a face. A standard argument (see [18]) shows that a limit point $p \in \mathfrak{Q}$ is a parabolic fixed point and furthermore $N \cap C_p(0)$, where N is a small neighborhood of p in $\mathfrak{B}^-$, is covered by a finite number of images of $\mathfrak{Q}$ under G. The desired result is clear if M_p is of rank two. If M_p is infinite cyclic, $N \cap C_p(0)^- \cap \partial\mathfrak{B} - \{p\}$ lies in $\Omega(G)$ and projects to $U_1 \cup U_2$, where U_i is a neighborhood of a puncture in $\partial\mathfrak{M}(G)$.

4.4. The proof above works for any non-euclidean fundamental polyhedron $\mathcal{P}$ which has the face-pairing properties of $\mathfrak{Q}$, the property that only a finite number of images of $\mathcal{P}$ under G intersect any given compact set in $B \cup \Omega(G)$, and the following property which was critical for our proof: If $p \in \mathcal{P}$ is a parabolic fixed point, then $\mathcal{P}$ is contained in a finite number of images of $C_p(0')$ under M_p for some, and hence any, $0' \in \mathfrak{B}$. No doubt this last condition can be weakened, perhaps entirely eliminated, but in any case it is satisfied by the usual "canonical" fundamental polyhedra.

We note that the Poincaré polyhedron with center at the origin of $\mathfrak{B}$ was called the isometric fundamental polyhedron by Ahlfors [3]. He described it as the set of those $x \in \mathfrak{B} \cup \Omega(G)$ for which $|A'(x)| \leqq 1$ for all $A \neq \mathrm{id}$ in G. Here $|A'(x)|$ denotes the linear ratio $|dA(x)|/|dx|$ which is independent of direction.

Proposition 4.3 can be expressed in terms of the classical isometric fundamental region of Ford [14]. We define the *closed* isometric fundamental set as

$$I(G) = \{z \in \mathbf{C}\colon |T'(z)| \leqq 1 \text{ for all } T \in G\}\ ,$$

assuming of course that $\infty \in \Omega(G)$.

COROLLARY 4.4. *$I(G)$ has the following two properties if and only if $\mathfrak{M}(G)$ has the structure described in Proposition* 4.3. (i) *$I(G)$ is the union of a finite number of finite sided circular polygons and at most a finite number of isolated points, and* (ii) *to each $p \in I(G)$ corresponds a neighborhood N on ∂B such that $N - N \cap I(G)$ can be covered by the interiors of a finite number of isometric circles.*

Proof. $I(G)$ is the set of points lying in the closed exterior of each isometric circle $\{z \in \mathbf{C}\colon |T'(z)| = 1,\ T \neq \mathrm{id},\ T \in G\}$. Identify $\mathbf{C} \cup \{\infty\}$ with $\partial\mathfrak{B}$. Each circle extends to a sphere orthogonal to $\partial\mathfrak{B}$. Let $I^*(G)$ denote the set of points in $\mathfrak{B}^-$ which lie in the closure of the exterior of every one of these spheres. With a natural interpretation $I^*(G)$ can be said to be a fundamental polyhedron for G. If $I(G) = I^*(G) \cap \partial\mathfrak{B}$ satisfies (i) and (ii), $I^*(G)$ can have only a finite number of faces in $\mathcal{B}$ and the converse is clear as well. The proof of Proposition 4.3 applies to $I^*(G)$ and Corollary 4.4 is true as a consequence.

4.5. COROLLARY 4.5. *If $\mathfrak{M}(G)$ has the structure described by Proposition* 4.3, *then the limit set $\Lambda(G)$ of G has* 2-*dimensional Lebesgue area zero.*

Proof. This follows from the remark in § 4.4 and the result of Ahlfors [3] that if the isometric fundamental polyhedron has a finite number of sides, then $\Lambda(G)$ has area zero.

For more general kleinian groups, the area of $\Lambda(G)$ is not known.

4.6. We conclude with an important negative result of L. Greenberg which is a Corollary of Lemma 1.3.

COROLLARY 4.6 (Greenberg [18]). *A degenerate group cannot have a finite sided fundamental polyhedron.*

Proof. If G is degenerate with a finite sided fundamental polyhedron then $\mathfrak{M}(G)$ must be compact. For the classical half of Proposition 4.3 shows that a parabolic fixed point corresponds to two punctures p_1, p_2 in $\Omega(G)/G = \partial\mathfrak{M}(G)$ (degenerate groups have no free abelian subgroups of rank 2). Furthermore, small circles c_1, c_2 about p_1, p_2 respectively are freely homotopic in $\mathfrak{M}$ and hence in $\partial\mathfrak{M}$ because the inclusion $\pi_1(\partial\mathfrak{M}) \to \pi_1(\mathfrak{M})$ is an isomorphism. This is impossible since G is not cyclic. Thus $\mathfrak{M}(G)$ is compact. Now Lemma 1.3 is applicable and shows that $\dim H_1(\mathfrak{M}) = g$, which is impossible.

4.7. Suppose more generally that G is a discrete group. Exactly as above a Poincaré fundamental polyhedron $\mathcal{P}$ can be constructed. The following result has been proven by Lie group methods.

LEMMA 4.7 (Selberg [48], Garland-Raghunathan [15]). *If $\mathcal{P}$ has finite hyperbolic volume then $\mathcal{P}$ has a finite number of faces.*

COROLLARY 4.8. *$\mathcal{P}$ has finite hyperbolic volume if and only if there is a compact submanifold $\mathfrak{M}_0$ of $\mathfrak{M}(G)$ with each component of $\mathfrak{M}(G) - \mathfrak{M}_0^0$ a canonical solid cusp torus* (§ 2.5).

5. Function groups

5.1. In this chapter we will analyze those manifolds $\mathfrak{M}(G)$ that arise from a function group G. This is of interest in itself but also the topological methods employed here are required in the more general situation of Chapter 6.

We recall that G is a *function group* if a component Ω_0 of $\Omega(G)$ is invariant under G. Special cases are quasi-fuchsian groups and B-groups defined below.

Definition. A function group G is a B-group if an invariant component Ω_0 of $\Omega(G)$ is simply connected.

The analysis of B-groups, which occupies most of this chapter, is the 3-dimensional analogue of Maskit's fundamental work [36].

Notation for Chapter 5. Given a function group G and an invariant component Ω_0 of $\Omega(G)$, set $S = \Omega_0/G$ and denote the remaining components $(\Omega(G) - \Omega_0)/G$ of $\mathfrak{M}(G)$ by $S_1, S_2, \cdots$. When only one group G is under discussion we will frequently write $\mathfrak{M}$ for $\mathfrak{M}(G)$.

5.2. The fact that G is a function group is characterized by the property that the inclusion map $\pi_1(S) \to \pi_1(\mathfrak{M}(G))$ is surjective. This basic property allows enormous simplification in the structure of $\mathfrak{M}(G)$.

It is also true that each of the inclusion maps $\pi_1(S_k) \to \pi_1(\mathfrak{M}(G))$ is injective, i.e., each component of $\pi^{-1}(S_k)$ is simply connected. To prove this one can either appeal to Accola [1] or use the following topological argument. If an inclusion $\pi_1(S_k) \to \pi_1(\mathfrak{M}(G))$ is not injective by Lemma 1.4 there exists a simple loop $\gamma \subset S_k$ which bounds a disk $D \subset \mathfrak{M}$ with $D \cap \partial\mathfrak{M} = \gamma$. Now D divides $\mathfrak{M}$ since $H_1(\mathfrak{M})$ is generated by cycles on S (which are therefore disjoint from D). Let R be the component of $S_k - \gamma$ which D separates from S. Then every loop in R is freely homotopic to a loop in S, hence to a loop on D, and therefore to a point in $\mathfrak{M}$. This implies that each component of $\pi^{-1}(R^0)$ is homeomorphic to R^0; the only possibility is that R is a disk.

5.3. *Case* 1. G is a B-group. This case is characterized by the fact that the inclusion homomorphism $\pi_1(S) \to \pi_1(\mathfrak{M})$ is actually an isomorphism. In the course of our analysis we shall prove in particular the following result.

PROPOSITION 5.1. *Let G be a B-group. Either G contains a degenerate group or $\mathfrak{M}(G)$ has the following structure. There are a finite number of mutually disjoint tubes $\mathcal{T}_i$ in $\mathfrak{M}(G)$ which pair some of the punctures of $\bigcup S_k$ such that if $\mathfrak{M}_0 = (\mathfrak{M} - \bigcup \mathcal{T}_i)^-$, then $\mathfrak{M}_0 \cong S \times [0, 1]$. In particular $\partial\mathfrak{M}_0$ is the union of S and a surface homeomorphic to S.*

We will first dispose of two special cases. The first is that some S_k is compact. In this case, G is a quasi-fuchsian group. Indeed, $\pi_1(S_k)$ is isomorphic to a subgroup of $\pi_1(S)$. Since subgroups of free groups are free, $\pi_1(S)$ cannot be a free group; therefore S is also a compact surface. Exactly the analysis used to investigate quasi-fuchsian groups shows that $\mathfrak{M} \cong S \times [0, 1]$ and therefore that G is quasi-fuchsian.

5.4. Before proceeding to the second special case we have to show that there are at most a finite number of components S_k and each of these is a finitely punctured compact surface. Of course this is a consequence of the finiteness theorem but the methods that are involved in the topological proof are also important in future work. By Lemma 1.2 the genus of $\bigcup S_k$ is finite.

The cylinder theorem (Lemma 1.6) implies that to each non-trivial simple loop γ' on S_k corresponds a simple loop γ on S which together with γ' bounds a cylinder C in $\mathfrak{M}$. Let M be a component of $\mathfrak{M} - C$. Each component M^* of $\pi^{-1}(M^0)$ is a topological ball (Lemma 1.12) and the component of $\pi^{-1}(\partial M \cap S)$ contained in ∂M^* is simply connected. The same is true of the components of $\pi^{-1}(\partial M \cap S_k)$ in ∂M^*. Therefore, if γ_1 is a non-trivial simple loop in $\partial M \cap S_k$, γ_1 is not only freely homotopic in $\mathfrak{M}$ to a loop in S, but is freely homotopic in M to a loop in $\partial M \cap S$. This argument may be extended so as to conclude that, given mutually disjoint, non-trivial simple loops $\gamma_1, \gamma_2, \cdots$ in S_k there exist mutually disjoint cylinders $C_1, C_2, \cdots$, in $\mathfrak{M}$ which are bounded by $\gamma_1, \gamma_2, \cdots$ together with simple loops on S.

Now there are only a finite number of mutually disjoint simple loops on S, no two of which are freely homotopic. Suppose γ_1, γ_2 are two non-trivial simple loops on S_k which are not freely homotopic on S_k. If γ_1 and γ_2 are freely homotopic to loops on S which are freely homotopic on S then γ_1 and γ_2 are freely homotopic in $\mathfrak{M}$. By Lemma 3.1 this implies γ_1 and γ_2 are retractible on S_k to distinct punctures. It follows that S_k is a finitely punctured compact surface.

The same argument shows there are only a finite number of punctures on $\bigcup S_k$ and therefore there are only a finite number of components S_k.

5.5. Before dealing with the second special case we will introduce the following definition.

Definition. A loop γ in S determines an *accidental parabolic transformation* $T \in G$ if T is a parabolic transformation determined by γ (cf. § 2.2), yet γ is not retractible to a puncture in S.

Suppose there is a component, say S_1, of $\partial\mathfrak{M}$, $S_1 \neq S$, with the following property: none of those loops on S which are freely homotopic to loops on S_1 determines an accidental parabolic transformation. Then G is a quasi-fuchsian group.

Assume S_1 has punctures, for otherwise § 5.3 shows that G is quasi-fuchsian. Since there are no accidental parabolic transformations determined on S, every puncture on S_1 is paired with a puncture on S. More generally every loop on S_1 is freely homotopic to a loop on S. Now the topological situation is identical to that involved in Lemma 3.3, Case 2. That proof applies to show $\mathfrak{M} \cong S \times [0, 1] \cong S_1 \times [0, 1]$.

5.6. We shall now return to our analysis of a general B-group G. Assume that no S_k is compact. Take a small simple loop c_j about each puncture on $\bigcup S_k$ and draw a cylinder c_j in $\mathfrak{M}$ bounded by c_j and a simple loop c'_j on S. c_j determines a conjugacy class of accidental parabolic transformations in G if and only if c'_j is not contractible in S to a puncture. We can take the C_j to be mutually disjoint.

If c'_i and c'_j are freely homotopic on S then they bound an annulus in S. Replace c'_j by c'_i so that $C_i \cap C_j \cap S = c'_i$. Observe that when it is moved slightly away from S, $C_i \cup C_j$ becomes a cylinder pairing two punctures on $\bigcup S_k$.

Let M be a component of $\mathfrak{M} - \bigcup C_j$. Note that it is conceivable that a C_j does not separate $\mathfrak{M}$ and therefore the two "sides" of C_j may comprise two of the relative boundary components of M in $\mathfrak{M}$. However, we will soon find that this possibility does not occur (although $\partial M \cap S$ may lie on both sides of the loop $C_j \cap S$ because of our convention setting $c'_i = c'_j$ as above).

Fix a component M^* of $\pi^{-1}(M^0)$. Then M^* is a topological ball because M^* is obtained from $\mathcal{B}$ by dividing $\mathcal{B}$, in general infinitely often, by the components of $\pi^{-1}(C_j^0)$ which are simply connected (Lemma 1.12). For the same reason we see that $\partial M^* \cap \Omega_0$ is connected and simply connected. Consequently $M \cap S$ is connected.

An immediate consequence is that the inclusion $\pi_1(M \cap S) \rightarrow \pi_1(M)$ is an isomorphism. For since $\partial M^* \cap \Omega_0$ is simply connected, this inclusion is injective. And because any element of G that preserves M^* must also preserve $\partial M^* \cap \Omega_0$, it is also surjective.

Moreover, $M \cap S$ is not a neighborhood of a puncture on S unless M^- itself is a tube pairing a puncture on S with a puncture on $\bigcup S_k$. For in this case $\partial(M \cap \mathcal{B})$ is one of the cylinders, say C_1^0. C_1 must separate $\mathfrak{M}$ since a set of generators for $\pi_1(\mathfrak{M})$ lies in $\mathfrak{M} - C_1$. Now apply Lemma 3.1.

Suppose then that M^- is not a pairing tube. Set $R = M \cap S$ and assume ∂M contains a component R_1 of $M \cap (\bigcup S_j)$ which is not the neighborhood of a puncture (the other case will be analyzed in § 5.8). For some k, $R_1 \subset S_k$. Each component of $S_k - R_1^-$ is the neighborhood of a puncture. We have yet to rule out the possibility that ∂M contains both "sides" of some C_j; in this case the neighborhood of a puncture on S_k may also appear as a component of $M \cap S_k$.

Denote by R^* the component of $\pi^{-1}(R)$ in ∂M^* and let R_1^* be a component of $\pi^{-1}(R_1)$ in ∂M^*; both are simply connected. Let G_0 be the subgroup of G which preserves M^*. G_0 is also the subgroup of G that preserves R^*. Consequently the regular set $\Omega(G_0)$ has a simply connected invariant component

ω_0 which contains R^*. We see that ω_0/G_0 is obtained from R by adjoining once punctured disks to the components of ∂R (two components of ∂R may consist of the opposite "sides" of some loop $C_j \cap S$).

Some component ω_1 of $\Omega(G_0)$ contains R_1^*. In fact ω_1 is identical with the component Ω_1 of $\pi^{-1}(S_k)$ that contains R_1^*. For every transformation of G that keeps Ω_1 invariant also keeps R_1^* invariant.

Denote the natural projection $\mathcal{B} \cup \Omega(G_0) \rightarrow \mathfrak{M}(G_0)$ by $\mathfrak{p}$. Every loop in $\mathfrak{p}(\omega_1)$ is freely homotopic in $\mathfrak{M}(G_0)$ to a loop in $\mathfrak{p}(\omega_0) = \omega_0/G_0$. In addition, by our construction, no loop in $\mathfrak{p}(\omega_1)$ determines an accidental parabolic transformation (with respect to $\mathfrak{p}(\omega_0)$) in G_0. Hence G_0 is a quasi-fuchsian group (§ 5.5).

We conclude that $\Omega(G_0) = \omega_0 \cup \omega_1$ and M is just $\mathfrak{M}(G_0)$ less a finite number of tubes which pair all the punctures. That is, $M \cong R \times [0, 1]$ in such a way that R corresponds to $R \times \{0\}$, R_1 to $R \times \{1\}$ and a union of C_j to $\partial R \times [0, 1]$.

In other words if S_k' is the component of $S_k - S_k \cap (\bigcup C_j)$ which is not the neighborhood of a puncture, then S_k' is paired with a uniquely determined component of $S - S \cap (\bigcup C_j)$.

5.7. Before analyzing those components M such that every component of $M \cap (\bigcup S_j)$ is the neighborhood of a puncture, we simplify the picture as follows: Each C_i either pairs a puncture on S with a puncture on S_k or else pairs a simple loop on S not retractible to a puncture with a puncture p on some S_k. Remove all cylinders which satisfy the former condition. For the latter condition there may also be another cylinder $C_j \neq C_i$ that pairs $\partial C_i \cap S$ with a puncture $p' \neq p$ on $\bigcup S_k$. (p and p' do not necessarily lie on different components of $\bigcup S_k$.) In this case replace the two cylinders C_i, C_j by a single canonical pairing cylinder and canonical tube $\mathcal{T}$ pairing p and p'. Repeating this process as often as possible, we end up with a finite number of canonical pairing tubes $\{\mathcal{T}_i\}$ pairing only punctures on $\bigcup S_k$. The $\mathcal{T}_i$ may be assumed to be mutually disjoint and disjoint also from those cylinders C_j which have not been eliminated.

Set $\mathfrak{M}' = (\mathfrak{M} - \bigcup \mathcal{T}_i)^-$. Let N be a component of $\mathfrak{M}' - \bigcup C_j$ (these are the cylinders remaining after the elimination process above). No component of $N \cap S$ is the neighborhood of a puncture. We claim that there are two possibilities for N.

The first is that ∂N is the union of a component R of $S - S \cap (\bigcup C_j)$, a component R_1 of $\partial\mathfrak{M}' - \partial\mathfrak{M}' \cap (\bigcup C_j)$ disjoint from R, and some of the cylinders C_j, in such a way that $N \cong R \times [0, 1]$ where R corresponds to $R \times \{0\}$ and R_1 to $R \times \{1\}$. The second possibility for N is that every component of $N \cap (\bigcup S_k)$ is the neighborhood of a puncture and ∂N is the union of these neighborhoods, a component R of $S - S \cap (\bigcup C_j)$, and some cylinders C_j.

Suppose not every component of $N \cap (\bigcup S_k)$ is the neighborhood of a puncture. The proof that the first possibility holds can be accomplished in either of two ways, neither of which we will carry out in detail. One method of proof is to show exactly how N is formed from the components M of § 5.6, namely by pasting together adjacent components of $\mathfrak{M} - \bigcup C_j$ (notation of § 5.6). Another method is to note that the topological methods of § 5.6 can be applied without change to N in place of M there. That is, fix a component N^* of $\pi^{-1}(N^0)$ and consider the subgroup G_0 of G that stabilizes N^*. One shows that $\partial N^* \cap (\mathcal{B} \cup \Omega(G))$ is the union of

(a) two simply connected regions invariant under G_0, one R^* a component of $\pi^{-1}(R)$, the other R_1^* of $\pi^{-1}(R_1)$, and

(b) a collection of components of $\{\pi^{-1}(C_j)\}$.

Furthermore, $N^* \cup R^* \cup R_1^*/G_0 \cong R \times [0, 1] \cong R_1 \times [0, 1]$. A close analogy with a quasi-fuchsian group can be made if one enlarges N, and correspondingly N^*, by adjoining to N along each of those cylinders C_j which appear in ∂N a tube pairing two punctures in $\partial\mathfrak{M}(G_0)$. Denoting the enlarged N by N_1, $\mathfrak{M}(G_0) - N_1$ consists of tubes pairing punctures on $\partial\mathfrak{M}(G_0)$. In general, G_0 is not itself quasi-fuchsian.

5.8. Finally we have to deal with the case that ∂N is the union of $N \cap S$ (which is connected), certain cylinders which we may label $C_1, \cdots, C_m$, and the neighborhoods U_i of the punctures p_i on $\bigcup S_k$ which are bounded by the simple loops $C_i \cap (\bigcup S_k)$, $1 \leqq i \leqq m$. For this case we must embark on a sequence of modifications. A rough description of the first of these is as follows. Shrink each U_i to p_i and correspondingly modify C_i, changing it to C_i', so as to end up with N' where $N' \cap S = N \cap S$, $N' \cap (\bigcup S_k) = \emptyset$ and each component C_i' of $\partial N' - N' \cap S$ is a half open cylinder $\{z \in \mathbf{C}: 0 < |z| \leqq 1\}$.

Specifically we can proceed as follows. Insert in $\mathfrak{M}$ mutually disjoint,

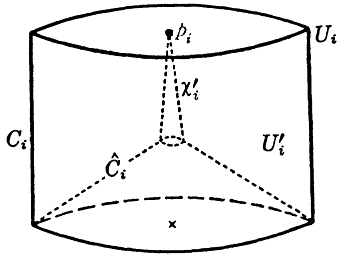

FIG. 3.

doubly infinite cylinders χ_i corresponding to the punctures p_i (§ 2.5). These can be made sufficiently small so as to lie in N. Change C_i to $\hat{C}_i \subset N^-$ which differs from C_i in that instead of terminating at $C_i \cap (\bigcup S_k)$ (which is contractible to p_i), it terminates at a simple loop $\hat{C}_i \cap \chi_i$. Take $\hat{C}_i$ so that $\hat{C}_i \cap C_i = C_i \cap S$. Then $\hat{C}_i \subset N^-$. There is a closed, half infinite section χ_i' of χ_i bounded by $\hat{C}_i \cap \chi_i$ such that $C_i' = \hat{C}_i \cup \chi_i'$ has the following property. $C_i \cup C_i'$ bounds a subregion U_i' of N which is retractible in N onto U_i. Set $N' = N - \bigcup U_i'$. Then $\partial N'$ is the union of $N \cap S$ and the half infinite cylinders C_i'.

Definition. $\partial N'$ will be called a *punctured* compact surface. The open cylinders $C_i'^0$ are *neighborhoods of the punctures* p_i' on $\partial N'$. The cylinder $C_i \cup \hat{C}_i$ *pairs* p_i and p_i'.

Now making changes only near $N' \cap S$ we want to "lift" N' slightly away from S so that N' becomes a non-compact submanifold $\mathfrak{M}_1$ of $\mathfrak{M}$ whose boundary $\partial\mathfrak{M}_1$ is contained in $\mathfrak{M}^0$. There is a minor complication in carrying this out because $N' \cap S$ itself may have punctures q_j. This is dealt with in exactly the same manner as were the punctures on $\bigcup S_k$ in the construction of the C_i' for N': i.e. doubly infinite cylinders associated with the q_j are inserted and $\partial\mathfrak{M}_1$ is so chosen that a half infinite portion of each of these cylinders becomes a neighborhood of each of the "punctures" q_j' on $\partial\mathfrak{M}_1$. The punctures q_j on S and q_j' on $\partial\mathfrak{M}_1$ can be paired by a cylinder indicated in the previous construction.

So we end up with a submanifold $\mathfrak{M}_1$ of $\mathfrak{M}$ such that

(i) $\partial\mathfrak{M}_1$ is a non-compact incompressible surface in $\mathfrak{M}^0$, and

(ii) the inclusion $\pi_1(\partial\mathfrak{M}_1) \to \pi_1(\mathfrak{M}_1)$ is an isomorphism.

Furthermore, we have defined the term "puncture" on $\partial\mathfrak{M}_1$ and each such puncture can be paired by a cylinder with a puncture on $\partial\mathfrak{M}$.

5.9. LEMMA 5.2. *Suppose G is a kleinian group and η is a submanifold of $\mathfrak{M}(G)$ such that*

(i) *$\partial\eta$ is a finitely punctured* (*in the sense of* § 5.8) *compact surface in $\mathfrak{M}^0$ which is incompressible and*

(ii) *the inclusion $\pi_1(\partial\eta) \to \pi_1(\eta)$ is an isomorphism. Suppose η^* is a component of $\pi^{-1}(\eta^0)$ and G_0 is the subgroup of G that preserves η^*. Then if G_0 is a B-group, G_0 is in addition a degenerate group* (*cf.* § 2.3).

Proof. The hypothesis implies that $\partial\eta^* \cap \Omega(G) = \varnothing$. Let $\mathfrak{p}\colon \mathfrak{B} \cup \Omega(G_0) \to \mathfrak{M}(G_0)$ denote the natural projection. Then $\mathfrak{p}\circ\pi^{-1}$ is a homeomorphism of η onto a submanifold of $\mathfrak{M}(G_0)$ which we shall also denote by η (here π^{-1} denotes the 1 to ∞ map $\eta \to \eta^{*-} \cap \mathfrak{B}$). Consider the submanifold $\eta_1 = \mathfrak{M}(G_0) - \eta^0$.

The inclusions $\pi_1(\partial\eta) \to \pi_1(\eta_1) \to \pi_1(\mathfrak{M}(G_0))$ are all isomorphisms since $\partial\eta$

is incompressible. Therefore, there is only one component of $\mathfrak{p}^{-1}(\partial\eta)$ and it is simply connected and contained in $\partial\eta_1^*$ where η_1^* is the single component of $\mathfrak{p}^{-1}(\eta_1^0)$. Since G_0 is a B-group, there is a simply connected invariant component $\Omega_0(G_0)$ of $\Omega(G_0)$ of necessity contained in $\partial\eta_1^*$. Thus the topological situation is identical to that of a quasi-fuchsian group with η_1^* replacing $\mathcal{B}$ and $\mathfrak{p}^{-1}(\partial\eta)$, $\Omega_0(G_0)$ serving as the invariant components for G_0. Furthermore, every ideal boundary component of $\partial\eta$ is a puncture so the analogy includes even this. Consequently Lemma 3.3 holds for this situation and we conclude that $\eta_1 \cong \partial\eta \times [0, 1]$. In particular $\Omega(G_0)$ has only one component, $\Omega_0(G_0)$, in $\partial\eta_1^*$. But since $\partial\eta^*$ does not contain any components of $\Omega(G_0)$ we see that $\Omega_0(G_0)$ must be the only component. That is, G_0 is degenerate.

Remark. If G_0 is not known to be a B-group, then all we can assert about η_1 is that it has the structure of a manifold associated with a B-group (and we are in the process of analyzing these). The component $\mathfrak{p}^{-1}(\partial\eta)$ of η_1^* plays the role of an invariant, simply connected component of the "B-group", G_0 acting in $\eta_1^* \cup \mathfrak{p}^{-1}(\partial\eta) \cup \Omega(G_0)$.

COROLLARY 5.3. *Let $\mathfrak{M}_1$ be the submanifold of G (here a B-group) constructed in* § 5.8. *Suppose G_0 is the subgroup that preserves the component $\mathfrak{M}_1^*$ of $\pi^{-1}(\mathfrak{M}_1^0)$. Then G_0 is a degenerate group.*

Proof. In order to apply Lemma 5.2 we need only check to see whether G_0 itself is a B-group. Referring back to our construction, $\mathfrak{M}_1^*$ is contained in a component N^* of $\pi^{-1}(N^0)$ which is also preserved under G_0 ($\mathfrak{M}_1^0$ is topologically the same as N^0 and in fact retractible onto N^0). ∂N^* contains one component $R^* \subset \pi^{-1}(R)$, $R = N \cap S$, and this too is preserved under G_0 since the inclusion $\pi_1(R) \to \pi_1(N)$ is an isomorphism (§ 5.6). Let ω_0 denote the component of $\Omega(G_0)$ that contains R^*.

The boundary ∂R of R in S consists of a number of simple loops $C_j \cap S$, for some cylinders $C_j \subset \partial N$. The surface ω_0/G_0 is obtained from $R = R^*/G_0$ by adjoining once punctured disks to the components of ∂R. More precisely, each component α of $\pi^{-1}(\partial R) \cap R^{*-}$ is an open Jordan arc with end points at the fixed point p of a parabolic transformation $T \in G_0$ $\big(T(\alpha) = \alpha\big)$. One of the components D^* on $\partial\mathcal{B}$ determined by $\alpha \cup \{p\}$ is disjoint from R^*. $D^*/\{T\}$ is the punctured disk to which we just referred. ω_0 is obtained from R^* by adjoining all such regions D^*. Consequently ω_0 is simply connected and invariant under G_0; that is, G_0 is a B-group.

5.10. It remains to combine all our information. From § 5.7 we have the manifold $\mathfrak{M}'$ which is the complement in $\mathfrak{M}$ of the (interior of) tubes pairing certain punctures on $\bigcup S_k$. In $\mathfrak{M}'$ there are cylinders C_j pairing simple loops on S, not retractible to punctures on S, with some punctures on $\partial\mathfrak{M}' \cap (\bigcup S_k)$. Given a component R' of $\partial\mathfrak{M}' - S$ the loops $(\bigcup C_j) \cap R'$ bound small neighborhoods of some of the punctures on R'; denote by R'' the result of deleting the closure of these neighborhoods from R'. Corresponding to R'' are a component R of $S - S \cap (\bigcup C_j)$ and a component N of $\mathfrak{M}' - \bigcup C_j$ such that R and R'' correspond to the levels $R \times \{0\}$, $R \times \{1\}$ in the homeomorphism $N \cong R \times [0, 1]$. The remaining components of $S - S \cap (\cup C_j)$ and adjacent components N of $\mathfrak{M} - \bigcup C_j$ have the same product structure after a non-compact submanifold $\mathfrak{M}_1$ of N is removed from N and each $C_j \subset \partial N$ is changed to $\hat{C}_j$ which pairs $C_j \cap S$ with a puncture on $\partial\mathfrak{M}_1$. Furthermore, the cylinder $C_j \cup \hat{C}_j$ pairs this puncture on $\partial\mathfrak{M}_1$ with a puncture on $\partial\mathfrak{M}' - S$.

Putting together the information about each component N of $\mathfrak{M}' - \bigcup C_j$, we obtain the following description of $\mathfrak{M}(G)$.

PROPOSITION 5.4. *Assume G is a B-group with invariant component Ω_0 of $\Omega(G)$. Set $S = \Omega_0/G$ and let $S_1, \cdots, S_n$ denote the remaining components (necessarily a finite number). Then*

(a) *There are a finite number of finitely punctured (cf. definition in* §5.8), *compact, incompressible surfaces W_i in $\mathfrak{M}(G)^0$ each of which is the boundary in $\mathfrak{M}(G)$ of a non-compact submanifold $\mathfrak{W}_i$. For each i, the inclusion $\pi_1(W_i) \to \pi_1(\mathfrak{W}_i)$ is an isomorphism and the subgroup of G which preserves a given component of $\pi^{-1}(\mathfrak{W}_i)$ is degenerate.*

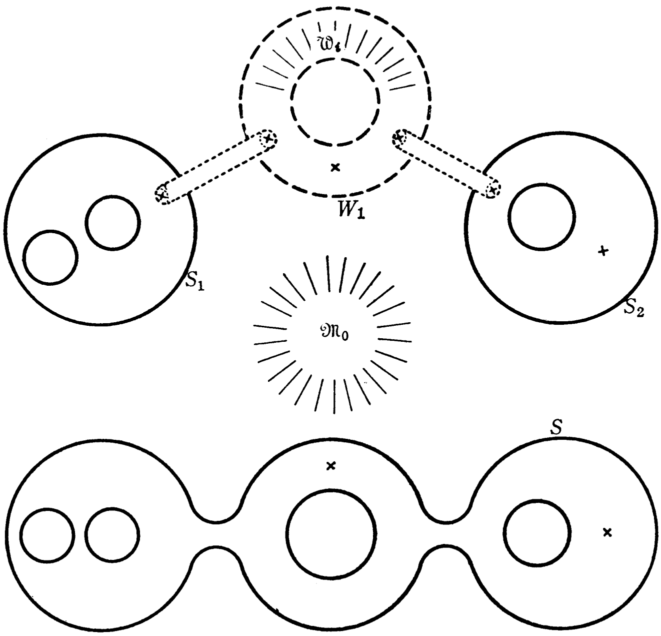

FIG. 4.

(b) *A certain number of punctures in* $(\bigcup S_k) \cup (\bigcup W_i)$ *can be joined by tubes* $\mathcal{T}_j$ *in* $\mathfrak{M}' = \mathfrak{M} - \bigcup \mathfrak{W}_i^\circ$ *such that the manifold* $\mathfrak{M}_0 = (\mathfrak{M}' - \bigcup \mathcal{T}_j)^-$ *is homeomorphic to* $S \times [0, 1]$.

In short, every B-group has a fuchsian-like structure which has been topologically recaptured above. See Fig. 4.

COROLLARY 5.5. *A B-group G has a finite sided fundamental polyhedron if and only if* $\mathfrak{M}(G)$ *has the following structure. There are a finite number of mutually disjoint tubes* $\mathcal{T}_i$ *pairing some of the punctures on* $\bigcup S_j$ *such that if* $\mathfrak{M}_0 = \mathfrak{M}(G) - \bigcup \mathcal{T}_i^\circ$, *then* $\mathfrak{M}_0 \cong S \times [0, 1]$.

5.11. We shall make one more refinement of Proposition 5.4.

COROLLARY 5.6. *The surfaces* $W_i \subset \mathfrak{M}(G)^0$ *of Proposition* 5.4 (a) *can be chosen so that a loop* $\alpha \subset W_i$ *determines a parabolic transformation in G if and only if* α *is retractible to a puncture on* W_i.

Proof. Suppose $\alpha \subset W_i$ determines an "accidental parabolic transformation" $T \in G$. That is, T is parabolic but α is not retractible to a puncture in W_i. α is freely homotopic to a loop $\alpha' \subset S$ with respect to which T is an accidental parabolic transformation. For each puncture on S is paired with a puncture on $(\partial\mathfrak{M}_0 - S)$. If α were freely homotopic in $\mathfrak{M}_0$ to a loop about a puncture on $(\partial\mathfrak{M}_0 - S)$ then α would be freely homotopic in $(\partial\mathfrak{M}_0 - S)$ to this loop. Since $\mathfrak{M}_0 \cong S \times [0, 1]$, this is impossible.

Let p denote the fixed point of T and insert in $\mathfrak{M}$ a doubly infinite cylinder χ associated with p (§2.5) so small that $\chi \cap W_i = \varnothing$. χ can only be located in $\mathfrak{W}_i$. Construct a cylinder C in $\mathfrak{W}_i$ with one component of ∂C a simple loop on χ, the other a simple loop on W_i not retractible to a puncture. Let $N(C)$ be a regular neighborhood of C in $\mathfrak{W}_i$ (i.e., the result of "thickening" C) and set $\mathfrak{W}_i' = \mathfrak{W}_i - N(C)^0$. $\mathfrak{W}_i'$ has one or two components depending on whether or not $\partial C \cap W_i$ separates W_i. Replace $\mathfrak{W}_i$ and W_i by $\mathfrak{W}_i'$ and $\partial\mathfrak{W}_i'$. After a finite number of changes of this sort, Corollary 5.6 is proved.

Remark 5.7. There is a set $\{\alpha_i\}$ of mutually disjoint simple loops α_i on S with the properties

(i) each α_i determines an accidental parabolic transformation,

(ii) if $\alpha \subset S$ is a loop which determines an accidental parabolic transformation, then α is freely homotopic to a power of some α_i.

Suppose S has genus g and b punctures. The set $\{\alpha_i\}$ has at most $3g + b - 3$ elements and $S - \bigcup \alpha_i$ has at most $2g + b - 2$ components.

5.12. We digress for a moment to use the decomposition obtained for B-groups G to compute the dimension $\sigma(G)$ of the complex vector space of cusp forms for G [7]. This will be required in Chapter 10 for those groups G with a finite-sided fundamental polyhedron so we will restrict ourselves here to that case (this computation also appears in [36]).

Suppose $S = \Omega_0(G)/G$ is a b-times punctured compact surface of genus g and there are $a \leqq 3g + b - 3$ mutually disjoint simple loops $\{\gamma_i\}$ dividing S into $n \leqq 2g + b - 2$ regions S_j' such that each γ_i determines an accidental parabolic transformation. Corresponding to each S_j' is exactly one component $S_j \neq S$ of $\partial\mathfrak{M}(G)$ which is topologically obtained from S_j' by attaching a once punctured disk to each boundary component of S_j' in S (if S_j' appears on both sides of a γ_i then punctured disks are attached to both sides of γ_i).

Suppose S_j' has genus g_j, b_j punctures, and is bounded by a_j of the loops $\{\gamma_i\}$; a γ_i both sides of which lie in S_j' is counted twice. Then $\sum a_j = 2a$. The a loops $\{\gamma_i\}$ bound n regions S_j' so they satisfy $(n - 1)$ relations in $H_1(S)$. Hence

$$g = \sum g_j + a - (n - 1) .$$

Recalling that the space of cusp forms for a y-times punctured compact surface of genus x has dimension $3x + y - 3$ we find

$$(1) \qquad \begin{aligned} \sigma(H) &= \textstyle\sum_{j=1}^{n} (3g_j + b_j + a_j - 3) + 3g + b - 3 \ , \quad \text{or} \\ \sigma(H) &= 6g + 2b - a - 6 \ . \end{aligned}$$

5.13. PROPOSITION 5.8. *If $\Omega_0(G)$ is not simply connected then G has a decomposition into a free product of subgroups*

$$G = G_1 * \cdots * G_m * \mathfrak{A}_1 * \cdots * \mathfrak{A}_r * \{A_1\} * \cdots * \{A_s\} * \{T_1\} * \cdots * \{T_n\}$$

where each G_i is a B-group, $\mathfrak{A}_j$ is a free abelian group of rank two with two parabolic generators, $\{A_k\}$ is the cyclic group generated by the parabolic transformation A_k, and $\{T_l\}$ is the cyclic group generated by the loxodromic transformation T_l. Furthermore, each parabolic transformation in G is conjugate in G to one in a listed subgroup. G has a finite-sided fundamental polyhedron if and only if all the groups G_i do.

Proof. Applying Dehn's lemma and the loop theorem (Lemma 1.4) there exists a simple loop $\gamma \subset S = \Omega_0(G)/G$, not homotopic to a point on S, which bounds a disk $D \subset \mathfrak{M}(G)$.

Case 1. D divides $\mathfrak{M}(G)$. Let $\mathfrak{M}_1$, $\mathfrak{M}_2$ denote the closure of the two components of $\mathfrak{M}(G) - D$. Then by Lemma 1.8, $\pi_1(\mathfrak{M}(G)) = \pi_1(\mathfrak{M}_1) * \pi_1(\mathfrak{M}_2)$. Let S_1, S_2 be the two components of $S - \gamma$ so labeled that $S_1 \subset \partial\mathfrak{M}_1$, $S_2 \subset \partial\mathfrak{M}_2$. Then one component of $\partial\mathfrak{M}_i$ is just S_i with a disk attached along γ, $i = 1, 2$.

Fix adjacent components $\mathfrak{M}_1^*$, $\mathfrak{M}_2^*$ of $\pi^{-1}(\mathfrak{M}_1^0)$, $\pi^{-1}(\mathfrak{M}_2^0)$ in $\mathfrak{B}$ and denote by G_1, G_2 the subgroups of G that preserve these two components. Then $G = G_1 * G_2$. Furthermore, we can regard $\mathfrak{M}_i$ as imbedded in $\mathfrak{M}(G_i)$, $i = 1, 2$, and then we see that $(\mathfrak{M}(G_i) - \mathfrak{M}_i)^0$ is connected and a topological ball.

Now G_i preserves $\mathfrak{M}_i^{*-} \cap \Omega_0(G)$ which is necessarily connected. So if G_i is not elementary, it is a function group. If neither G_1 nor G_2 is elementary repeat the decomposition process with each of them separately. If no further decomposition is possible for G_1, say, then G_1 is a B-group.

Next suppose for example that G_1 is cyclic. Then $\partial\mathfrak{M}(G_1)$ is either a torus or a twice punctured sphere. In the former case there is a simple loop γ' in $\partial\mathfrak{M}(G_1)$ that does not separate $\partial\mathfrak{M}(G_1)$ but bounds a disk D' in $\mathfrak{M}(G_1)$. We may assume D' lies in $\mathfrak{M}_1 \subset \mathfrak{M}(G_1)$. In this case we will replace the original γ by γ' and the separating disk D with the non-separating D'. This puts us in Case 2.

If $\partial\mathfrak{M}(G_1)$ is a twice punctured sphere then G_1 is generated by a parabolic transformation A. In this case and also in the situation that G_1 is free abelian of rank two, repeat the decomposition process with G_2 (if it is not an elementary group).

Case 2. D does not divide $\mathfrak{M}(G)$. Then $\pi_1(\mathfrak{M}) = \pi_1(\mathfrak{M} - D)*\mathbf{Z}$. Interpreting this decomposition in $\mathfrak{B}$ we find that $G = G_1 * \{T\}$ where T is loxodromic and G_1 is a function group. Note that in this case $[\mathfrak{M}(G_1) - (\mathfrak{M}(G) - D)]^0$ is the union of two topological balls.

Continue this decomposition process in the obvious way. The process ends after a finite number of steps because each decomposition reduces the ranks of the groups (Lemma 1.9).

The assertion in Proposition 5.8 about parabolic transformations is true because the conjugacy class of a maximal parabolic subgroup M_p corresponds to either a cusp torus or a doubly infinite cylinder in $\mathfrak{M}(G)$. Neither of these need be disturbed by the introduction of the cutting disks D.

The necessary and sufficient condition for $\mathfrak{M}(G)$ to have a finite-sided polyhedron is a direct consequence of Proposition 4.2.

5.14. Finally in preparation for Chapter 10 we will extend the computation of § 5.11 to compute the dimension $\sigma(G)$ of the space of cusp forms for an arbitrary function group G which has a finite-sided fundamental polyhedron.

Referring back to § 5.13, let $\{S_i\}$ be the components obtained by cutting $S = \Omega_0(G)/G$ along the simple loops ∂D_i where $\{D_i\}$ is a complete set of cutting disks used in the decomposition of G (we can assume these are mutually disjoint). If S_i^* is a component of $\pi^{-1}(S_i)$ in $\Omega_0(G)$, the subgroup H_i of G that preserves S_i^* is either a cyclic parabolic group, free abelian of rank two, or a B-group. In each case an invariant region $\Omega_0(H_i)$ for H_i is obtained by adding to S_i^* the disks in $\partial\mathcal{B}$ which are bounded by the relative boundary components of S_i^* in $\Omega_0(G)$. H_i is conjugate to one of the subgroups G_i, $\mathfrak{A}_j$, or $\{A_k\}$ constructed in Proposition 5.8. Label the $\{S_i\}$ so that S_i, $1 \leqq i \leqq m$, corresponds to the group G_i while S_i, $m+1 \leqq i \leqq m+r$, corresponds to $\mathfrak{A}_{i-m}$, and S_i, $m+r+1 \leqq i \leqq m+r+s$, corresponds to $\{A_{i-m-r}\}$.

Suppose S_i has genus g_i, b_i punctures, and is bounded by t_i of the simple loops ∂D_j, counting a loop twice if S_i lies on both sides of it. Then $\sum t_i = 2t$ where t is the number of disks D_j. The t loops ∂D_j satisfy $m+r+s$ relations in $H_1(S)$ because they bound the S_i. One of these relations is a consequence of the others, hence from Proposition 5.8 if S has genus g,

$$g = \sum_{i=1}^{m+r+s} g_i + t - (m+r+s-1) = \sum_{i=1}^{m+r+s} g_i + n \ .$$

Now each S_i, $1 \leqq i \leqq m$, corresponds to a cluster of components $\neq S$ of $\partial\mathfrak{M}(G)$ which are joined together by cylinders pairing some of their punctures in the manner of § 5.10. Refer back to § 5.11 and replace the group G there by G_i, a by a_i, and b by b_i for $1 \leqq i \leqq m$. Since $b = (\sum_{i=1}^{m} b_i) + 2s$ is the total number of punctures on S we obtain (cf. (1))

$$\sigma(G) = \sum_{i=1}^{m+r} (3g_i + b_i - a_i - 3) + (3g + b - 3)\ , \ \text{or} \tag{2}$$
$$\sigma(G) = 6g + 2b - 3(m + r + n + 1) - 2s - a$$

(define $a_i = 0$ for $i > m$, and $a = \sum_1^m a_i$).

6. The assumption and resulting structure theorem

6.1. From a number of possibilities we have chosen the following assumption on the grounds of its simplicity and ease of interpretation in the ball $\mathcal{B}$.

Assumption 6.1. There exists a set of generators K_0 of $\pi_1(\mathfrak{M}; p)$ for some p and a compact set $K \supset K_0$ such that every loop in $\mathfrak{M} - K$ which bounds a disk in $\mathfrak{M}$ bounds a disk in $\mathfrak{M} - K_0$. (By a disk we mean a non-singular disk.)

It will be clear from the analysis below that if $\mathfrak{M}(G)$ satisfies Assumption 6.1 for some K_0 and if $\gamma_1, \cdots, \gamma_n$ is any set of generators of $\pi_1(\mathfrak{M}; p)$, there exists a set $\{\gamma_i'\}$, γ_i' homotopic to γ_i, $1 \leqq i \leqq n$, which also can be taken as K_0.

Topologically it is easy to find examples which violate the assumption; paste two 3-manifolds together across a disk in the boundary of each and then remove the joined boundary component. The disk remains and becomes an infinite disk which separates the fundamental group into a free product. We shall generalize our assumption in § 12.3 to include this case. What we are really forced to exclude is the situation in a manifold where $\gamma_n = \partial D_n$ is a sequence of simple loops approaching the "ideal boundary" of $\mathfrak{M}$ while the D_n are non-singular disks which oscillate with ever-increasing frequency through a set of generators of π_1. (However it seems unlikely that this occurs; see § 13.1).

Function groups clearly satisfy Assumption 6.1 as do groups with a finite-sided fundamental polyhedron, as we see from Proposition 4.3. We know of no groups which fail to satisfy either Assumption 6.1 or its generalization in § 12.3. (Also see Chapter 13.)

This chapter is devoted to a topological analysis of those manifolds $\mathfrak{M}(G)$ which satisfy Assumption 6.1. We will discover exactly how they are related to compact manifolds. In particular we will prove

THEOREM 6.2. *If G is a finitely generated group which satisfies Assumption* 6.1, *then either G has a finite-sided fundamental polyhedron or G contains a degenerate B-group* (*cf.* § 6.4).

6.2. Before beginning the analysis of $\mathfrak{M}(G)$ we wish to point out the relation of our work to Ahlfors' finiteness theorem (§ 2.4), part of which we are going to use. If G satisfies Assumption 6.1, we will prove independently of Ahlfors' theorem that $\mathfrak{M}(G)$ has a finite number of boundary components, each of finite topological type. But without Ahlfors' theorem we cannot say that every isolated ideal boundary component of $\partial\mathfrak{M}(G)$ is a puncture, although in the special case that two ideal boundary components are "paired" it is not hard to give a geometric proof. On the other hand, it will follow from our analysis that there are at most a finite number of conjugacy classes of maximal parabolic subgroups of G, a fact that the finiteness theorem does not reveal.

We begin our analysis of $\mathfrak{M}(G)$ by inserting a complete, mutually disjoint collection of (a) doubly infinite cylinders $\{\chi_i\}$ (cf. § 2.5), one corresponding to each conjugacy class of cyclic maximal parabolic subgroups, and (b) canonical cusp tori $\{\sigma_j\}$ (cf. § 2.5), one corresponding to each conjugacy class of rank two maximal parabolic subgroups. Let $\hat{\chi}_i$, $\hat{\sigma}_j$ denote the component of $\mathfrak{M}(G) - \chi_i$, $\mathfrak{M}(G) - \sigma_j$ respectively which does not meet $\partial\mathfrak{M}(G)$. Each component of $\{\pi^{-1}(\hat{\chi}_i)\}$, $\{\pi^{-1}(\hat{\sigma}_j)\}$ is a topological ball. Let $\mathcal{B}' = \mathcal{B} - \bigcup\{\pi^{-1}(\hat{\chi}_i)\} \cup \{\pi^{-1}(\hat{\sigma}_j)\}$, where the union is over all components of $\pi^{-1}(\cdot)$ and all i, j, and set $\mathfrak{M}' = \mathcal{B}' \cup \Omega(G)/G$. $\mathfrak{M}(G) - \mathfrak{M}'$ is the union of the regions $\hat{\chi}_i$ and $\hat{\sigma}_j$.

By choosing the χ_i and σ_j sufficiently small K will lie in $\mathfrak{M}'$. If Assumption 6.1 holds in $\mathfrak{M}(G)$ with respect to K and K_0 it also holds in $\mathfrak{M}'$. For if D is a disk in $\mathfrak{M}(G) - K$ with $\partial D \subset \mathfrak{M}'$ then ∂D also bounds a disk in $\mathfrak{M}' - K$.

This is seen by analyzing $D \cap \chi_i$ and $D \cap \sigma_j$ in the manner of the proof of Lemma 1.12. The converse is also true.

We can find a compact submanifold η of $\mathfrak{M}'$ which has the following properties: (i) $K \subset \eta^0$. (ii) The relative boundary $\partial_0\eta$ of η in $\mathfrak{M}'^0$ has a finite number of components each of which is a closed surface in $\mathfrak{M}'^0$ or the interior of a compact bordered surface S with border $S \cap \partial\mathfrak{M}' = \partial S \cap \partial\mathfrak{M}'$. (iii) $(\partial_0\eta)^-$ is disjoint from the cusp tori in $\partial\mathfrak{M}'$. Because $H_1(\mathfrak{M}')$ is generated in K, each component of $\partial_0\eta$ divides $\mathfrak{M}'$.

Now eliminate the superfluous handles on $\partial_0\eta$ by repeated use of Lemma 1.4 in the light of Assumption 6.1. In outline form this process is as follows. Suppose $\gamma \subset \partial_0\eta$ is a non-trivial loop on $\partial_0\eta$ which bounds a disk in $\mathfrak{M}' - \partial_0\eta$ and hence a disk D in $\mathfrak{M}' - K_0 - \partial_0\eta$. Thicken D to D' (i.e., let D' be a regular neighborhood of D in $\mathfrak{M}' - \eta^0$ or in η) and form η' by adding or deleting D' from η depending on whether D' extends into the exterior or the interior of η. Denote by η_1 the component of η' which contains K_0. After a finite number of repetitions of this process (the $(j+1)^{\text{st}}$ step either decreases the total genus of $\partial_0\eta_j$ or increases the number of components of $\partial_0\eta_j$) we obtain a submanifold $\mathfrak{M}_0$ of $\mathfrak{M}'$ with the property that no non-trivial loop on $\partial_0\mathfrak{M}_0$ bounds a disk in $\mathfrak{M}' - K_0 - \partial_0\mathfrak{M}_0$. In carrying this out we have made use of the fact that each succeeding simple loop can be pushed off the previously adjoined disks so as to lie in the original $\partial_0\eta$.

Now we can apply Lemma 1.11 to conclude that for each component R of $\partial_0\mathfrak{M}_0$, $\ker\big(\pi_1(R) \to \pi_1(\mathfrak{M}')\big) = 0$ and each component of $\pi^{-1}(R)$ in $\mathfrak{B}$ is simply connected. Consequently each component of $\pi^{-1}(\mathfrak{M}_0^0)$ is a topological ball (Lemma 1.12).

Finally we can assume that no component of $\mathfrak{M}' - \mathfrak{M}_0^0$ is compact. Otherwise it can be added to $\mathfrak{M}_0$.

6.3. Let η be a component of $\mathfrak{M}' - \mathfrak{M}_0^0$. The relative boundary $S_0 = \partial_0\eta$ of η in $\mathfrak{M}'^0$ is either a closed surface or the interior of a compact bordered surface S_0^- with $\partial S_0 \subset \partial\mathfrak{M}'$. It is important to recognize two basic properties of η which follow directly from an application of the argument of Corollary 1.13.

(1) Two loops in η which are freely homotopic in $\mathfrak{M}'$ (or in $\mathfrak{M}(G)$) are freely homotopic in η.

(2) The inclusion $\pi_1(\partial_0\eta) \to \pi_1(\eta)$ is an isomorphism.

From these two properties we deduce that the inclusion $\pi_1(\partial\eta - S_0) \to \pi_1(\eta)$ is injective. The argument required to show this is exactly that used for B-groups in § 5.2.

Let S be the component of $\partial\eta$ that contains S_0. Each component α of ∂S_0 either lies on $\partial\mathfrak{M}(G)$ or on one of the doubly infinite cylinders inserted in $\mathfrak{M}(G)^0$. In the first case α bounds a component R of $S - S_0$ which lies in $\partial\mathfrak{M}(G)$. We can assume R^0 is conformally a once punctured disk. For since the inclusion $\pi_1(R) \to \pi_1(\eta)$ is injective, by (1) and (2) every non-trivial loop in R is freely homotopic in S to a non-trivial loop in S_0, and therefore to a loop in α (as a point set); that is, either $\pi_1(R)$ is infinite cyclic or is trivial in which case R^0 is a disk. We can assume the latter possibility does not occur by a small modification of our construction.

In the second case α is a simple loop on some doubly infinite cylinder χ_i and exactly one of the two components, say R, determined by α on this cylinder lies entirely in η and hence in S. Consistent with the terminology of § 5.8 we will say that α surrounds a *puncture* on S.

6.4. Fix a component η^* of $\pi^{-1}(\eta^0)$ (η a fixed component of $\mathfrak{M}' - \mathfrak{M}_0$) and let $G_0 = \{T \in G \colon T\eta^* = \eta^*\}$. Then G_0 acting on η^* preserves the component S^* of $\pi^{-1}(S)$ which lies in $\partial\eta^*$. Since S^* is simply connected we can think of G_0 as a "B-group" with S^* playing the role of the invariant component of the "regular set" $S^* \cup (\Omega(G) \cap \eta^{*-})$ and the union of η^* and those components of $\mathfrak{B} - \mathfrak{B}'$ adjacent to $\partial\eta^* - S^*$ playing the role of $\mathfrak{B}$. That is, the *topological* analysis of Chapter 5 as summarized in Proposition 5.4 applies without change. However, the assertion in part (a) of Proposition 5.4 concerning certain degenerate subgroups becomes more complicated. For this, the following definition is needed.

Definition 6.2. *A degenerate B-group* is a kleinian group H such that there exists a B-group H' with a finite-sided fundamental polyhedron, an isomorphism $\varphi \colon H \to H'$, and a quasiconformal homeomorphism f of $\Omega(H)$ onto a *proper* subset of $\Omega(H')$ which induces φ.

The hypothesis on f in Definition 6.2 insures that f projects to a quasiconformal homeomorphism $f_* \colon \partial\mathfrak{M}(H) \to \partial\mathfrak{M}(H') - \bigcup S'_j$, where $\{S'_j\}$ is a non-empty, proper subset of the set of components of $\partial\mathfrak{M}(H')$. A topological model for $\mathfrak{M}(H)$ might be conjectured to be $\mathfrak{M}(H') - \bigcup S'_j$.

In particular a degenerate group is also a degenerate B-group. A degenerate B-group which is not also a degenerate group is sometimes referred to as a partially degenerate group.

LEMMA 6.3. *A degenerate B-group H does not have a finite-sided fundamental polyhedron.*

Outline of proof. Let H' be as in Definition 6.2 and f_* be as in the paragraph following it. Also set $S' = \Omega_0(H')/H'$ where $\Omega_0(H')$ is an invariant region for H'. If the set $\{S'_j\}$ contains all the components of $\partial\mathfrak{M}(H')$ except S' then H' is a degenerate group and Lemma 6.3 reduces to Corollary 4.6. At the other extreme, if there is only one S'_j and this one is S' then H again has the topolopical structure of a degenerate group. For a subset of the punctures on $\partial\mathfrak{M}(H)$ can be joined by pairing cylinders so as to obtain a 3-manifold with one boundary component. (This pairing is dictated by the pairing in $\partial\mathfrak{M}(H') - S'$). In the other cases we see from looking at $\partial\mathfrak{M}(H')$ that not all punctures on $\partial\mathfrak{M}(H)$ are paired.

6.5. Rather than restate Proposition 5.4 for each η obtained in § 6.3 we will simply present the final result which is obtained by putting together the results for each η. The details are omitted.

We recall that the definition of a finitely punctured compact surface in some $\mathfrak{M}(G)^0$ was given in § 5.8 (also see § 6.3).

THEOREM 6.4 (BASIC STRUCTURE THEOREM). *Assume that the kleinian group G satisfies Assumption* 6.1. *There is a submanifold* $\mathfrak{M}_1 \subset \mathfrak{M}(G)$ *with* $\partial\mathfrak{M}(G) \subset \partial\mathfrak{M}_1$ *and a compact submanifold* $\mathfrak{M}_c \subset \mathfrak{M}(G)$ *with the following properties*:

(i) *The inclusion map* $\pi_1(\mathfrak{M}_1) \rightarrow \pi_1(\mathfrak{M}(G))$ *is an isomorphism.*

(ii) *Each relative boundary component W of* $\mathfrak{M}_1$ *in* $\mathfrak{M}(G)^0$ *bounds a non-compact component* $\mathfrak{W}$ *of* $\mathfrak{M}(G) - \mathfrak{M}_1^0$. *There are at most a finite number of the surfaces W.*

(iii) *W is a finitely punctured compact surface which is incompressible in* $\mathfrak{M}(G)^0$ *and the inclusion* $\pi_1(W) \rightarrow \pi_1(\mathfrak{W})$ *is an isomorphism. The universal covering surface of W is the disk.*

(iv) *The subgroup* G_0 *of G that stabilizes a component of* $\pi^{-1}(\mathfrak{W})$ *is a degenerate B-group.* G_0 *is a degenerate group if and only if every parabolic transformation in* G_0 *is determined by a puncture on W.*

(v) *The punctures on* $\partial\mathfrak{M}_1$ (= *the union of* $\partial\mathfrak{M}(G)$ *and the surfaces W*) *are arranged in a finite number of distinct pairs* (p_i, p'_i). *The pairs are in* 1–1 *correspondence with the conjugacy classes of maximal parabolic subgroups of G which are cyclic.* p_i *and* p'_i *are paired by a pairing tube* $\mathcal{T}_i$ *in* $\mathfrak{M}_1$.

(vi) *There are a finite number of solid cusp tori* $\mathfrak{C}_j$ *in* $\mathfrak{M}_1^0$ *which are in* 1–1 *correspondence with the conjugacy classes of maximal parabolic subgroups of G which are free abelian of rank two.*

(vii) *The* $\mathcal{T}_i$ *and* $\mathfrak{C}_j$ *can be taken to be mutually disjoint and the complement in* $\mathfrak{M}_1$ *of their interiors is a compact submanifold* $\mathfrak{M}_c$ *of* $\mathfrak{M}(G)$. *The* $\mathcal{T}_i$ *and* $\mathfrak{C}_j$ *account for all the parabolic elements of G.*

The last assertion of (v) is true because the conjugacy class of each maximal parabolic subgroup which is cyclic, determines a doubly infinite cylinder in $\partial\mathfrak{M}'$, all of which are accounted for in the process described in §§ 6.3 and 5.11.

It follows from Theorem 6.4 that G has a finite-sided fundamental polyhedron if and only if none of the surfaces W appear.

6.6. We close this chapter with a result that will be required in Chapter 10. Although formulated as a corollary, it could have been proven at a much earlier stage.

COROLLARY 6.5. *Suppose G is a kleinian group with a finite-sided fundamental polyhedron, Ω_1 is a component of $\Omega(G)$, and $G_1=\{T\in G: T\Omega_1=\Omega_1\}$. Then G_1 also has a finite-sided fundamental polyhedron.*

Ouline of proof. Assume $G_1 \neq G$. G_1 is a function group with invariant region Ω_1.

Case 1. Ω_1 is simply connected. If G_1 has no parabolic transformations then G_1 is a quasi-fuchsian group since $\Omega_1 \neq \Omega(G)$ implies the complement on $\partial\mathfrak{B}$ has interior points. More generally there is a set $\{\alpha_i\}$ of mutually disjoint simple loops on Ω_1/G_1, not retractible to punctures, which determine accidental parabolic transformations in G and G_1 (Chapter 5). To each α_i correspond two punctures (p_i, p_i') on $\partial\mathfrak{M}(G)$ (Theorem 6.4) and two cylinders C_i, C_i' in $\mathfrak{M}(G)$, one boundary component of each being α_i, the other retractible to p_i or p_i'. Furthermore, $C_i \cap C_i' = \alpha_i$. In addition all the punctures on Ω_1/G_1 are paired by cylinders in $\mathfrak{M}(G)$ with punctures on $\partial\mathfrak{M}(G)$. This implies that the $\{\alpha_i\}$ and the punctures on Ω_1/G_1 are correspondingly paired in $\mathfrak{M}(G_1)$. In fact all the pairing cylinders can be interpreted as imbedded in $\mathfrak{M}(G_1)$ as well as in $\mathfrak{M}(G)$. Since all the parabolic transformations of G_1 are accounted for, Corollary 6.5 follows from Proposition 5.4.

Case 2. Ω_1 is not simply connected. In this case the corollary can be proved by applying the method of Proposition 5.6.

7. The boundary estimate

7.1. We begin with the decompositions $\mathfrak{M}_1$, $\mathfrak{M}_c$ of $\mathfrak{M}(G)$ obtained in Theorem 6.4. The relative boundary of $\mathfrak{M}_1$ in $\mathfrak{M}(G)^0$ consists of finitely many finitely punctured compact surfaces. The punctures on $\partial\mathfrak{M}_1$ are arranged in pairs and $\mathfrak{M}_c$ is the compact manifold resulting from the introduction into $\mathfrak{M}_1$ of cylinders joining the paired punctures and of cusp tori corresponding to the free abelian subgroups of rank 2. We need the following list of notation.

N is the number of generators of G.

g_i is the genus of the i^{th} component of $\partial\mathfrak{M}(G)$; b_i is the number of its punctures; χ is the number of components of $\partial\mathfrak{M}(G)$; χ_{pct} is the number that have punctures; $g = \sum g_i$, $b = \sum b_i$.

g_i' is the genus of the i^{th} *relative* boundary component of $\mathfrak{M}_1$ in $\mathfrak{M}(G)^0$; b_i' is the number of its punctures; χ^1 is the number of relative boundary components of $\mathfrak{M}_1$ in $\mathfrak{M}(G)^0$; χ^1_{pct} is the number that have punctures; $g' = \sum g_i'$; $b' = \sum b_i'$.

t is the number of cusp tori in $\partial\mathfrak{M}_c$.

χ^* is the number of components of $\partial\mathfrak{M}_c$.

7.2. The total genus of $\partial\mathfrak{M}_c$ is the total of the number of handles on $\partial\mathfrak{M}_1$, the number of cusp tori, and something contributed by the adjoined cylinders. The totality of simple loops, one around each adjoined cylinder, is subject to one relation for each punctured surface in $\partial\mathfrak{M}_1$. These relations are not completely independent however: If S is a component of $\partial\mathfrak{M}_c$ formed by joining several punctured surfaces, then the relation contributed by one punctured surface involved in S is a consequence of the remaining ones. Therefore, from Lemma 1.2 we obtain

$$\sum g_i + \sum g_i' + (1/2)(\sum b_i + \sum b_i') + \chi^* - \chi - \chi^1 \leqq N . \tag{1}$$

This can be written as

$$\begin{aligned}[\textstyle\sum_{b_i=0} g_i + \sum_{b_i'=0} g_i' + t] + [\textstyle\sum_{b_i\neq 0}(g_i + b_i/2 - 1) + \sum_{b_i'\neq 0}(g_i' + b_i'/2 - 1)] \\ + \chi^* - t - (\chi - \chi_{\text{pct}}) - (\chi^1 - \chi^1_{\text{pct}}) \leqq N .\end{aligned} \tag{2}$$

From (2) we also obtain

$$\chi + \chi^1 - (1/2)(\chi_{\text{pct}} + \chi^1_{\text{pct}}) + \chi^* \leqq N . \tag{3}$$

For given N, χ is largest when $\chi^1 = t = 0$, $\chi = \chi_{\text{pct}}$, and $\chi^* = 1$. We have proved

THEOREM 7.1. *If G satisfies Assumption* 6.1 *then $\partial\mathfrak{M}(G)$ has at most* $2N - 2$ *components. In the maximal case each component is either a triply punctured sphere or a once punctured torus. The punctures can be pairwise joined by cylinders so as to form a compact surface of genus N.*

We also note that in the maximal case there are always at least $N - 2$ triply punctured spheres. Furthermore, two punctures on the same triply punctured sphere are not joined by one of the cylinders (in a non-maximal situation this is possible however; for an example see [34]). For any N it is not difficult to construct G so that $\partial\mathfrak{M}(G)$ is a union of $2(N - 1)$ triply punctured spheres.

Inequality (2) is a sharpened form of Bers' inequality (§ 2.4). From it we deduce that there is equality (in the case G satisfies Assumption 6.1) in Bers' estimate

$$\sum (g_i + b_i/2 - 1) = N - 1$$

if and only if the punctures on $\partial\mathfrak{M}$ can be pairwise joined by cylinders to form a compact surface of genus N (which bounds a compact submanifold).

8. The isomorphism theorem

8.1. In view of the fact that we do not have a topological characterization of the manifolds corresponding to degenerate groups, it is natural to restrict our attention to those manifolds which can be compactified in our sense. Proposition 4.2 tells us that these come from kleinian groups with a finite-sided fundamental polyhedron. With our problems reduced to those involving compact manifolds, some deep results of Waldhausen [51] become available. Making use of these we obtain the following theorem.

THEOREM 8.1. *Suppose G and H are kleinian groups such that*

(i) *G has a finite-sided fundamental polyhedron,*

(ii) *There exists an orientation preserving homeomorphism*[1] $f: \Omega(G) \to \Omega(H)$ *which induces an isomorphism* $\varphi: G \to H$.

Then there exists a quasiconformal homeomorphism of the closed ball $g: \mathfrak{B}^- \to \mathfrak{B}^-$ *which induces φ. If f is quasiconformal, f has a quasiconformal extension to $\partial\mathfrak{B}$. If f is conformal, then φ is an inner automorphism.*

8.2. *Proof.* Hypothesis (ii) implies that f projects to a homeomorphism $f_*: \partial\mathfrak{M}(G) \to \partial\mathfrak{M}(H)$. Under f_* the ideal boundary components of $\partial\mathfrak{M}(G)$ correspond to those of $\partial\mathfrak{M}(H)$. Using Ahlfors' finiteness theorem we can say that f_* maps each puncture p on $\partial\mathfrak{M}(G)$ to a puncture $f_*(p)$ on $\partial\mathfrak{M}(H)$ (Ahlfors' theorem is not necessary here; an alternative proof can be given using some of the methods below). It is also true that if p_1, p_2 are paired punctures on $\partial\mathfrak{M}(G)$, then $f_*(p_1)$, $f_*(p_2)$ are paired on $\partial\mathfrak{M}(H)$. For $f_*(p_1)$ and $f_*(p_2)$ are distinct and φ preserves the algebraic characterization of Lemma 2.3.

Let $\{\mathfrak{S}_i\}$ be a complete set of mutually disjoint (smoothed) canonical pairing tubes in $\mathfrak{M}(G)$; $\mathfrak{S}_i$ pairs two punctures p_i and q_i on $\partial\mathfrak{M}(G)$. Choose a set of mutually disjoint canonical tubes $\{\mathfrak{S}_i'\}$ in $\mathfrak{M}(H)$ so that $\mathfrak{S}_i'$ pairs $f_*(p_i)$ and $f_*(q_i)$. These tubes can be so chosen that $\mathfrak{S}_i$ is conformally equivalent to $\mathfrak{S}_i'$ in such a way that p_i is mapped to $f_*(p_i)$ (Lemma 2.2).

In addition G may contain free abelian subgroups of rank two. Corre-

[1] The proof will in fact show it suffices to assume that f maps $\Omega(G)$ into $\Omega(H)$.

sponding to each of these, there is a canonical solid cusp torus $\mathcal{T}_j$ in $\mathfrak{M}(G)$ and, via the isomorphism φ, a canonical solid cusp torus $\mathcal{T}'_j$ in $\mathfrak{M}(H)$. There is a quasiconformal map $\mathcal{T}_j \to \mathcal{T}'_j$ (§ 2.5). Our choices can be made so that the total collection of submanifolds $\{\mathcal{S}_j, \mathcal{T}_j\}$, $\{\mathcal{S}'_j, \mathcal{T}'_j\}$ are mutually disjoint in $\mathfrak{M}(G)$ and $\mathfrak{M}(H)$ respectively.

Then $\mathfrak{M}_0(G) = \mathfrak{M}(G) - \bigcup \mathcal{S}_i^0 - \bigcup \mathcal{T}_i^0$ is compact. We do not yet know that $\mathfrak{M}_0(H) = \mathfrak{M}(H) - \bigcup \mathcal{S}_i'^0 - \bigcup \mathcal{T}_j'^0$ is also compact. In any case we will work with these manifolds, the corresponding "regular sets" $\Omega_r(G) = \pi^{-1}(\partial\mathfrak{M}_0(G))$, and $\Omega_r(H) = \pi^{-1}(\partial\mathfrak{M}_0(H))$, and the topological balls $\mathcal{B}_0 = \pi^{-1}(\mathfrak{M}_0(G)^0)$ and $\mathcal{B}'_0 = \pi^{-1}(\mathfrak{M}_0(H)^0)$ (here π denotes the respective natural projections $\mathcal{B} \cup \Omega(G) \to \mathfrak{M}(G)$, $\mathcal{B} \cup \Omega(H) \to \mathfrak{M}(H)$).

Referring to Lemma 1.17 we see that there is a C^∞ triangulation of $\mathfrak{M}(G)$ which induces a triangulation of $\mathfrak{M}_0(G)$. The same is true of $\mathfrak{M}(H)$ and $\mathfrak{M}_0(H)$ but we must show that the triangulation can be chosen so that in addition the conformal maps $\sigma_i: \mathcal{S}_i \to \mathcal{S}'_i$ and the C^∞ quasiconformal maps $\tau_j: \mathcal{T}_j \to \mathcal{T}'_j$ are PL. To do this observe that σ_i extends conformally and τ_j C^∞ quasiconformally to neighborhoods of $\mathcal{S}_i$, $\mathcal{T}_j$ respectively in $\mathfrak{M}_0(G)$. Making use of f_* we can obtain a homeomorphism ζ of a tubular neighborhood of $\partial\mathfrak{M}_0(G)$ onto one of $\partial\mathfrak{M}_0(H)$ which agrees with σ_i, τ_j near $\mathcal{S}_i$, $\mathcal{T}_j$ respectively, for all i and j. In addition we can assume ζ is a diffeomorphism (Lemma 1.18). ζ induces a C^∞ triangulation of a neighborhood of $\partial\mathfrak{M}_0(H)$. Applying [44, 10.7] this triangulation near $\partial\mathfrak{M}_0(H)$ can be extended to a C^∞ triangulation of $\mathfrak{M}_0(H)$. To obtain the desired triangulation of $\mathfrak{M}(H)$, carry the given triangulations of $\mathcal{S}_i$, $\mathcal{T}_j$ over to $\mathcal{S}'_i$, $\mathcal{T}'_j$ by the maps σ_i, τ_j.

Now that the triangulations of $\mathfrak{M}_0(G)$ and $\mathfrak{M}_0(H)$ are fixed, change f_* as follows (cf. ζ above): f_* is homotopic to a PL homeomorphism $\partial\mathfrak{M}(G) \to \partial\mathfrak{M}(H)$ which restricts to a homeomorphism $g^1_*: \partial\mathfrak{M}_0(G) \cap \partial\mathfrak{M}(G) \to \partial\mathfrak{M}_0(H) \cap \partial\mathfrak{M}(H)$ with the proper boundary values so as to have the following property. There exists a PL homeomorphism $g_*: \partial\mathfrak{M}_0(G) \to \partial\mathfrak{M}_0(H)$ which is equal to g^1_* on $\partial\mathfrak{M}_0(G) \cap \partial\mathfrak{M}(G)$, to σ_i on the canonical cylinder bounding $\mathcal{S}_i$, and to τ_j on the torus bounding $\mathcal{T}_j$, for all i and j. g_* has a lift $g: \Omega_r(G) \to \Omega_r(H)$ which induces φ.

8.3. *Case* 1. Each component of $\Omega_r(G)$ is simply connected. We will first show that $\mathfrak{M}_0(H)$ is compact. The double $D(\mathfrak{M}_0(G))$ is compact without boundary and $\pi_1(D(\mathfrak{M}_0(G)))$ can be obtained purely algebraically from $\pi_1(\mathfrak{M}_0(G))$ and $\pi_1(\partial\mathfrak{M}_0(G))$ by application of van Kampen's theorem (for a proof see [56, Prop. 2.1]). Since $\pi_1(D(\mathfrak{M}_0(H)))$ of the double $D(\mathfrak{M}_0(H))$ of $\mathfrak{M}_0(H)$ is obtained in exactly the same manner by way of g_*, φ extends to an isomorphism

$\pi_1(D(\mathfrak{M}_0(G)))\to\pi_1(D(\mathfrak{M}_0(H)))$. Now the homotopy groups π_2 and π_3 of $D(\mathfrak{M}_0(H))$ and $D(\mathfrak{M}_0(H))$ are zero: $\pi_2 = 0$ because of the sphere theorem and $\pi_3 = 0$ because the universal covering space is not compact [19, pp. 154, 167]. By Lemma 1.7, $D(\mathfrak{M}_0(H))$ is compact.

We are now in a position to apply Waldhausen's result [51] that the isomorphism $\varphi_*: \pi_1(\mathfrak{M}_0(G)) \to \pi_1(\mathfrak{M}_0(H))$ is induced by a PL homeomorphism $h: \mathfrak{M}_0(G) \to \mathfrak{M}_0(H)$. But h and g_* are homotopic on $\partial\mathfrak{M}_0(G)$ since all components of $\Omega_r(G)$ are simply connected. Consequently by an application of Lemma 1.14 we can assume $h|\partial\mathfrak{M}_0(G) = g_*$. Rename h to be g_*.

Because of our special choice for the boundary values of g_* on $\partial\mathfrak{M}_0(G)$, g_* can be extended to a map $\partial\mathfrak{M}(G) \to \partial\mathfrak{M}(H)$ which is conformal in the regions $\mathcal{S}_i$ and quasiconformal in the $\mathcal{T}_j$.

If g is the lift of g_* to $\mathcal{B}$ which induces $\varphi: G \to H$, then g is quasiconformal in $\mathcal{B}$. By Gehring's theorem (§ 2.8), g extends to be quasiconformal on $\partial\mathcal{B}$ as well.

8.4. *Case* 2. Not all components of $\Omega_r(G)$ are simply connected. If R is one of those which are not, then by Dehn's lemma and the loop theorem, there is a disk D_1 in $\mathfrak{M}_0(G)$ with $\partial D_1 \subset R$ a non-trivial simple loop. By Lemma 1.8, D_1 splits $\pi_1(\mathfrak{M}_0(G))$: $G = G_1 * G_2$ where neither G_1 nor G_2 is trivial. Perform this construction as often as possible. In the step-by-step cutting down of $\mathfrak{M}_0(G)$, the boundary of an added disk can always be pushed off the previously added disks. Since $\mathfrak{M}_0(G)$ is compact the process terminates after a finite number of steps. It will be convenient for us to write $\mathfrak{M}_0(G) = \bigcup_{i=1}^{m} \mathfrak{M}_i(G)$ where the $\mathfrak{M}_i(G)$ are the closures of the components of $\mathfrak{M}_0$ determined by the totality of disks, although this notation requires some explanation: i.e., the boundary $\partial\mathfrak{M}_i(G)$ may contain both sides of one or more of the disks D.

Each component in $\mathcal{B}_0^- = (\pi^{-1}(\mathfrak{M}_0(G))^0)^-$ of $\pi^{-1}(\partial\mathfrak{M}_i(G))$ is simply connected (otherwise the reduction process could be carried further, see Lemma 1.11). Fix a component $\mathfrak{M}_i(G)^*$ of $\pi^{-1}(\mathfrak{M}_i(G)^0)$ in $\mathcal{B}_0$. The relative boundary $\partial_0\mathfrak{M}_i(G)^*$ in $\mathcal{B}_0$ is the union of a set $\Sigma_d(G_i)$ of topological disks, none of which is invariant under the subgroup G_i of G that preserves $\mathfrak{M}_i(G)^*$ (if $G_i \neq \mathrm{id}$). The boundaries of the disks $\Sigma_d(G_i)$ form a set $\Sigma(G_i)$ of mutually disjoint simple loops on $\Omega_r(G)$.

We will soon need the following three facts.

1. A limit point p of G lies in $\mathfrak{M}_i(G)^{*-}$ only if p is also a limit point of G_i. For if p were not a limit point of G_i then p would not be the limit of a sequence of disks in $\Sigma_d(G_i)$. At the same time p would be the limit of a sequence of distinct components of $\{\pi^{-1}(\mathfrak{M}_i(G)^0)\}$. This is an impossible situa-

tion since each of these components is separated from $\mathfrak{M}_i(G)^*$ by a disk in $\Sigma_d(G_i)$.

2. Let R be a component of $\Omega_r(G)$, Σ_0 the subset of $\Sigma(G_i)$ lying in R, and assume $\Sigma_0 \neq \emptyset$. Then Σ_0 is a finite set if and only if $\mathfrak{M}_i(G)^0$ is a ball and $G_i = \{\mathrm{id}\}$. To see this, set $R_0 = R \cap \partial\mathfrak{M}_i(G)^*$. If Σ_0 is finite no element $T \neq \mathrm{id}$ of the subgroup of G that preserves R also preserves R_0 since $\Sigma_0 = \partial_0 R_0$. For no such T to exist, each component of $\partial\mathfrak{M}_i(G)$ that meets $\pi(R_0)$ must be a sphere (since each component of $\pi^{-1}(\partial\mathfrak{M}_i(G))$ is simply connected). The only possibility is that $\partial\mathfrak{M}_i(G)$ is a sphere bounding the ball $\mathfrak{M}_i(G)^0$.

3. If $\mathfrak{M}_i(G)^0$ is not a ball then a component R_1 of $\Omega_r(G) - \Sigma(G_i)$ lies in $\partial\mathfrak{M}_i(G)^*$ if and only if R_1 is preserved by an infinite subgroup of G_i.

8.5. Carrying on with the proof of Case 2, g maps $\Sigma(G_i)$ onto a set $\Sigma(H_i)$ of mutually disjoint loops in $\Omega_r(H)$; $H_i = \varphi(G_i)$ preserves $\Sigma(H_i)$. Each $\gamma \in \Sigma(H_i)$ projects to a non-trivial simple loop on $\partial\mathfrak{M}_0(H)$ which therefore bounds a disk in $\mathfrak{M}_0(H)$. More generally the set $\Sigma(H_i)$ bounds a set of mutually disjoint disks $\Sigma_d(H_i)$ in $\mathcal{B}_0' = \pi^{-1}(\mathfrak{M}_0(H)^0)$ each of which projects 1–1 to a disk in $\mathfrak{M}_0(H)$. We have to prove that $\Sigma_d(H_i)$ is the relative boundary in $\mathcal{B}_0'$ of a region $\mathfrak{M}_i(H)^*$ topologically the same as $\mathfrak{M}_i(G)^*$ in the sense that the methods of Case 1 can be applied to $\mathfrak{M}_i(G)^*$ and $\mathfrak{M}_i(H)^*$.

We will first dispose of the case that $\mathfrak{M}_i(G)^0$ is a ball. In this case $\partial\mathfrak{M}_i(G)^* \cap \Omega_r(G)$ is a connected, compact region R bounded by the finite set $\Sigma(G_i)$. Then $g(R) \cup \Sigma_d(H_i)$ is the boundary in $\mathcal{B}_0'^-$ of a subregion $\mathfrak{M}_i(H)^*$. Extend g to $\Sigma_d(G_i)$ to give a (PL) homeomorphism $\partial\mathfrak{M}_i(G)^* \to \partial\mathfrak{M}_i(H)^*$, but g must be subjected to the following restriction: If $TD_1 = D_2$ for $D_1, D_2 \in \Sigma_d(G_i)$, $T \in G$, then $g|D_2 = \varphi(T) \circ g|D_1$. Then g extends to a homeomorphism $\mathfrak{M}_i(G)^* \to \mathfrak{M}_i(H)^*$ which projects to $g_*: \mathfrak{M}_i(G) \to \mathfrak{M}_i(H)$, where $\mathfrak{M}_i(H)^0$ is a ball in $\mathfrak{M}_0(H)$ and $\partial_0\mathfrak{M}_i(H)$ is the union of (possibly both sides of) the disks $\pi(\Sigma_d(H_i))$.

Return now to the general case and assume that $\mathfrak{M}_i(G)$ is not a ball. First we will show that no disk in $\Sigma_d(H_i)$ separates in $\mathcal{B}_0'$ two other disks of $\Sigma_d(H_i)$. To do this consider the manifold $\mathcal{H}_i = \mathcal{B}_0' \cup \Omega_r(H)/H_i$ which satisfies $\pi_1(\mathcal{H}_i) \cong \pi_1(\mathfrak{M}_i(G))$. Each $D \in \Sigma_d(H_i)$ projects 1–1 to a disk D_* in $\mathcal{H}_i$ with ∂D_* a simple loop on $\partial\mathcal{H}_i$. By Lemma 1.10, D_* cannot cut $\mathcal{H}_i$ so as to decompose $\pi_1(\mathcal{H}_i)$ into a non-trivial free product because $\mathfrak{M}_i(G)$ can be reduced no further. Therefore, one of the two components of $\mathcal{H}_i - D_*$ is a ball $b_*(D)$. We have yet to show that $b_*(D)$ does not contain the projection of any disk in $\Sigma_d(H_i)$.

Let $\{b(D_j)\}$ denote the lifts of $b(D_*)$ to $\mathcal{B}_0'$; the relative boundary $\partial_0 b(D_j)$

of $b(D_j)$ in $\mathcal{B}_0'$ is a disk $D_j \in \Sigma_d(H_i)$. The balls $b(D_j)$ are mutually disjoint and the subset $\{D_j\}$ of $\Sigma_d(H_i)$ consists of precisely those disks which project to D_* in $\mathcal{H}_i$. Furthermore, the closure $b(D_j)^-$ in $\mathcal{B}_0'^-$ contains no limit point of H_i for such a point would have to lie on ∂D_j.

Suppose for $j = 1$, $b(D_1)$ contains $D' \in \Sigma_d(H_i)$ (necessarily $D' \neq D_k$ for all k). Examine the position of the simple loops $g^{-1}(\partial D')$, $g^{-1}(\partial D_1) \in \Sigma(G_i)$ in $\Omega_r(G)$.

On the one hand, these two loops cannot lie in different components of $\Omega_r(G)$. For by facts (1) and (2) of §8.4, $g^{-1}(\partial D')$ together with infinitely many other loops $\{\alpha_j\}$ of $\Sigma(G_i)$ bound a subregion of $\Omega_r(G)$. Consequently applying g we see that ∂D_1 does not separate $\partial D'$ from the $\{g(\alpha_j)\}$ in $\Sigma(H_i)$. That is $b(D_1)^-$ contains limit points of H_i, a contradiction.

On the other hand, $g^{-1}(\partial D')$ and $g^{-1}(\partial D_1)$ cannot lie in the same component of $\Omega_r(G)$ either. Again this follows from fact (2) but this time we have to use the assumption that $\mathfrak{M}_i(G)$ is not a ball.

We conclude that the union of the disks in $\Sigma_d(H_i)$ is the relative boundary in $\mathcal{B}_0'$ of a topological ball $\mathfrak{M}_i(H)^*$ which projects to the interior of a submanifold $\mathfrak{M}_i(H)$ of $\mathfrak{M}_0(H)$. Fact (3) of § 8.4 holds for $\mathfrak{M}_i(H)^*$ for the same reason it holds for $\mathfrak{M}_i(G)^*$. Therefore, g is a homeomorphism $\partial\mathfrak{M}_i(G)^* \cap \Omega_r(G) \to \partial\mathfrak{M}_i(H)^* \cap \Omega_r(H)$. Extend g to a homeomorphism $\Sigma_d(G_i) \to \Sigma_d(H_i)$ which satisfies the additional requirement that if $T(D_1) = D_2$ for D_1, $D_2 \in \Sigma_d(G_i)$, $T \in G$, then $g|D_2 = \varphi(T)g|D_1$.

Now we have exactly the situation of Case 1. Using the result of that case we can extend g_* from a homeomorphism $\partial\mathfrak{M}_i(G) \to \partial\mathfrak{M}_i(H)$ to a homeomorphism $\mathfrak{M}_i(G) \to \mathfrak{M}_i(H)$. Note that g_* is also a homeomorphism of the disks $\pi(\Sigma_d(G_i))$ in $\mathfrak{M}_0(G)$ onto $\pi(\Sigma_d(H_i))$.

Now that g has been extended to $\mathfrak{M}_i(G)^*$, take a region $\mathfrak{M}_j(G)^*$ which is adjacent to $\mathfrak{M}_i(G)^*$ and repeat this process until $\mathcal{B}_0$ is completely filled up. In doing this make sure that once g has been extended to a disk D, its extension to $T(D)$ for any $T \in G$ is determined by the equation $g|T(D) = \varphi(T)g|D$. Then down in $\mathfrak{M}_0(G)$, as we pass from $\mathfrak{M}_i(G)$ to an adjacent $\mathfrak{M}_j(G)$, the projection g_* will dictate how to attach $\mathfrak{M}_j(H)$ to $\mathfrak{M}_i(H)$. After joining all the $\mathfrak{M}_i(H)$ we must obtain $\mathfrak{M}_0(H)$, because the relative boundary components of $\mathfrak{M}_i(H)^0$ in $\mathfrak{M}_0(H)^0$ are disks all of which are accounted for in this joining process. Thus the original $g_*: \partial\mathfrak{M}_0(G) \to \partial\mathfrak{M}_0(H)$ can be extended to a (PL) homeomorphism $g_*: \mathfrak{M}_0(G) \to \mathfrak{M}_0(H)$.

The final extension of g_* to a homeomorphism $\mathfrak{M}(G) \to \mathfrak{M}(H)$ proceeds exactly as in Case 1. After that is done, the lift of g_* which induces φ has a quasiconformal extension to $\mathcal{B}^-$.

8.6. We can now prove the second part of Theorem 8.1. If f is the given quasiconformal homeomorphism $\Omega(G) \rightarrow \Omega(H)$ and g is the quasiconformal map $\mathcal{B}^- \rightarrow \mathcal{B}^-$ obtained in § 8.5 then $h = g^{-1}f$ restricted to $\Omega(G)$ is a quasiconformal map $\Omega(G) \rightarrow \Omega(G)$ which induces the identity automorphism of G. f has a quasiconformal extension to $\partial\mathcal{B}$ if and only if h does.

That h does have a quasiconformal extension is a consequence of a theorem of Maskit [37]. Maskit's proof proceeds by showing first that h has a continuous extension which is the identity on $\Lambda(G)$ and second that, by a direct calculation, the extension is quasiconformal. However, we will provide another proof of this fact which from our point of view is natural and casts further light on Maskit's theorem. The goal is to extend h to $\mathfrak{M}(G)$ in order to apply Gehring's theorem. But the fact that h is not necessarily smooth forces us to take a small detour.

Let $\mu(z)$ be the complex dilation of h in $\Omega(G)$ and let h_0 be a solution of the Beltrami equation $(\partial\cdot/\partial\bar{z}) = \mu(\partial\cdot/\partial z)$ which is a quasiconformal homeomorphism on $\partial\mathcal{B}$. Then h_0h^{-1} induces an isomorphism between kleinian groups $\psi: G \rightarrow K$ and is a conformal homeomorphism $\Omega(G) \rightarrow \Omega(K)$. Rename h_0h^{-1} to be h; we will show that h has a quasiconformal extension to $\partial\mathcal{B}$.

Assume first that $\partial\mathfrak{M}(G)$ is compact. It will be clear from our proof that it is no additional restriction to assume $\partial\mathfrak{M}(G)$ is connected as well. The projection $h_*: \partial\mathfrak{M}(G) \rightarrow \partial\mathfrak{M}(K)$ is a diffeomorphism. By the collaring theorem [44, Theorem 5.9] there are compact, C^∞ submanifolds $U \subset \mathfrak{M}(G)$, $V \subset \mathfrak{M}(K)$ with $\partial\mathfrak{M}(G) \subset \partial U$, $\partial\mathfrak{M}(K) \subset \partial V$, and diffeomorphisms $f: U \rightarrow R \times I$, $g: V \rightarrow R \times I$ where $R \cong \partial\mathfrak{M}(G)$, $I = [0, 1]$.

Set $M = \mathfrak{M}(G) - U^0$, $N = \mathfrak{M}(K) - V^0$. h_* determines a diffeomorphism $h'_*: \partial M \rightarrow \partial N$ and by the first part of Theorem 8.1 there is a PL homeomorphism $\chi_1: M \rightarrow N$ whose restriction to ∂M is homotopic to h'_*. Smoothing χ_1 [43] we obtain a diffeomorphism $\chi: M \rightarrow N$ which restricts to a diffeomorphism $\partial M \rightarrow \partial N$ homotopic to h'_*.

Then the maps gh_*f^{-1} and $g\chi f^{-1}$ are diffeomorphisms of the surfaces, say $R \times \{0\}$ and $R \times \{1\}$ respectively. Applying Lemma 1.13 there is a diffeomorphic extension of them to $R \times I$. This in turn yields a diffeomorphic extension of h_* to all $\mathfrak{M}(G)$ and Gehring's theorem shows that any lift of h_* to $\mathcal{B} \cup \Omega(G)$ has an extension to $\mathcal{B}^-$.

Now consider the case that $\mathfrak{M}(G)$ is not compact. For simplicity we will assume $\mathfrak{M}(G)$ contains no cusp tori (cusp tori can be dealt with by a simpler version of the technique that will be used below). There are a finite number of mutually disjoint canonical pairing tubes $\mathcal{S}_i$ in $\mathfrak{M}(G)$ such that $M = \mathfrak{M}(G) - \bigcup \mathcal{S}_i^0$ is compact (Proposition 4.2). The $\mathcal{S}_i$ can be taken smoothly attached to

$\partial\mathfrak{M}(G)$ (cf. § 2.7). If these tubes are sufficiently small there are corresponding tubes $\mathfrak{S}_i$ in $\mathfrak{M}(K)$ and conformal maps $\chi_i\colon \mathfrak{S}_i - \mathfrak{S}'_i$. Also by the first part of Theorem 8.1, $N = \mathfrak{M}(K) - \bigcup \mathfrak{S}'^0_i$ is compact.

It is not hard to show that the projection $h_*\colon \partial\mathfrak{M}(G) \to \partial\mathfrak{M}(K)$ of h can be changed in a small neighborhood of $(\bigcup \mathfrak{S}_i) \cap \partial\mathfrak{M}(G)$ so as to obtain a diffeomorphism $h'_*\colon \partial\mathfrak{M}(G) \to \partial\mathfrak{M}(K)$ which is homotopic to h_* and agrees with χ_i in a neighborhood of $\mathfrak{S}_i \cap \partial\mathfrak{M}(G)$, for all i. Extend h'_* to each $\mathfrak{S}_i$ by setting it equal to χ_i there. In particular we then obtain a diffeomorphism $\partial M \to \partial N$. The argument above using Lemma 1.13 allows us to extend h'_* to a diffeomorphism $M \to N$.

Let h' be the lift of h'_* that agrees with h on $\Omega(G)$ except on a neighborhood of $\pi^{-1}(\bigcup \mathfrak{S}_i) \cap \Omega(G)$. By Gehring's theorem, h' has a quasiconformal extension to the limit set $\Lambda(G)$. But the only limit point on the closure of each component R_i of $\pi^{-1}(\bigcup \mathfrak{S}_i) \cap \Omega(G)$ is a parabolic fixed point. It follows that the extension of h defined by setting $h = h'$ on $\Lambda(G)$ is quasiconformal (change h' to h first near R_1^-, then R_2^-, etc.).

To complete the proof of Theorem 8.1, if f is conformal on $\Omega(G)$ it also has a quasiconformal extension to $\partial\mathfrak{B}$ as above. Since, however, $\Lambda(G)$ has 2-dimensional measure zero, f is conformal everywhere.

8.7. In particular we have proved the following result.

COROLLARY 8.2. *Suppose G, H are kleinian groups, $\mathfrak{M}(G)$ is compact and $\varphi\colon \pi_1(\mathfrak{M}(G)) \to \pi_1(\mathfrak{M}(H))$ is an isomorphism. If $f\colon \partial\mathfrak{M}(G) \to \partial\mathfrak{M}(H)$ is a diffeomorphism such that*

$$i'\big(f(\pi_1(\partial\mathfrak{M}(G)))\big) = \varphi\big(i(\pi_1(\partial\mathfrak{M}(G)))\big)$$

for some canonical inclusions $i'\colon \pi_1(\partial\mathfrak{M}(H)) \to \pi_1(\mathfrak{M}(H))$, $i\colon \pi_1(\partial\mathfrak{M}(G)) \to \pi_1(\mathfrak{M}(G))$, then f can be extended to a diffeomorphism $\mathfrak{M}(G) \to \mathfrak{M}(H)$ which induces φ.

Proof. By a canonical inclusion i, we understand the following. Fix a point p_i on each component S of $\partial\mathfrak{M}(G)$ which is to serve as the origin of $\pi_1(S)$. Fix a point $0 \in \mathfrak{M}(G)$ as the origin of $\pi_1(\mathfrak{M}(G))$ and join each p_i to 0 by an arc. The choice of these arcs determines an inclusion i. The hypothesis enables us to lift f to a diffeomorphism $\Omega(G) \to \Omega(H)$.

Remark 1. With simple modifications, Theorem 8.1 holds as well if f is orientation reversing. One merely adds the terms orientation reversing and anti-conformal at the appropriate places in the hypotheses and conclusions. The statement that φ is an inner automorphism must then be interpreted in a larger group than SL(2, C)/$\pm$ 1.

Remark 2. There is an analogue of Theorem 8.1 for "constructible groups" due to Maskit.

8.8. Theorem 8.1 should be contrasted with the following result which is an elaboration of Mostow's rigidity theorem.

THEOREM 8.3. *Suppose G and H are discrete subgroups of* SL(2, C)/ ± 1 *such that*

(i) *G has finite volume* (i.e., *G has a fundamental polyhedron with finite hyperbolic volume*), *and*

(ii) *there exists an isomorphism φ: $G \to H$.*

Then φ is an inner automorphism (*by a conformal or anticonformal map of* $\partial\mathfrak{B}$).

Proof. The isomorphism φ determines a one-to-one correspondence between cusp tori in $\mathfrak{M}(G)$ and in $\mathfrak{M}(H)$. By Corollary 4.8, $\mathfrak{M}(G)$ can be compactified by the insertion of cusp tori. By Lemma 1.7 as applied in § 8.3, the same is true of $\mathfrak{M}(H)$.

If $H_1(\mathfrak{M}(G))$ is not a finite group then Waldhausen's work is applicable and we conclude as in § 8.3 that φ is induced by a PL homeomorphism f: $\mathfrak{M}(G) \to \mathfrak{M}(H)$. We have also seen that f can be taken to be quasiconformal (this requires adjustment in cusp tori). Now f liftes to a quasiconformal homeomorphism $\mathfrak{B} \to \mathfrak{B}$ and Mostow's rigidity theorem [41] can be applied to complete the proof for this case.

If $H_1(\mathfrak{M}(G))$ is finite then in general Waldhausen's construction does not work. However, in this case $\mathfrak{M}(G)$ must be compact so some recent results obtained independently by Margulis [30] and Mostow [42] can be applied to finish the proof. Their results are obtained as follows. Since $\mathfrak{M}(G)$ and $\mathfrak{M}(H)$ are $K(\pi, 1)$ spaces, φ is induced by a PL map g which is not necessarily a homeomorphism. A lift g^* of g maps $\mathfrak{B} \to \mathfrak{B}$. It can be shown by elementary geometrical methods (see references listed in [30]) that even though g^* is not a homeomorphism, it can be extended to $\partial\mathfrak{B}$ to be a homeomorphism and even to be quasiconformal there. This is all that is needed for Mostow's theorem.

Remark. Garland and Raghunathan [15, § 10] have extended Mostow's theorem to the non-compact case for hyperbolic space of dimension $\geqq 6$.

9. Stability

9.1. *Definition*. Let $A_1, \cdots, A_n$ be a set of generators for the kleinian group G. An ε-deformation (with respect to $A_1, \cdots, A_n$) is a homomorphism φ: $G \to$ SL(2, C)/ ± 1 satisfying

(i) $|\varphi(A_i) - A_i| < \varepsilon$, $1 \leqq i \leqq N$, in a suitable matrix representation, and

(ii) if $T \in G$ is parabolic, then $\varphi(T)$ is parabolic.

Definition. G is *quasiconformally stable* if given a basis $A_1, \cdots, A_n$ of G there exists $\varepsilon_0 > 0$ such that each ε-deformation of G with $\varepsilon < \varepsilon_0$ is induced by a quasiconformal homeomorphism f of the 2-sphere $\partial\mathfrak{B}$ (that is, $\varphi(G) = fGf^{-1}$).

Clearly, if G is quasiconformally stable with respect to one set of generators, it is stable with respect to any other set. The goal of this chapter is to prove

PROPOSITION 9.1. *If G has a finite-sided fundamental polyhedron then G is quasiconformally stable.*

9.2. Before beginning the proof we will fix our notation and terminology as follows. Given a finite-sided hyperbolic polyhedron $\mathfrak{Q}$ in $\mathfrak{B}^-$, the faces lying on $\partial\mathfrak{B}$ are called *free faces*; without the adjective free, a face lies in $\mathfrak{B}$ except perhaps for its edges or vertices. Likewise vertices or edges contained in $\partial\mathfrak{B}$ are called *free vertices* and *free edges* to distinguish between those objects lying in $\mathfrak{B}$ except perhaps for their boundary. All these objects, polyhedra, faces, etc., are taken to be closed in $\mathfrak{B}^-$.

Thus $\mathfrak{Q} \cap \partial\mathfrak{B}$ is a finite union of circular polygons and *isolated free vertices*. An isolated free vertex lies on at least three edges as does an ordinary vertex of $\mathfrak{Q}$. But a free vertex p which is also a vertex of one of the polygons $\mathfrak{Q} \cap \partial\mathfrak{B}$ may not lie on any edge of $\mathfrak{Q}$. In this case two faces of $\mathfrak{Q}$ are tangent at p and p lies on two free edges. We single this situation out for special attention only in the following circumstances. The faces f, f' are called *cusp faces* with *cusp point p with respect to* a Möbius transformation T if T is parabolic with fixed point p and $Tf' = f$. $\mathfrak{Q}$ has an *open cusp* at p if p lies on no other face of $\mathfrak{Q}$. If there are two pairs of cusp faces at p with respect to T_1, T_2 respectively and p lies on no other face of $\mathfrak{Q}$, then the four faces are said to form a *closed cusp* at the *closed cusp point p* with respect to T_1 and T_2.

9.3. We begin by proving

LEMMA 9.2. *For all sufficiently small ε, if φ is an ε-deformation then $\varphi(G)$ is a discrete group and φ is an isomorphism.*

First we will prove Lemma 9.2 under the assumption that G does not contain parabolic transformations and then show that, with a minor elaboration, the proof works for the general case as well.

A rough idea of the proof is as follows. G has a finite-sided fundamental polyhedron $\mathcal{P}$. When G is deformed slightly then in particular the transformations pairing the opposite faces of $\mathcal{P}$ are deformed and consequently the faces of $\mathcal{P}$ change slightly but still form a polyhedron $\mathcal{P}_\varphi$ close to $\mathcal{P}$. We are going to show that for all small ε, $\mathcal{P}_\varphi$ is a fundamental polyhedron for $\varphi(G)$. What makes this true is that the relations in G force a certain rigidity on $\mathcal{P}$ with respect to its property of being a fundamental set. To describe how this is so we must analyze the relationship between a fundamental polyhedron for a group and the relations in a group. For this reason we must proceed rather formally as follows.

9.4. *Definition.* Given N, fix two sets of letters $f_1, f_1', \cdots, f_N, f_N'$ and $F_1, \cdots, F_N$. A *formal polyhedron* $\mathcal{Q}$ is a non-euclidean polyhedron in $\mathcal{B}^-$ such that

(i) $\mathcal{Q}$ has $2m$ faces (free faces are not included), $m \leqq N$, which are labeled from the set of letters $(f_1, \cdots, f_N')$. This labeling has the property that if f_j (resp. f_j') is one face then f_j' (resp. f_j) is another face.

(ii) Corresponding to a pair of faces (f_j, f_j') is a Möbius transformation labeled F_j such that $F_j(f_j') = f_j$. F_j and F_j^{-1} are called face pairing transformations.

(iii) None of the transformations F_i associated with $\mathcal{Q}$ are elliptic or parabolic.

Suppose $\mathcal{Q}$ is a formal polyhedron, e_1 an edge of $\mathcal{Q}$, and g_1 $(=$ some f_j or $f_j')$ a face of $\mathcal{Q}$ containing e_1. There is a face pairing transformation G_1 $(= F_j$ or $F_j^{-1})$ which maps g_1' $(= f_j'$ or $f_j)$ onto g_1 and an edge e_2 of g_1' with $G_1 e_2 = e_1$. Since G_1 is not elliptic $e_2 \neq e_1$. There is exactly one face $g_2 \neq g_1'$ which also contains e_2. Find the face g_2' and the side pairing transformation G_2 for which $G_2(g_2') = g_2$. There is an edge e_3 of g_2' for which $G_2(e_3) = e_2$, etc. In this specific manner we obtain a sequence of edges $(e_1, e_2, \cdots)$ such that for each $(k+1)$ there is a face pairing transformation G_k with $G_k(e_{k+1}) = e_k$. Then $G_1 G_2 \cdots G_k$ (G_k first) maps e_{k+1} onto e_1 and the polyhedra $\mathcal{Q}$, $G_1(\mathcal{Q}), \cdots, G_1 G_2 \cdots G_k(\mathcal{Q})$ are arranged in cyclic order about e_1.

LEMMA 9.3. *If $e_k = e_i$ for $1 < i < k$ then $e_j = e_1$ for some $1 < j < k$.*

Proof. Choose the smallest possible k with the requisite property. e_k is the common edge of g_{k-1}' and g_k while e_i is the common edge of g_{i-1}' and g_i. Since $e_k = e_i$, G_{k-1} maps one of the faces containing e_i onto the face g_{k-1} containing e_{k-1} with $G_{k-1}(e_i) = G_{k-1}(e_k) = e_{k-1}$. That is, either $g_{k-1}' = g_{i-1}'$ and $G_{k-1} = G_{i-1}$, or $g_{k-1}' = g_i$ and $G_{k-1} = G_i^{-1}$ (since $g_{k-1} = g_i'$). In the former case $e_{k-1} = e_{i-1}$ contradicting the minimal choice of k unless $i = 2$ in which case $e_{k-1} = e_1$. In the latter case $e_{k-1} = e_{i+1}$. Again this contradicts the minimal choice of k provided $k-1>i+1$. Trivially we cannot have $k-1<i+1$. If $k=i+2$ then $e_{i+2}=e_k=e_i$. For this to occur either $g'_{i+1}=g_i$ or $g'_{i+1}=g'_{i-1}$. The first case is clearly impossible since $g_{i+1}\neq g'_i$. In the second case the four sides g'_{i-1}, g_i, g'_i, g_{i+1} form a closed loop which is also not possible.

In view of Lemma 9.3 there is a smallest $n>1$ for which $e_{n+1}=e_1$, that is, for which the transformation $G_1\cdots G_n$ maps e_1 onto itself. The n *words* G_1, $G_1G_2,\cdots,G_1G_2\cdots G_n$ when expressed in terms of the $2N$ letters F_1, $F_1^{-1},\cdots,F_N^{-1}$ are called *edge pairing words* corresponding to e_1. The word $G_1G_2\cdots G_n$, when expressed in the letters F_j, F_j^{-1}, is also called an *edge relation* at e_1. There is a finite set of edge pairing words and one edge relation for each edge of $\mathfrak{Q}$.

9.5. We now apply the same process to vertices (not free vertices) of $\mathfrak{Q}$. Choose a vertex v_1 and one of the least three faces g_1 ($=$ some f_j or f'_j) that share v_1. The face g'_1 ($=f'_j$ or f_j) paired with g_1 by G_1 ($=F_j$ or F_j^{-1}) contains a vertex v_2 with $G_1(v_2)=v_1$. Since G_1 is not elliptic, $v_2\neq v_1$. There are at least two choices for a face $g_2\neq g'_1$ which also contains v_2. A choice for g_2 determines the vertex v_3, etc. We thus obtain a sequence of vertices (v_1, $v_2,\cdots$) where $G_kv_{k+1}=v_k$ for a face pairing transformation G_k, for each k. The transformation $G_1\cdots G_k$ maps v_{k+1} onto v_1 and the polyhedra $\mathfrak{Q}$, $G_1(\mathfrak{Q}),\cdots$, $G_1G_2\cdots G_k(\mathfrak{Q})$ all have v_1 as a common vertex; one is adjacent to the next along a common face.

Note that if $v_i=v_j$ for some $i<j$ then the sequence $(v_i, v_{i+1},\cdots,v_{j-1})$ arises from a face pairing procedure as above starting at v_i instead of v_1. Furthermore, if we obtain a sequence $(v_1,\cdots,v_n)$ with $v_{n+1}=v_1$ then there is a least k for which there is a subset $(v_1, v'_2,\cdots,v'_k)$ such that $v'_{k+1}=v_1$ but $v'_i\neq v'_j$ for $1\leqq i<j\leqq k$. For if $v_i=v_j$ with $1\leqq i<j\leqq n$ then the shorter sequence $(v_1,\cdots,v_i,v_{j+1},\cdots,v_n)$ also arises from a face pairing procedure above. (However, we cannot assert that as transformations $G_1\cdots G_n=G_1\cdots G_iG_{j+1}\cdots G_n$.)

We will admit a sequence $(v_1,\cdots,v_n)$ only if $v_{n+1}=v_1$ and $v_i\neq v_j$ for $1\leqq i<j\leqq n$. The corresponding *words* $G_1, G_1G_2,\cdots,G_1G_2\cdots G_n$, when expressed in the letters F_j, F_j^{-1}, $1\leqq j\leqq N$, are called *vertex pairing words* for v_1. The word $G_1\cdots G_n$ in addition is called a *vertex relation* of v_1. In general there are many, but still a finite number, of admissible sequences at a given vertex v_1. Each of these sequences has a corresponding set of vertex pairing words. On the other hand, we do not assert that every vertex of $\mathfrak{Q}$ has an admissible sequence as above. However, it is clear that at least some vertices of $\mathfrak{Q}$ do.

Define the following two *finite* sets of words in the $2N$ letters F_1, $F_1^{-1}, \cdots, F_N^{-1}$.

$$\mathfrak{W}(\mathfrak{Q}) = \{\text{the set of all vertex pairing words and edge pairing words for all vertices and edges of } \mathfrak{Q}\},$$

$$\mathfrak{R}(\mathfrak{Q}) = \{\text{the set of all vertex relations and edge relations in } \mathfrak{W}(\mathfrak{Q})\}.$$

9.6. *Definition.* Two formal polyhedra $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are *equivalent* if there exists a homeomorphism $\mathfrak{Q}_1 \to \mathfrak{Q}_2$ which maps faces onto faces, edges onto edges, vertices onto vertices, such that a face named f_i (resp. f_i') of $\mathfrak{Q}_1$ is mapped onto a face of $\mathfrak{Q}_2$ which is also named f_i (resp. f_i').

Obviously if $\mathfrak{Q}_1$ is equivalent to $\mathfrak{Q}_2$ then $\mathfrak{W}(\mathfrak{Q}_1) = \mathfrak{W}(\mathfrak{Q}_2)$ and $\mathfrak{R}(\mathfrak{Q}_1) = \mathfrak{R}(\mathfrak{Q}_2)$. Also, there are only a finite number of equivalence classes of formal polyhedra. Hence we can consider the large, but still finite, sets of words in the $2N$ letters F_1, $F_1^{-1}, \cdots, F_N^{-1}$,

$$\mathfrak{W} = \bigcup \mathfrak{W}(\mathfrak{Q}), \quad \mathfrak{R} = \bigcup \mathfrak{R}(\mathfrak{Q})$$

where the unions are taken over all equivalence classes of formal polyhedra.

9.7. Now we are ready to begin the proof of Lemma 9.2. Fix a point $0 \in \mathfrak{B}$ and let $\mathcal{P}$ denote the finite-sided fundamental polyhedron for G with center 0 ($\mathcal{P}$ is closed in B^-). Set $\mathfrak{V} = \{T \in G\colon T(\mathcal{P}) \cap \mathcal{P} \neq \varnothing,\ T \neq \text{identity}\}$. Then $\mathfrak{V}$ is a finite set of transformations and $T \in \mathfrak{V}$ if and only if $T^{-1} \in \mathfrak{V}$. Hence $\mathfrak{V}$ contains an even number $2N$ of elements. Enumerate these elements: S_1, $S_1^{-1}, \cdots, S_N$, S_N^{-1}.

Now given this number N consider all formal polyhedra as in § 9.5. We will interpret $\mathcal{P}$ as one of these formal polyhedra in the following specific way. Label a pair of faces of $\mathcal{P}$ as (f_i, f_i') if and only if S_i is the transformation such that $S_i(f_i') = f_i$. Then set $F_i \equiv S_i$.

Let $\mathcal{F}$ denote the abstract free group generated by N letters, $\mathcal{F} = \langle F_1, \cdots, F_N\rangle$. Extend the correspondence

$$\sigma(F_i) = S_i, \qquad 1 \leqq i \leqq N,$$

to $\mathcal{F}$ to create a homomorphism $\mathcal{F} \to G$. A word $W \in \mathfrak{W}(\mathcal{P})$ is also in $\mathfrak{R}(\mathcal{P})$ if and only if $\sigma(W) = \text{id}$. ($\mathfrak{W}$ and $\mathfrak{R}$ are subsets of $\mathcal{F}$.)

9.8. We will now deform G. Recall first the definition

$$H(T)^0 = \{p \in \mathfrak{B}\colon d(p, 0) = d(p, T(0))\}$$

and $H(T)$ is the closure in $\mathfrak{B}^-$ of $H(T)^0$. It is important to observe that $H(T) \cap \mathcal{P} \neq \varnothing$, $T \in G$, if and only if $T \in \mathfrak{V}$. In particular, for $W \in \mathfrak{W}$, $H(\sigma(W)) \cap \mathcal{P} \neq \varnothing$ if and only if $\sigma(W) \in \mathfrak{V}$, if $\sigma(W) \neq \text{id}$.

Moreover, there exists a neighborhood U of $\mathcal{P}$ in $\mathfrak{B}^-$ for which

$$H(T) \cap U = \varnothing\ , \qquad\qquad T \neq \mathrm{id} \in G,\ T \notin \mathcal{V}.$$

Fix this neighborhood.

If φ is an ε-deformation we will consider the finite-sided polyhedron

$$\mathcal{P}'_\varphi = \{p \in \mathcal{B}\colon d(p, 0) \leqq d(p, \varphi(T)(0))\ ,\ \text{for}\ \ T = S_j^{\pm 1},\ 1 \leqq j \leqq N\},$$

and its closure $\mathcal{P}_\varphi$ in $\mathcal{B}^-$. For $\varphi = \mathrm{id}$, $\mathcal{P}_\varphi$ reduces to $\mathcal{P}$. We will consider deformations so small that $\mathcal{P}_\varphi$ looks like $\mathcal{P}$ except perhaps for additional faces arising in small neighborhoods of the vertices and edges of $\mathcal{P}$. It suffices to impose the following conditions.

The admissible ε-deformations. There exists ε_1 sufficiently small so that all ε-deformations, determined with respect to $S_1, \cdots, S_N$, with $\varepsilon < \varepsilon_1$, satisfy the following conditions:

(i) $\mathcal{P}_\varphi \subset U$.

(ii) $H(\varphi(\sigma(W))) \cap U = \varnothing$ for all words $W \in \mathcal{W}$ such that $\sigma(W) \neq \mathrm{id}$, $\sigma(W) \notin \mathcal{V}$.

(iii) (a) If there is a face of $\mathcal{P}$ in $H(T)$ there is a face of $\mathcal{P}_\varphi$ in $H(\varphi(T))$. (b) If $f \subset H(\varphi(T^{-1}))$, $T \in \mathcal{V}$, is a face of $\mathcal{P}_\varphi$ and ρ is any half open ray from 0 tending to a point on $\varphi(T)(f)$, then $\rho \cap H(\varphi(V)) = \varnothing$ for all those $V \in \mathcal{V}$ such that $V^{-1}(H(T) \cap \mathcal{P})$ is disjoint from $\mathcal{P}$.

(iv) $\varphi(S_j)$ is not elliptic, parabolic, or the identity, $1 \leqq j \leqq N$.

Note the admissible V in (iii) (b) are just those for which $H(V)$ does not meet the face, edge, or vertex $H(T) \cap \mathcal{P}$. It is possible to satisfy (i) and (iii) because $H(T)$ depends continuously on the parameters of T. For condition (ii) we also need the fact that $\mathcal{W}$ is a finite set of words.

LEMMA 9.4. *The faces of $\mathcal{P}_\varphi$ are arranged in pairs, paired by some of the transformations $\varphi(S_j)$, $1 \leqq j \leqq N$.*

Proof. Consider a face $f \subset H(\varphi(T^{-1}))$ of $\mathcal{P}_\varphi$, $T \in \mathcal{V}$, and let x be any point in the interior of f. We will show that $\varphi(T)(x)$ also lies on a face of $\mathcal{P}_\varphi$. From this it follows that $\varphi(T)(f) \subset H(\varphi(T))$ is also a face of $\mathcal{P}_\varphi$.

Let v, v' be any two vertices of $\mathcal{P}$ such that $T(v) = v'$. Let $T_1 \in \mathcal{V}$, $T_1 \neq T^{-1}$, be any transformation such that $T_1(v_1) = v$ for some vertex v_1 of $\mathcal{P}$. Obviously $T\,T_1 \in \mathcal{V}$. Conversely any $V \in \mathcal{V}$, with $V(v_1) = v'$ for some vertex $v_1 \neq v$, is of the form $T\,T_1$ for some such T_1. Note also that $v' \in H(T\,T_1)$. Similar observations hold in case there are edges e, e' of $\mathcal{P}$ with $T(e) = e'$. If $T_1 \neq T^{-1}$ maps an edge e_1 onto e then $T\,T_1 \in \mathcal{V}$ and any $V \in \mathcal{V}$ which maps an edge $\neq e$ onto e' is of this form. Of course $e' \subset H(T\,T_1)$.

Since $d(0, x) = d(0, \varphi(T)(x))$ and $d(0, x) < d(x, \varphi(V)(0))$ for all $V \in \mathcal{V}$, $V \neq T^{-1}$, we see in particular that

$$d(0, \varphi(T)(x)) < d(\varphi(T)(x), \varphi(T T_1)(0))$$

for all T_1 as above. This says that $\varphi(T)$ (x) is not separated from 0 by any of the planes $H(\varphi(T T_1))$.

Now because of (iii) (b), the ray ρ from 0 to $\varphi(T)$ (x) does not cross any planes $H(\varphi(V))$ for those $V \in \mathcal{V}$ for which V maps neither an edge nor a vertex into $H(T) \cap \mathcal{P}$. But the remaining $V \in \mathcal{V}$, $V \neq T$, are of the form $T T_1$ considered above. We have just seen that ρ cannot cross any of their planes $H(T T_1)$ either. Hence $\varphi(T)$ (x) lies on a face of $\mathcal{P}_\varphi$ as asserted.

Make $\mathcal{P}_\varphi$ into a formal polyhedron in the following way. A pair of faces of $\mathcal{P}_\varphi$ is labeled (f_i, f_i') if and only if they are paired by $\varphi(S_i)$ in the manner $\varphi(S_i)$ $(f_i') = f_i$. Then set $F_i = \varphi(S_i)$. If $\varphi = \text{id}$, $\mathcal{P}_\varphi$ reduces to the formal polyhedron $\mathcal{P}$.

Recall that $\mathcal{F}$ denotes the free group $\langle F_1, \cdots, F_N \rangle$. We have already introduced the homomorphism $\sigma: \mathcal{F} \to G$ defined by $\sigma(F_i) = S_i$. A homomorphism $\mathcal{F} \to \varphi(G)$ is determined by $\varphi \circ \sigma$: $F_i \to \varphi(S_i)$. To each word $W \in \mathcal{W}(\mathcal{P}_\varphi)$ corresponds the transformation $W_* = \varphi(\sigma(W)) \in \varphi(G)$ and this correspondence is the natural one determined by the construction of W.

9.9. Preliminary to the lemma below are the following two observations.

(a) $\varphi(\sigma(W))(\mathcal{P}_\varphi) \cap \mathcal{P}_\varphi = \varnothing$ for all $W \in \mathcal{W}$ such that $\sigma(W) \neq \text{id}$, $\sigma(W) \notin \mathcal{V}$.

(b) $\varphi(S)(\mathcal{P}_\varphi^0) \cap \mathcal{P}_\varphi^0 = \varnothing$ for all $S \in \mathcal{V}$.

The first is true because of condition (ii) in the choice of ε_1 and the second is true because of the definition of $\mathcal{P}_\varphi$.

LEMMA 9.5. *If φ is an ε-deformation of G, $\varepsilon < \varepsilon_1$ where ε_1 is given above, then $\varphi(G)$ is a discrete group and $\mathcal{P}_\varphi$ is the fundamental polyhedron for $\varphi(G)$ with center at* 0.

Proof. The proof is based on the following two facts.

(1) If $W \in \mathcal{W}(\mathcal{P}_\varphi)$ then either $\sigma(W) = \text{id}$ or $\sigma(W) \in \mathcal{V}$.

(2) If $W \in \mathcal{R}(\mathcal{P}_\varphi)$ then $\sigma(W) = \text{id}$.

The first is true because of (a) above since $W_*(\mathcal{P}_\varphi) \cap \mathcal{P}_\varphi \neq \varnothing$ where $W_* = \varphi \circ \sigma(W)$. If the second assertion is false then from (1), $\sigma(W) = S_j^{\pm 1}$ for some j and $W_* = \varphi(S_j)^{\pm 1}$. But since W_* must preserve a vertex of $\mathcal{P}_\varphi$ or map an edge onto itself, either $W_* = \text{id}$ or W_* is elliptic. Both possibilities are impossible by condition (iv) of § 9.8.

Consider an edge of $\mathcal{P}_\varphi$ and the corresponding cyclic arrangement of polyhedra about e_1; in the notation of § 9.4 this cycle is $\mathcal{P}_\varphi$, $G_1(\mathcal{P}_\varphi)$, $\cdots$, $G_1 \cdots G_n(\mathcal{P}_\varphi) = \mathcal{P}_\varphi$ (the last equality is true because of (2)). The interiors of the polyhedra of this cycle are mutually disjoint. For suppose $G_1G_2 \cdots G_i(\mathcal{P}_\varphi^0) \cap G_1G_2 \cdots G_j(\mathcal{P}_\varphi^0) \neq \emptyset$, $i < j$. Then $G_{i+1} \cdots G_j(\mathcal{P}_\varphi^0) \cap \mathcal{P}_\varphi^0 \neq \emptyset$. But as a word, $G_{i+1} \cdots G_j \in \mathcal{W}(\mathcal{P}_\varphi)$. Again as a transformation by (1), $G_{i+1} \cdots G_j = \varphi(S_k)$ or $\varphi(S_k^{-1})$ for some k. Thus we have a contradiction to (b) above.

The analogous property is true at the vertices but requires that we re-examine § 9.5 in the light of the more extensive information available on $\mathcal{P}_\varphi$ than on arbitrary formal polyhedra. The extra information we now have is that $\varphi(\sigma(W))$, $W \in \mathcal{W}(\mathcal{P}_\varphi)$, fixes a point, in particular a vertex of $\mathcal{P}_\varphi$, only if $\varphi(\sigma(W)) = \text{id}$. This implies that there is at least one admissible cycle $(v, v_2, \cdots, v_n)$ with $v_{n+1} = v$ at each vertex v; one such sequence arises from the edge cycle of an edge of $\mathcal{P}_\varphi$ containing v.

Moreover the following is true. In the notation of § 9.5 suppose $T_1 = G_1 \cdots G_i$, $T_2 = G_1' \cdots G_j'$ are two vertex pairing transformations at v arising from the sequences $(v, v_2, \cdots, v_i)$, $(v, v_2', \cdots, v_j')$ respectively. We also can write $T_1 = \varphi(\sigma(W_1))$, $T_2 = \varphi(\sigma(W_2))$ for W_1, $W_2 \in \mathcal{W}(\mathcal{P}_\varphi)$. Then

$$T_1^{-1}T_2 = \varphi(\sigma(W_1^{-1})\sigma(W_2)) = \varphi(\sigma(W_1^{-1}W_2))$$

but it is not necessarily true that $W_1^{-1}W_2 \in \mathcal{W}(\mathcal{P}_\varphi)$. However, it is true that $T_1^{-1}T_2$ is a vertex pairing transformation arising from a "reduced" version of $W_1^{-1}W_2$ in $\mathcal{W}(\mathcal{P}_\varphi)$. This reduction is done by eliminating all redundant vertices in the sequence $(v_i, v_{i-1}, \cdots, v_2, v, v_2', \cdots, v_j')$ to obtain a sequence $(v_i, v_2'', \cdots, v_{k-1}'', v_j')$ and corresponding transformation $G_1'' \cdots G_k''$. Now we use the extra information about $\mathcal{P}_\varphi$ to deduce that

$$G_1'' \cdots G_k'' = G_i^{-1} \cdots G_1^{-1}G_1' \cdots G_j' = T_1^{-1}T_2 \,.$$

Suppose now that T_1, T_2 are two vertex pairing transformations at v: $T_1(v_1) = v = T_2(v_2)$. We claim that unless $T_1 = T_2$, $T_1(\mathcal{P}_\varphi^0) \cap T_2(\mathcal{P}_\varphi^0) = \emptyset$. For we have just shown that $T_1^{-1}T_2$ is a vertex pairing transformation at v_1 if it is not the identity. This implies that $T_1^{-1}T_2 = \varphi(S_k)$ or $\varphi(S_k^{-1})$ for some k and hence $T_1^{-1}T_2(\mathcal{P}_\varphi^0) \cap \mathcal{P}_\varphi^0 = \emptyset$.

In other words we have just shown that under application of the vertex pairing transformations at a vertex v of $\mathcal{P}_\varphi$, the images of $\mathcal{P}_\varphi$ are arranged in a non-overlapping manner about v. Furthermore, it is easy to see that these images of $\mathcal{P}_\varphi$ completely cover a sufficiently small neighborhood of v in $\mathcal{B}$. We have also proved the analogous statements about edges.

9.10. All that is needed now to complete the proof of Lemma 9.5 is to apply what is sometimes known as Poincaré's theorem (see [38]). According to this theorem, since $\mathcal{P}_\varphi$ satisfies the vertex and edge conditions that we have verified above, the face pairing transformations of $\mathcal{P}_\varphi$ generate a discrete group necessarily $\varphi(G)$, which has $\mathcal{P}_\varphi$ as a fundamental polyhedron. For the readers' convenience we will sketch a proof of this as follows.

Since hyperbolic distance is preserved under Möbius transformations, it follows that the orbit of $\mathcal{P}'_\varphi = \mathcal{P}_\varphi - \mathcal{P}_\varphi \cap \partial\mathcal{B}$ under $\varphi(G)$ covers $\mathcal{B}$. Therefore, the abstract configuration $\hat{\mathcal{B}}$ resulting from successively joining images of $\mathcal{P}'_\varphi$ together as dictated by the face pairing transformations can be regarded as an unlimited covering of $\mathcal{B}$ with the natural projection map. What we have shown above is that every point on an edge or a vertex has a neighborhood in $\hat{\mathcal{B}}$ homeomorphic to a ball. The simple connectivity of $\mathcal{B}$ implies that $\hat{\mathcal{B}} = \mathcal{B}$. Therefore, $\varphi(G)$ is discrete and $\mathcal{P}_\varphi$ is a fundamental polyhedron.

We remark that the free edges and vertices of $\mathcal{P}_\varphi$ play no role in this result.

9.11. In this section we will complete the proof of Lemma 9.2 for the case that G has no parabolic transformations. In view of Lemma 9.5 it remains only to show that φ is an isomorphism for all ε-deformations φ with $\varepsilon < \varepsilon_1$. This is a consequence of the following known fact.

If H is a discrete group of Möbius transformations in $\mathcal{B}$, each relation in H is a consequence of the edge relations for a fundamental polyhedron $\mathfrak{Q}$ (which need not be finite-sided). To see this, note that a relation in H corresponds to a simple loop γ in $\mathcal{B}$ not passing through the orbit of the vertices of $\mathfrak{Q}$. If for example the relation is $F_1F_2F_3 = 1$, we might take for γ a path from C (= center of $\mathfrak{Q}$) to $F_3(C)$ to $F_2F_3(G)$ to $F_1F_2F_3(C) = C$. γ bounds a singular disk: There is a continuous map $f: \Delta \to \mathcal{B}$ of the unit disk Δ, with $f(\partial\Delta) = \gamma$ and $f(\Delta)$ transverse to all edges and not passing through any vertex in the orbit of $\mathfrak{Q}$ under H. If Γ denotes the network of edges in $\mathcal{B}$ (= orbit of edges of $\mathfrak{Q}$) then $f^{-1}(\Gamma \cap f(\Delta))$ is a finite number of points. The relation determined by γ is a consequence of the edge relations determined by small loops around these points. That is, $\partial\Delta$ (with the proper base point) is homotopic in $\Delta - f^{-1}(\Gamma \cap f(\Delta))$ to a product of loops, each surrounding only one of these points. It is of course possible that $\mathfrak{Q}$ has no edges so that each face is a full hyperbolic plane. In this case H is a free group. (For more details of this sort of analysis see [22, p. 233].)

Returning to our discrete group $\varphi(G)$, the above result can be applied as follows. The kernel N of the homomorphism $\varphi\circ\sigma: \mathcal{F} \to \varphi(G)$ is the normal subgroup generated by the words $W \in \mathcal{R}(\mathcal{P}_\varphi)$ such that W is an edge relation. If φ is not an isomorphism there is a transformation $T \in G$ such that $T \neq \text{id}$ yet $\varphi(T) = \text{id}$. Express T as a word in the elements $S_i^{\pm 1}$ of $\mathcal{V}$. Replacing

$S_i^{\pm 1}$ by $F_i^{\pm 1}$ we obtain a word $T^* \in \mathcal{F}$ such that $\sigma(T^*) = T$. But $T^* \in N$. Now fact (2) of § 9.9 implies that $\sigma(W) = \text{id}$ for all $W \in N$ since this is true of the generators. Hence $T = \sigma(T^*) = \text{id}$, a contradiction.

In other words, because every edge relation of $\mathcal{P}_\varphi$ comes from a relation in $\mathcal{R}(\mathcal{P})$, the homomorphism φ is an isomorphism.

9.12. The case in which G contains parabolic transformations. A number of small modifications are necessary so that the preceding proof also applies to the general case. These are as follows.

1. Choose the center 0 of the fundamental polyhedron $\mathcal{P}$ of G so that a parabolic fixed point $p \in \mathcal{P}$ only if p is a cusp point of an open or closed cusp of $\mathcal{P}$ (in the terminology of § 9.2). The associated parabolic transformation(s) are to be generators of M_p. This choice of 0 is possible by Lemma 4.1.

Suppose $\mathcal{P}$ has r open cusps and s closed cusps.

2. The definition of formal polyhedra in § 9.4 must be revised as follows. Each formal polyhedron $\mathcal{Q}$ must have r open cusps and s closed cusps. As before, the faces of $\mathcal{Q}$ are labeled from the letters $f_1, \cdots, f'_N$ but now we make the convention that the letters $f_{N-r-2s+1}, \cdots, f'_{N-2s}$ be used for the faces of the open cusps, and $f_{N-2s+1}, \cdots, f'_N$ for the faces of closed cusps. In the latter case each sequence of four starting with $(f_{N-2s+1}, \cdots, f'_{N-2s+2})$ is to be used for one closed cusp. Of course the parabolic transformations associated with the cusps are also to be face pairing transformations of $\mathcal{Q}$.

The condition to replace (iii) of § 9.4 is that no face pairing transformation F_i of $\mathcal{Q}$ is elliptic or parabolic, except those associated with a cusp.

3. Let $\mathcal{P}^+$ denote the result of deleting the cusp points from $\mathcal{P}$. Then define (cf. § 9.7)

$$\mathcal{V} = \{T \in G\colon T(\mathcal{P}^+) \cap \mathcal{P}^+ \neq \varnothing,\ T \neq \text{id}\}$$

and correspondingly enumerate the elements of $\mathcal{V}$: $S_1, \cdots, S_N^{-1}$ so that $S_{N-r-2s+1}, \cdots, S_{N-2s}$ pair the faces of the open cusps and $S_{N-2s+1}, \cdots, S_N$ pair the faces of the closed cusps. Each sequence of two starting with (S_{N-2s+1}, S_{N-2s+2}) is associated with a single cusp.

4. The changes required in § 9.8 to determine the allowable deformations are as follows. First suppose that $\mathcal{Q}$ is a formal polyhedron and $p \in \mathcal{Q}$ is a cusp point. A *cusp neighborhood* $\beta(p)$ for p is an open ball in $\mathcal{B}$, internally tangent to $\partial\mathcal{B}$ at p, so small that $\beta(p)^-$ intersects only those faces of $\mathcal{Q}$ that are cusp faces at p. A *cusp neighborhood* β *for* $\mathcal{Q}$ is the union of cusp neighborhoods, one for each cusp point.

Now recall from Lemma 4.1 that for each cusp point p of $\mathcal{P}$,

$$H(T) \cap \{p\} = \varnothing\ , \qquad \text{all } T \in G,\ T \notin M_p. \tag{1}$$

Therefore, there exists a neighborhood U^+ of $\mathcal{P}^+$ such that

$$H(T) \cap U^+ = \varnothing \,, \qquad \text{all } T \neq \text{id in } G,\ T \notin \mathcal{V}. \tag{2}$$

There exists $\varepsilon_0 > 0$ so small that any ε-deformation φ with $\varepsilon < \varepsilon_0$ satisfies the following conditions (cf. (i)–(iv) of § 9.8).

$(\mathrm{i})_1^+$ $\mathcal{P}_\varphi$ has r open and s closed cusps, the associated parabolic transformations being the image under φ of the corresponding ones in $\mathcal{V}$.

$(\mathrm{i})_2^+$ For each cusp point q of $\mathcal{P}_\varphi$, $H(\varphi\sigma(W)) \cap \{q\} = \varnothing$ for all $W \in \mathcal{W}(\mathcal{P}_\varphi)$ such that $\sigma(W) \notin M_p$, for the corresponding cusp point p of $\mathcal{P}$.

$(\mathrm{i})_3^+$ There exists a cusp neighborhood β_φ for $\mathcal{P}_\varphi$ such that $\mathcal{P}_\varphi - \mathcal{P}_\varphi \cap \beta_\varphi \subset U^+$.

$(\mathrm{ii})^+$ $H(\varphi\sigma(W)) \cap U^+ = \varnothing$ for all words $W \in \mathcal{W}$ such that $\sigma(W) \neq \text{id}$, $\sigma(W) \notin \mathcal{V}$, and $\sigma(W) \notin M_p$ for all cusp points p of $\mathcal{P}$.

$(\mathrm{iii})^+$ This is the same as (iii) of § 9.8.

$(\mathrm{iv})^+$ $\varphi(S_j)$ is not elliptic, parabolic, or the identity, $1 \leqq j \leqq N - r - 2s$.

These conditions can be satisfied using (1) and (2) becasue $\mathcal{V}$ and $\mathcal{W}$ are finite sets. An immediate consequence of $(\mathrm{i})_2^+$ is that if $W \in \mathcal{W}(\mathcal{P}_\varphi)$ then $\varphi\sigma(W)$, the element of $\varphi(G)$ determined by W, cannot fix a cusp point of $\mathcal{P}_\varphi$ unless it pairs two faces or edges of that cusp. With this fact in mind the proof proceeds exactly as in the more restrictive case already discussed.

9.13. Proposition 9.1 will be deduced from Lemma 9.2 by applying Theorem 8.1. To do this we have to analyze the relation of $\partial\mathfrak{M}(\varphi(G))$ to $\partial\mathfrak{M}(G)$ for an ε-deformation φ (small ε). First we will show that $\partial\mathfrak{M}(\varphi(G))$ is homeomorphic to $\partial\mathfrak{M}(G)$. We will begin the necessary analysis by taking note of the structure of $P = \mathcal{P} \cap \partial\mathcal{B}$, where $\mathcal{P}$ is the fixed fundamental polyhedron for G.

P is the union of a finite number of finite-sided circular polygons and a finite number of isolated points. Each of these is either a cusp point of a closed cusp of $\mathcal{P}$ or is what will be called an *isolated vertex* of P. In addition a vertex v of P may be a *tangent vertex*. This is the case that two faces of $\mathcal{P}$ are tangent at v; however, v may or may not be an open cusp point of $\mathcal{P}$. In P we will interpret this situation to be that of two sides s_1, s_2 of P being tangent at v.

The sides of P are arranged in pairs but more than one pair of sides may be associated with a given side-pairing transformation (a face of $\mathcal{P}$ may determine more than one side of P). The side-pairing transformations of P are the restrictions to $\partial\mathcal{B}$ of the relevant face-pairing transformations of $\mathcal{P}$. If v is a tangent vertex for the sides s_1, s_2 then the image v_1 on the side s_1' paired with s_1 may or may not also be a vertex of P. If v is a cusp point then $s_1' = s_2$ and there is nothing more to be said. Otherwise, if v_1 is also a vertex (necessarily a tangent vertex), find v_2, etc. Ultimately, since $\pi(v)$ is an interior point of $\partial\mathfrak{M}(G)$, we must reach a point v_n which is not a vertex of P. However, it will be convenient in what follows to regard such a point v_n as a vertex too. That is, any point on P equivalent under G to a vertex of P will also be called a vertex.

$P_\varphi = \mathcal{P}_\varphi \cap \partial\mathcal{B}$ is similar to P allowing for the following possibilities: (a) a polygon which is a component of P may gain additional sides near a vertex, (b) an isolated vertex of P may disappear, (c) instead of an isolated vertex a small polygon may appear, or (d) a tangent vertex may disappear or be replaced by ordinary vertices. It is helpful to keep these possibilities in mind. Rather than show how each of these changes can arise topologically in $\partial\mathfrak{M}(G)$, we will proceed as follows.

9.14. Consider the vertices v of P which are not cusp points of $\mathcal{P}$. Fix open euclidean disks $D(v)$, $D_1(v)$ centered at v such that (a) $D(v)^- \subset D_1(v)$, (b) the disks $\{D_1(v)\}$ for all v are mutually disjoint, and (c) if $v_1 \neq v_2$ then $T(D_1(v_1)) \cap D_1(v_2) = \emptyset$ for all $T \in G$ unless $T(v_1) = v_2$ in which case $D_1(v_2) = T(D_1(v_1))$. In other words, $D_1(v)$ is the lift of a small neighborhood of $\pi(v) \in \partial\mathfrak{M}(G)$.

Choose $\varepsilon_2 < \varepsilon_0$ (ε_0 as chosen in § 9.12) so that all ε-deformations φ with $\varepsilon < \varepsilon_2$ satisfy

(i) $\varphi(T)(D_1(v_1)) \cap D_1(v_2) = \emptyset$ for all $T \in \mathcal{V}$ (cf. § 9.7) with $T(v_1) \neq v_2$,

(ii) a side of P extending between $D(v_1)$ and $D(v_2)$ determines a side of P_φ which also extends between $D(v_1)$ and $D(v_2)$,

(iii) a side of P_φ which does not arise from a side of P lies in some disk $D(v)$.

Since each side pairing transformation of P_φ is $\varphi(T)$ for some $T \in \mathcal{V}$, we see these conditions can be satisfied.

If $V(v_1') = v_2'$ for (non-cusp) vertices $v_1' \neq v_2'$ of P_φ, $V \in \varphi(G)$, then we know that $V = \varphi(T)$ for some $T \in \mathcal{V}$. Moreover, it is important to observe that if $v_1' \in D(v_1)$ and $v_2' \in D(v_2)$ it follows from (i) that $T(v_1) = v_2$ and $v_1 \neq v_2$.

Fix a vertex of P which is not an isolated or tangent vertex or equivalent to a tangent vertex. There is a cyclic arrangement of images of P about v and a corresponding sequence of transformations $G_1, G_1, G_2, \cdots, G_1G_2 \cdots G_n =$ id, where each G_i is a side pairing transformation of P. Suppose φ is an ε-deformation with $\varepsilon < \varepsilon_2$ and v' is any vertex of P_φ which lies in $D(v)$. Since each $\varphi(G_i)$ is a side pairing transformation of P_φ, the sequence

$$P_\varphi, \varphi(G_1)(P_\varphi), \cdots, \varphi(G_1 \cdots G_n)(P_\varphi) = P_\varphi$$

is a cyclic arrangement of distinct images of P_φ, each adjacent to the previous and to the succeeding region along a common side. Let $Q_\varphi(v)$ denote the union of these n regions. We have to make one more restriction on ε_2 as follows: ε_2 must be so small that for all ε-deformations φ with $\varepsilon < \varepsilon_2$,

(iv) $Q_\varphi(v)$ covers $D_1(v) - D(v)$ for all vertices of P not isolated or tangent vertices. The exceptional cases will be included after the discussion below.

Note that (iv) is the extension of (ii) to the polygons in the cycle at v.

We claim that $Q_\varphi(v)$ must in fact cover $D(v)$ as well as $D_1(v) - D(v)$. If this is not the case there is a point $v' \in D(v)$, $v' \in \partial Q_\varphi(v)$, such that v' is a vertex of $Q_\varphi(v)$. By replacing $Q_\varphi(v)$ by $Q_\varphi\big((G_1 \cdots G_k)^{-1}(v)\big)$ for a suitable k if necessary, we can assume that v' is a vertex of P_φ. There is a cyclic arrangement of images of P_φ about v' and a corresponding sequence of transformations $V_1, V_1V_2, \cdots, V_1V_2 \cdots V_m = \mathrm{id}$, where each V_j is a side pairing transformation of P_φ and $V_1 \cdots V_k$ maps a vertex $v'_k \neq v'$ of P_φ onto v', $k < m$. But by (i), $V_1 \cdots V_k = \varphi(T)$ for some $T \in \mathcal{V}$ and moreover if $v'_k \in D(v_1)$, $T(v_1) = v$. This implies $T = G_1 \cdots G_j$ for some j. But then $V_1 \cdots V_k(P_\varphi)$ is already contained in $Q_\varphi(v)$. Since this is true for all k, we have a contradiction.

An isolated vertex v of P has a cycle about it and there is an analogous $Q_\varphi(v)$. Since there is a $T \in \mathcal{V}$ such that $T(v)$ is an ordinary vertex of P, the proof above shows that for small enough ε_2, $Q_\varphi(\pi)$ covers $D_1(v) \supset D(v)$.

On the other hand, if v is a tangent vertex of P, but not a cusp point, there are two vertices w_1, w_2 which are interior points of the sides s_1, s_2 of P which are equivalent to v by transformations in $\mathcal{V}$. It suffices to examine the situation at w_1. There is a relation of the form $G_1 \cdots G_n T = \mathrm{id}$ where each G_i is a side pairing transformation of P and $T(w_1) = w_2$, $T(s_1) = s_2$. Each $G_1 \cdots G_i$, $i < n$, maps a tangent vertex of P to w_1. For sufficiently small ε_2, the analysis of $Q_\varphi(w_1)$ and hence $Q_\varphi(v)$ proceeds as before so that $Q_\varphi(v)$ covers not only $D_1(v) - D(v)$ but also $D(v)$.

The fact that the cycles about the vertices of P_φ cover the disks $D(v)$ together with property (i) of ε_2 imply the following statement. For any ε-deformation φ with $\varepsilon < \varepsilon_2$ the natural projection π_φ of $D(v)$ into $\partial\mathfrak{M}\big(\varphi(G)\big)$ is injective.

9.15. Let P^+ denote the result of removing the cusp points and isolated vertices from P. The orbit Ω^+ of P^+ under the subgroup G^+ of G generated by the side pairing transformations of P^+ is a union of components of $\Omega(G)$ and has the property that $\Omega^+/G^+ = \partial\mathfrak{M}(G)$. The complement in Ω^+ of the orbit of $R = P^+ - P^+ \cap \big(\bigcup_v D(v)\big)$ under G^+ is a countable union of mutually disjoint disks.

Compare R and $R_\varphi = P_\varphi^+ - P_\varphi^+ \cap (\bigcup_v D(v))$, where P_φ^+ is obtained from P_φ by removing cusp points and isolated vertices. Each component of R is a finite-sided circular polygon and is homeomorphic as a polygon (sides correspond to sides) to a component of R_φ which is close to it. This is true because of properties (ii) and (iii) of φ. Furthermore, this homeomorphism $R \to R_\varphi$ induces a 1-1 correspondence between side pairing transformations. In fact we can choose the homeomorphism $f: R \to R_\varphi$ so that if S is a side pairing transformation of P and p, $S(p)$ are points of R then $f(S(p)) = \varphi(S)(f(p))$.

Let Ω_φ^+ be the union of those components Ω' of $\Omega(\varphi(G))$ for which $T(R_\varphi) \subset \Omega'$ for some $T \in \varphi(G^+)$. We claim that f extends to a homeomorphism $\Omega^+ \to \Omega_\varphi^+$. It is clear that the isomorphism $\varphi: G^+ \to \varphi(G^+)$ determines an extension, again called f, to a homeomorphism between the orbits $G^+(R) \to \varphi(G^+)(R_\varphi)$. Each component D of $\Omega^+ - G^+(R)$ is a disk and the circle ∂D is the union of simple arcs, one each on say $T_1(R), \cdots, T_k(R)$, for $T_j \in G^+$, $1 \leqq j \leqq k$. Since the natural projection $D \to \partial\mathfrak{M}(\varphi(G))$ is injective (as observed above), D is also a component of $\Omega_\varphi^+ - \varphi(G^+)(R_\varphi)$. Our restrictions on ε which force R_φ to be close to R and the analysis in § 9.14 show that ∂D is also the union of simple arcs on $\varphi(T_1)(R_\varphi), \cdots, \varphi(T_k)(R_\varphi)$ with the order and orientation the same as when R is used. Thus we see f can be extended to D, and since D is arbitrary, to all Ω^+.

Finally f is extended to all $\Omega(G)$ as follows. G is a countable union of right cosets of G^+. Each coset G^+T can be identified with $T(\Omega^+)$ since $T(\Omega^+) \cap \Omega^+ = \emptyset$ unless $T \in G^+$ (otherwise $T(P) = T_1(P)$ for some $T_1 \in G^+$). Similarly the distinct right cosets of $\varphi(G^+)$ are in 1-1 correspondence with disjoint regions $\varphi(T)(\Omega_\varphi^+)$. Consequently by the use of φ, f can be extended to a homeomorphism $\Omega(G) \to \Omega(\varphi(G))$.

The conditions of Theorem 8.1 are now all fulfilled and Proposition 9.1 follows at once.

9.16. The methods developed above can also be used to prove

COROLLARY 9.5. *Let G be a discrete subgroup of* SL(2, C)/$\pm$ 1 *which has a finite-sided fundamental polyhedron of finite volume. Then there exists $\varepsilon_0 > 0$ such that for every ε-deformation φ with $\varepsilon < \varepsilon_0$, the homomorphism $G \to \varphi(G)$ is induced by a quasiconformal homeomorphism $\mathfrak{M}(G) \to \mathfrak{M}(\varphi(G))$.*

Our proof of Proposition 9.1 shows that for some ε_0, φ is an isomorphism for each $\varepsilon < \varepsilon_0$. If for example $H_1(\mathfrak{M}(G))$ is not a finite group then Waldhausen's theorem [51] implies that $\mathfrak{M}(G)$ and $\mathfrak{M}(\varphi(G))$ are quasiconformally equivalent. If however $H_1(\mathfrak{M}(G))$ is finite the quasiconformal map must be explicitly constructed. However, the proof following the lines introduced for Proposition 9.1 works in all cases. It is based on the fact that $\mathcal{P}_\varphi$ of § 9.8 is geometrically the same as $\mathcal{P}$ except in a small neighborhood of the edges. This implies that $\mathfrak{M}(\varphi(G))$ is homeomorphic to $\mathfrak{M}(G)$, using $\mathcal{P}$ and $\mathcal{P}_\varphi$ to construct the homeomorphism. We will omit the details since Corollary 9.5 is also a special case of Theorem 8.3 once it is known that φ is an isomorphism.

Remark. In the case that $\mathfrak{M}(G)$ is compact the assertion that φ is an isomorphism is a special case of a much more general result contained in A. Weil's classic papers [52]. From a somewhat different point of view it is also contained in Macbeath's paper [23]. Of course Theorem 8.3 implies that φ is in fact an inner automorphism. But for small deformations, this fact too is contained in [52]. Weil's work approaches the subject from the point of view of cohomology.

10. The deformation space

10.1. Terminology: A K-quasiconformal deformation f of the kleinian group G is a K-quasiconformal homeomorphism of $\partial\mathfrak{B}$ that induces an isomorphism φ of G onto another kleinian group. That is, the correspondence $\varphi: T \to fTf^{-1}$, $T \in G$, is an isomorphism between kleinian groups.

Definition. The kleinian group G is *strongly stable* if (i) given $K > 1$ there exists $\varepsilon_0 > 0$ such that every ε-deformation φ of G with $\varepsilon < \varepsilon_0$ is induced by a K-quasiconformal deformation of G, and (ii) each quasiconformal deformation H of G also satisfies (i).

THEOREM 10.1. *If G has a finite-sided fundamental polyhedron then G is strongly stable.*

Proof. We have already proved (Proposition 9.1) that for ε sufficiently small each ε-deformation φ is induced by a quasiconformal deformation g_φ^* of G. We need only show that for ε small enough g_φ^* can be taken with maximal dilatation $< K$. Now the projection g_φ of g_φ^* to $\partial\mathfrak{M}(G)$ is a quasiconformal map $\partial\mathfrak{M}(G) \to \partial\mathfrak{M}(\varphi(G))$. Let S be a component of $\partial\mathfrak{M}(G)$. It is enough to show that for all sufficiently small ε there is a K-quasiconformal map $f_\varphi: S \to S_\varphi = g_\varphi(S)$ which is homotopic to g_φ. For the result of replacing g_φ^* by the corresponding lift f_φ^* of f_φ in each component of $\Omega(G)$ over S again yields a quasiconformal homeomorphism of $\partial\mathfrak{B}$ (Theorem 8.1). And then, using the fact that the limit set $\Lambda(G)$ has 2-dimensional measure zero, when this process is completed for all components S of $\partial\mathfrak{M}(G)$ we obtain a K-quasiconformal deformation of G.

Furthermore, if we can verify condition (i) of the definition of strong stability then (ii) follows automatically. For G has a finite-sided fundamental polyhedron if and only if fGf^{-1} does, where f is a quasiconformal deformation of G.

Thus consider a component Ω_1 of $\Omega(G)$ over a component S of $\partial\mathfrak{M}(G)$ and the subgroup H of G that keeps Ω_1 invariant. In view of what has been said above, Theorem 10.1 will follow if, in particular, we can prove that the function group H is itself strongly stable. For Ω_1 is an arbitrary component of $\Omega(G)$ and the initial ε_0 obtained needs to be reduced at most once for each component of $\partial\mathfrak{M}(G)$. In applying what follows it is necessary to recall that Corollary 6.5 says that the subgroup H also has a finite-sided fundamental polyhedron.

10.2. LEMMA 10.2. *Suppose H is a B-group with a finite-sided fundamental polyhedron. Then H is strongly stable.*

Proof. As usual, $\Omega_0(H)$ denotes an invariant component of $\Omega(H)$. We have proved that for ε_0 sufficiently small, there exists a quasiconformal map $g\colon \Omega(H) \to \Omega(\varphi(H))$ which induces φ. Set $\Omega_0(\varphi(H)) = g(\Omega_0(H))$. In view of the discussion in § 10.1 it suffices to show that for sufficiently small ε_0, there is a K-quasiconformal map $f\colon \Omega_0(H) \to \Omega_0(\varphi(H))$ which induces φ. For if we can show this, the argument can be repeated for each component $\neq \Omega_0(H)$ of $\Omega(H)$ and the corresponding quasi-fuchsian subgroup of H.

Before continuing the proof, we will consider the following lemma.

LEMMA 10.3. *Given a compact set ω in $\Omega_0(H)$ there exists $\varepsilon_0 > 0$ so small that $\omega \subset \Omega_0(\varphi(H))$ for all ε-deformations φ with $\varepsilon < \varepsilon_0$.*

Proof. If $\mathcal{P}$ is a fundamental polyhedron for H, set $P = \mathcal{P} \cap \Omega_0(H)$. Then ω can be covered by a finite number of images $\{P_j\}$ of P under H. We see from our analysis in § 9.14 that P is covered by the union of $P_\varphi = \mathcal{P}_\varphi \cap \Omega_0(\varphi(H))$ and those translates of P_φ under $\varphi(H)$ that adjoin P_φ, for all sufficiently small ε-deformations φ. The same is true of each P_j and we choose ε_0 so small as to serve for all j. This proves Lemma 10.3.

10.3. Continuing the proof of Lemma 10.2, in the sense of Lemma 10.3, $\lim \Omega_0(\varphi(H)) = \Omega_0(H)$ as $\varepsilon_0 \to 0$. This is the kind of convergence required to apply the Carathéodory convergence theorem. The application of this theorem is as follows.

For sufficiently small ε_0, we can find a point $\zeta \in \Omega_0(H)$ such that $\zeta \in \Omega_0(\varphi(H))$ for all ε-deformations φ with $\varepsilon < \varepsilon_0$. Let f_φ be the conformal map $\Omega_0(\varphi(H)) \to \Delta = \{z\colon |z| < 1\}$ normalized by $f_\varphi(\zeta) = 0$, $f'_\varphi(\zeta) > 0$; the corresponding conformal map for $\Omega_0(H)$ is denoted by f. Each fuchsian group $F_\varphi = f_\varphi \varphi(H) f_\varphi^{-1}$ is isomorphic to $F = fHf^{-1}$ under the correspondence

$$\psi(A) = f_\varphi \varphi(f^{-1}Af)f_\varphi^{-1}, \qquad A \in F.$$

Clearly $\psi(T)$ is parabolic if and only if T is, since these T are induced by punctures on $\Omega_0(H)/H$. Moreover, $Q = f(P)$ and $Q_\varphi = f_\varphi(P_\varphi)$ are fundamental sets for F and F_φ respectively in Δ.

Suppose then ω is a given compact set of $\Omega_0(H)$ which contains P in its interior ω^0, except for small neighborhoods of its cusps. For sufficiently small ε_0, ω^0 also contains P_φ, except for small neighborhoods of its cusps. It follows from the Carathéodory convergence theorem that given ε_1, ε_0 can be chosen so small that

$$\begin{gathered} |f_\varphi(z) - f(z)| < \varepsilon_1, \ \ |f'_\varphi(z) - f'(z)| < \varepsilon_1, \\ |f_\varphi^{-1}(w) - f^{-1}(w)| < \varepsilon_1, \ \ |(f_\varphi^{-1})'(w) - (f^{-1})'(w)| < \varepsilon_1 \end{gathered} \tag{1}$$

for all $z \in \omega$, $w \in f(\omega)$, and all ε-deformations φ with $\varepsilon < \varepsilon_0$.

F is generated by the side pairing transformations $A_1, \cdots, A_N$ of Q, the images under f of the side pairing transformations of P (which, it is easily seen, generate H). Using the set $\{A_i\}$ as generators consider the ε-deformations of F as a fuchsian group (essentially homomorphisms into $\mathrm{SL}(2, \mathbf{R})/\pm 1$). It is known [7], [8] that $\varepsilon_2 > 0$ can be chosen so small that every ε-deformation of F with $\varepsilon < \varepsilon_2$ is induced by a K-quasiconformal homeomorphism of Δ onto itself. We will show below that for small enough ε_0, for every ε-deformation φ with $\varepsilon < \varepsilon_0$, the isomorphism $\psi: F \to F_\varphi$ is an ε-deformation of F as a fuchsian group with $\varepsilon < \varepsilon_2$. Assuming this, if ψ is induced by the K-quasiconformal map h, then φ is induced by the K-quasiconformal map $f_\varphi h f^{-1}: \Omega_0(H) \to \Omega_0(\varphi(H))$.

To prove our assertion above, recall that given p, $q \in \Delta$ and $0 \leqq \Theta < 2\pi$, a Möbius transformation T (which preserves Δ) is uniquely determined by the requirements $T(p) = q$ and $\arg T'(p) = \theta$. Furthermore, T depends continuously on the parameters p, q and $\exp(i\theta)$. In particular given $p \in \Delta$ and a Möbius transformation A, there is a neighborhood N of p and a number $\mathfrak{p}$ such that any Möbius transformation T which satisfies

$$T(N) \cap A(N) \neq \varnothing \ , \ \ |T'(p) - A'(p)| < \mathfrak{p} \tag{2}$$

also satisfies (in the matrix sense) $|T - A| < \varepsilon_2$.

We will apply this fact to show that ε_0 may be chosen so small that $|\psi(A_i) - A_i| < \varepsilon_2$, $1 \leqq i \leqq N$. Consider for example $A = A_1$ and choose $p \in f(\omega^0)$ so that $A(p) \in f(\omega^0)$. Find a neighborhood $N \subset \omega^0$ (and $A(N) \subset \omega^0$) about p and a number $\mathfrak{p}$ which fulfill the conditions above with respect to A. Examine the expressions

$$\psi(A) = f_\varphi \varphi(B) f_\varphi^{-1}, \quad B = f^{-1}Af ,$$
$$\psi(A)'(p) = [f_\varphi'(\varphi(B))(f_\varphi^{-1}(p))][\varphi(B)'(f_\varphi^{-1}(p))][(f_\varphi^{-1})'(p)] .$$

In view of (1) it is clear that for ε_1, and ultimately ε_0, small enough, the element $T = \psi(A)$ will satisfy (2). This completes the proof of Lemma 10.2.

10.4. In order to complete the proof of Theorem 10.1 we have to make use of the following object.

Definition 10.4. Suppose G is a kleinian group. The *deformation variety* $V(G)$ for G is the algebraic variety constructed as follows. Describe G group-theoretically in terms of its generators $A_1, \cdots, A_N$ and relations

(1) $$W_i(A_1, \cdots, A_N) = \text{id}, \qquad i = 1, 2, \cdots$$

where each W_i is a word in the A_j, $1 \leqq j \leqq N$. Furthermore, there are conditions involving the trace

(2) $$\text{tr}^2 W_j^*(A_1, \cdots, A_N) = 4 , \qquad j = 1, 2, \cdots$$

that arise from the parabolic elements of G. The coefficients in a matrix representation of the A_i, $1 \leqq i \leqq N$, can be interpreted as the homogeneous coordinates of a point in the product complex projective 3-space $\mathbf{P}_3(\mathbf{C})^N$. As we range through every ordered set of N normalized matrices $\{B_i\}$ whose entries satisfy (1) and (2) we describe a Zariski open subset $V(G)$ of an algebraic variety in $\mathbf{P}_3^N$.

The points of $V(G)$ are in 1-1 correspondence with homomorphisms $\varphi: G \rightarrow \text{SL}(2, \text{C})/\pm 1$ which send parabolic transformations of G to parabolic transformations. For this reason the notation $\text{Hom}_a(G, \text{SL}(2, \text{C})/\pm 1)$ is also used for $V(G)$.

$V(G)$ is said to be *locally irreducible* at a point $x \in V(G)$ if x has a neighborhood in $V(G)$ which is manifold.

10.5. LEMMA 10.5. *Suppose H is a function group which is not a B-group. If H has a finite-sided fundamental polyhedron, then H is strongly stable.*

Proof. According to Bers [8, Props. 1, 2], a necessary and sufficient condition for the uniform stability ($\equiv$ condition (i) in the definition of strong stability) of a kleinian group G is that $V(G)$ be locally irreducible at the point corresponding to G and have there complex dimension $\sigma(G) + 3$. Here $\sigma(G)$ is the complex dimension of the space of cusp forms for G (cf. [8]). We will prove Lemma 10.5 by applying this result.

The first step is to decompose H as dictated by Proposition 5.8 to obtain

$$H = H_1 * \cdots * H_m * \mathfrak{A}_1 * \cdots * \mathfrak{A}_r * \{A_1\} * \cdots * \{A_s\} * \{T_1\} * \cdots * \{T_n\}$$

where each H_i is a B-group with a finite-sided fundamental polyhedron, $\mathfrak{A}_j$ is a free abelian group of rank two with two parabolic generators, $\{A_k\}$ is the cyclic group generated by the parabolic A_k, and $\{T_l\}$ is the cyclic group generated by the loxodromic transformation T_l. By Lemma 10.2 and Bers' criterion, $V(H_i)$ is locally irreducible at H_i for each i. The varieties $V(\mathfrak{A}_j)$, $V(\{A_k\})$, $V(\{T_l\})$ are all manifolds. The deformation variety $V(H)$ is the product of these individual varieties and therefore is locally irreducible at H. Lemma 10.4 will follow once we prove that $\dim_C V(H) = \sigma(H) + 3$.

Now $\dim V(\mathfrak{A}_j) = 3$ since the two parabolic generators have a common fixed point, and $\dim V(\{A_k\}) = 2$, $\dim V(\{T_l\}) = 3$. Therefore, from the fact that $\dim V(H_i) = \sigma(H_i) + 3$ using § 5.12 (1) we find, in the notation of § 5.12 except introducing the subscript i for H_i,

$$\dim V(H) = \textstyle\sum_{i=1}^{m} (6g_i + 2b_i - a_i - 3) + 3n + 3r + 2s \; .$$

But if g is the genus of $\Omega_0(H)/H$ and b is the number of punctures, then

$$g = \textstyle\sum_{i=1}^{m} g_i + r + n \; , \quad b = \textstyle\sum_{i=1}^{m} b_i + 2s \; .$$

Consequently if $a = \Sigma a_i$,

$$\dim V(H) = 6g + 2b - 3(r + n + m) - 2s - a \; .$$

On the other hand referring back to § 5.14 (2) we find

$$\sigma(H) = \dim V(H) - 3 \; .$$

This proves Lemma 10.5 and completes the proof of Theorem 10.1.

10.6. The main application of Theorem 10.1 is to the theory of moduli.

Definition 10.6. If G is a kleinian group, the deformation space $T(G)$ for G is the set of all pairs $(h, \mathfrak{M}(H))$, where $h: \mathfrak{M}(G) \to \mathfrak{M}(H)$ is a homeomorphism, with the identification $(h_1, \mathfrak{M}(H_1)) = (h_2, \mathfrak{M}(H_2))$ if and only if $h_1 h_2^{-1}: \mathfrak{M}(H_2) \to \mathfrak{M}(H_1)$ is homotopic to a conformal map.

Bers defines $T(G)$ as the set of equivalence classes of isomorphisms $\psi: G \to \psi(G) \in \mathbf{M} \equiv \mathrm{SL}(2, \mathbf{C})/\pm 1$ which are induced by quasiconformal homeomorphisms of $\partial\mathfrak{B}$. Two isomorphisms are called equivalent if they differ by an inner automorphism of $\mathbf{M}$.

LEMMA 10.7. *If G has a finite-sided fundamental polyhedron, the Definition* 10.6 *of $T(G)$ is equivalent to Bers' definition.*

Proof. A homeomorphism $h: \mathfrak{M}(G) \to \mathfrak{M}(H)$ is homotopic to a PL homeomorphism h_1. By the method of Proposition 3.4, h_1 is homotopic to a quasiconformal map h_2 which is conformal or quasiconformal outside a compact set (i.e., conformal in the solid cusp tori, quasiconformal in the pairing tubes). A lift of $h_2|\mathfrak{M}(G)^0$ to $\mathcal{B}$ is quasiconformal and consequently has a quasiconformal extension to $\partial\mathcal{B}$ (Gehring's theorem).

The converse is a consequence of Theorem 8.1.

Recall that the dimension $\sigma(G)$ of the complex vector space of cusp forms for G is

$$\sigma(G) = \Sigma(3g_i + b_i - 3)$$

where g_i is the genus of the i^{th} component of $\partial\mathfrak{M}(G)$ and b_i is the number of its punctures.

On combining Theorem 10.1 with some results of Bers we obtain

THEOREM 10.8. *Assume G is a kleinian group with a finite-sided fundamental polyhedron. Then $T(G)$ is a $\sigma(G)$-dimensional complex analytic manifold and there is a canonical holomorphic bijection of $T(G) \times$* **M** *onto an open subregion of the deformation variety $V(G)$ (defined in* § 10.4).

Other definitions of $T(G)$ are due to Maskit [37] and quite recently Kra [21]. Their theories are more general in that they apply to all kleinian groups and show that $T(G)$ is always a complex analytic manifold. However, the price paid is that these results do not say how $T(G)$ appears in the deformation variety $V(G)$. For further details of this work which show in particular how $T(G)$ is related to the Teichmüller spaces of the individual components of $\partial\mathfrak{M}(G)$, we must refer to the references cited above.

11. Composition of groups

11.1. In a series of papers [31, 32, 35], Maskit has presented some procedures for building up kleinian groups from simpler ones. These results extend the original Klein combination theorems. It is interesting to express his techniques in terms of 3-manifolds. We will not seek the most general formulation but will be content with the most practical, namely, combination techniques involving circles.

Suppose G is a kleinian group (without torsion) and π is the natural map $\pi\colon \mathcal{B} \cup \Omega(G) \to \mathfrak{M}(G)$. An oriented hyperbolic plane h divides $\mathcal{B}$ into two parts which we distinguish as the right $R(h)$ and left $L(h)$ sides of h. $R(h) \cap T(R(h)) = \varnothing$ for all $T \in G$ if and only if $\pi\colon R(h) \to \mathfrak{M}(G)$ is injective. In this case $\pi(h)$ is an open disk in $\mathfrak{M}(G)$. In the case $\partial h \subset \Omega(G)$, $\pi(\partial h)$ bounds a disk in $\partial\mathfrak{M}(G)$ and $\pi(R(h))$ will be referred to as a *trivial ball*.

If a parabolic $A \in G$ with fixed point p is induced by a puncture on $\partial\mathfrak{M}(G)$, we can find a plane h with $p \in \partial h \subset \Omega(G) \cup \{p\}$ so small that for a suitable orientation of h, $R(h) \cap T(R(h)) = \varnothing$ for all $T \in G$, $T \notin M_p$ (the maximal parabolic subgroup corresponding to p). If $T \in M_p$ we require $T(R(h)) = R(h)$. Then $\pi(h)^-$ is a half-infinite cylinder in $\mathfrak{M}(G)$; $R(h)$ is referred to as an *invariant ball* corresponding to M_p.

If a component Ω_1 of $\Omega(G)$ is a euclidean disk on $\partial\mathcal{B}$, we can erect a plane h on $\partial\Omega_1$. One of the regions, say $R(h)$, determined by h has the property that $T(R(h)) \cap R(h) = \varnothing$ unless $T \in G_1$, the subgroup that preserves Ω_1 (there are two choices for $R(h)$ only if G is fuchsian). $\pi(R(h)) \cong (\Omega_1/G_1) \times (0, 1)$ and $R(h)$ is called the *invariant ball* determined by G_1.

The three types of combinations, in order of the frequency with which they can be applied, are as follows. In each case the proofs are immediate. G, G_1, G_2 denote kleinian groups and h, h_1, h_2 oriented hyperbolic planes. The notation $\langle X, Y\rangle$ is used to denote the group generated by the groups X and Y.

(1a) Suppose $R(h_1)$, $R(h_2)$ are disjoint trivial balls for G. Let T be any Möbius transformation such that $T(R(h_2)) = L(h_1)$. Then $H = \langle G, \{T\}\rangle$ is kleinian.

(1b) Suppose $R(h)$ is a trivial ball for G_1 and $L(h)$ is trivial for G_2. Then $H = \langle G_1, G_2\rangle$ is kleinian.

(2a) Suppose $R(h_1)$, $R(h_2)$ are disjoint invariant balls corresponding to the maximal parabolic subgroups M_p, M_q of G. Find a Möbius transformation T for which $T(R(h_2)) = L(h_1)$ and $TM_qT^{-1} = M_p$. Then $H = \langle G, \{T\}\rangle$ is kleinian.

(2b) Suppose M_p is a maximal parabolic subgroup for both G_1 and G_2 and $R(h)$, $L(h)$ corresponding invariant balls for G_1 and G_2 respectively. Then $H = \langle G_1, G_2\rangle$ is kleinian.

(3a) Suppose Ω_1, Ω_2 are two components of $\Omega(G)$ which are euclidean disks and let $R(h_1)$, $R(h_2)$ denote the invariant balls corresponding to the subgroups G_1 and G_2 which preserve Ω_1 and Ω_2 respectively. Assume there is a Möbius transformation T such that $G_1 = TG_2T^{-1}$ and $T(R(h_2)) = L(h_1)$. Then $H = \langle G, \{T\}\rangle$ is kleinian.

(3b) Suppose Ω_1 is a component of $\Omega(G_1)$ which is a euclidean disk and likewise Ω_2 of $\Omega(G_2)$. Assume $R(h)$ is an invariant ball for the subgroup K_1 of G_1 that preserves Ω_1 and likewise $L(h)$ and K_2 for Ω_2. If $K_1 = K_2$ then $G = \langle G_1, G_2\rangle$ is kleinian.

In (1a), $H = G * \{T\}$. In the cases (2a) and (3a) the relations in H are consequences of the relations in G and the relations arising from the conjugation. For the cases (b), $H = G_1 *_K G_2$ where $K \subset G_1$, G_2 is the common subgroup that preserves h.

The manifold $\mathfrak{M}(H)$ is obtained *topologically* (not analytically) from the original ones as follows. In the cases (a), let Δ_1, Δ_2 be the open euclidean disks in $\partial\mathcal{B}$ which are bounded by ∂h_1, ∂h_2 and are adjacent to $R(h_1)$, $R(h_2)$ respectively. To obtain $\mathfrak{M}(H)$, identify Δ_1/K_1 with Δ_2/K_2 where K_1, K_2 are the subgroups of G preserving h_1, h_2 respectively. $\mathfrak{M}(H)$ contains the non-dividing surface $h_1/K_1 \equiv h_2/K_2$. For the cases (b), to obtain $\mathfrak{M}(H)$ identify $\Delta_1/K \subset \partial\mathfrak{M}(G_1)$ with $\Delta_2/K \subset \partial\mathfrak{M}(G_2)$, where Δ_1, Δ_2 are the complementary components on $\partial\mathcal{B}$ which are bounded by ∂h and K is the common subgroup preserving h. $\mathfrak{M}(H)$ is divided by the surface h/K.

11.2. PROPOSITION 11.1. *Let $\mathcal{C}$ be the class of kleinian groups which have a finite-sided fundamental polyhedron. Then $\mathcal{C}$ is preserved by the operations* (1), (2), (3) *above.*

Proof. If $G \in \mathcal{C}$, there exists a compact $\mathfrak{M}_c \subset \mathfrak{M}(G)$ such that $\mathfrak{M}(G)^0 - \mathfrak{M}_c^0$ is a finite union of pairing tubes and solid cusp tori. In view of Proposition 4.2 and the topological interpretation of the various procedures (1)–(3), Proposition 11.1 will follow once we recognize the fate of the pairing tubes under these operations. Combinations of type (1) do not affect these tubes so Proposition 11.1 is immediate in this case. For the remaining cases it suffices to illustrate the situation for a combination of type (2a) since the others are dealt with in the same way.

Combination (2a) results in the conjugation of a puncture p_1 in a component S_1 of $\partial\mathfrak{M}(G)$ and a puncture $p_2 \neq p_1$ of S_2 (possibly $S_2 = S_1$). There also results identification of a punctured disk $\Delta_1 \subset S_1$ about p_1 with a punctured disk $\Delta_2 \subset S_2$ about p_2. S_1 and S_2 become a single component $(S_1 - \Delta_1) \cup_{\partial\Delta_1 \equiv \partial\Delta_2} (S_2 - \Delta_2)$ of $\partial\mathfrak{M}(H)$. In addition there is a cylinder C_1 pairing p_1 with another puncture p_1' of $\partial\mathfrak{M}(G)$ with $\partial\Delta_1 \subset \partial C_1$ and C_2 pairing p_2 with p_2', $\partial\Delta_2 \subset \partial C_2$. If p_1 and p_2 determine the same conjugacy class of parabolic transformations, take $C_1 = C_2$. Otherwise we can assume $C_1 \cap C_2 = \varnothing$. In the case $C_1 \neq C_2$, $C = C_1 \cup_{\partial\Delta_1 \equiv \partial\Delta_2} C_2$ becomes a new cylinder pairing p_1' and p_2'. If $C_1 = C_2$, then C is a torus corresponding to the conjugacy class of a maximal parabolic subgroup which is free abelian of rank two. In either case $\partial\mathfrak{M}(H)$ contains two fewer punctures than $\partial\mathfrak{M}(G)$ but has one higher genus (if $S_1 = S_2$) or one fewer component (if $S_1 \neq S_2$). Using these facts we see that the operation can be carried out in $\mathfrak{M}_c$ (properly constructed), the result is compact in $\mathfrak{M}(H)$, and the complementary components are (the interiors of) pairing tubes and solid cusp tori. Q.E.D.

Incidently, if $\partial\mathfrak{M}(G)$ has $2n$ paired punctures, we have shown how to eliminate all of them by adding n generators to G and creating n conjugacy classes of free abelian subgroups of rank two of H.

Maskit [31] has used combinations (1) and (2) to construct all possible function groups.

COROLLARY 11.2. *Suppose G and H are kleinian groups such that*

(i) $G \in \mathcal{C}$,

(ii) *all components of* $\Omega(G)$ *and* $\Omega(H)$ *are euclidean disks,*

(iii) *the components of* $\partial\mathfrak{M}(G)$, $\partial\mathfrak{M}(H)$ *are arranged in distinct pairs* (R_i, R_i') *and* (S_i, S_i') *where* R_i *and* R_i', S_i *and* S_i' *are anticonformally equivalent,*

(iv) *there is an orientation preserving homeomorphism* $f\colon \Omega(G) \to \Omega(H)$ *which induces an isomorphism* $\varphi\colon G \to H$ *and a injection between the pairs* $(R_i, R_i') \to (S_i, S_i')$.

Then G and H are conjugate groups.

Proof. Combination (3a) allows us to identify the paired components of $\partial\mathfrak{M}(G)$ to obtain a manifold $\mathfrak{M}'(G)$ without boundary but of finite hyperbolic volume. Similarly we can obtain $\mathfrak{M}'(H)$ from $\mathfrak{M}(H)$. With the help of f, the isomorphism φ can be extended to an isomorphism $\pi_1(\mathfrak{M}'(G)) \to \pi_1(\mathfrak{M}'(H))$. Now apply Theorem 8.3.

11.3. It is not unreasonable to ask whether in some sense an arbitrary $G \in \mathcal{C}$ can be built up from simpler groups by repeated application of combinations (1)–(3). For the case that G has no parabolic elements, Waldhausen [51] has shown how to break up $\mathfrak{M}(G)$ into a union of balls by successively introducing incompressible surfaces in $\mathfrak{M}(G)$. This process could indeed be formulated in terms of a combination process for kleinian groups in $\mathcal{C}$ although it would require a significantly more complicated version of combination 3. Furthermore, this process would extend to all groups of $\mathcal{C}$. However, the more interesting question is whether there is some *a priori* way of doing this for each G. This would depend on knowing a "canonical" form for each manifold $\mathfrak{M}(G)$. In contrast to the situation for surfaces, such a detailed classification for 3-manifolds is not yet known.

12. An extension of the assumption

12.1. The assumption in § 6.1 is not broad enough to include all known groups. For Maskit has a method whereby he can "degenerate" the invariant component of the regular set for some function groups which are not B-groups. In order to include Maskit's groups, we shall make a more general assumption which evidently encompasses all known groups.

Definition. A kleinian group G is *reducible* if $G = G_1 * G_2$ is a non-trivial free product of subgroups with the property that if Ω_1 is a component of $\Omega(G)$ and H the subgroup of G that preserves Ω_1, then H is conjugate in G to a subgroup of either G_1 or G_2.

Note that at least one of the groups G_1, G_2 appearing in a reducible decomposition of G is kleinian; the other is either kleinian, cyclic, or free abelian of rank two. For given a component Ω_1 of $\Omega(G)$, for some $T \in G$, $T(\Omega_1)$ is a component of either $\Omega(G_1)$ or $\Omega(G_2)$. Obviously function groups are not reducible nor are groups for which $\mathfrak{M}(G)$ is compact and each component of $\Omega(G)$ simply connected (Lemma 1.10). Probably no group with a finite-sided fundamental polyhedron is reducible but we shall not pursue this question here.

12.2. Given a kleinian group H, denote the number of components of $\partial\mathfrak{M}(H)$ by $b(H)$. If H is cyclic or free abelian of rank two, set $b(H) = 1$.

LEMMA 12.1. *If $G = G_1 * G_2$ is a reducible decomposition,*

$$b(G) \leqq b(G_1) + b(G_2) - 2 .$$

Proof. Assume G_1 is kleinian and denote by $G(\Omega')$ the subgroup of G that preserves the component Ω' of $\Omega(G)$. If $TG(\Omega')T^{-1} \subset G_1$, $T \in G$, then $G(T(\Omega')) \subset G_1$. Hence there is a maximal set of components $\Omega_1, \cdots, \Omega_n$ of $\Omega(G)$ such that (i) the groups $G(\Omega_i)$ are non-conjugate (in G_1) subgroups of G_1, and (ii) if $G(\Omega')$ is conjugate in G to a subgroup of G_1, then $T(\Omega') = \Omega_i$ for some i and $T \in G$. That condition (ii) can be satisfied depends on the fact that two elements of G_1 which are conjugate in G are actually conjugate in G_1 [24, Corollary 4.1.5].

Let Γ be the orbit of $\cup\, \Omega_i$ under G_1 and let Γ^* denote the exterior of Γ with respect to $\partial\mathfrak{B}$. If $T \in G$, $T \notin G_1$, then $T(\Omega_1) \subset \Gamma^*$. For otherwise $T(\Omega_1) = S(\Omega_i)$ for some i and $S \in G_1$ so that $S^{-1}TG(\Omega_i)T^{-1}S = G(\Omega_1)$ and $T^{-1}S \in G_1$, a contradiction. We conclude that the open set $\Gamma^* \neq \emptyset$ and $\Gamma^* \cap \Omega(G_1)/G_1$ is the union of the components of $\partial\mathfrak{M}(G_1)$ which do not appear in $\partial\mathfrak{M}(G)$.

Q.E.D.

For any kleinian group G we can consider reducible decompositions into subgroups

(1) $$G = G_1 * \cdots * G_m * \{T_1\} * \cdots * \{T_n\} * \mathfrak{A}_1 * \cdots * \mathfrak{A}_r , \qquad m \geqq 1,$$

where each G_i is kleinian, each $\{T_i\}$ is cyclic, and each $\mathfrak{A}_i$ free abelian of rank two. The ranks satisfy $r(G) = \Sigma r(G_i) + n + 2r$ (Lemma 1.9) and Lemma 12.1 implies that

(2) $$b(G) \leqq \textstyle\sum_{i=1}^{m} b(G_i) - 2m - n - r + 2 .$$

Suppose Δ is an infinite polyhedral disk in $\mathfrak{M}(G)$ whose ideal boundary "approaches" an end of $\mathfrak{M}(G)$ (i.e., Δ has no limit points in $\mathfrak{M}(G)$). Then Δ

splits $\pi_1(\mathfrak{M}(G))$ into a free product $G = G_1 * G_2$. One sees that the set of limit points of Δ viewed as a subset of $\mathfrak{M}(G_i)$ is contained in a component of $\partial\mathfrak{M}(G_i)$. It is not known whether all reducible decompositions are determined by such disks Δ. Even for Maskit's example this is unclear.

12.3. *Assumption* 12.2. G has a reducible decomposition (1) where every subgroup G_i satisfies Assumption 6.1.

The next result follows at once from (2) and Theorem 7.1 which says that if H satisfies Assumption 6.1 and has N generators then $b(H) \leqq 2N - 2$.

PROPOSITION 12.3. *Assume G satisfies Assumption* 12.2 *and has N generators. Then if G has the decomposition* (1) *and $\partial\mathfrak{M}(G)$ has $b(G)$ components,*

$$b(G) \leqq 2N - 4m - 3n - 5r + 2 .$$

If $b(G) = 2N - 2$ then G is not reducible.

We hasten to add that Bers' inequality shows that Proposition 12.3 holds without Assumption 12.2. Of course we have more information on the structure of $\mathfrak{M}(G)$ since Theorem 6.4 tells us about each $\mathfrak{M}(G_i)$.

13. Appendix

13.1. Recently G. P. Scott proved a result which has very important implications for (finitely generated) kleinian groups G. As applicable to our situation his result is as follows.

THEOREM 13.1 (Scott [55]). *There is a compact submanifold M of $\mathfrak{M}(G)$ such that the inclusion map $\pi_1(M) \to \pi_1(\mathfrak{M}(G))$ is an isomorphism.*

The following result, which is an immediate consequence, has also been obtained earlier by Scott [54] and independently by Shalen.

COROLLARY 13.2. *G is finitely presented.*

COROLLARY 13.3. *If G is not a* (*non-trivial*) *free product then Assumption* 6.1 *is satisfied.*

Proof. Either apply the techniques of Chapter 6 or refer directly to Scott's paper [54].

13.2. Particularly in view of the results cited above, the basic reasons for the validity of Ahlfors' finiteness theorem and Bers' inequality seem much clearer. Of course there is still the very important problem of extending the above results to the general case that G is a free product. But one cannot resist posing in addition the following two problems which appear insurmountable at this time.

1. Is $\mathfrak{M}(G)^0$ homeomorphic to the interior of a compact manifold?

2. Is there a necessary and sufficient condition on $\pi_1(\eta)$ that the 3-manifold η be homeomorphic to $\mathfrak{M}(G)$ for some G (of course, η is orientable, irreducible, aspherical and $\pi_1(\eta)$ has trivial center)?

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