Mostow's Rigidity Theorem

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1. Introduction

In [9] Mostow proved that two closed hyperbolic manifolds having the same fundamental group are isometric.

Theorem 1 (**Mostow's Rigidity Theorem**). *Let M_1 and M_2 be connected, compact, oriented hyperbolic manifolds of dimension $n \geq 3$ and $\phi \colon \pi_1(M_1) \to \pi_1(M_2)$ a group isomorphism. Then there exist an isometry $f \colon M_1 \to M_2$ such that the morphism f_* induced between the fundamental groups is ϕ.*

It is equivalent to show that $\pi_1(M_1)$ and $\pi_1(M_2)$ are conjugate in the group of isometries of the hyperbolic space. Note that in dimension 2 the result is false. In fact, there is a $6g - 6$ dimensional space of possible hyerbolic metrics on each closed orientable surface of genus $g \geq 2$ (see B.4, [1]). On the other hand, Mostow's Rigidity Theorem is also true for the finite volume case.

Our aim is to give three different proofs of Mostow's Rigidity Theorem: a proof given by Gromov and Thurston [12], a proof by Besson, Courtois, and Gallot [2], and a proof by Tukia [13].

The first step is common to all proofs of Mostow's Theorem. It involves showing that any isomorphism between the fundamental groups of two closed hyperbolic manifolds induces a homotopy equivalence $f \colon M_1 \to M_2$ which lifts to a quasi-isometry $\bar{f} \colon \mathbb{H}^n \to \mathbb{H}^n$ between the universal cover. The map $\bar{f}$ gives rise to a homeomorphism of the boundary of the hyperbolic space to itself,

$$\partial \bar{f} \colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n,$$

which satisfies the equivariance condition,

$$\partial \bar{f} \circ \gamma = \phi(\gamma) \circ \partial \bar{f} \quad \text{for all } \gamma \in \pi_1(M_1).$$

For the Gromov-Thuruston and Tukia's proofs, the objective is to show that $\partial \bar{f}$ is the trace of an isometry of the hyperbolic space. Then by using the equivariance we can extend it to $\bar{\mathbb{H}}^n$ and passing it to the quotient, obtain the wanted isometry .

In order to achieve this objective, the Gromov-Thurston's basic strategy is to show that the $\partial \bar{f}$ sends regular ideal simplices to regular ideal simplices. To see that this is the case, one uses the fact that the Gromov norm of the fundamental class is proportional to the volume of the manifold.

Theorem 2 (Gromov-Thurston). *If M is an oriented compact hyperbolic manifold then*

$$\|M\| = \frac{vol(M)}{v_n}.$$

This theorem is of independent interest because it says that the volume of any oriented compact hyperbolic manifold is bounded above from above by the number of simplexes of any triangulation.

Tukia's proof of Mostow's Theorem uses that if $\bar{f}$ is a quasi-isometry then $\partial\bar{f}$ is quasi-conformal and by the Radamacher-Stepanov Theorem we have that it is also a.e. differentiable. On the other hand, since the manifolds in question are cocompact, every point in $\partial\mathbb{H}^n$ is a conical limit point. Mostow's Rigidity Theorem follows then if we apply the next theorem to $h = \partial\bar{f}$:

Theorem 3 (Tukia). *Suppose Γ_1 is any discrete subgroup in $SO_0(n,1)$, with ξ a conical point, $h\colon S^{n-1} \to S^{n-1}$ a homeomorphism which is differentiable at ξ with nonzero derivative and $h\Gamma_1 h^{-1} \subset SO_0(n,1)$. Then h is a Möbius transformation.*

Finally, Besson, Courtois, and Gallot obtain Mostow's Theorem as a consequence of a more general result. Recall that the volume entropy $h(g)$ for (Y,g) a compact connected Riemannian $n-$manifold is:

$$h(g) := \lim_{R\to\infty} \frac{1}{R} \log Vol_p^g(R).$$

Where $Vol_p^g(R)$ is the volume of the ball of radius R centered at the point p in $\tilde{Y}$, the universal covering of Y.

Theorem 4 (Besson, Courtois, and Gallot). *Let (Y,g) and (X,g_0) be two compact and negatively curved Riemannian manifolds, such that (X,g_0) is hyperbolic. Suppose that X and Y are homotopically equivalent. Then, if $dimX = dimY = n \geq 3$ we have*

i) $h(g)^n Vol(Y,g) \geq h(g_0)^n Vol(X,g_0)$.

ii) Equality holds if and only if X is isometric to Y (after rescaling g by $\frac{h(g)}{h(g_0)}$).

Mostow's Theorem follows immediately by noticing that $h(g) = n-1$ for hyperbolic manifolds. The existence of the boundary map is crucial in the proof of Theorem 4, as $\tilde{X}$ and $\tilde{Y}$ are $\delta-$hyperbolic spaces, we can also obtain $\partial\bar{f}$ homeomorphic, $\phi-$equivariant map between the respective boundaries of their universal covers. In [2] Besson, Courtois, and Gallot's proof relies on the fact that $\partial\bar{f}$ extends to a smooth map $\bar{F}\colon \tilde{Y} \to \tilde{X}$ (barycentric extension or the Douady-Earle extension) which descends to $F\colon Y \to X$ homotopic to $\bar{f}$ and also by giving tight estimates on the Jacobian of F.

This text is organized as follows:

Section 2 is a collection of definitions and facts about hyperbolic geometry, isometries of the hyperbolic space and hyperbolic manifolds.

In section 3 we prove the first common step of Mostow's Theorem, which involves relating by quasi-isometry an hyperbolic manifold with its fundamental group (Milnor-Svarc Theorem). And constructing the boundary map $\partial \bar{f}$ using Morse Lemma.

In section 4 we give the Gromov-Thurston proof of Mostow's Theorem. We introduce the topological invariant, the Gromov norm and by using measure homology we proof Theorem 2 which relates the topology and geometry of the manifold.

In section 5 we show the Besson, Courtois, and Gallot's proof of Mostow's Theorem by constructing the barycentric extension using the Paterson-Sullivan measures and the barycenter of a non atomic measure on the boundary.

And in section 6 we present the Tukia's proof of Mostow's Theorem, using quasi-conformal maps on the sphere of dimension $n \geq 2$ and the Radamacher-Stevanov Theorem.

2. Preliminar

The $n-$dimensional hyperbolic space $\mathbb{H}^n$ is the unique simply connected $n-$manifold with sectional curvature -1. The models used in this text to represent the hyperbolic space will be the upper half-space model and the Poincaré model.

The group of orientation-preserving isometries $Iso_+(\mathbb{H}^n)$ is isomorphic to the identity component of the Lorenz group $SO(n,1)$. For n small there are far identifications, $Iso_+(\mathbb{H}^2) = PSL(2,\mathbb{R})$ and $Iso_+(\mathbb{H}^3) = PSL(2,\mathbb{C})$. In general the group $Iso(\mathbb{H}^n)$ acts on $\partial\mathbb{H}^n = S^{n-1}$ as the group of *Möbius transformations*, i.e. compositions of inversions in round spheres.

An *hyperbolic manifold* is a Riemannian manifold with sectional curvature -1, or equivalently, a manifold which is the quotient of $\mathbb{H}^n$ by a torsion free and discrete subgroup of $SO_0(n,1)$. Emphasize that many geometric and topological properties of hyperbolic manifolds also hold in the case where the sectional curvature is non-positive. For example, every complete, simply connected manifold of non-positive curvature is diffeomorphic to $\mathbb{R}^n$ and its distance function is convex.

Furthermore, the *ideal boundary of* X the universal covering of a Riemannian manifold with non-positive sectional curvature, can be characterized in a very concrete way as follows. We say that two unit speed geodesics $\beta_1, \beta_2\colon [0,\infty) \to X$ are *asymptotic* if

$$\limsup d(\beta_1(t), \beta_2(t)) < \infty.$$

The ideal boundary ∂X is the set of equivalence classes in the geodesics rays space under the relation of being asymptotic. So we denoted the class of β_1 by $[\beta_1]$.

Given $z_1, z_2 \in \partial X$ and $p \in X$, there exist a unique geodesic ray β_1 and β_2 from p, with unit speed, such that $z_i = [\beta_i]$. Define

$$\measuredangle_p(z_1, z_2) := \text{ angle between } \beta_1'(0) \text{ and } \beta_2'(0).$$

Define similarly the angle between $z \in \partial X$ and $q \in X$. And the cone

$$C_p(z,\epsilon) := \{q \in X \cup \partial X \mid p \neq q,\ \measuredangle_p(z,q) < \epsilon\}.$$

The *cone topology* on $X \cup \partial X$ is the topology generated by the open sets in X and these cones. The induced topology on ∂X is called the *sphere topology.* We have that q_i converges to $z \in \partial X$ if and only if for every $p \in X$, $d(p, q_i) \to \infty$ and $\measuredangle_p(z, q_i) \to 0$.

There exist a homeomorphism between the unit sphere $S^{n-1} \subset T_pX$ and ∂X which associates to each unit vector v at p the class $[\beta_v]$ represented by the geodesic ray issuing from p with initial velocity v. One shows that $X \cup \partial X$ is homeomorphic to the unit closed ball in $\mathbb{R}^n$. Moreover, if γ is an isometry of X. There is a natural extension of γ to ∂X sending $[\beta]$ to $[\gamma \circ \beta]$.

Another remark is that if we have $f\colon \mathbb{H}^n/\Gamma_1 \to \mathbb{H}^n/\Gamma_2$ a homotopy equivalence between closed, connected, orientable, hyperbolic $n-$manifolds. Let $\pi_1\colon \mathbb{H}^n \to \mathbb{H}^n/\Gamma_1$ and $\pi_2\colon \mathbb{H}^n \to \mathbb{H}^n/\Gamma_2$ be the quotient maps. Let γ be an element of Γ_1 and let $\bar{f}\colon \mathbb{H}^n \to \mathbb{H}^n$ be the lift of f. Then we have

$$\pi_2 \circ \bar{f} \circ \gamma = f \circ \pi_1 \circ \gamma = f \circ \pi_1 = \pi_2 \circ \bar{f}.$$

Hence, there is a unique element $f_*(\gamma)$ of Γ_2 such that

$$\bar{f} \circ \gamma = f_*(\gamma) \circ \bar{f}.$$

In this case we say that f is equivariant with respect to the action of Γ_1 and Γ_2 on $\mathbb{H}^n$, or simply that f is f_*-equivariant. Moreover, $f_*\colon \Gamma_1 \to \Gamma_2$ is a morphism.

This gives us another way of thinking about isometries between hyperbolic manifolds. If f is an isometry then f_* is certainly a isomorphism where f lifts to an isometry $\bar{f}$ that is equivariant with respect to the action of the respective fundamental groups on $\mathbb{H}^n$. Going back, any isometry $\bar{f}$ that is equivariant with respect to the action of the fundamental groups descends to an isometry f.

In the case where the fundamental groups are isomorphic we have an homotopy equivalence thanks to the next result.

Theorem 5 (Whitehead's theorem). *If a map $f\colon X \to Y$ between connected $CW-$complexes induces isomorphisms $f_*\colon \pi_n(X) \to \pi_n(Y)$ for all n then f is a homotopy equivalence.*

Since we can represent any closed connected hyperbolic manifold as a $CW-$complex by considering the fundamental domain, and that $\mathbb{H}^n$ is contractible so the high homotopy groups of M are trivial. Then we can apply the Theorem 5 in the follow result whose proof can be found in (Thm.C.5.2, [1]).

Corollary 6. *If M_1 and M_2 are closed hyperbolic manifolds with $\phi\colon \pi_1(M_1) \to \pi_1(M_2)$ a group isomorphism, then there exist a homotopy equivalence $f\colon M_1 \to M_2$ such that the morphism $f_* = \phi$. Moreover, f is unique up to homotopy.*

3. Boundary map

The goal of this section is to prove that an isomorphism between the fundamental groups of closed hyperbolic manifolds induces a quasi-isometric homeomorphism between the boundary of their universal covers, which is equivariant under the action of the fundamental groups. The standard references typically prove this by constructing a homotopy equivalence out of the isomorphism between fundamental groups. Our argument is a little diferent we avoid this step, instead the idea of the proof is that ϕ induce a quasi-isometry $\bar{f}$ by using the Milnor-Svacr Theorem 8 which induces the homeomorphism $\partial\bar{f}$.

Theorem 7. *Suppose Γ_1 and Γ_2 are subgroups of $SO_0(n,1)$ such that $\mathbb{H}^n/\Gamma_1$ and $\mathbb{H}^n/\Gamma_2$ be $n-$dimensional connected, compact, oriented manifolds endowed with a hyperbolic structure. If $\phi\colon \Gamma_1 \to \Gamma_2$ is a isomorphism then there is a homeomorphism $\partial\bar{f}\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ that is $\phi-$equivariant.*

3.1. **Quasi-isometry and $\delta-$hyperbolic spaces.** We have the groups Γ_i acting properly discontinuously cocompactly on $\mathbb{H}^n$. In order to induced the desired isometry, firstly we will approach by defining a metric to any finitely generated group G, known as the *Cayley Graph* of G.

Let $F_G \subset G$ be a finite generating set for G. Let ${F_G}^*$ denote the set of words in the elements of F_G and their inverses. Then we can define a metric on G (depending on F_G) as follows:

$$|g|_{F_G} = \min\{length(w) \mid w \in {F_G}^*\ w =_G g\}.$$

Then for any $g, h \in G$ define

$$d_{F_G}(g,h) = |gh^{-1}|_{F_G}.$$

Notice that G acts on the right by isometries on the Cayley Graph.

Of course, the generating sets changes the metric. But there is a notion of equivalence between metric spaces such that if F_G and F'_G are generating set of the same group G. Then (G, d_{F_G}) and $(G, d_{F'_G})$ are equivalent in a sense that we precise next.

Given two metric spaces (X, d_X) and (Y, d_Y), a map $f\colon X \to Y$ is a $(\lambda,\epsilon)-$*quasi-isometry* if there are constants $\lambda \geq 1$ and $\epsilon \geq 0$ such that for all $x_1, x_2 \in X$

$$\frac{1}{\lambda}d_X(x_1,x_2) - \epsilon \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1,x_2) + \epsilon$$

And if,

$$Y \subset N_\epsilon(f(X)).$$

One of the ideas to proof Theorem 7 is to construct a quasi-isometry between $\mathbb{H}^n$ and a group of isometries that acts freely, properly discontinuously and cocompactly on $\mathbb{H}^n$. This will be given by the next result.

Theorem 8 (**Milnor-Svarc Theorem**). *Let X be a geodesic metric space. Suppose that Γ acts properly discontinuously and cocompact by isometries of X. Then Γ is finite generated. If A is any finite generating set for Γ and x_0 is any base point, the map $(\Gamma, d_A) \ni \gamma \to \gamma * x_0 \in (X,d)$ is a quasi-isometry. In particular, if M is a compact, connected Riemannian manifold with his universal covering $\tilde{M}$ then there exist a quasi-isometry between $\pi_1(M)$ and $\tilde{M}$.*

Proof. Since Γ acts cocompactly on X there is some compact $K \subset X$ such that $x_0 \in K$ and $X = \Gamma * K$. Let κ be the diameter of K. For every $x \in X$ there is $\gamma \in \Gamma$ such that $d_X(x, \gamma x_0) \leq \kappa$. Thus the map $\gamma \to \gamma * x_0$ is quasi-surjective.

Let $A = \{\gamma \in \Gamma \mid d(x_0, \gamma * x_0) \leq 4\kappa\}$, we have that it is finite by properly discontinuity using the compact set $B(x_0, 4\kappa)$.

To see that A generates Γ, suppose by contradiction that it generates H a proper subgroup and construct the sets $V = H * U$ and $V' = (\Gamma/H) * U$ where U is an open neighborhood of K of diameter 2κ. Clearly $V \cup V' = X$ and if $V \cap V' \neq \emptyset$ then there will be $x \in X$ and $\gamma \in H$ $\gamma' \notin H$ such that $d_X(x_0, \gamma * x_0) \leq 2\kappa$ and $d_X(\gamma' * x_0, x) \leq 2\kappa$ and that would say that $d_X(x_0, \gamma^{-1}\gamma' * x_0) \leq 4\kappa$ which implies $\gamma^{-1}\gamma' \in H$ so we have a contradiction.

By one way using λ as the maximal distance of x_0 to the elements of its orbit by elements of $A \cup A^{-1}$ we can prove that by writing each element of $\gamma \in \Gamma$ as a finite product of elements of $A \cup A^{-1}$ the next result:

$$d_X(\gamma * x_0, \gamma' * x_0) \leq \lambda d_A(\gamma, \gamma').$$

For the other inequality we consider $\gamma * x_0, \gamma' * x_0$ and a geodesic joining them, then we make a partition of that geodesic in sizes of lenght less than κ.

Recall that the map is quasi-surjective, thus there is some $\gamma_i \in \Gamma$ such that $\gamma_i * x_0$ is $\kappa-$near of the $i-$th point of the partition then,

$$d_X(\gamma_i * x_0, \gamma_{i+1} * x_0) \leq 3\kappa.$$

Consequently $\gamma_i^{-1}\gamma_{i+1} \in A$ and

$$\gamma^{-1}\gamma' = (\gamma_0^{-1}\gamma_1)(\gamma_1^{-1}\gamma_2)...(\gamma_{N-1}^{-1}\gamma_N),$$

it follows that

$$d_A(\gamma, \gamma') \leq \frac{1}{\kappa} d_X(\gamma * x_0, \gamma' * x_0).$$

$\square$

Corollary 9. *If F_G and F'_G are two finite generating sets of the group G, then the metric spaces (G, d_{F_G}) and $(G, d_{F'_G})$ are quasi-isometric.*

Corollary 10. *Suppose that X and X' are proper geodesic metric spaces, G, G' acting properly discontinuously cocompactly on X and X' respectively and $\phi\colon G \to G'$ be an isomorphism. Then there exists a ϕ-equivariant quasi-isometry $f\colon X \to X'$.*

The negative curved spaces can be regarded as $\delta-$hyperbolic spaces or Gromov hyperbolic spaces. Roughly speaking, a geodesic space X is called 'hyperbolic' if all geodesic triangles in X are 'slim'. More precisely, a geodesic metric space X is called *$\delta-$hyperbolic*, if for every geodesic triangle each side is contained in the $\delta-$neighborhood of the union of the two sides. As an example, $\mathbb{H}^n$ is $2-$hyperbolic.

Although we will not need this fact, it is interesting to note that two geodesic metric spaces (X, d_X) and (Y, d_Y) are quasi-isometric, then X is $\delta-$hyperbolic for some δ if and only if Y is $\delta'-$hyperbolic for some δ' (see Thm. III.H.1.9 [3]). This is a further evidence that quasi-isometry is a right notion of equivalence in this context.

3.2. **Quasi-geodesics and Morse Lemma.** Quasi-geodesics play an important role in $\delta-$hyperbolic spaces because they are not far away from being geodesics. This statement about stability of quasi-geodesics is made precise by the Morse Lemma.

Proposition 11 (**Morse Lemma**). *If X is a $\delta-$hyperbolic space then there is some constant $R = R(\delta, \lambda, \epsilon)$ such that for every $(\lambda, \epsilon)-$quasi-geodesic $\beta\colon [a, b] \to X$.*

$$d_H(\beta([a, b]), [\beta(a), \beta(b)]) \leq R$$

Proof. We are going to make the proof for $X = \mathbb{H}^n$, the $\delta-$hyperbolic space proof can be found in (Thm. 6, Chap. 5, [5]). First we can substitute $(\lambda, \epsilon)-$quasi-geodesic with a continuous one whose distance from the original one is a constant r that depends only of λ and ϵ. This is made by making partitions of the original quasi-geodesic and joining the extremal point with geodesic segments.

Suppose that $\beta([a, b]) \not\subset N_r([\beta(a), \beta(b)])$ and let $[a'', b'']$ be maximal such that $\beta([a'', b''])$ is contained in the complement of the open r neighborhood of $[\beta(a), \beta(b)]$. Let $x = \beta(a'')$ and $y = \beta(b'')$, $\gamma = [\beta(a), \beta(b)]$, $proj_\gamma(x) = x'$ and $proj_\gamma(y) = y'$. Then using triangle inequality we get $d(x, y) \leq d(x', y') + 2r$.

$$length(\beta([a'', b'']) \geq length(proj_{\partial N_r(\gamma)}(\beta([a'', b'']))) = cosh(r)d(x', y')$$

and

$$\lambda d(x', y') + \epsilon' \geq \lambda d(x, y) + \epsilon \geq length(\beta([a'', b'']).$$

Then the distance r is bounded and so on the length of $\beta|_{[a'',b'']}$. And β is contained in a $R-$neighborhood of γ. □

It is a fact, that if $\beta\colon [0, \infty) \to X$ a quasi-geodesic there is a unique geodesic ray $A(\beta)\colon [0, \infty) \to X$ starting in the same point and such that the Hausdorff distance between the image of β and the image of $A(\beta)$ is finite. Indeed, for $p = \beta(0)$ we construct a function $V_p\colon X \to UT_pX$ that for every point x give the unitary vector of the geodesic joining p to x. For every $t \in [0, \infty)$ let B^t_R denote the closed ball of radius R given by the Morse Lemma and centered on $\beta(t)$. Set

$$U = \cap V_p(B^t_R).$$

Notice that U is at most one point because two distinct ray geodesics not remain within a bounded distance of each other. And is not empty cause of the election of R and compactness. Thus U is a singleton and gives us the velocity of the ray geodesic $A(\beta)$.

3.3. **Homeomorphic boundary map.** In this section we will prove that every quasi-isometry on $\mathbb{H}^n$ extends uniquely to a self homeomorphism to the boundary. It is important to notice that this result is true for manifolds with negative sectional curvature, moreover for $\delta-$hyperbolic spaces. A proof of this generalization can be found in (Thm. 2.2, Chap. 3, [4]).

Now, given a quasi-isometry $\bar{f}\colon \mathbb{H}^n \to \mathbb{H}^n$ we can define $\partial\bar{f}\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ by

$$\partial\bar{f}([\beta]) = [\bar{f}\circ\beta]$$

It is well defined because the images of any two rays in the same class are at finite distance of each other. Actually, $\partial\bar{f}$ is a homeomorphism thanks to the next result that is related to the fact that $\partial\bar{f}$ is quasi-conformal, and we will discus that in section 6.

Lemma 12. *If β is a geodesic line in $\mathbb{H}^n$, H is a hyperbolic hyperplane orthogonal to β and $\bar{f}\colon \mathbb{H}^n \to \mathbb{H}^n$ is a $(\lambda,\epsilon)-$quasi-isometry. Let $\pi_{A(\bar{f}\circ\beta)}$ denote the orthogonal projection onto the geodesic line $A(\bar{f}\circ\beta)$. Then there exist some constant c depending only on λ and ϵ such that*

$$\operatorname{diam} \pi_{A(\bar{f}\circ\beta)} f(H) \leq c.$$

Proof. Let x be the point of intersection of H and β, the take $y \in H/\{x\}$. Let l be the geodesic ray contained in H that joins x to y with endpoint θ, also let η_1 and η_2 the other endpoints of β. Let l_i the geodesic joining θ and η_i,and let x_i the closes point in l_i to x, for $i = 1, 2$. Notice that $\cosh(d(x, x_i)) = \sqrt{2}$.

Let z be the point on $A(\bar{f}\circ\beta)$ that is closest to $\bar{f}(x)$ and z_0 the foot perpendicular from $\partial\bar{f}(\theta)$ to $A(\bar{f}\circ\beta)$. First note that z is a uniformly bounded distance from each $A(\bar{f}\circ l_i)$ since

$$d(A(\bar{f}\circ l_i), z) \leq d(A(\bar{f}\circ l_i), \bar{f}(x_i)) + d(f(x_i), \bar{f}(x)) + d(\bar{f}(x), z) \leq R + k\lambda + \epsilon + R.$$

Now let a_i be the closest points on $A(\bar{f}\circ l_i)$ to z. One of the geodesic segments $[z, a_i]$ intersects the geodesic ray emanating from z_0 with end point $\partial\bar{f}(\theta)$. Without loss of generality assume it is $[z, a_2]$ and let the point of intersection be a. Then azz_0 is a right angled hyperbolic triangle so

$$d(z, z_0) \leq d(z, a) \leq d(z, A(\bar{f}\circ l_2)) \leq 2R + k\lambda + \epsilon.$$

Now suppose $w \in A(\bar{f}\circ l)$. Then the projection of w onto $A(\bar{f}\circ\beta)$ lies on the geodesic segment $[z, z_0]$. So $d(\pi_{A(f\circ\beta)}(w), z) \leq 2R + \lambda k + \epsilon$. If w is the closest point on $A(\bar{f}\circ l)$ to $\bar{f}(y)$ then because orthogonal projection reduces distances, we have

$$\begin{aligned} d(\pi_{A(\bar{f}\circ\beta)}(\bar{f}(y)), z) &\leq d(\pi_{A(\bar{f}\circ\beta)}(\bar{f}(y)), \pi_{A(\bar{f}\circ\beta)}(w)) + d(\pi_{A(\bar{f}\circ\beta)}(w), z) \\ &\leq d(\bar{f}(y), w) + d(\pi_{A(\bar{f}\circ\beta)}(\bar{f}(y)), z) \\ &\leq R + 2R + k\lambda + \epsilon. \end{aligned}$$

Finally, $c = 2(3R + k\lambda + \epsilon)$ complete the proof. □

Lemma 13. *If $\bar{f}\colon \mathbb{H}^n \to \mathbb{H}^n$ is a quasi-isometry then $\partial\bar{f}\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is a homeomorphism.*

Proof. It is easy to show that $\partial\bar{f}$ is a bijection whose inverse is $\partial\bar{g}$ where $\bar{g}\colon \mathbb{H}^n \to \mathbb{H}^n$ is any quasi-inverse of $\bar{f}$. Hence we have only to show that $\partial\bar{f}$ is continuous. Fix Q a neighborhood of $\partial\bar{f}([\beta])$ and let $t_0 \in [0,\infty)$, such that for $t > t_0$ the ball of radius R (given by the Morse Lemma) around $\bar{f}(\beta(t))$ is contained in Q.

Let H be the hyperplane orthogonal to β passing through $\beta(t_0)$ and let H_t be the hyperplane orthogonal to β passing through $\beta(t)$. By the previous Lemma 12 $\bar{f}(H_t) \subset Q$. The same if we take the closure of each hyperplane, which in union we call Q'. It follows that $\bar{f}(Q') \subset Q$. □

We now proceed to prove the main Theorem of this section.

Proof of Theorem 7 . Since $M_1 = \mathbb{H}^n/\Gamma_1$ and $M_2 = \mathbb{H}^n/\Gamma_2$ are compact manifolds then Γ_1 and Γ_2 are subgroups of $SO_0(n,1)$ that act properly discontinuously and co-compactly.

Fix a base point $x_0 \in \mathbb{H}^n$. By the Corollary 10 of Milnor-Svarc Theorem we have a ϕ-equivariant quasi-isometry $\bar{f}\colon \mathbb{H}^n \to \mathbb{H}^n$. And by the Proposition 13 there is a homeomorphism $\partial\bar{f}\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$. It remains to show that $\partial\bar{f}$ satisfies the equivariance condition.

$$(\partial\bar{f}\circ\gamma)([\beta]) = \partial\bar{f}([\gamma\circ\beta]) = [\bar{f}\circ\gamma\circ\beta] = [\phi(\gamma)\circ\bar{f}\circ\beta] = \phi(\gamma)([\bar{f}\circ\beta]) = (\phi(\gamma)\circ\partial\bar{f})([\beta]).$$

□

4. The Gromov-Thurston proof

In this section we outline Gromov and Thurston's proof of Mostow's Theorem that appeared in Thurston's Princeton lecture notes (Chap.6, [12]), and in Gromov's survey [6].

Our aim is to prove that $\partial\bar{f}$ is a trace of an isometry of the hyperbolic space. To achieve this fact, we will proof that $\partial\bar{f}$ send regular ideal simplices to themselves. At the same time we will also prove that simplicial volume is proportional to the geometric volume (Theorem 2).

4.1. **Gromov Invariant and Measure homology.** Let X be a topological space and let $S(X;\mathbb{R})$ be the *singular chain complex* of X. More concretely, $S_k(X;\mathbb{R})$ is the set of formal linear combination of $k-$simplices. We denote by $\|c\|$ the l_1-norm of the $k-$chain c. If α is a homology class in $H_k^{sing}(X;\mathbb{R})$, the *Gromov norm of* α is defined

$$\|\alpha\| = \inf_{[c]=\alpha} \{\|c\| = \sum_{\sigma} |r_\sigma| \text{ such that } c = \sum_{\sigma} r_\sigma \sigma\}.$$

Let $f\colon X \to Y$ be a continuous function, α a homology class in $H_k^{sing}(X;\mathbb{R})$ and $c = \sum_\sigma r_\sigma\sigma$. be a $k-$cycle representing α. Then $f(c) = \sum_\sigma r_\sigma f\circ\sigma$ represents the homology class of $f_*(\alpha)$. So,

$$\|f_*(c)\| \leq \sum_{\sigma} |r_\sigma| = \|c\|, \quad \text{so} \quad \|f_*(\alpha)\| \leq \|\alpha\|.$$

The *simplicial volume* of a closed, connected, orientable manifold M is the Gromov norm of a fundamental class of M in $H_n^{sing}(M;\mathbb{R})$ and is denoted by $\|M\|$. Note that is not a norm but a pseudo-norm, because every closed, connected, orientable, spherical or Euclidean $n-$manifold, with $n \geq 1$, has null simplicial volume.

Recall that the fundamental class is characterize by the fact that if Ω_M is the volume form of a Reimannian metric in M, then $Vol(M) =< \Omega_M, [M] >$ where this pairing is defined such that if $c = \sum_\sigma r_\sigma \sigma \in S_n(X, \mathbb{R})$ then,

$$< \Omega_M, c >= \int_c \Omega_M := \sum_\sigma r_\sigma \int_{\Delta^n} \sigma^* \Omega_M. \tag{4.1}$$

Note that is well defined because the pairing depends only on the homology class of c by Stoke's Theorem.

It turns out that defining the simplicial volume in terms of a generalization of singular homology, called measure homology, makes it more user-friendly. As follows we develop the theory of measure homology for the hyperbolic manifold $M = \mathbb{H}^n/\Gamma$.

For each integer $k \geq 0$, let $C^\infty(\Delta^k, M)$ be the space of C^∞ singular $k-$simplexes in M topologized with the C^1 topology. Let $C_k(M)$ be the real vector space of all compactly supported, signed, Borel measures μ of bounded total variation $\|\mu\|$ on the space $C^\infty(\Delta^k, M)$. Where

$$\|\mu\| = \mu_+(C^\infty(\Delta^k, M)) + \mu_-(C^\infty(\Delta^k, M)),$$

and $\mu_+ - \mu_-$ is the Jordan decomposition of μ into its positive and negative variation.

For each $i = 0, ..., k$ let $\eta_i \colon \Delta^{k-1} \to \Delta^k$ be the ith face map. The η_i induces a continuous function

$$\eta_i^* \colon C^\infty(\Delta^k, M) \to C^\infty(\Delta^{k-1}, M),$$

and the linear transformation

$$(\eta_i^*)_* \colon C_k(M) \to C_{k-1}(M).$$

We define a linear transformation $(\delta_k)_* \colon C_k(M) \to C_{k-1}(M)$ by the formula

$$\delta_k = \sum_{i=0}^{k} (-1)^i (\eta_i^*)_*.$$

Making the usual calculation $\delta_{k-1} \circ \delta_k = 0$, then the system $\{C_k(M), \delta_k\}$ is a chain complex. Thus we can define $H_k^{med}(M)$ the *measure homology of M*. And we have the analogue of the Gromov norm. Let α is a homology class in $H_k^{med}(M)$, then

$$\|\alpha\| = \inf_{[\mu]=\alpha} \{\|\mu\|\},$$

where $\|\mu\|$ is the total variation of μ.

Let $S^\infty(M)$ be the subchain complex of $S(M;\mathbb{R})$ of C^∞ singular chains in M. It is known that the inclusion chain map of $S^\infty(M)$ into $S(M;\mathbb{R})$ induces en isomorphism on homology.

Given a C^∞ $k-$simplex σ, represent the Dirac measure on $C^\infty(\Delta^k, M)$ at σ as δ_σ. And define a linear transformation,

$$m_k \colon S_k^\infty(M) \to C_k(M),$$

by the formula,

$$m_k\left(\sum_\sigma r_\sigma \sigma\right) = \sum_\sigma r_\sigma \delta_\sigma.$$

The family $\{m_k\}$ of linear transformations is a chain map from $S^\infty(M)$ to $C(M)$.

For all oriented, closed, connected manifold M the inclusion of the singular chain complex into the measure chain complex induces a natural isomorphism

$$H_*^{sing}(M) \simeq H_*^{med}(M).$$

This isomorphism is isometric with respect the Gromov norm on singular homology and the analogue norm on measure homology induced by the total variation. Indeed, we can prove that the inclusion m induces a natural isomorphism. The idea is to use a similar method as the Poncairé duality in defining for each ω a C^∞ $k-$form on M the function

$$I_\omega \colon C^\infty(\Delta^k, M) \to \mathbb{R}$$

by

$$I_\omega(\sigma) = \int_{\Delta^k} \sigma^*\omega,$$

and proof that it is continuous.

Then for each measure μ in $C_k(M)$ with K the compact support of μ, we have that the set $I_\omega(K)$ is bounded for each ω in $\Omega^k(M)$ and as μ has bounded total variation, the integral $\int_K I_\omega d\mu$ is finite for each ω. And this induces a linear functional d_μ element of the de Rham chain complex such that

$$d_\mu(\omega) = \int_{C^\infty(\Delta^k,M)} \left(\int_{\Delta^n} \sigma^*\omega\right) d\mu(\sigma).$$

Also induces a linear map l between the measure chains and the de Rham chain complex by $l(\mu) = d_\mu$. Finally it is easy to verify that $l_* m_* = I_*$, so we have an isomorphism on homology.

And the part of the isometry is given by proving that the operator in homology does not increase the respective norm. A reference for a clear and general study of this result can be found in [10].

An analogous definition of the pairing, used to characterize the fundamental class, is given as follows. If $\mu \in C_n(M)$ is a measure $n-$cycle then,

$$< \Omega_M, \mu >= d_\mu(\Omega_M) := \int_{C^\infty(\Delta^n,M)} \left(\int_{\Delta^n} \sigma^*\Omega_M\right) d\mu(\sigma). \tag{4.2}$$

Notice that if we think of smooth singular $n-$chains as weighted combinations of point masses then 4.2 reduces to 4.1 .

4.2. **Simplices in Hyperbolic space.** Let S_n denote the set of all $n-$simplices in $\bar{\mathbb{H}}^n$ having totally geodesic faces. An element $\sigma \in S_n$ is called *ideal* if all the vertices lie on $\partial\mathbb{H}^n$ and it is called *regular* if every permutation of its vertices can be obtained as a restriction of an isometry in $\mathbb{H}^n$.

A remarkable fact is that all elements in S_n have finite volume. To see that this is the case note that there is an ideal simplex which contains the original one. It suffices thus, to prove that ideal simplices have finite volume. Suppose that one of its vertices is at ∞ and cut your ideal simplex with the plane $\mathbb{R}^2 \times \{t\}$ and calculate the volume using the volume formula:

$$\int \frac{dxdydz}{z^3},$$

which converges thanks to the fact that the variable z goes to inifity. By using an isometry that sends one vertex to ∞ and repeat the calculation on each vertex of the ideal simplex. Finally the piece of simplex that is left is compact, so it has finite volume.

It is known that in $\bar{\mathbb{H}}^2$ the area of all ideal triangles is π. Moreover the volume function over S_n has a maximum v_n that is achieved in all and only regular ideal simplices (see [11]). Another important fact is that if σ is an ideal simplex in $\mathbb{H}^n$ with vertices $\infty, p_1, p_2, ..., p_n$ then σ is regular if and only if τ the Euclidean $(n-1)-$simplex having vertices $p_1, p_2, ..., p_n$ is regular. Indeed, let σ be regular. If we make a permutation of their vertices such that it fixes ∞ then there exist an isometrie ϕ such that the restriction to $\mathbb{R}^n \times \{0\}$ results a multiple of an Euclidean isometry and its clear that the scale factor must be 1. Conversely, if τ is a regular Euclidean $(n-1)-$simplex then every permutation of their vertices keeping fix ∞ induces a isometry on $\mathbb{H}^n$. Just take the inversion given by sphere with center at any vertex p_i and radius the distance to the other vertices. This induces a transposition between ∞ and p_i. Hence σ is regular too.

It is a fact that the isometry classes of ideal simplices in $\mathbb{H}^n$ is parametrized by the similarity classes of triangles in $\mathbb{R}^{n-1}$. For example, the volume of the ideal simplexes in $\mathbb{H}^3$ is given by

$$Vol(\sigma) = \Lambda(\alpha(\sigma)) + \Lambda(\beta(\sigma)) + \Lambda(\gamma(\sigma))$$

where α, β, γ are the dihedral angles and Λ is the Lobatchevski function (see Sect. C.2, [1]).

4.3. **Simplicial volume proportional to the volume.** In order to prove Theorem 2 we are going to use the two definitions of homology.

Proof of Theorem 2 . First we are going to prove

$$\|M\| \geq \frac{Vol(M)}{v_n}.$$

One of the key ideas of the proof is the *straightening* and we explain this next. Take a simplex $k-$simplex σ in M, lift it to $\tilde{\sigma}$ a singular $k-$simplex in $\mathbb{H}^n$ and construct $str(\tilde{\sigma})$ the unique totally geodesic singular $k-$simplex with vertices $\tilde{\sigma}(e_0), ..., \tilde{\sigma}(e_k)$

in $\mathbb{H}^n$. Now let $str(\sigma) = \pi(str(\tilde{\sigma}))$ this definition doesn't depends on the lift since any two lift differ by an element of $SO_0(n, 1)$. Remarkably, if $\sum_\sigma r_\sigma \sigma$ represents some class, then $\sum_\sigma r_\sigma str(\sigma)$ also represents it because the inclusion of the chain subcomplex generated by all straight singular chains into $S(M, \mathbb{R})$ induces isomorphism on homology.

Let Ω_M be the volume form for M and let $c = \sum_\sigma r_\sigma \sigma$ be any straight singular $n-$cycle representing the fundamental class of M. Then

$$\int_c \Omega_M = \sum_\sigma r_\sigma \int_{\Delta^n} \sigma^* \Omega_M = \sum_\sigma \pm r_\sigma Vol(\sigma(\Delta^n)) \leq \sum_\sigma |r_\sigma| Vol(\sigma(\Delta^n))$$

Now as $Vol(\sigma(\Delta^n)) \leq v_n$. Therefore we have

$$Vol(M) = \int_c \Omega_M \leq \sum_\sigma |r_\sigma| v_n,$$

then taking the infimum,

$$Vol(M) \leq \|[M]\| v_n.$$

Let us show the other inequality, or in other words prove that the fundamental cycle of M can be represented efficiently by a cycle using simplices which have (on the average) nearly maximal volume.

Given any positively oriented straight $n-$simplex σ in $\mathbb{H}^n$, let σ_- denote the image of σ under some reflection. Let us construct a measure chain $SMR_M(\sigma)$ that is essentially, a measure uniformly supported on all the projections of isometric copies of σ in M. To do this, let $\pi\colon \mathbb{H}^n \to M$ be the universal projection and by taking the function

$$\alpha(\sigma)(g\Gamma) = \pi \circ g \circ \sigma.$$

Define

$$SMR_M(\sigma) := \alpha(\sigma)_*[\lambda],$$

the push-forward of the Haar measure on $SO_0(n-1)/\Gamma$ by $\alpha(\sigma)$, such that,

$$\lambda(\{\gamma \mid \gamma * x_0 \in U\}) = Vol(U),$$

for every open set U.

The first assertion follow from the fact that $SMR_M(\sigma)$ is a positive measure and satisfies,

$$SMR_M(C^\infty(\Delta^n, M)) = \alpha(\sigma)_*[\lambda](S_n) = \lambda(\alpha(\sigma)^{-1}(S_n)) = \lambda(SO_0(n{-}1)/\Gamma) = Vol(M).$$

Also using the fact that $\pi^*\Omega_M = \Omega$ and $g^*\Omega = \Omega$ for any $g \in G$,

$$\begin{aligned}
< SMR_M(\sigma), \Omega_M > &= \int_{C^\infty(\Delta^n, M)} \left(\int_{\Delta^n} \tau^* \Omega_M \right) d(SMR_M(\sigma))(\tau) \\
&= \int_{SO_0(n,1)/\Gamma} \left(\int_{\Delta^n} [\alpha(\sigma)(g\Gamma)]^* \Omega_M \right) d\lambda(\Gamma g) \\
&= \int_{SO_0(n,1)/\Gamma} \left(\int_{\Delta^n} \sigma^* g^* \pi^* \Omega_M \right) d\lambda(\Gamma g) \\
&= \int_{SO_0(n,1)/\Gamma} \left(\int_{\Delta^n} \sigma^* \Omega \right) d\lambda(\Gamma g) \\
&= Vol(\sigma) Vol(M).
\end{aligned}$$

Now to give the cycle, let

$$\mu = \frac{1}{2}(SMR_M(\sigma) - SMR_M(\sigma_-)) \in C_n(M).$$

Then since $SMR_M(\sigma)$ and $SMR_M(\sigma_-)$ have disjoint support,

$$\|\mu\| = \frac{1}{2}\|SMR_M(\sigma)\| + \frac{1}{2}\|SMR_M(\sigma_-)\| = Vol(M).$$

Next we argue that μ is actually a cycle. En efect, for every face of every isometric copy of $\pi \circ \sigma$, there is a face of an isometric copy of $\pi \circ \sigma_-$, that matches the first face, but with opposite orientation. Hence the faces cancel out in pairs and so $\partial(\mu) = 0$. Then $[\mu] = k[M]$. And also,

$$\begin{aligned}
kVol(M) &= < \mu, \Omega_M > \\
&= \frac{1}{2}(< SMR_M(\sigma), \Omega_M > - < SMR_M(\sigma_-), \Omega_M >) \\
&= \frac{1}{2}(Vol(\sigma)Vol(M) - Vol(\sigma_-)Vol(M)) \\
&= Vol(\sigma)Vol(M).
\end{aligned}$$

So for every straight simplex σ in $\mathbb{H}^n$, $[\mu]/Vol(\sigma)$ represents $[M]$ and so

$$\|M\| \leq \frac{\|\mu\|}{Vol(\sigma)} = \frac{Vol(M)}{Vol(\sigma)}.$$

Taking the infimum over straight simplices σ in $\mathbb{H}^n$ gives

$$\|M\| \leq \frac{Vol(M)}{v_n}.$$

$\square$

Let us emphasize the fact that if M_1 and M_2 are two hyperbolic, closed, orientable manifolds homotopy equivalent then there exist g a homotopy inverse. And if we take k the fundamental class of M_1 then $f_*(k)$ is the fundamental class of M_2.

$$\|k\| = \|g_*(f_*(k))\| \leq \|f_*(k)\| \leq \|k\|.$$

Therefore, we have the next result:

Corollary 14. *If M_1 and M_2 are two hyperbolic compact orientable manifolds homotopy equivalent. Then,*

$$\textit{if} \ \ \|M\| = \|N\| \ \ \textit{then} \ \ Vol(M_1) = Vol(M_2).$$

4.4. **Ideal simplexes correspondence of the boundary map.** The key argument to the Mostow's Theorem proof lays on the next result.

Proposition 15. *If $u_0, ..., u_n$ are vertices of a maximal volume simplex then the same holds for $\bar{f}(u_0), ..., \bar{f}(u_n)$.*

Proof. By contradiction, suppose that $u_0, ..., u_n$ are vertices of an ideal simplex of maximal volume but the ideal $n-$simplex spanned by $\bar{f}(u_0), ..., \bar{f}(u_n)$ is not. Then, by continuity, simplexes that are sufficiently close to the given simplex have images under $\bar{f}$ which after straightening, have volumes that are bounded away from v_n.

More precisely, there is a $r > 0$ such that if σ is a straight regular $n-$simplex, with $|u_i - \sigma(e_i)| < r$ for each i, then

$$Vol(str(\bar{f}\sigma(\Delta^n))) < v_n - \epsilon, \quad \text{for some} \quad \epsilon > 0.$$

Let

$$U_i = \{x \in \mathbb{H}^n \mid |u_i - x| < r\} \ \ \text{and} \ \ K_i = \{x \in \mathbb{H}^n \mid |u_i - x| < r/2\}$$

so that

$$U = \{g \in SO_0(n-1) \mid gK_i \subset U_i\}.$$

Then U is open in $SO_0(n-1)$, since $SO_0(n-1)$ has the compact-open topology. Also the quotient map $\kappa\colon SO_0(n-1) \to SO_0(n-1)/\Gamma_1$ is an open map, since it is a covering projection. Hence $\kappa(U)$ is an open subset of $SO_0(n-1)/\Gamma_1$ and $Vol(\kappa(U)) > 0$.

Let σ be a straight singular $n-$simplex in M_1 such that $|u_i - \sigma(e_i)| < r/2$ for each i and

$$Vol(\sigma) > v_n - \delta \ \ \text{where} \ \ \delta = \frac{\epsilon Vol(\kappa(U))}{2Vol(M_1)}.$$

Now if g is in U, then

$$Vol(str(\bar{f}g\sigma)) < v_n - \epsilon < Vol(\sigma) + \delta - \epsilon,$$

whereas if g is not in U, then

$$Vol(str(\bar{f}g\sigma)) < v_n < Vol(\sigma) + \delta.$$

We now assume that M_1 and M_2 are oriented so that $\bar{f}$ is orientation-preserving. By switching the index of u_0 and u_1, if necessary, we may assume that σ is orientation preserving. Observe that

$$\begin{aligned}
< \Omega_{M_2}, (str_* \circ \bar{f}_*)(SMR_{M_1}(\sigma)) > &= \int_{C^\infty(\Delta^n, M_2)} \left(\int_{\Delta^n} \sigma^* \Omega_{M_2} \right) d((str \circ \bar{f}_*)(SMR_{M_1}(\sigma))(\tau) \\
&= \int_{SO_0(n,1)/\Gamma} (\int_{\Delta^n} (str(\bar{f}\pi g\sigma))^* \Omega_{M_2}) d(\Gamma g)) \\
&= \int_{SO_0(n,1)/\Gamma} Vol(str(\bar{f} \circ g\sigma)) d(\Gamma g)) \\
&< (Vol(\sigma) + \delta - \epsilon) Vol(\kappa(U)) \\
&\quad + (Vol(\sigma) + \delta)(Vol(M_2) - Vol(\kappa(U)) \\
&= (Vol(\sigma) - \delta) Vol(M_2).
\end{aligned}$$

Analogus if we take σ_- denote the image of τ under some reflection. Then we have

$$- < \Omega_{M_2}, (str_* \circ \bar{f}_*)(SMR_{M_1}(\sigma)) >< (Vol(\sigma) + \delta) Vol(M_2).$$

Therefore, if $\mu = \frac{1}{2}(SMR_{M_1}(\sigma) - SMR_{M_1}(\sigma_-)) \in C_n(M_1)$ then,

$$< \Omega_{M_2}, (str_* \circ \bar{f}_*)(\mu) >< Vol(\sigma) Vol(M_2),$$

but as str is homotope to the identity and $\bar{f}$ is an homotopy equivalance, we have

$$< \Omega_{M_2}, (str_* \circ \bar{f}_*)(\mu) >=< \Omega_{M_2}, (\bar{f}_*)(\mu) >=< (\bar{f}_*)(\Omega_{M_2}), \mu >= Vol(\sigma) Vol(M_2),$$

which is a contradiction. $\square$

4.5. **End of the G-T proof.** Knowing that $\partial \bar{f} \colon \partial \mathbb{H}^n \to \partial \mathbb{H}^n$ is a continuous one-to-one mapping such that given an ideal geodesic simplex with vertices $u_0, ..., u_n$ has volume v_n then the ideal geodesic simplex with vertices $\partial \bar{f}(u_0), ..., \partial \bar{f}(u_n)$ has volume v_n too. Then ∂f is the trace of a $\gamma_f \in SO_0(n,1)$. Indeed, as all ideal regular simplexes are conjugate in $SO_0(n,1)$ we can assume that $\partial \bar{f}$ fixes a regular ideal simplex with vertex $u_0 = \infty$, so $u_1, ..., u_n$ is an equilateral triangle in $\mathbb{R}^n \times \{0\}$. Then we shall prove that ∂f is the identity. This fact comes from using that the projection to the boundary, of regular ideal simplexes are regular simplex in $\mathbb{R}^n$, then we can prove that the vertices of the tessellation of $\mathbb{R}^n \times \{0\}$ by the regular simplex $u_1, ..., u_n$ are fixed.

The next step is to use the inversion that preserves the triangle $u_1, ..., u_n$ and sends the center to infinity to show that the barycentric points of the tessellation are also fixed. We continue by fixing the middle points of the edges of $u_1, ..., u_n$. Iterating the process we obtain a dense subset which is fixed by $\partial \bar{f}$.

Proof of Theorem 1. According to the partial results we have obtained by now, we can assume that there exist a lift $\bar{f} \colon \mathbb{H}^n \to \mathbb{H}^n$ such that:

1) $\bar{f}$ extends in a continuous way to $\partial \mathbb{H}^n$,
2) $\bar{f} \circ \gamma = f_*(\gamma) \circ \bar{f}$ for every $\gamma \in \Gamma_1$ holds in the whole of $\bar{\mathbb{H}}^n$,
3) there exist $\gamma_f \in SO_0(n,1)$ such that $\bar{f}|_{\partial \mathbb{H}^n} = \gamma_f|_{\partial \mathbb{H}^n}$.

These facts imply that

$$\gamma_f \circ \gamma = f_*(\gamma) \circ \gamma_f \text{ on } \bar{\mathbb{H}}^n.$$

We set now for $x \in \mathbb{H}^n$, $F\colon M_1 \to M_2$ as, $F(\pi_1(x)) = \pi_2(\gamma_f(x))$. Thanks to the equivariance and the fact that π_1, π_2 and γ_f are local isometries we have that F is well define, is onto and is an isometry. To see that F is homotopic to f, take

$$H(t, \pi_1(x)) = \pi_2(t\bar{f}(x) + (1-t)\gamma_f(x)),$$

then by (2) easily imply that H is a well define homotopy. □

5. The proof of Besson, Courtois, and Gallot

Besson, Courtois, and Gallot's proof of Mostow Rigidity arises as corollary of a much more general result. In this section, we outline this more general setting and discuss the arguments to proof Mostow's Theorem and its generalization.

Let X be a compact connected smooth $n-$manifold, and g a Riemannian metric on X. Let $\tilde{X}$ be the universal covering of X and denote by $V_p^g(R) := Vol(B_p^g(R))$ the volume of the ball of radius R centered at the point $p \in \tilde{X}$, the *volume entropy of g* is defined as

$$h(g) := \lim_{R\to\infty} \frac{1}{R} \log V_p^g(R).$$

It turns out that this limit exists and is independent of the choice of p (See [8]). Essentially it measures the growth rate of balls in the universal cover of X. For example, if X is a hyperbolic manifold and in this case we know $Vol(B_p^g(R)) = e^{(n-1)R}$, then $h(g) = n-1$.

Clearly Theorem 1 is a consequence of Theorem 4 . Indeed, since the closed hyperbolic manifolds M_1 and M_2 are homotopic there exist $f\colon M_1 \to M_2$ of degree 1, the volume entropy is the same because both are hyperbolic and thanks to the degree 1 of f they have the same volume. Thus by condition *ii*) they are isometric.

To have a complete proof of Theorem 1, we will construct the isometry F of Theorem 4 by extending $\partial\bar{f}$ to a smooth map f_*-equivariant which is often called the barycentric extension (Douady-Earle extension). Then, actually Theorem 4 is a consequence of the next result.

Proposition 16. *For $dimX = dimY \geq 3$ the barycentric extension F is of class C^1. Furthermore, one has*

i) $|Jac_y F| \leq (\frac{h(g)}{h(g_0)})^n$

ii) If for some $y \in Y$, $|Jac_y F| = \left(\frac{h(g)}{h(g_0)}\right)^n$ *then D_yF is homothety of ratio* $\frac{h(g)}{h(g_0)}$.

Proof of Theorem 4. Let us assume that F is a homotopy equivalence and hence is a map of degree one. Let ω_0 be the volume form of the oriented manifold (X, g_0) and ω the volume form of (Y, g), then

$$\int_Y F^*(\omega_0) = \deg F \int_X \omega_0 = vol(X, g_0),$$

and the inequality of i) gives

$$vol(X,g_0) \leq \int_Y |F^*(\omega_0)| = \int_Y |(JacF)\omega| \leq \left(\frac{h(g)}{h(g_0)}\right)^n \int_Y \omega = \left(\frac{h(g)}{h(g_0)}\right)^n Vol(Y,g),$$

which proves the first part. In the equality case then $|Jac_y F| = \left(\frac{h(g)}{h(g_0)}\right)^n$ for all y and hence $D_y F$ is a homothety of ratio $(\frac{h(g)}{h(g_0)})^n$ for all y. $\square$

The principal tool to construct the barycentric extension are the Busemann functions, which give a way to measure the distances between points in X and points in ∂X. This will allow to introduces the Petterson-Sullivan measures and the barycenter of a measure, fundamental for the construction of F.

5.1. **Busemann functions.** Let X be a simply connected Riemannian manifold with non-positive sectional curvature. It seems to make no sense in defining the distance between a point in X and a point in ∂X. But we can define a sensible notion of relative distance between those points that capture this notion nicely as follows, given a point $x_0 \in X$, the *Buseman function normalized at* x_0 is the function $B^{x_0}\colon X \times \partial X \to \mathbb{R}$ given by

$$B^{x_0}(x,[\beta]) := \lim_{t\to\infty} d(\beta(t),x) - d(\beta(t),x_0),$$

where the representative β is chosen to satisfy $\beta(0) = x_0$.

In the sequel, we fix $x_0 \in X$, and write either $B^{x_0}_x\colon \partial X \to \mathbb{R}$ or $B^{x_0}_\theta\colon X \to \mathbb{R}$ whenever we want to think of B as a function of one variable with the other left fixed. Notice that, this functions are invariant under isometries. And due to the fact that X is a $CAT(0)$ space we have that the limit exist (see Lemma 8.18, [3]) and is a convex C^1 function with $\|D_x B_\theta\| = 1$ (see Prop.8.22, [3]).

In the case of $\mathbb{H}^n$ we have a geometric interpretation of the level sets of the Busemann function, if H denote the horosphere passing through $\theta \in \partial\mathbb{H}^n$ and $x_0 \in \mathbb{H}^n$. Then

$$B^{x_0}(x,\theta) = \pm d(x,H)$$

where the sign is chosen according if x is inside (minus sign) or outside (plus sign) the horoball whose boundary is H. And the Hessian of B_θ is given by:

$$Hess_x(B_\theta)(u,v) = <u,v> - <D_xB_\theta,u><D_xB_\theta,v>.$$

5.2. **Patterson-Sullivan measures and visual measures.** Let (Y,g) be a connected compact $n-$dimensional Riemannian manifold, where the metric g is assumed to have negative curvature. Let (arbitrary) choice of an origin 0 in the universal covering $\tilde{Y}$ of Y that allows to identify $\tilde{Y}$ with the unit ball in $\mathbb{R}^n$, thus the geometric boundary $\partial\tilde{Y}$ being identified with the unit sphere. To each $y \in \tilde{Y}$ we associate a measure on $\tilde{Y}$, denoted by ν_y. For y and y' in $\tilde{Y}$ the measures $\nu_{y'}$ and ν_y are in the same class of density measures and, for $\theta \in \partial\tilde{Y}$

$$\frac{d\nu_y}{d\nu_{y'}}(\theta) = e^{-h(g)B_{y'}(y,\theta)}.$$

Let $\frac{1}{c(y)} = \int_{\partial \tilde{Y}} e^{-h(g)B_{y'}(y,\theta)} d\nu_0(\theta)$, then $\mu_y = c(y)\nu_y$ is a probability measure on $\partial \tilde{Y}$. Furthermore, the map

$$\mu \colon y \in Y \to \mu_y \in \mathcal{M}_1(\partial \tilde{Y})$$

is equivariant which means that for any isometry γ of $\partial \tilde{Y}$ one has

$$\mu_{\gamma(y)} = \gamma_*(\mu_y).$$

The construction of this family of measures briefly it goes as follows: let $g_s(y,z) = \sum_{\gamma \in \Gamma} e^{-sd(y,\gamma(z))}$ be the Poincaré series of Γ. It converges for $s > h(g)$ and diverges for $s \leq h(g)$. Now for $s > h(g)$ let us define

$$\nu_{y,z}(s) = \frac{\sum_{\gamma \in \Gamma} e^{-sd(y,\gamma(z))} \delta_{\gamma(z)}}{\sum_{\gamma \in \Gamma} e^{-sd(y,\gamma(y))}}$$

where d is the distance in $\tilde{Y}$ associated to $\tilde{g}$ the pullback metric form the metric g on Y and $\delta_{\gamma(z)}$ is the Dirac measure. This defines a family of measures on $\tilde{Y} \cup \partial \tilde{Y}$ and we obtain ν_y by taking a weak limit for the subsequence when s goes to $h(g)$. The fact that the denominator diverges when $s = h(g)$, ensures that ν_y is concentrated on the set of accumulation points of the orbit $\Gamma(z)$, i.e. on the boundary $\partial \tilde{Y}$. Let us remark that as Γ is cocompact then the Patterson-Sullivan measure is unique and has no atom.

In the case of $\mathbb{H}^n$, we recall that we can think of $\partial \mathbb{H}^n$ as being identified with $UT_{y_0}\mathbb{H}^n$ and define the visual map $V_y \colon UT_{y_0}\mathbb{H}^n \to UT_y\mathbb{H}^n$ as $V_y(u) = -D_y B_u$, so that every ray geodesic form y_0 has a corresponding geodesic ray form y such that they converge to the same point at infinity. Then if λ_y is the canonical measure on the unit sphere $UT_y\mathbb{H}^n$. Then the *visual measure at* $y \in \mathbb{H}^n$ is

$$\mu_y = (V_y^{-1})_*(\lambda_y),$$

the push-forward of λ_y to $UT_{y_0}\mathbb{H}^n$ by V_y^{-1}, which satisfies the same properties of the Patterson-Sullivan measures.

5.3. **The barycenter.** As before (X, g_0) denotes a compact negatively curved manifold, $\tilde{X}$ is identified with the unit ball in $\mathbb{R}^n$ and $\partial \tilde{X}$ with the unit sphere by choosing an origin 0. Now if λ is a measure on $\partial \tilde{X}$, let us define the function

$$\beta_\lambda(x) = \int_{\partial \tilde{X}} B^0_\theta(x) d\lambda(\theta).$$

Proposition 17. *If λ has no atom, the function β is strictly convex on $\tilde{X}$. Furthermore $\beta(x)$ goes to infinity when x goes to $\theta \in \partial \tilde{X}$ along a geodesic. Hence β has a unique critical point in $\tilde{X}$ which is a minimum which will be called the barycenter of the measure λ and denoted by $bar(\lambda)$.*

Proof. Since the metric $\tilde{g}$ on $\tilde{X}$ is negatively curved for each θ then the function $x \to B^0_\theta(x)$ is convex and therefore β_λ which is an average of such functions is also convex. It is in fact not difficult to show that β is strictly convex, indeed

$$Hess_x\beta_\lambda(-,-) = \int_{\partial\tilde{X}} Hess_x B^0_\theta(-,-)d\lambda(\theta).$$

is positive definite at each $x \in \tilde{X}$ if λ is no atomic.

One shows furthermore that $\beta(x) \to +\infty$ as x goes to infinity along a geodesic. $\square$

The final property of the barycenter that we will need is that it is equivariant with respect to isometries, i.e., if λ is an atomless probability measure on $\partial\tilde{X}$ and $\gamma \in SO_0(n,1)$ then

$$bar(\gamma_*(\lambda)) = \gamma(bar(\lambda)).$$

Indeed, let $y = \gamma(bar(\lambda))$ and let $x = \gamma^{-1}(y) = bar(\lambda)$. The the implicit equation for the barycenter tells us that,

$$\begin{aligned} 0 &= \int_{\partial\tilde{X}} D_x B^0_\theta(-)d(\lambda)(\theta) \\ &= \int_{\partial\tilde{X}} D_y B^0_{\gamma(\theta)}(D_x\gamma(-))d(\lambda)(\theta) \\ &= \left(\int_{\partial\tilde{X}} D_y B^0_\theta(-)d\gamma_*(\lambda(\theta)\right) \circ D_x\gamma. \end{aligned}$$

Since $D_x\gamma$ is invertible,

$$\int_{\partial\tilde{X}} D_y B^0_\theta(-)d\gamma_*(\lambda(\theta)) = 0.$$

so by uniqueness of solutions to the implicit equation, $y = bar(\gamma_*(\lambda))$.

5.4. **The barycentric extension.** Let (Y,g) and (X,g_0) be two $n-$dimensional compact and negatively curved manifolds. We assume that they are homotopically equivalent, i.e. that there exist two continuous maps

$$f\colon X \to Y \text{ and } h\colon Y \to X,$$

such that $f\circ h$ is homotopic to id_X and $h\circ f$ is homotopic to id_Y. Since both Y and X are negatively curved this hypothesis is equivalent to saying that their fundamental groups are isomorphic as abstract groups.

We are going to construct a smooth map $\bar{F}\colon \tilde{Y} \to \tilde{X}$ which we will call the *barycentric extension*, its definition is highly geometric and hence it becomes the most natural candidate for being an isometry between (Y,g) and (X,g_0).

It is know for the section 3 that if X and Y are homotopically equivalent one can obtain a quasi-isometry $\bar{f}$ between the respective universal covers $\tilde{X}$ and $\tilde{Y}$ which give rise to an homeomorphism between the boundaries at infinity

$$\partial\bar{f}\colon \partial\tilde{Y} \to \partial\tilde{X}$$

satisfying both

$$\partial \bar{f} \circ \gamma = \phi(\gamma) \circ \partial \bar{f}.$$

The Patterson-Sullivan measure described previously gives an equivariant map from $\tilde{Y}$ to the space $M_1(\partial\tilde{Y})$ of probability measures on $\partial\tilde{Y}$. As mentioned before μ_y has no atoms and can be push forward each measure by the continuous map $\bar{f}$ and thereby construct a map

$$y \in \tilde{Y} \to \partial \bar{f}_*(\mu_y) \in \mathcal{M}_1(\partial\tilde{Y}).$$

The $\phi-$equivariance property of $\partial\bar{f}$ shows that this map is $\phi-$equivariant with respect to the actions of $\pi_1(Y)$ on $\tilde{Y}$ and on $\mathcal{M}_1(\partial\tilde{Y})$. Finally, since $\partial\bar{f}$ is an homeomorphism, the measures $\bar{f}_*(\mu_y)$ are well defined and have not atom.

We can now define the map $\bar{F}$ by

$$\bar{F}(y) = bar(\bar{f}_*(\mu_y)).$$

which satisfy the equivariant relation. And gives rice to $F\colon Y \to X$ and let us notice that F induces also the isomorphism ρ between the two fundamental groups and hence is homotpic to f.

5.5. **Jacobian of the barycentric extention and the BCG proof.** Let (Y, g) and (X, g_0) be two compact negatively curved Riemannian $n-$manifolds. We assume that (X, g_0) is hyperbolic and that Y and X are homotopically equivalent. Let us proceed with the proof of the main result of this section.

Proof of Proposition 16 . For the sake of simplicity we shall use the same notation for the natural map F and its pull-back to the universal cover. Let us call $\{\mu_y\}_{y\in\tilde{Y}}$ the family of Patterson measures on $\partial\tilde{Y}$. We have that the natural map is defined by the implicit equation

$$0 = \int_{\partial\tilde{X}} D_{F(y)} B^0_\theta(-) d(\tilde{f}_*(\mu_y))(\theta) = \int_{\partial\tilde{Y}} D_{F(y)} B^0_{\bar{f}_*(\alpha)}(-) d(\mu_y)(\alpha) =$$

which is a vector valued equation. Equivalently one has,

$$0 = \int_{\partial\tilde{Y}} D_{F(y)} B^0_{\bar{f}_*(\alpha)}(-) e^{-h(g)B_\alpha(y)} d(\mu_o)(\alpha).$$

Let us insist on the fact that B_0 (B) is the Busemann function of $(\tilde{X}, g_0)$ $((\tilde{Y}, g))$. we choose a frame $\{E_i(x)\}_{i=1\ldots n}$ of $T_z\tilde{X}$ depending smoothly on x. Let us define the functions:

$$G_i(x, y) = \int_{\partial\tilde{Y}} D_x B^0_{\bar{f}_*(\alpha)}(E_i(x)) e^{-h(g)B_\alpha(y)} d(\mu_o)(\alpha)$$

$$G\colon \tilde{X} \times \tilde{Y} \to \mathbb{R}^n \quad \text{and} \quad proj_i \circ G = G_i$$

Then

$$G(F(y), y) = 0.$$

Since the Busemann functions are smooth with respect the first variable and $\partial\tilde{Y}$ is compact, we have that G is a smooth map. Then the proof of the fact that F is C^1

is a simple application of the implicit function theorem. In fact for some $\epsilon > 0$ let $\gamma\colon (-\epsilon, \epsilon) \to \mathbb{H}^n$ be a geodesic segment with $\gamma(0) = x_0$ and $\gamma'(0) = u$. Then

$$\partial_t G_i(\gamma(t), y)|_{t=0} = \int_{\partial\tilde{Y}} D_{E_i(x_0)} D_{x_0} B^0_{\bar{f}_*(\alpha)}(E_i(x)) d(\mu_y)(\alpha) = Hess_{x_0}(\beta_{\mu_y})(E_i(x_0), u)$$

where the last equality holds because $E_i(\gamma(t))$ is parallel along γ. It follows that $Hess_{x_0}(\beta_{\mu_y})$ is positive definitive. So for all non-zero $u \in T_{x_0}\tilde{X}$,

$$\partial_t G_i(\gamma(t), y)|_{t=0} > 0.$$

Hence the derivative of each $G_i(y, x)$ with respect the first variable has trivial kernel, so the same holds for the derivative of G with respect the first variable.

As usual the implicit function theorem gives the existence of the differential of the implicitly defined function F and a formula for its differential.

$$D^1_{(F(y),y)} G \circ D_y F = D^2_{(F(y),y)} G$$

and for $u \in T_{F(y)}\tilde{X}$ and $v \in T_y\tilde{Y}$,

$$\begin{aligned} Hess_{F(y)}(\beta_{\bar{f}_*(\mu_y)})(D_yF(v), u) &= \int_{\partial\tilde{Y}} D_{D_yF(v)} D_{F(y)} B^0_{\bar{f}_*(\alpha)}(u) d(\mu_y)(\alpha) \\ &= h(g) \int_{\partial\tilde{Y}} D_{F(y)} B^0_{\bar{f}_*(\alpha)}(u) D_y B_\alpha(v) d(\mu_y)(\alpha). \end{aligned}$$

This equality is to be understood as an equality between bilinear forms. Let us introduce the following quadratic forms whose induced symetric endomorphism with respect to the metric $g_0 :=<,>_0$ are K and H on $T_{F(y)}\tilde{X}$.

$$< K_{F(y)}u, u >_0 := Hess_{F(y)}(\beta_{\bar{f}_*(\mu_y)})(u, u) = \int_{\partial\tilde{X}} D_u D_{F(y)} B^0_\theta(u) d(\bar{f}_*(\mu_y)(\theta),$$

$$< H_{F(y)}u, u > := \int_{\partial\tilde{X}} [D_{F(y)} B^0_\theta(u)]^2 d(\bar{f}_*(\mu_y)(\theta).$$

Remark that we have the following induce function:

$$u \in T_{F(y)}\tilde{X} \longrightarrow (\theta \to D_{F(y)} B^0_\theta(u)) \in L^2(\partial\tilde{X}, \bar{f}_*(\mu_y)).$$

For $u \in T_{F(y)}\tilde{X}$ and $v \in T_y\tilde{Y}$, the Cauchy-Schwarz in $L^2(\partial\tilde{Y}, \mu_y)$ inequality gives

$$\begin{aligned} &< K_{F(y)} D_yF(v), u >_0 = h(g) \int_{\partial\tilde{Y}} D_{F(y)} B^0_{\bar{f}_*(\alpha)}(u) D_y B_\alpha(v) d(\mu_y)(\alpha) \\ &\le h(g) \left(\int_{\partial\tilde{Y}} [D_{F(y)} B^0_{\bar{f}_*(\alpha)}(u)]^2 d(\mu_y)(\alpha) \right)^{\frac{1}{2}} \left(\int_{\partial\tilde{Y}} [D_y B_\alpha(v)]^2 d(\mu_y)(\alpha) \right)^{\frac{1}{2}} \\ &= h(g) \left(< H_{F(y)}u, u >_0\right)^{\frac{1}{2}} \left(\int_{\partial\tilde{Y}} [D_y B_\alpha(v)]^2 d(\mu_y)(\alpha) \right)^{\frac{1}{2}}. \end{aligned}$$

We remark that K is invertible since the induced bilinear form is the Hessian of the strictly convex function $\beta_{\bar{f}_*(\mu_y)}$

Lemma 18. *With the above notation*

$$|Jac_y F| \leq \frac{h(g)^n}{n^{n/2}} \frac{(detH)^{1/2}}{detK}.$$

Proof of Lemma 18. If $D_y F$ has not maximal rank, the the inequality is clear. We can assume that $D_y F$ is invertible. Let us take $\{u_i\}$ orthonormal basis of $T_{F(y)}\tilde{X}$ which diagonalizes the endomorphism H. And take and orthonormal basis $\{v_i\}$ of $T_y\tilde{Y}$ such that the matrix $K \circ D_y F$ written in the basis $\{v_i\}$ for $T_y\tilde{Y}$ and $\{u_i\}$ for $T_{F(y)}\tilde{X}$ is triangular. Then

$$det(K \circ D_y F) = (detK)(Jac_y F) = \prod_{i=1}^{n} < K_{F(y)} D_y F(v_i), u_i >_0 .$$

Here we identify endomorphisms with matrices using the basis involved.

$$(detK)(Jac_y F) \leq h(g) \prod_{i=1}^{n} \left(< H_{F(y)} u_i, u_i >_0\right)^{\frac{1}{2}} \prod_{i=1}^{n} \left(\int_{\partial\tilde{Y}} [D_y B_\alpha(v_i)]^2 d(\mu_y)(\alpha)\right)^{\frac{1}{2}} .$$

By the choice of the basis

$$\prod_{i=1}^{n} \left(< H_{F(y)} u_i, u_i >_0\right)^{\frac{1}{2}} = (detH)^{1/2},$$

$$\prod_{i=1}^{n} \left(\int_{\partial\tilde{Y}} [D_y B_\alpha(v_i)]^2 d(\mu_y)(\alpha)\right)^{\frac{1}{2}} \leq \left(\frac{\int_{\partial\tilde{Y}} [D_y B_\alpha(v_i)]^2 d(\mu_y)(\alpha)}{n}\right)^{\frac{n}{2}} \leq \frac{1}{n^{n/2}},$$

since $\sum [D_y B_\alpha(v_i)]^2 = \|D_y B_\alpha\|^2 = 1$ and μ_y is a probability measure, one gets the desired inequality:

$$|Jac_y F| \leq \frac{h(g)^n}{n^{n/2}} \frac{(detH)^{1/2}}{detK}.$$

As we have showed in the section 5.1,

$$Hess_x(B_\theta)(u,v) =< u, v >_0 - D_x B_\theta^0(u) D_x B_\theta^0(v).$$

which gives after integration

$$K = I - H.$$

Also notice that

$$trace(H) = \sum_{i=1}^{n} < H_{F(y)} u_i, u_i >_0 = \int_{\partial\tilde{X}} \sum_{i=1}^{n} [D_{F(y)} B_\theta^0(u_i)]^2 d(\tilde{f}_*(\mu_y)(\theta) = 1.$$

The Proposition 16 follows from the next result:

If H is a positive symmetric matrix with $trace(H) = 1$ and $n \geq 3$. Then

$$\frac{detH}{(det(I-H))^2} \leq (\frac{n}{(n-1)^2})^n \quad \text{and equality holds if and only if} \quad H = \frac{1}{n} I.$$

If $|Jac_y F| = (\frac{h(g)}{h(g_0)})^n$ then

$$H_{F(y)} = \frac{1}{n} I \quad \text{and} \quad K_{F(y)} = \frac{n-1}{n} I = \frac{h(g_0)}{n} I.$$

It follows that

$$\frac{h(g_0)}{n} < D_y F(v), u >_0 \leq \frac{h(g)}{n^{1/2}} \|u\|_0 \left(\int_{\partial \tilde{Y}} [D_y B_\alpha(v)]^2 d(\mu_y)(\alpha) \right)^{\frac{1}{2}}$$

for all $u \in T_{F(y)} \tilde{X}$ and $v \in T_y \tilde{Y}$. By taking the supremum in $u \in T_{F(y)} \tilde{X}$ such that $\|u\|_0 = 1$, one gets

$$\|D_y F(v)\|_0 \leq n^{1/2} \frac{h(g)}{h(g_0)} \left(\int_{\partial \tilde{Y}} [D_y B_\alpha(v)]^2 d(\mu_y)(\alpha) \right)^{\frac{1}{2}}$$

for all $v \in T_y \tilde{Y}$. Let L be the endomorphism of $T_y \tilde{Y}$ defined by

$$L = (D_y F)^* \circ (D_y F)$$

and $\{v_i\}$ a $g-$orthonormal basis of $T_p X$. Then we have

$$trace(L) = \sum_{i=1}^{n} < L v_i, v_i >_g = \sum_{i=1}^{n} < D_y F(v_i), D_y F(v_i) >_0 \leq n \left(\frac{h(g)}{h(g_0)} \right)^2,$$

where we have again used the fact that $\|D_y B_\alpha\|_g = 1$. We have know

$$\left(\frac{h(g)}{h(g_0)} \right)^{2n} = |Jac_y F|_0^2 = detL \leq \left(\frac{trace(L)}{n} \right)^n \leq \left(\frac{h(g)}{h(g_0)} \right)^{2n}$$

Therefore the determinant of L is $\left(\frac{trace(L)}{n}\right)^n$ and $L = \left(\frac{h(g)}{h(g_0)}\right)^2 I$. This precisely means that $D_y F$ is an isometry composed with a homothety of ratio $\frac{h(g)}{h(g_0)}$.

$\square$

6. Quasi-conformal proof

Let X and Y be metric spaces with $f \colon X \to Y$ an homeomorphism. The mapping f is called *quasi-conformal* if the function

$$H_f(x) := \limsup_{r \to 0} \frac{\sup\{\|f(x) - f(y)\| \mid \|x - y\| = r\}}{\inf\{\|f(x) - f(y)\| \mid \|x - y\| = r\}},$$

is bounded from above in X. A quasi-conformal mapping is called $K-$quasi-conformal if the function H_f is bounded from above by K in X. In this section we are going to work the case when $X = Y = S^{n-1} = \partial \mathbb{H}^n$.

Proposition 19. *If $\bar{f} \colon \mathbb{H}^n \to \mathbb{H}^n$ is a $(\lambda, \epsilon)-$quasi-isometry then $\partial \bar{f} \colon \partial \mathbb{H}^n \to \partial \mathbb{H}^n$ defined in section 3.3 is a quasi-conformal homeomorphism.*

Proof. By Lemma 13 we have that $\bar{f}$ is an homeomorphism. According to the definition, it is enough to verify quasi-conformity at each particular point x with uniform estimates on the function $H_f(x)$. Thus, after composing $\partial\bar{f}$ with a Möbius transformation, we can take $x = 0 = \partial\bar{f}(x)$ and $\partial\bar{f}(\infty) = \infty$.

Take a euclidean sphere $S_r(0)$ in $\mathbb{R}^{n-1}$ with the center at the origin. This sphere is the ideal boundary of a hyperplane $P_r \subset \mathbb{H}^n$ which is orthogonal to the vertical segment L connecting 0 to ∞. By Lemma 12,

$$\operatorname{diam} \pi_L \bar{f}(P_r) \leq c,$$

where π_L denote the orthogonal projection onto the geodesic line L and c some constant depending only on λ and ϵ. The projection π_L extends naturally to $\partial\mathbb{H}^n$. We conclude

$$\operatorname{diam} \pi_L \partial\bar{f}(S_r(0)) \leq c.$$

Thus $\partial\bar{f}(S_r(0))$ is contained in a spherical shell

$$\{x \in \mathbb{R}^{n-1} \mid k_1 \leq \|x\| \leq k_2\},$$

where $\log(k_1/k_2) \leq c$. This implies that the function $H_{\partial\bar{f}}(0)$ is bounded from above by $K := e^c$. We conclude that the mapping $\partial\bar{f}$ is $K-$quasi-conformal. □

The following theorem is due to Radmacher-Stepanov (see [14]). It is a deep result in geometric measure theory, it establishes strong regularity properties for quasi-conformal homeomorphism.

Theorem 20. *Any $K-$quasi-conformal homeomorphism of S^{n-1} is absolutely continuous with respect to Lebesgue measure and is differentiable a.e. with $K-$quasi-conformal differential.*

6.1. **Mostow rigidity proof.** Now we can prove Mostow's Rigidity Theorem. First we start by proving the main thoerem of this section.

Proof of Theorem 3. We will give the proof reported in (Thm. 8.34, [7]). The limit point ξ is a conical point if it has the following property. Let γ be a geodesic ray ending at ξ. Given a point $y_0 \in \mathbb{H}^n$ there exist $r > 0$ such that there is an infinite subsequence of the orbit $\Gamma_1(y_0)$ that lies in a $r-$tubular neighborhood about γ and hence converges to ξ. A loxodromic fixed point is always a conical limit point but a parabolic fixed point is not.

We may assume that $\xi = 0 = h(\xi)$ and that y_0 lies on the vertical axis rising from $x = 0$ in $\mathbb{H}^n$. Let γ be a vertical segment descending from y_0 to $z = 0$. There is an infinite sequence $g_n \in \Gamma_1$ such that for $r > 0$ and each large index n, the hyperbolic distance $d(g_n(y_0), \gamma) < r$. Find the point $y_n \in \gamma$ that is closest to $g_n(y_0)$, it is with in distance r. Then find $a_n > 0$ such that the hyperbolic transformation $A_n(\bar{x}) = a_n\bar{x}$ with in γ that takes y_0 to y_n, further $\lim_{n\to 0} a_n = 0$. Passing to a subsequence if necessary we may also assume that the $\lim_{n\to 0} g_n^{-1}A_n = B$ exist as a Möbius transformation (because the distance of $g_n^{-1}A_n(y_0)$ to y_0 is uniformly bounded by r).

Set

$$h_n(x) := a_n^{-1}h(a_n(x)) = A_n^{-1} \circ h \circ A_n(x) \quad \text{and} \quad x \in \mathbb{R}^n.$$

As h is differentiable at $x = 0$ with nonzero derivative means that there is a linear transformation $L \in GL(n-1, \mathbb{R})$, such that:

$$h(x) = L(x) + \epsilon(x)x \quad \text{and} \quad \lim_{x \to 0} \frac{\epsilon(x)}{x} = 0.$$

Using the commutativity of the diagonal matrix A_n with L, we have:

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} A_n^{-1} \circ h \circ A_n(x) = L(x) \quad \text{uniformly on compact subsets.}$$

It now follows that

$$\lim_{n\to\infty} A_n^{-1}\Gamma_1 A_n = \lim_{n\to\infty} A_n^{-1} g_n \Gamma_1 g_n^{-1} A_n = B^{-1}\Gamma_1 B.$$

This implies that the sequence groups $\{A_n^{-1}\Gamma_1 A_n\}$ converges geometrically, i.e. every $B^{-1}gB$ is the limit of elements of the approximations $\{A_n^{-1}\Gamma_1 A_n\}$, namely $B^{-1}gB = \lim_{n\to\infty}(A_n^{-1}g_n)g(g_n^{-1}A_n)$ Conversely, the limit of any convergent sequence of elements of $\{A_n^{-1}\Gamma_1 A_n\}$ lies in $B^{-1}\Gamma_1 B$, namely

$$h := \lim_{n\to\infty} A_n^{-1} h_n A_n = \lim_{n\to\infty} A_n^{-1} g_n (g_n^{-1} h_n g_n) g_n^{-1} A_n = B^{-1}(\lim_{n\to\infty} g_n^{-1} h_n g_n) B.$$

Recall that $h\Gamma_1 h^{-1} \subset SO_0(n, 1)$. Given $g \in \Gamma_1$,

$$L \circ B^{-1} g B \circ L^{-1} = \lim_{n\to\infty} A_n^{-1} h g_n g g_n^{-1} \circ h^{-1} A_n$$

The element on the left is therefore a Möbius transformation. We have established that LkL^{-1} is a Möbius transformation for any $k \in B^{-1}\Gamma_1 B$. Since not all elements of Γ_1 fix $B(\infty)$, there exist $k \in B^{-1}\Gamma_1 B$ with $k(\infty) \notin \{\infty, 0\}$.

Lemma 21. *Suppose that $k \in SO_0(n, 1)$ is such that $k(\infty) \notin \{\infty, 0\}$, $L \in GL(n-1, \mathbb{R})$ is an element which conjugates k to $LkL^{-1} \in SO_0(n, 1)$. Then L is a Euclidean similarity.*

Proof. Suppose that L is not a similarity. According to our assumption , $Lk^{-1}(\infty) \neq 0$. Let P be a hyperplane in $\mathbb{R}^{n-1}$ which contains the origin but does not contain $Lk^{-1}(\infty)$. Then $kL^{-1}(P)$ is a round sphere Σ in $\mathbb{R}^{n-1}$. Since L is not a similarity, the image $L(\Sigma)$ is an ellipsoid which is not round sphere. Hence the composition LkL^{-1} does not send planes to round spheres and therefore it is not Möbius. $\square$

Thus we have to prove that h is conformal at the point 0. To conclude that h is Möbius we need to use the fact that $h\Gamma_1 h^{-1} \subset SO_0(n, 1)$ once again.

Pick three distinct points $p_1, p_2, p_3 \in S^{n-1}$. For any homeomorphism $F\colon S^{n-1} \to S^{n-1}$ set a normalization $N(F) = \bar{F} \circ F$ where the Möbius transformation $\bar{F}$ is the uniquely chosen so that $N(F)$ fixes each p_i. The normalization is uniquely determined by F up to post-composition with an element of the compact subgroup $K_{p_i} \subset SO_0(n, 1)$ which fixes the round circle in S^{n-1} containing $\{p_1, p_2, p_3\}$. We let $\tilde{N}(F)$ denote the projection of $N(F)$ to the quotient $K_{p_i}/Homeo(S^{n-1})$. Thus, $\tilde{N}(F) = \tilde{N}(g \circ F)$ for all $g \in SO_0(n,1)$. Upon setting $u_n = g_n^{-1}A_n$ so that $\lim u_n = B$.

$$\tilde{N}(h_n) = \tilde{N}(A_n^{-1}hA_n) = \tilde{N}(hA_n) = \tilde{N}(hg_nu_n) = \tilde{N}(g_n'hu_n) = \tilde{N}(hu_n).$$

Going to the limit,

$$\tilde{N}(L) = \lim_{n\to\infty} \tilde{N}(h_n) = \tilde{N}(hB).$$

Since L and B are Möbius transformation, h must be one as well. $\square$

To prove Mostow's Rigidity Theorem by using the Theorem 3, take $\Gamma_1 = \pi_1(M_1)$. By Proposition 13, Proposition 19 and Theorem 20 there exist a $\phi-$equivariant quasi-conformal homeomorphism $\partial\bar{f}\colon S^{n-1} \to S^{n-1}$ a.e. differentiable on S^{n-1}, which has a.e. non-zero Jacobian determinants because $n \geq 3$. Moreover, as $\pi_1(M_1)$ is cocompact, every point in $\partial\mathbb{H}^n$ is a conical limit point. And the $\phi-$equivariant condition implies that $(\partial\bar{f})\pi_1(M_1)(\partial\bar{f})^{-1} \subset SO_0(n,1)$. So $\partial\bar{f}$ is a Möbius transformation. Finally, the Mostow's Theorem proof ends as in Gromov-Thurston's proof (see section 4.5).

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