QUASI-CONFORMAL MAPPINGS IN n-SPACE AND THE RIGIDITY OF HYPERBOLIC SPACE FORMS

by G. D. MOSTOW

INTRODUCTION

The phenomenon that we wish to present in this paper can perhaps best be introduced by directing attention to the familiar case of compact Riemann surfaces.

Let Y and Y′ be diffeomorphic compact Riemann surfaces of genus greater than 1. From the classical theory of uniformizing parameters, we may regard Y and Y′ as two dimensional Riemannian manifolds having constant negative curvature. We know moreover that Y and Y′ need *not* be conformally equivalent. A special case of our main theorem asserts (cf. Corollary 12.2):

Let Y *and* Y′ *be diffeomorphic compact Riemannian manifolds having constant negative curvature. Then* Y *and* Y′ *are conformally equivalent, provided their dimension is greater than two.*

The conformal mapping Ψ of Y to Y′ can be chosen so that it is homotopic to any given diffeomorphism Φ of Y to Y′, and this conformal mapping Ψ is unique.

Corollary 12.2 deals with the more general case that Y and Y′ have *finite measure.* To conclude that they are conformally equivalent, we have to make the hypothesis that there is a homeomorphism $\Phi : Y \to Y'$ which is *quasi-conformal.* Indeed, the backbone of the method in this paper is the application of the theory of quasi-conformal mappings to the study of the following question:

Suppose Φ is a homeomorphism of n-dimensional hyperbolic space X onto itself, and suppose Γ and Γ' are groups of isometries of X onto X. (Γ and Γ' correspond to the fundamental groups of Y and Y′.) Let $\theta : \Gamma \to \Gamma'$ be an isomorphism and assume

$$(*) \qquad \Phi(\gamma x) = \theta(\gamma)\Phi(x)$$

for all $x \in X$, $\gamma \in \Gamma$. Then $\Gamma' = \Phi\Gamma\Phi^{-1}$ so that by hypothesis Γ and Γ' are conjugate in the group of homeomorphisms of X onto X. Are Γ and Γ' conjugate in the *group* G *of isometries* of X?

The central idea of our method is to study the effect of Φ " at infinity ". That is, if we look at the two dimensional case from the point of view of transformation groups, what distinguishes the case that Γ and Γ' are conjugate from the contrary case? If we think of X as the interior of the unit disc, and the compact Riemann surfaces as $Y=\Gamma\backslash X$ and $Y'=\Gamma'\backslash X$, then the map Φ will be smooth on X if Φ is a diffeomorphism, but it will *not* be smoothly extendible to the *boundary* unless Γ is conjugate to Γ' in the group G of isometries.

This last assertion is valid more generally for arbitrary symmetric Riemannian spaces X (cf. Mostow, On the conjugacy of subgroups of semi-simple groups, *Proc. of Symposia on Pure Math*, v. 9 (1966), pp. 413-419) and that general fact motivates the viewpoint adopted here.

Indeed, onde our attention is directed to the behaviour of Φ at the boundary of X, the relevance of quasi-conformal mappings comes into view. The young Japanese mathematician, A. Mori, in a posthumous article: On quasi-conformality and pseudo-analyticity, *Trans. Amer. Math. Soc.*, v. 84 (1957), pp. 56-77, proved in dimension 2:

Theorem **(10.1)**. — *A quasi-conformal mapping of the open n-ball induces a homeomorphism on the boundary.*

Mori's paper contains the seeds of not only the generalization to n dimensions, but even of the more striking Theorem (10.2) which states in effect:

A quasi-conformal mapping of the open n-ball induces a quasi-conformal mapping on the boundary. This result was first proved in dimension 3 by F. W. Gehring in " Rings and quasi-conformal mappings in space ", *Trans. Amer. Math. Soc.*, v. 103 (1962), pp. 353-393. Unable, as was Mori, to use an inequality of Teichmüller on the moduli of rings because the proof rested on planar conformal mapping theory, Gehring achieved the requisite inequality in dimension 3 by proceeding from Loewner's " conformal capacity " (cf. Gehring, Symmetrization of rings in space, *Trans. Amer. Math. Soc.*, v. 101 (1961), pp. 449-519). In this paper we develop the theory of quasi-conformal mappings in n-space adapting in large measure the procedure of Gehring.

The proof of the main theorem of this paper, Theorem (12.1), comes in the last section. It depends on an analysis of the boundary mapping induced by a quasi-conformal mapping of hyperbolic n-space which satisfies condition $(*)$ above. A key role in this analysis is played by an ergodic theorem for semi-groups in a semi-simple Lie group due essentially to F. I. Mautner. We have tried to reduce to the minimum the mathematics that our proof depends on, and thereby we have arrived at a relatively self-contained presentation. The expert on quasi-conformal mappings may recognize here and there some new proofs of known theorems.

The author lectured on quasi-conformal mappings at the University of Paris in the Fall of 1966, and wishes to acknowledge his debt to Professor C. Chevalley whose critical comments added to the clarity of this account.

CONTENTS

§ 1. THE MOEBIUS GROUP

Definition. — *Moebius n-space is the one-point compactification of euclidean n-space* $\mathbf{R}^n$; *it is denoted by* $\mathbf{R}^n \cup \infty$.

$GM(n)$, *the Moebius group of Moebius n-space is the group of transformations generated by reflections in the spheres and* $(n-1)$ *planes of* $\mathbf{R}^n$.

The composition of the reflection $\sigma_\lambda : p \to \dfrac{\lambda^2 p}{|p|^2}$ of the sphere with center at the origin and radius λ and the reflection σ_1 is

$$p \to \frac{\lambda^2 p}{|p|^2} \to \frac{\lambda^2 p}{|p|^2}\left|\frac{\lambda^2 p}{|p|^2}\right|^{-2} = \lambda^{-2} p.$$

Similarly the composition of reflections in two parallel $(n-1)$-planes at a distance d gives a translation by $2d$ along the normal to the planes. Since the group generated by translations and stretchings permute the spheres of $\mathbf{R}^n$ transitively, and since σ_1 carries spheres through the origin into planes, we see easily that $GM(n)$ is generated by σ_1 together with the subgroup of translations and stretchings.

Moebius space can be given a conformal structure invariant under $GM(n)$ and compatible with the euclidean metric structure of $\mathbf{R}^n$.

Let S^n be a sphere in $\mathbf{R}^{n+1}$ i.e. $(\xi_1 - a_1)^2 + \ldots + (\xi_{n+1} - a_{n+1})^2 = r^2$, and let p_∞ be a point of S^n. Let F be an n-plane in $\mathbf{R}^{n+1}$ parallel to the tangent plane to S^n at p_∞ and different from it. *Stereographic projection from* p_∞ *of* S^n *onto* F is the mapping π of $S^n - p_\infty$ onto F given by projecting from p_∞. The remarkable property of stereographic projection is that it preserves angles.

For let L and L′ be any two lines in F through a point x. Let N and N′ denote the 2-planes determine by $p_\infty \cup L$ and $p_\infty \cup L'$ respectively. Then N and N′ cut S^n in two circles C and C′ which pass through p_∞ and $\pi^{-1}(x)$. The angle formed by C and C′ at $\pi^{-1}(x)$ equals the angle formed by them at p_∞. (This fact involves looking only at the three dimensional plane spanned by $p_\infty \cup L \cup L'$ and its intersection with S^n which is an ordinary 2-sphere.) On the other hand, the angle formed by C and C′ at p_∞ equals the angle formed by L, L′, since F is parallel to the tangent plane to S^n at p_∞. Hence $\sphericalangle(L, L')$ equals the angle formed by C and C′ at $\pi^{-1}(x)$.

It is useful to have the analytic expression for stereographic projection. Let S^n be the n-sphere $\xi_1^2+\xi_2^2+\ldots+\xi_n^2+\left(\xi_{n+1}-\frac{1}{2}\right)^2=1/4$, let $\mathbf{R}^n$ be the plane $\xi_{n+1}=0$, let $\xi=(\xi_1, \ldots, \xi_{n+1})$ be a point on S^n, and let $x=(x_1, \ldots, x_n, 0)=\pi(\xi)$, π denoting stereographic projection onto $\mathbf{R}^n$ from $p_\infty=(0, \ldots, 1)$. Then

$$\frac{\xi_i}{x_i}=\frac{1-\xi_{n+1}}{1} \quad \text{and} \quad \frac{1-\xi_{n+1}}{1}=\frac{\cos\theta}{(1+|x|^2)^{1/2}}=\frac{\dfrac{1}{(1+|x|^2)^{1/2}}}{(1+|x|^2)^{1/2}}=\frac{1}{1+|x|^2}.$$

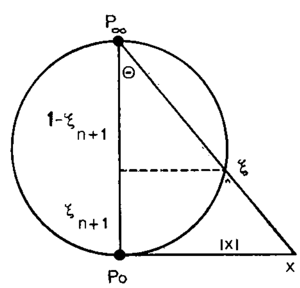
Thus

$$[\pi] \qquad x_i=\frac{\xi_i}{1-\xi_{n+1}} \qquad (i=1, \ldots, n)$$

and

$$[\pi^{-1}] \qquad \xi_i=\frac{x_i}{1+|x|^2}, \qquad \xi_{n+1}=\frac{|x|^2}{1+|x|^2}.$$

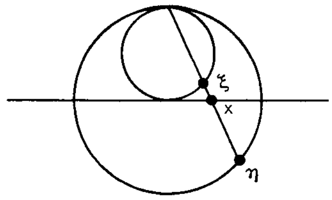
Note. — Stereographic projection from $(0, 0, \ldots, 1)$ of the unit sphere $\sum_{i=1}^{n+1}\xi_i^2=1$ onto the plane $\xi_{n+1}=0$ is given by the same formula $[\pi]$:



$$\begin{cases} \xi_i=\dfrac{\eta_i}{2} & (i=1, \ldots, n) \\ \xi_{n+1}=\dfrac{\eta_{n+1}+1}{2} \end{cases}$$

$$[\pi] \qquad x_i=\frac{\xi_i}{1-\xi_{n+1}}=\frac{\dfrac{\eta_i}{2}}{1-\dfrac{\eta_{n+1}+1}{2}}=\frac{\eta_i}{1-\eta_{n+1}}.$$

But for π^{-1}:



$$\begin{cases} \eta_i = 2\xi_i = \dfrac{2x_i}{1+|x|^2}, & i=1, \ldots, n \\ \eta_{n+1} = 2\xi_{n+1} - 1 = \dfrac{2|x|^2}{1+|x|^2} - 1 = \dfrac{|x|^2-1}{|x|^2+1}. \end{cases}$$

If σ_1 is the reflection in the unit sphere of $\mathbf{R}^n$, then $\pi\sigma_1\pi^{-1}$ is the reflection of S^n in its equatorial plane $\xi_{n+1}=\frac{1}{2}$ and is thus *conformal*, that is, it preserves angles. The stretching $h_\lambda : x \to \lambda x$ of $\mathbf{R}^n$ corresponds to the mapping of S^n given by $\xi_i \to \dfrac{\lambda x_i}{1+\lambda^2|x|^2}$. The effect of this mapping for large $|x|$ is seen by expressing

$$\frac{\lambda x_i}{1+\lambda^2|x|^2} = \frac{\lambda x_i}{1+|x|^2}\,\frac{1+|x|^2}{1+\lambda^2|x|^2} = \lambda\xi_i \cdot \frac{1+|x|^2}{1+\lambda^2|x|^2}.$$

Thus at p_∞, this mapping is approximately $\xi_i \to \lambda^{-1}\xi_i$. In particular it is analytic and conformal at p_∞.

Regarding $GM(n)$ as a transformation group on the n-sphere S^n, one sees at once that it is the group generated by the group of rotations of S^n, the reflection in some equatorial sphere, and the one parameter group corresponding to the stretchings h_λ.

Theorem **(1.1)**. — $GM(n)$ *is isomorphic as a transformation group to* $O(1, n+1)/(\pm 1)$, *the orthogonal group of the quadratic form* $y_0^2 - y_1^2 \ldots - y_{n+1}^2$, *operating on the projective variety* $y_0^2 - y_1^2 - \ldots - y_{n+1}^2 = 0$. *Moreover, if we set* $\eta_i = \dfrac{y_i}{y_0}$ $(i=1, \ldots, n+1)$, *then* $O(1, n+1)/(\pm 1)$ *is the Moebius group of* $\eta_1^2 + \ldots + \eta_{n+1}^2 = 1$.

Proof. — Introducing non-homogeneous coordinates, we can identify $O(1, n+1)/(\pm 1)$ with a group of transformations of the n-sphere $S^n : \eta_1^2 + \ldots \eta_{n+1}^2 = 1$. $O(1, n+1)$ clearly contains the group $O(n+1)$, the orthogonal group of an S^n; also $O(1, n+1)$ contains the reflection $\eta_1 \to -\eta_1$. To prove that $O(1, n+1)/(\pm 1)$ contains $GM(n)$, it suffices only to verify that it contains the one parameter group corresponding to stretchings of $\mathbf{R}^n$.

Let h'_λ denote the element of $GL(n+2, \mathbf{R})$ given by

$$\begin{aligned} y_0 + y_{n+1} &\to \lambda(y_0 + y_{n+1}) \\ y_0 - y_{n+1} &\to \lambda^{-1}(y_0 - y_{n+1}) \\ y_i &\to y_i \qquad (i=1, \ldots, n). \end{aligned}$$

Taking stereographic projection from (0, 0, ..., 1) onto the plane $\eta_{n+1}=0$, we get

$$x'_i=\frac{\eta'_i}{1-\eta'_{n+1}}=\frac{y'_i}{y'_0-y'_{n+1}}=\lambda.\frac{y_i}{y_0-y_{n+1}}=\lambda x_i \qquad (i=1, \ldots, n)$$

which is the stretching h_λ on $\mathbf{R}^n$.

Theorem (**1.2**). — *The subgroup* G′ *of* GM(n) *which stabilizes the hemisphere* $S_-:\eta_{n+1}<0$ *is isomorphic to* GM($n-1$) *under the restriction homomorphism into its action on the equatorial* ($n-1$)*-sphere*: $\eta_{n+1}=0$. *Moreover* G′ *operates transitively on* S_- *and keeps invariant a positive definite quadratic differential form* ds^2. *Under stereographic projection from* (0, 0, ..., 1), S_- *maps onto the unit ball* $x_1^2+\ldots+x_n^2<1$ *and its invariant metric* ds^2 *becomes* $\dfrac{dx_1^2+\ldots+dx_n^2}{(1-|x|^2)^2}$ (up to a constant factor).

Proof. — We continue the notations employed above. Then G′ can be identified modulo a factor ± 1, with the subgroup of $O(1, n+1)$ which keeps invariant the half-space $y_{n+1}<0$ in the $n+2$ dimensional cartesian space $(y_0, \ldots, y_{n+1})$. It follows that any element g in this subgroup G′ sends y_{n+1} into cy_{n+1}, $c>0$. Since g keeps invariant the plane $y_{n+1}=0$ as well as its orthogonal complement, it follows from the invariance of $y_0^2-\ldots-y_{n+1}^2$ that $c=1$ and G′ preserves $y_0^2-y_1^2-\ldots-y_n^2$; that is, G′ can be identified with $O(1, n)/(\pm 1)$; by the preceding theorem, $G'\approx GM(n-1)$.

We next define the G′-invariant metric on S_-. In $\mathbf{R}^{n+2}$, let A denote one of the two connected components of the n-dimensional surface

$$\begin{cases} y_0^2-y_1^2-\ldots-y_{n+1}^2=0 \\ y_{n+1}=-1 \end{cases}$$

say the half defined by $y_0>0$. Then A is a cross-section for the fiber map of the half-cone

$$\begin{cases} y_0^2-y_1^2-\ldots-y_{n+1}^2=0 \\ y_{n+1}<0 \end{cases}$$

into the corresponding projective variety, and indeed A maps onto the hemisphere S_-:

$$\begin{cases} \eta_1^2+\ldots+\eta_{n+1}^2=1 \\ \eta_{n+1}<0 \end{cases}.$$

The differential form $ds_L^2=dy_0^2-dy_1^2-\ldots-dy_n^2$ is clearly invariant under $G'=O(1, n)\bmod(\pm 1)$. We compute the form induced by ds_L^2 on S_-:

From $y_i=\eta_i y_0$ $(i=1, \ldots, n)$, and $y_0^2-\sum_{i=1}^n y_i^2=1$, we get

$$y_0^2(1-\sum_{i=1}^n \eta_i^2)=1$$

$$y_0=(1-\sum_{i=1}^n \eta_i^2)^{-1/2}$$

$$dy_0 = \frac{\sum_i \eta_i d\eta_i}{(1-\sum_i \eta_i^2)^{3/2}}$$

$$dy_i = y_0 d\eta_i + \eta_i dy_0 = \frac{d\eta_i}{(1-\sum_j \eta_j^2)^{1/2}} + \frac{\eta_i \sum_j \eta_j d\eta_j}{(1-\sum_j \eta_j^2)^{3/2}}$$

$$= (1-\sum_j \eta_j^2)^{-3/2}(\eta_i \sum_j \eta_j d\eta_j + (1-\sum_j \eta_j^2) d\eta_i)$$

$$dy_0^2 - \sum_{i=1}^n dy_i^2 = (1-\sum_j \eta_j^2)^{-3}[(\sum_j \eta_j d\eta_j)^2 - \sum_i (\eta_i \sum_j \eta_j d\eta_j + (1-\sum \eta_j^2) d\eta_i)^2]$$

$$= (1-\sum_j \eta_j^2)^{-3}[(\sum \eta_j d\eta_j)^2 - (\sum_i \eta_i^2)(\sum_j \eta_j d\eta_j)^2 - (1-\sum_j \eta_j^2)^2 (\sum_i d\eta_i^2) - 2(\sum_i \eta_i d\eta_i)(\sum_j \eta_j d\eta_j) + 2(\sum_i \eta_i^2)(\sum_i \eta_i d\eta_i)^2]$$

$$= (1-\sum_j \eta_j^2)^{-3}[(\sum_i \eta_i d\eta_i)^2(-1+\sum_j \eta_j^2) - \sum_j d\eta_j^2 (1-\sum_j \eta_j^2)^2]$$

$$= -(1-\sum_j \eta_j^2)^{-1}[(1-\sum_{j=1}^n \eta_j^2)^{-1}(\sum_{i=1}^n \eta_i d\eta_i)^2 + \sum_{i=1}^n d\eta_i^2].$$

The term in the [ ] is exactly $\sum_{i=1}^{n+1} d\eta_i^2$ on the sphere $\eta_1^2 + \ldots + \eta_{n+1}^2 = 1$ since $\eta_{n+1} = (1-\sum_{i=1}^n \eta_i^2)^{1/2}$. Hence ds_L^2 becomes $-\eta_{n+1}^{-2}(d\eta_1^2 + \ldots + d\eta_{n+1}^2)$.

Now

$$\eta_i = \frac{2x_i}{1+|x|^2}, \qquad i = 1, \ldots, n$$

$$\eta_{n+1} = \frac{|x|^2 - 1}{|x|^2+1}.$$

Hence

$$\sum_{i=1}^{n+1} d\eta_i^2 = 4\sum_{i=1}^n \left(\frac{(1+|x|^2)dx_i - 2x_i \sum_j x_j dx_j}{(1+|x|^2)^2}\right)^2 + \left(\frac{(1+|x|^2)2\sum_j x_j dx_j - (|x|^2-1)2\sum_j x_j dx_j}{(1+|x|^2)^2}\right)^2$$

$$= 4(1+|x|^2)^{-4}(\sum_{i=1}^n [(1+|x|^2)^2 dx_i^2 - 4(1+|x|^2)(\sum x_j dx_j) x_i dx_i + 4(\sum_j x_j dx_j)^2 x_i^2)] + 16(1+|x|^2)^{-4}(\sum_j x_j dx_j)^2$$

$$= 4(1+|x|^2)^{-2}\sum_{i=1}^n dx_i^2$$

and $\eta_{n+2}^{-2}(\sum_{i=1}^{n+1} d\eta_i^2) = \frac{4}{(1-|x|^2)^2}\sum_{i=1}^n dx_i^2.$

Remark 1. — We will prove later that GM(n) is the group of *all* conformal mappings of S^n to S^n for $n > 1$.

Remark 2. — The hemisphere $\eta_{n+1}<0$ with metric $\eta_{n+1}^{-2}\sum_{i=1}^{n+1} d\eta_i^2$ or equivalently the unit n-ball $|x|<1$ with metric $(1-|x^2|)^{-2}\sum_{i=1}^{n} dx_i^2$ is a Riemannian space in which the isotropy subgroup at a point is $O(n)$. Hence these spaces have constant curvature.

Definition. — *n-dimensional hyperbolic space is the Riemannian space described in Remark* 2.

§ 2. CONFORMAL CAPACITY OF A SHELL

Definition. — *A shell in Moebius n-space is a connected open set* D *whose complement consists of two connected components* C_0 *and* C_1. *The subsets* $\Delta_0 = C_0 \cap \overline{D}$ *and* $\Delta_1 = C_1 \cap \overline{D}$ *are called the boundaries of* D.

A shell not containing the point ∞ is called a shell *in* $\mathbf{R}^n$, the component C_i of its complement which contains ∞ is the *unbounded* component and Δ_i is called the *boundary* of the unbounded component of the complement.

Let D be a shell in the Moebius space S^n. Injection of the pair $(\overline{D}, \overline{D}-D)$ into (S^n, S^n-D) gives rise to the following commutative diagram of exact integral cohomology sequences based on compact supports:

$$\begin{array}{ccccccc}
\mathbf{Z} & & \mathbf{Z}+\mathbf{Z} & & & & 0 \\
\| & & \| & & & & \| \\
H^0(S^n) & \longrightarrow & H^0(S^n-D) & \longrightarrow & H^1_c(D) & \longrightarrow & H^1(S^n) \\
\downarrow\approx & & \downarrow & & \downarrow\approx & & \downarrow \\
H^0(\overline{D}) & \longrightarrow & H^0(\overline{D}-D) & \longrightarrow & H^1_c(D) & \longrightarrow & H^1(\overline{D})
\end{array}$$

It follows that $H^1_c(D)=\mathbf{Z}$, the ring of integers, that

$$H^0(\Delta_0 \cup \Delta_1) = H^0(\overline{D}-D) \approx H^0(S^n-D) \approx \mathbf{Z}+\mathbf{Z},$$

and thus each Δ_i is connected.

Note. — From the fact that $H^1_c(D)=\mathbf{Z}$ and the invariance of domain theorem, it follows that any homeomorphism of a shell D into S^n has a shell for its image. The homeomorphism need not, of course, extend to a homeomorphism on the boundaries.

We adopt the following conventions for notation. If f is a C^1 mapping of manifolds, $\dot{f}_p$ denotes the differential of f at the point p, or merely $\dot{f}$ when the intended point p is clear from the context. In the special case of C^1 maps of $\mathbf{R}^n$ to $\mathbf{R}^m$, we identify $\dot{f}_p$ with a map of $\mathbf{R}^n$ to $\mathbf{R}^m$ in the standard way; that is, by identifying the tangent space at any point of Cartesian space $\mathbf{R}^n$ with $\mathbf{R}^n$ itself. We denote by $\langle X, Y \rangle$ the standard inner product on either $\mathbf{R}^n$ or its dual space. We write $|X| = \langle X, X \rangle^{1/2}$.

Definition. — *Let* D *be a shell in Moebius n-space and let* Δ_0, Δ_1 *denote its boundaries. The conformal capacity* $\mathscr{C}(\mathrm{D})$ *is defined as*

$$\operatorname*{Inf}_u \int_{\mathrm{D}} |\nabla u|^n \, d\mathrm{D}$$

where u is a continuous real valued function on $\overline{\mathrm{D}}$ *which is of class* C^1 *on* D *and takes the constant value* 0 *on* Δ_0 *and the constant value* 1 *on* Δ_1; ∇u *denotes the gradient of u with respect to the standard metric of* $\mathbf{R}^n$, *and* $d\mathrm{D}$ *denotes the standard measure on* $\mathbf{R}^n$.

Note. — The definition of $\mathscr{C}(\mathrm{D})$ does not depend on designation of the boundaries since $|\nabla u| = |\nabla(1-u)|$.

Given a diffeomorphism of shells

$$f : \mathrm{D} \to \mathrm{D}'$$

and given u, a C^1 function on D taking boundary values 0 and 1 on Δ_0 and Δ_1, we define the function u' on D′ so that

$$u = u' \circ f.$$

It is easily seen that u' is a C^1 function on the shell D′ taking the value 0 on one boundary and the value 1 on the other. Writing $y = f(x)$, we have

$$\nabla u = \left(\frac{\partial u}{\partial x_i}\right), \qquad \frac{\partial u}{\partial x_i} = \sum_j \frac{\partial u'}{\partial y_j}\frac{\partial y_j}{\partial x_i},$$

so that $\nabla u = {}^t\dot{f}(\nabla u')$, where ${}^t\dot{f}$, the transpose of $\dot{f}$, sends covectors at $f(p)$ to covectors at p. Hence

$$\begin{aligned}\int_{\mathrm{D}'} |\nabla u'|^n \, d\mathrm{D}' &= \int_{\mathrm{D}} (|\nabla u'|^n \circ f)\,|\det \dot{f}\,|\, d\mathrm{D} \\ &= \int_{\mathrm{D}} (|{}^t\dot{f}^{-1}(\nabla u)|^n \circ f)\,|\det \dot{f}\,|\, d\mathrm{D}.\end{aligned}$$

Suppose now that f is *conformal*, that is, for all points $p \in \mathrm{D}$ and for all non-zero tangent vectors X and Y at p,

$$\frac{\langle \dot{f}_p \mathrm{X}, \dot{f}_p \mathrm{Y} \rangle}{|\dot{f}_p \mathrm{X}| \, . \, |\dot{f}_p \mathrm{Y}|} = \frac{\langle \mathrm{X}, \mathrm{Y} \rangle}{|\mathrm{X}| \, . \, |\mathrm{Y}|}.$$

For any two orthogonal unit vectors X_i, X_j we deduce from $\langle \mathrm{X}_i + \mathrm{X}_j, \mathrm{X}_i - \mathrm{X}_j \rangle = 0$ that

$$0 = \langle \dot{f}_p(\mathrm{X}_i + \mathrm{X}_j), \dot{f}_p(\mathrm{X}_i - \mathrm{X}_j) \rangle = |\dot{f}_p(\mathrm{X}_i)|^2 - |\dot{f}_p(\mathrm{X}_j)|^2.$$

It follows that $\dot{f}_p$ maps the unit ball in the tangent space at p to a ball of radius $\lambda(p) = |\dot{f}_p(\mathrm{X}_i)|$; equivalently, ${}^t\dot{f}_p \dot{f}_p = \lambda^2(p)\,.\,\text{identity}$.

Hence if $f : \mathrm{D} \to \mathrm{D}'$ is a conformal mapping, we have

$$|\det \dot{f}_p|^2 = \det\,({}^t\dot{f}_p . \dot{f}_p) = \lambda^{2n}(p)$$

and

$$|\nabla u|^2 = \langle {}^t\dot{f}(\nabla u'), {}^t\dot{f}(\nabla u') \rangle = \langle \dot{f} . {}^t\dot{f}(\nabla u'), \nabla u' \rangle.$$

But $\dot{f}\cdot{}^t\dot{f}={}^t\dot{f}^{-1}({}^t\dot{f}\cdot\dot{f}){}^t\dot{f}={}^t\dot{f}^{-1}\cdot\lambda^2\cdot{}^t\dot{f}=\lambda^2$. Hence $|\nabla u|^2(p)=\lambda^2(p)|\nabla u'|^2(f(p))$, that is

$$|\nabla u|^n=|\det \dot{f}|(|\nabla u'|^n\circ f).$$

Hence

$$\int_{D'}|\nabla u'|^n dD'=\int_D|\nabla u|^n dD$$

if f is *conformal.*

Note. — Conformal capacity in dimension 2 is a classical notion identical with *Newtonian* capacity. For $n>2$, it was introduced by C. Loewner (*Journal of Mathematics and Mechanics*, vol. 8, 1959) who proved that $\mathscr{C}(D)>0$ if and only if neither boundary of D degenerates to a point. We shall prove this later.

We shall first present some technical lemmas about the functions u that define $\mathscr{C}(D)$; these are needed to establish the important fact that $\mathscr{C}(D)$ varies continuously with D.

§ 3. ADMISSIBLE FUNCTIONS

A continuous function f on the interval $a\leqslant x\leqslant b$ is called *absolutely continuous* if its derivative $\frac{df}{dx}$ exists almost everywhere and is integrable and $\int_{x_0}^{x_1}\frac{df}{dx}dx=f(x_1)-f(x_0)$ for all $a\leqslant x_0, x_1\leqslant b$. It should be pointed out that if $\frac{df}{dx}$ exists almost everywhere, then it is measurable (Saks, p. 112). This condition is equivalent to the assertion: Given an $\varepsilon>0$, there is a $\delta>0$ such that for any union of intervals A of measure less than δ, we have $F(A)<\varepsilon$, where F is the set function formed from f by the rule:

$$F[x_0, x_1]=f(x_1)-f(x_0).$$

A continuous function f defined on a domain $D\subset\mathbf{R}^n$ is called ACL in D if in any closed ball lying in D it is absolutely continuous on almost all lines parallel to the coordinate axes. (If in addition we assume that $\int_D|\nabla f|dD<\infty$, the condition is invariant under diffeomorphisms, but we shall not need this fact.)

The partial derivates of an ACL function exist almost everywhere, as is seen by applying Fubini's theorem to the set on which the derivatives do not exist.

With ACL functions having integrable derivatives, one can differentiate under an integral sign.

Lemma (3.1). — *Let* D *be a domain in* $\mathbf{R}^n$. *Let* f *be continuous on* $[a, b]\times D$, *and absolutely continuous in* x *for almost all* y *in* D. *Assume* $\frac{\partial f}{\partial x}(x,y)$ *is integrable on* $[a, b]\times D$. *Then* $\frac{d}{dx}\int_D f(x,y)dy=\int_D\frac{\partial f}{\partial x}(x,y)dy$ *for almost all* x *in* $[a, b]$.

Proof. — By hypothesis ACL, $\left(\frac{\partial f}{\partial x}(x,y)\right.$ is integrable in x for almost all $y\in D$, and$\left.\right)$ $\int_{x_0}^{x}\frac{\partial f}{\partial x}(x,y)dx=f(x,y)-f(x_0,y)$ for almost all y. Since by hypothesis $\frac{\partial f}{\partial x}(x,y)$ is integrable on $[a, b]\times D$, we find by Fubini's theorem (Saks, p. 81) that $g(x)=\int_D \frac{\partial f}{\partial x}(x, y)dy$ exists for almost all x, is integrable on $[a, b]$ and that

$$\int_{x_0}^{x} g(x)dx=\int_D \left(\int_{x_0}^{x} \frac{\partial f}{\partial x}(x,y)dx\right)dy=\int_D (f(x,y)-f(x_0,y))dy.$$

Since (Saks, p. 117), $g(x)=\frac{d}{dx}\int_{x_0}^{x} g(x)dx$ for almost all x, we get

$$\frac{d}{dx}\int_D f(x,y)dy=\int_D \frac{\partial f}{\partial x}(x,y)dy$$

for almost all x in $[a, b]$.

The next lemma collects some elementary observations that will be used repeatedly.

Lemma (**3.2**). — *Let v be a continuous,* ACL *function in an open set* $R\subset \mathbf{R}^n$. *Let* R' *be an open subset with compact closure and* $\overline{R}'\subset R$. *Let* U *be the ball* $|y|<\varepsilon$ *in* $\mathbf{R}^n$, *and suppose that* 2ε *does not exceed the distance from* R' *to the complement of* R. *Assume that* $|\nabla v|$ *is integrable on* R. *Set* $w(x)=\frac{1}{m(U)}\int_U v(x+y)dy$ *for* $x+U\subset R$, *where* $m(U)$ *denotes the n-dimensional measure of* U.

Then

1) w *is of class* C^1 *on* R'.
2) $\lim_{\varepsilon\to 0} w=v$ *uniformly on compact subsets of* R.
3) $\nabla w(x)=\frac{1}{m(U)}\int_U \nabla v(x+y)dy$ *for almost all* x.
4) $\int_{R'}|\nabla w(x)|^p dx\leqslant \int_R |\nabla v(x)|^p dx \qquad p\geqslant 1$.

Proof. — Assertions 1) and 2) are well-known facts about continuous functions. Assertion 3) follows directly from Lemma (3.1).

We have $|\nabla w(x)|\leqslant \frac{1}{m(U)}\int_U |\nabla v(x+y)|dy$. Hence

$$\left(\int_{R'}|\nabla w(x)|^p dx\right)^{\frac{1}{p}}\leqslant \left(\int_{R'}\left(\frac{1}{m(U)}\int_U |\nabla v(x+y)|dy\right)^p dx\right)^{\frac{1}{p}}$$
$$\leqslant \frac{1}{m(U)}\int_U \left(\int_{R'}|\nabla v(x+y)|^p dx\right)^{\frac{1}{p}}dy$$

by the integral form of Minkowski's inequality $||\sum_i f_i||_p\leqslant \sum_i ||f_i||_p$.

Now $\int_{R'}|\nabla v(x+y)|^p dx\leqslant \int_R |\nabla v(x)|^p dx$; hence we get

$$\left(\int_{R'}|\nabla w(x)|^p dx\right)^{\frac{1}{p}}\leqslant \left(\int_R |\nabla v(x)|^p dx\right)^{\frac{1}{p}}\frac{1}{m(U)}\int_U dy$$

from which assertion 4) follows.

Let D be a shell in $\mathbf{R}^n$, let C_1 be the unbounded component of its complement in Moebius space, and let C_0 be the bounded component.

A real valued function u defined on $\overline{D}$ is called *admissible* if it is continuous, ACL in D, takes the value 0 on $C_0 \cap \overline{D}$ and the value 1 on $C_1 \cap \overline{D}$ and if moreover $|\nabla u|^n$ is integrable on D. The function u is called smoothly admissible if in addition it is of class C^1 on D and its gradient ∇u has compact support in D.

The shell D has an admissible function. To see this, cover C_1 by a finite set of closed balls in Moebius space whose union C_1' does not meet C_0 (a closed ball with center at ∞ being $\{y; |y| \geqslant r\}$, and then cover C_0 with a finite number of closed balls whose union C_0' does not meet C_1'. The functions $d(x, C_0')$ and $d(x, C_1')$ are then easily seen to be ACL. In fact, for any set C, the function $f(x) = d(x, C)$ satisfies $|f(x+\Delta x) - f(x)| \leqslant |\Delta x|$ by the triangle inequality, and from this it is seen that $f(x)$ is absolutely continuous on all lines. Hence $u(x) = \dfrac{d(x, C_0')}{d(x, C_0') + d(x, C_1')}$ is ACL and its gradient is a.e. bounded and of compact support. Thus $\int_D |\nabla u|^n dD < \infty$ and u is admissible.

Lemma (3.3). — *The conformal capacity of the shell* D *may be defined by*

$$\operatorname*{Inf}_u \int_D |\nabla u|^n dD$$

where u ranges over the class of admissible functions or over the class of smoothly admissible functions.

Proof. — Let u be an admissible function.

Fix a: $0 < a < \frac{1}{2}$. Set

$$v(x) = \begin{cases} 0 & \text{if } u(x) \leqslant a \\ \dfrac{u(x) - a}{1 - 2a} & \text{if } a < u(x) < 1 - a \\ 1 & \text{if } 1 - a \leqslant u(x) \end{cases}$$

and extend v to be 0 on C_0 and 1 on C_1. Then ∇v has compact support K, which lies in D at a positive distance, say 2ε, from the complement of D. Let $U = \{y; |y| < \varepsilon\}$. Set

$$w(x) = \frac{1}{m(U)} \int_U v(x+y)\, dy.$$

Then $w = 0$ on C_0 and $w = 1$ on C_1. Since ∇v has compact support K, we get by Hölder's inequality

$$\int_D |\nabla v| dD = \int_K |\nabla v| . 1 dD \leqslant \left(\int_K |\nabla v|^n dD\right)^{\frac{1}{n}} \left(\int_K 1 . dD\right)^{\frac{n-1}{n}}$$
$$\leqslant (1-2a)^{-1} \left(\int_D |\nabla u|^n dD\right)^{\frac{1}{n}} m(K) < \infty.$$

Thus $|\nabla v|$ is integrable on D. Hence for almost all x,

$$\nabla w(x) = \frac{1}{m(U)} \int_U \nabla v(x+y)\, dy$$

by Lemma 3.2, and moreover w is of class C^1 on $K+\overline{U}$, which contains the support of ∇w. Thus w is a smoothly admissible function on D. Hence

$$\mathscr{C}(D) \leqslant \int_D |\nabla w|^n dx \leqslant \int_D |\nabla v|^n dx$$

by assertion 4) of Lemma (3.2), and on the other hand

$$\int_D |\nabla v|^n dx \leqslant \int_D |\nabla u|^n (1-2a)^{-n} dx.$$

Hence

$$\mathscr{C}(D) \leqslant \int_D |\nabla w|^n dx \leqslant (1-2a)^{-n} \int_D |\nabla u|^n dx.$$

Letting $a \to 0$, we find

$$\mathscr{C}(D) \leqslant \operatorname*{Inf}_u \int_D |\nabla u|^n dx$$

as n varies over either the class of admissible functions or the class of smoothly admissible function.

Note. — We can now assert that $\mathscr{C}(D) < \infty$ for any shell D in Moebius space. For, performing a Moebius transformation, we can assume $D \subset \mathbf{R}^n$. Then we can find an admissible function for D as remarked above. Hence $\mathscr{C}(D) < \infty$ by Lemma (3.3).

Lemma **(3.4)**. — *Let* D *be a shell in Moebius space, let* C_0 *and* C_1 *denote the components of its complement. Let* u *be an admissible function on* D. *Extend* u *to a function on Moebius space by assigning it the constant values* 0 *and* 1 *on* C_0 *and* C_1. *Then* u *is continuous and* ACL *everywhere on* $\mathbf{R}^n$, *and* $|\nabla u| = 0$ *on* $C_0 \cup C_1$ *a.e.*

Proof. — Clearly u is continuous.

Let X denote a variable line parallel to the x_1-axis. Then

$$\int_{\operatorname{Proj} D} \left(\int_{X \cap D} |\nabla u|^n dx_1 \right) dx_2 \ldots dx_n = \int_D |\nabla u|^n dx < \infty.$$

Hence

1) $\int_{X \cap D} |\nabla u|^n dx_1 < \infty$ for almost all X. Morever, since u is ACL in D,

2) u is *absolutely continuous* on any compact interval in $X \cap D$ for almost all X. Choose X satisfying 1) and 2).

Then given any interval $I = [p, q] \subset X$, we show

$$|u(p) - u(q)| \leqslant \int_{I \cap D} |\nabla u| \, dx_1.$$

Clearly no generality is lost in assuming $p \in \overline{D}$ and $q \in \overline{D}$. Let p' be the nearest point of $(C_0 \cup C_1) \cap I$ to p and let q' be the nearest point of $(C_0 \cup C_1) \cap I$ to q. Then

$$|u(p) - u(q)| \leqslant |u(p) - u(p')| + |u(p') - u(q')| + |u(q') - u(q)|.$$

Then

$$|u(p) - u(p')| \leqslant \int_{[p, p']} |\nabla u| \, dx_1 \quad \text{and} \quad |u(q') - u(q)| \leqslant \int_{[q', q]} |\nabla u| \, dx_1.$$

If $u(p')=u(q')$, the assertion follows at once.

If $u(p')\neq u(q')$, we can find in $[p', q']$ a shortest interval $[p'', q'']$ joining the two closed disjoint subsets $C_0 \cap [p', q']$ and $C_1 \cap [p', q']$. The interior of $[p'', q'']$ lies in D and hence

$$\int_{[p'', q'']} |\nabla u|\, dx_1 = 1 = |u(p')-u(q')|.$$

This completes the proof of the assertion.

Given now any union E of disjoint intervals $[p_k, q_k]$, we have

$$\sum_k |u(p_k)-u(q_k)| \leqslant \int_{E\cap D} |\nabla u|\, dx_1 \leqslant \left(\int_E |\nabla u|^n dx_1\right)^{\frac{1}{n}} \left(\int_E 1 \,.\, dx_1\right)^{\frac{n-1}{n}}$$
$$\leqslant \left(\int_{X\cap D} |\nabla u|^n dx_1\right)^{\frac{1}{n}} (\text{length E})^{\frac{n-1}{n}}.$$

Hence u is absolutely continuous on X. It follows that u is ACL everywhere in $\mathbf{R}^n$.

It is known that almost all points of a measurable set in $\mathbf{R}^n$ are points of linear density in the direction of the coordinate axes (Saks, p. 298). Thus if ∇u exists at such a point p of $C_1 \cup C_2$, we have $\nabla u(p)=0$. Hence $\nabla u=0$ a.e. on $C_0 \cup C_1$.

Apart from the smoothly admissible approximations to a given admissible function u on a shell D, there is another kind of approximation that will be needed, namely *piecewise linear approximations.*

. Let D be an open set such that $\overline{D} \subset \mathbf{R}^n$.

Let u be a continuous function on $\mathbf{R}^n$.

Let τ be a triangulation of $\mathbf{R}^n$ into n-simplices.

We let u_τ denote the piecewise linear function whose value at a point $\sum_{i=0}^{n} t_i p_i$ $(t_i>0, \sum_i t_i=1)$ of a simplex with vertices at $p_0, \ldots, p_n$ is $\sum_{i=0}^{n} t_i u(p_i)$.

We denote by $|\tau|$ the supremum of the diameters of the simplices of τ. Clearly as $|\tau|\to 0$, $u_\tau \to u$ uniformly on $\overline{D}$.

It will be convenient to " standardize " the triangulation τ. Thus for any positive number a, we denote by $\tau(a)$ the following triangulation:

Partition $\mathbf{R}^n$ into the cubes

$$[k_1, \ldots, k_n] : k_i a \leqslant x_i \leqslant (k_i+1)a \qquad (i=1, \ldots, n;\ k_i=0, 1, \ldots),$$

and then triangulate each cube in the same way.

Finally, there is one other kind of admissible function which will be used. The idea of using such functions goes back to Lebesgue (*Rend. Circ. Mat. Palermo*, v. 24 (1907), p. 382) who employed them in a paper on Dirichlet's problem of minimizing $\int_D |\nabla u|^2 dD$ for plane regions D.

Let D be an open set in a locally connected topological space and let f be a continuous function on $\overline{D}$. The function f is called *monotone* on D if for every connected open subset $U \subset D$, $\sup_{x \in U} f(x) \leqslant \sup_{y \in \overline{U} - U} f(y)$ and $\inf_{x \in U} f(x) \geqslant \inf_{y \in \overline{U} - U} f(y)$ if $U \neq \overline{U}$. Thus, for any connected open subset U, if f is constant on the non-empty $\overline{U} - U$, then f is constant on U. This property is easily seen to be equivalent to the monotonicity of f. And that observation leads to the following method of Lebesgue for " straightening " any function into a monotone function having the same values on $\overline{D} - D$.

For any open set U and any point x in U, let U_x denote the connected component of U containing x. For any real number c, and any continuous function f on D, let cDf denote the union of all connected components of $D - f^{-1}(c)$ whose closures lie in D.

For any real number a, let $f.a$ denote the function on $\overline{D}$ defined by

$$(f.a)(x) = \begin{cases} a & \text{if } x \in aDf \\ f(x) & \text{if } x \notin aDf \end{cases}.$$

Thus, f and $f.a$ have the same values on $\overline{D} - D$. Moreover, on the boundary of any connected component of aDf, the function f takes on the constant value a and for any non-empty connected subset $T \not\subset aDf$, we have $f.a(T) \subset f(T)$ provided $\overline{D}_x \neq D_x$ for all $x \in D$.

Note. — According to our definition, the only monotone function on a connected topological space X is the constant function. Thus the function $f(x_1, \ldots, x_n) = x_1$ on $\mathbf{R}^n$ is not monotone in our sense. However, it is monotone on any relatively compact open set D. If, in the definition of monotone, we required that $f|U$ take on its maximum and minimum values on $\overline{U} - U$ for only relatively *compact* open subsets U, then we would come out with the usual monotone functions on $\mathbf{R}^1$. We have not done so since compactness does not enter the proof below. The function $f(x) = x$ on $\mathbf{R}^1$ is thus monotone in our sense on all bounded intervals, but not on $\mathbf{R}^1$.

Lebesgue's Straightening Lemma. — *Let* D *be an open proper subset of a connected locally connected topological space and let* f *be a continuous real valued function on* $\overline{D}$ *with values in the bounded interval* $[m, M]$. *Let* $a_1, a_2, \ldots, a_n, \ldots$, *be an enumeration of the rational numbers in* $[m, M]$. *Set*

$$f_n = (\ldots((f.a_1).a_2)\ldots).a_n.$$

Then f_n *converges uniformly on* $\overline{D}$ *to a monotone function.*

Proof. — Let a and c be distinct real numbers and let x be a point of the subset $cD(f.a)$. Then on the boundary of $(cD(f.a))_x$, the function $f.a$ has the constant value c. From the definition of $f.a$ and the fact that $a \neq c$, it follows at once that f has the constant value c on the boundary of $(cD(f.a))_x$ and $x \in cDf$ if $f(x) \neq c$. Thus $cD(f.a) \subset cDf \cup f^{-1}(c)$.

Suppose $a > c$. Then the subset S of elements y in $(cD(f.a))_x$ such that $f(y) > a$ clearly has its boundary in $(cD(f.a))_x$ and hence $S \subset aDf \cap cD(f.a)$. Consequently $(f.a)(y) = a$ for all $y \in S$. Equivalently, we can assert

$$\sup\ (f.a)(y) \leqslant a$$

for all $y \in c\mathrm{D}(f.a)$ if $a>c$. Similarly

$$\inf\,(f.a)(y) \geqslant a$$

for all $y \in c\mathrm{D}(f.a)$ if $a<c$.

Clearly $(c\mathrm{D}(f.a))_x \not\subset a\mathrm{D}f$. Hence the values assumed by $f.a$ on $(c\mathrm{D}(f.a))_x$ are assumed by f on $(c\mathrm{D}(f.a))_x$. Thus

$$\sup\{(f.a)(y)\,;\, y\in c\mathrm{D}(f.a)\} \leqslant \sup\{f(y)\,;\, y\in c\mathrm{D}f\}$$
$$\inf\{(f.a)(y)\,;\, y\in c\mathrm{D}(f.a)\} \geqslant \inf\{f(y)\,;\, y\in c\mathrm{D}f\}.$$

Given now any distinct real numbers $a_1, a_2, \ldots, a_{n+1}$, we write

$$\mathrm{D}_k = a_{n+1}\mathrm{D}(f.a_1.a_2.\ldots.a_k)$$

and we have

$$(3) \qquad \begin{aligned} \sup_{y\in \mathrm{D}_n}(f.a_1a_2\ldots a_n)(y) &\leqslant \sup_{y\in\mathrm{D}_{n-1}}(f.a_1a_2\ldots a_{n-1})(y) \\ &\leqslant \sup_{y\in\mathrm{D}_k}(f.a_1.a_2\ldots a_k)(y) \\ &\leqslant a_k \quad \text{if} \quad a_k>a_{n+1}. \end{aligned}$$

Hence $\sup\limits_{y\in\mathrm{D}_n}(f_n(y)-f_{n+1}(y)) \leqslant a_k - a_{n+1}$ if $a_{n+1}<a_k$.

Similarly,

$$\inf_{y\in\mathrm{D}_n}(f_n(y)-f_{n+1}(y)) \geqslant a_k - a_{n+1} \quad \text{if} \quad a_k<a_{n+1}.$$

Since D_n is the set on which f_{n+1} differs from f_n, we find

$$\sup_{y\in\mathrm{D}}|f_n(y)-f_{n+1}(y)| \leqslant d(n+1) \leqslant d(n),$$

where $d(n)$ is the length of the longest interval in $[m, \mathrm{M}]-\{a_1, a_2, \ldots, a_n\}$.

Suppose now that r is an integer less than $n+1$. We shall show by induction on $n-r$ that

$$(4) \qquad \sup_{y\in\mathrm{D}}|f_r(y)-f_{n+1}(y)| \leqslant d(r).$$

Observe first that (4) is true if $n-r=0$, by the assertion above. Next, from $c\mathrm{D}(f.a) \subset c\mathrm{D}f \cup f^{-1}(c)$, we obtain

$$\begin{aligned} \mathrm{D}_n &\subset \mathrm{D}_{n-1} \cup f_n^{-1}(a_{n+1}) \\ &\subset \mathrm{D}_r \cup f_{r+1}^{-1}(a_{n+1}) \cup \ldots \cup f_n^{-1}(a_{n+1}) \\ &\subset \mathrm{D}_r \cup f_{r+1}^{-1}(a_{n+1}). \end{aligned}$$

Since f_{n+1} differs from f_n on at most D_n, we find by the inductive hypothesis that $\sup\limits_{y\in\mathrm{D}}|f_r - f_{n+1}|$ is the maximum of $d(r)$ and

$$\sup_{y\in\mathrm{D}_n}|f_r(y)-f_{n+1}(y)|.$$

The latter term is majorized by

$$\sup_{y\in\mathrm{D}_r\cup f_r^{-1}(a_{n+1})}|f_r(y)-a_{n+1}| = \sup_{y\in\mathrm{D}_r}|f_r(y)-a_{n+1}| \leqslant d(r)$$

by the inequality (3) above (and the corresponding inequality for $\inf_{y\in D_r} f_r$). Thus assertion (4) is proved.

Since $\lim_{r\to\infty} d(r)=0$, it follows immediately that $\{f_n\}$ is uniformly convergent on D to a limit function $\bar{f}$.

The function $\bar{f}$ is monotone on D. Otherwise there would be a real number c such that $cD\bar{f}$ is not empty. Let $b=\sup_y \bar{f}(y)-c$ as y ranges over one of the connected components in $cD\bar{f}$, say $(cD\bar{f})_x$ and for the sake of definiteness we can suppose that $b=\sup_y |\bar{f}(y)-c|$, y varying over $(cD\bar{f})_x$. Thus for any n with $\sup|f_n-\bar{f}|<\frac{b}{3}$ we have

$$\sup\{f_n(y)-c';\ y\in c'\,Df_n\}>\frac{b}{3}$$

whenever $c<c'<c+\frac{b}{3}$. Choosing n so that $d(n)>\frac{b}{3}$ and $c<a_{n+1}<c+\frac{b}{3}$, we would have

$$\sup\{f_n(y)-a_{n+1};\ y\in a_{n+1}\,Df_n\}>d(n);$$

that is, $\sup|f_n-f_{n+1}|>d(n)$ contradicting (4).

The proof of the lemma is now complete.

§ 4. Oscillation of functions with L^n-integrable gradient

Lemma (**4.1**). — *Let* S *be the boundary of a ball of radius r in* $\mathbf{R}^n$ *for* $n\geqslant 2$, *and let u be a function of class* C^1. *Then*

$$(\operatorname{osc}_S u)^n \leqslant Ar\int_S |\nabla u|^n\, dS$$

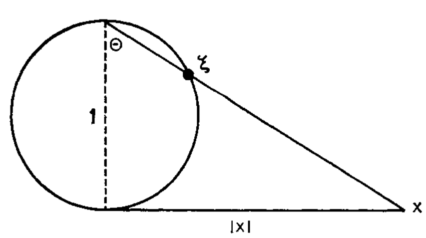
where A *is a constant depending only on the dimension n, but independent of u and r, and the gradient is taken along* S.

Proof. — Since each side of the inequality is invariant under stretchings, we can assume $r=1$.

Let p and q be any two distinct points on S.

Let π denote the stereographic projection from p onto the tangent plane E to S at the antipodal point of p. Set $a=\pi(q)$, $u'=u\circ\pi^{-1}$, $v=|\nabla u|\circ\pi^{-1}$.

Let ξ denote a variable point on S and set $x=\pi(\xi)$. Then $|x|=\tan\theta$, where θ is defined in the diagram



Since π is conformal,

$$\det \dot{\pi} = \left|\frac{d|x|}{d(2\theta)}\right|^{n-1} = \left(\frac{\sec^2\theta}{2}\right)^{n-1} = \left(\frac{1+|x|^2}{2}\right)^{n-1}.$$

We have

$$|u(p)-u(q)| = |u'(\infty)-u'(a)| = \left|\int_0^\infty \frac{du'}{dt}(a+ty)\,dt\right| \leqslant \int_0^\infty |\nabla u'|\,dt$$

where y is any unit vector in E, which we identify with $\mathbf{R}^{n-1}$.

Let Y denote the unit sphere in $\mathbf{R}^{n-1}$, that is the set of all y with $|y|=1$.

Then

$$|u(p)-u(q)| \leqslant \frac{1}{c_{n-2}}\int_Y\int_0^\infty |\nabla u'|(a+ty)\,dt\,dy = \frac{1}{c_{n-2}}\int_E |\nabla u'|\frac{dE}{|x-a|^{n-2}},$$

where c_{n-2} is the $(n-2)$-measure of the unit $n-2$-sphere. We have seen, for any conformal mapping π, that

$$|\nabla u'| = (|\nabla u|\circ\pi^{-1})(\det\dot{\pi})^{-\frac{1}{n-1}}.$$

Hence $|\nabla u'| = \dfrac{2v}{1+|x|^2}$ and we can write

$$|u(p)-u(q)| \leqslant \frac{2}{c_{n-2}}\int_E \frac{v}{(1+|x|^2)^{\frac{n-1}{n}}}\cdot\frac{1}{|x-a|^{n-2}(1+|x|^2)^{\frac{1}{n}}}\,dE.$$

By Hölder's inequality, the right side is no more than

$$\frac{2}{c_{n-2}}\left(\int_E \frac{v^n}{(1+|x|^2)^{n-1}}\,dE\right)^{\frac{1}{n}}\left(\int_E \frac{1}{|x-a|^{\frac{(n-2)n}{n-1}}(1+|x|^2)^{\frac{1}{n-1}}}\,dE\right)^{\frac{n-1}{n}}.$$

The denominator of the integral on the right at $x=\infty$ has its order of magnitude $|x|^{-e}$ with

$$e = \frac{(n-2)n}{n-1} + \frac{2}{n-1} = \frac{n^2-2n+2}{n-1} = \frac{(n-1)^2+1}{n-1} = n-1+\frac{1}{n-1} > n-1$$

and is therefore convergent at ∞. At $x=a$, the denominator is essentially $|x-a|^{\frac{(n-2)n}{n-1}}$, the exponent being

$$\frac{(n-2)n}{n-1} = \frac{n^2-2n}{n-1} = \frac{(n-1)^2-1}{n-1} = n-1-\frac{1}{n-1} < n-1.$$

Thus the integral is finite at $x=a$.

Set

$$A(a) = \left(\int_E\left[|x-a|^{n-1-\frac{1}{n-1}}(1+|x|^2)^{\frac{1}{n-1}}\right]^{-1}dE\right)^{\frac{n-1}{n}}.$$

Then

$$|u(p)-u(q)|^n \leqslant \left(\frac{2A(a)}{c_{n-2}}\right)^n \int_E \frac{v^n}{(1+|x|^2)^{n-1}} dE = A\int_S |\nabla u|^n dS.$$

It remains to obtain a bound for $A(a)$. From $|x| \leqslant |x-a|+|a|$, we see that if $|x-a| \geqslant \frac{1}{2}|a|$, then $|x| \leqslant 3|x-a|$ and thus $|x-a| \geqslant \frac{1}{3}|x|$. If on the other hand $|x-a| \leqslant \frac{|a|}{2}$, then $|x| \geqslant |a|-|x-a| \geqslant \frac{1}{2}|a|$. Now set $b=\sup\left(\frac{1}{2}|a|, 1\right)$. Then write

$$\int_E \frac{1}{|x-a|^{n-1-\frac{1}{n-1}}(1+|x|^2)^{\frac{1}{n-1}}} dE = \int_{|x-a|\leqslant b} + \int_{|x-a|\geqslant b}$$

and denote the integrals on the right by I and II.

The first integral, in case $b \leqslant 1$, is bounded by

$$\begin{aligned} I &\leqslant \int_{|x-a|\leqslant b} \frac{1}{|x-a|^{n-1-\frac{1}{n-1}}} dE = c_{n-2}\int_0^b r^{-1+\frac{1}{n-1}} dr = (n-1)c_{n-2}b^{\frac{1}{n-1}} \\ &\leqslant (n-1)c_{n-2}. \end{aligned}$$

On the other hand if $b>1$, then $b=\frac{1}{2}|a|$ and $|x| \geqslant b$ on $|x-a| \leqslant b$. Thus

$$\begin{aligned} I &\leqslant (1+b^2)^{-\frac{1}{n-1}} \int_{|x-a|\leqslant b} \frac{1}{|x-a|^{n-1-\frac{1}{n-1}}} dE \leqslant b^{-\frac{2}{n-1}}(n-1)c_{n-2}b^{\frac{1}{n-1}} \\ &\leqslant (n-1)c_{n-2}. \end{aligned}$$

For the second integral, we have

$$\begin{aligned} II &\leqslant \int_{|x-a|\geqslant b} \frac{1}{\left(\frac{1}{3}|x|\right)^{n-1-\frac{1}{n-1}}(1+|x|^2)^{\frac{1}{n-1}}} dE = \int_{\substack{|x-a|>b \\ |x|<1}} + \int_{\substack{|x-a|>b \\ |x|>1}} \\ &\leqslant 3^{n-1}\int_{|x|<1} \frac{1}{|x|^{n-1-\frac{1}{n-1}}} dE + 3^{n-1}\int_{|x|>1} \frac{1}{|x|^{n-1+\frac{1}{n-1}}} dE \\ &\leqslant 3^{n-1}c_{n-2}\int_0^1 r^{-1+\frac{1}{n-1}} dr + 3^{n-1}c_{n-2}\int_1^\infty r^{-1-\frac{1}{n-1}} dr = 2.3^{n-1}c_{n-2}(n-1). \end{aligned}$$

Hence $A \leqslant \dfrac{2^n}{c_{n-2}^n}((n-1)c_{n-2}+2(n-1)c_{n-2}.3^{n-1})^{n-1} = \dfrac{2^n(n-1)^{n-1}(1+2.3^{n-1})^{n-1}}{c_{n-2}}.$

The proof of Lemma (4.1) is now complete.

As a corollary of Lemma (4.1), we can prove the following result of Loewner on the non-vanishing of conformal capacity.

Lemma **(4.2)**. — *Let* D *be a shell in Moebius space* $\mathbf{R}^n$ *and let* C_0, C_1 *denote the connected components of the complement of* D. *Then* $\mathscr{C}(D)>0$ *if neither* C_0 *nor* C_1 *consists of a single point.*

Proof. — Choose a point o in $\mathbf{R}^n$ such that the two spheres with center at o and radius r_1 and r_2 $(0<r_1<r_2)$ intersect C_0 and C_1. Let S_r denote the sphere with center at o and radius r. Since C_0 and C_1 are connected, S_r meets both C_1 and C_2 for all r, $r_1\leqslant r\leqslant r_2$. Let D′ denote the spherical shell: $r_1\leqslant|x|\leqslant r_2$. Let u be any smoothly admissible function in D. Then

$$\int_D|\nabla u|^n dD=\int_{\mathbf{R}^n}|\nabla u|^n dx\geqslant\int_{D'}|\nabla u|^n dx=\int_{r_1}^{r_2}\int_{S_r}|\nabla u|^n d\sigma dr$$

where $d\sigma$ denotes $(n-1)$-measure on S_r.

Now the gradient ∇u at any point x in S_r has length no less than the gradient along S_r. Hence by Lemma (4.1)

$$\int_{S_r}|\nabla u|^n d\sigma\geqslant A^{-1}r^{-1}(\operatorname*{osc}_{S_r} u)^n=A^{-1}r^{-1}$$

and thus

$$\int_D|\nabla u|^n dD\geqslant A^{-1}\int_{r_1}^{r_2}\frac{dr}{r}=A^{-1}\log\frac{r_2}{r_1}$$

for all smoothly admissible functions u. Taking the infimum over such u, we find

$$\mathscr{C}(D)\geqslant A^{-1}\log\frac{r_2}{r_1}>0.$$

Lemma **(4.3)**. — *Let u be continuous and* ACL *in the shell* D : $a<|x|<b$ *of* $\mathbf{R}^n$. *Then*

$$\int_a^b(\operatorname*{osc}_{|x|=r} u)^n\frac{dr}{r}\leqslant A\int_D|\nabla u|^n dD.$$

Proof. — We can assume that $|\nabla u|^n$ is integrable on D, otherwise the result is trivially true. From Hölder's inequality it follows that $|\nabla u|$ is integrable (cf. Proof of Lemma (3.3)). Let D′ denote the shell $a+\varepsilon<|x|<b-\varepsilon$ and set

$$v(x)=\frac{1}{m(U)}\int_U u(x+y)dy,\qquad x\in D'$$

where $U=\{y;\ |y|<\varepsilon/2\}$. Then by Lemma 3.3,

$$\nabla v(x)=\frac{1}{m(U)}\int_U\nabla u(x+y)dy$$

and v is of class C^1 on D′. Hence by Lemma (4.1)

$$\int_{a+\varepsilon}^{b-\varepsilon}(\operatorname*{osc}_{|x|=r} v)^n\frac{dr}{r}\leqslant A\int_{D'}|\nabla v|^n dx\leqslant A\int_D|\nabla u|^n dx$$

by Lemma (3.3).

Thus

$$\int_{a+\varepsilon}^{b-\varepsilon}(\underset{|x|=r}{\operatorname{osc}} v)^n \frac{dr}{r} \leqslant A \int_D |\nabla u|^n dD.$$

As $\varepsilon \to 0$, v converges to u uniformly and hence

$$\int_a^b (\underset{|x|=r}{\operatorname{osc}} u)^n \frac{dr}{r} \leqslant A \int_D |\nabla u|^n dD.$$

Lemma **(4.4)**. — *Let u be continuous and* ACL *in the half-space* $E_+ : x_n > 0$ *of* $\mathbf{R}^n$, *and let* S_+ *be the hemisphere*: $|x| = r, x_n > 0$. *Then*

$$\int_0^\infty (\underset{S_+}{\operatorname{osc}} u)^n \frac{dr}{r} \leqslant 2A \int_{E_+} |\nabla u|^n dx.$$

Proof. — For each $\varepsilon > 0$, let v be the function on $\mathbf{R}^n$:

$$v(x_1, x_2, \ldots, x_n) = u(x_1, x_2, \ldots, \varepsilon + |x_n - \varepsilon|).$$

Then v equals u on $x_n \geqslant \varepsilon$ and is symmetric with respect to the plane $x_n = \varepsilon$. Thus v is continuous and ACL in $\mathbf{R}^n$. By the lemma above,

$$\int_0^\infty (\underset{|x|=r}{\operatorname{osc}} v)^n \frac{dr}{r} \leqslant A \int_{\mathbf{R}^n} |\nabla v|^n dx \leqslant 2A \int_{E_+} |\nabla u|^n dx.$$

For each $r > 0$, we have $\underset{|x|=r}{\operatorname{osc}} v$ monotone increasing as ε decreases and

$$\underset{S_+}{\operatorname{osc}} u = \lim_{\varepsilon \to 0} \underset{|x|=r}{\operatorname{osc}} v.$$

Hence

$$\int_0^\infty (\underset{S_+}{\operatorname{osc}} u)^n \frac{dr}{r} = \lim_{\varepsilon \to 0} \int_0^\infty (\underset{|x|=r}{\operatorname{osc}} v)^n \frac{dr}{r} \leqslant 2A \int_{E_+} |\nabla u|^n dx.$$

Lemma **(4.5)**. — *Let u be a monotone admissible function on a shell* D *in* $\mathbf{R}^n$, *and set* $M = \int_D |\nabla u|^n dD$. *Let* C_0, C_1 *denote the bounded and unbounded components of the complement of* D, *and extend u to be* 0 *on* C_0 *and* 1 *on* C_1. *Set*

$$a = \text{diameter of } C_0$$
$$b = d(C_0, C_1)$$
$$c(x, y) = \inf(d(x, C_0), d(y, C_0)).$$

Then

$$|u(x) - u(y)|^n \leqslant AM \left(\log \frac{a}{d(x, y)}\right)^{-1}$$

for all x, y in $\mathbf{R}^n$ *with* $d(x, y) < a$, *and*

$$|u(x) - u(y)|^n \leqslant AM \left(\log \frac{c(x, y)}{b}\right)^{-1}$$

for all x, y with $c(x, y) > b$.

Proof. — By hypothesis, u is monotone on D and therefore on every open subset of D. Given any connected open subset V in $\mathbf{R}^n$ not contained in or containing C_0 or C_1, set $W = V - (C_1 \cup C_0)$. Clearly the boundary of W meets C_i if V does (because $V \not\subset C_i$ and V is connected) and thus

$$\sup_{x\in V} u(x) = \sup_{x\in W} u(x), \qquad \inf_{x\in V} u(x) = \inf_{x\in W} u(x)$$

because u is constant on each C_i $(i = 0, 1)$. Furthermore

$$u(\overline{V} - V) = u(\overline{W} - W)$$

since $\overline{V} - V \subset (\overline{W} - W) \cup C_0 \cup C_1$ and the boundary of V meets C_i if V does (because $C_i \not\subset V$ and C_i is connected). Thus

$$\sup_{x\in V} u(x) = \sup_{x\in \overline{V}-V} u(x), \quad \inf_{x\in V} u(x) = \inf_{x\in \overline{V}-V} u(x)$$

because u is monotone on W. It follows that u is monotone on any connected open subset V not containing C_0 or C_1.

We apply this observation to the open ball V with center at x and radius a. The oscillation of u on the sphere $S_r : |z - x| = r$ increases monotonically with r, $0 \leqslant r \leqslant a$. We have therefore $|u(x) - u(y)| \leqslant \operatorname{osc}_{S_r} u$ for $d(x, y) < r < a$ and thus, by Lemma (4.3)

$$|u(x) - u(y)|^n \int_{d(x,y)}^{a} \frac{dr}{r} \leqslant AM.$$

On the other hand, we can apply the observation above to the exterior V of the closed ball with center p and radius $r > d(p, C_1)$ where p is a point in C_0 with $d(p, C_1) = d(C_0, C_1) = b$. The fact that u is monotone on V implies here that $\operatorname{osc}_{S_r} u$ is monotonically *decreasing* in r, where S_r is the sphere $|z - p| = r$. Thus $|u(x) - u(y)| \leqslant \operatorname{osc}_{S_r} u$ if $r < d(p, x), d(p, y)$. By Lemma (4.3)

$$|u(x) - u(y)|^n \int_{b}^{c(x,y)} \frac{dr}{r} \leqslant AM.$$

These are the inequalities of our lemma.

§ 5. EXTREMAL ADMISSIBLE FUNCTIONS

The following lemma is due to J. A. Clarkson (On uniformly convex spaces, *Trans. Amer. Math. Soc.*, v. 40 (1936), pp. 396-414).

Lemma (5.1). — *For any two* $\mathbf{C}^n$*-valued functions* x *and* y *on a measure space* E *whose component functions are in* L^p, $p \geqslant 2$,

$$(1) \qquad 2(\|x\|^p + \|y\|^p) \leqslant \|x+y\|^p + \|x-y\|^p \leqslant 2^{p-1}(\|x\|^p + \|y\|^p)$$

where $\|x\|$ *denotes* $\| |x| \|_p = \left(\int_E (|x_1|^2 + \ldots + |x_n|^2)^{p/2}\right)^{\frac{1}{p}}$.

Proof. — Observe first that the right inequality in (1) is equivalent to the left inequality; to see this, set $x'=x+y$, $y'=x-y$. Thus it suffices to prove only the left inequality.

First we establish the left inequality when E is a point; that is

$$(1') \qquad 2(|x|^p+|y|^p)\leqslant|x+y|^p+|x-y|^p$$

for any two complex n-tuples x and y. We may assume without loss of generality that $|y|\leqslant|x|$. Dividing by $|x|$, the inequality (1') is equivalent to

$$(1'') \qquad 2(1+|c|^p)\leqslant|u+c|^p+|u-c|^p$$

for any complex n-tuples u and c with $|u|=1$, $|c|\leqslant 1$. Set $2a=|u+c|+|u-c|$, $1+b=\sup\left(\frac{|u+c|}{a},\frac{|u-c|}{a}\right)$. Then $2a\geqslant|2u|=2$, $b\geqslant 0$, and $1-b=\inf\left(\frac{|1+c|}{a},\frac{|1-c|}{a}\right)$. Hence $|u+c|^p+|u-c|^p=a^p((1+b)^p+(1-b)^p)$ where $a\geqslant 1$ and $0\leqslant b\leqslant 1$. For $p=2$, we have equality in (1') and (1''). As p increases, the left side of (1'') decreases or stays constant. On the other hand, the right side of (1'') increases with p, for the derivative of $f(t)=(1+b)^t+(1-b)^t$ is, for $t\geqslant 1$,

$$\begin{aligned}f'(t)&=(1+b)^t\log(1+b)+(1-b)^t\log(1-b)\geqslant(1+b)\log(1+b)+(1-b)\log(1-b)\\&=-2\left(\frac{b^2}{2}+\frac{b^4}{4}+\dots\right)+2b\left(b+\frac{b^3}{3}+\dots\right)\\&=2\left(b^2\left(1-\frac{1}{2}\right)+b^4\left(\frac{1}{3}-\frac{1}{4}\right)+\dots\right)>0.\end{aligned}$$

Hence (1'') and therefore (1') is established.

Integrating the inequality (1') which holds at each point for any $\mathbf{C}^n$-valued functions on E, we obtain the left side of (1).

Theorem **(5.1)**. — *Let* D *be a shell whose boundary components each contains more than one point. Then there exists a unique admissible function u for* D *such that*

$$\mathscr{C}(\mathrm{D})=\int_{\mathrm{D}}|\nabla u|^n\,d\mathrm{D}.$$

The function u is monotone on D.

Proof. — Let C_0 and C_1 denote the components of the complement of D in Moebius space. Since $\mathscr{C}(\mathrm{D})$ is invariant under conformal mappings, we can assume without loss of generality that $\infty\in C_1$. The hypothesis of the theorem assures that the constants $a=$ diameter C_0 and $b=d(C_0, C_1)$ of Lemma (4.5) are both positive. The inequalities of Lemma (4.5) then imply that the family of monotone admissible functions u such that $\int_{\mathrm{D}}|\nabla u|^n\,d\mathrm{D}\leqslant\mathrm{M}$ is an equicontinuous family of functions on the entire Moebius space $\mathbf{R}^n\cup\infty$ (without further warning we shall denote by the same letter an admissible function and its trivial extension to C_0 and C_1). Given therefore any sequence of functions in this family, we can, by Ascoli's theorem, select a uniformly convergent subsequence.

Consider now any sequence $u_1, u_2, \ldots,$ of monotone admissible functions such that

$$\lim_{r\to\infty}\int_D |\nabla u_r|^n \, dD = \mathscr{C}(D).$$

By Clarkson's Lemma, we have

$$\left\|\frac{\nabla u_r + \nabla u_s}{2}\right\|^n + \left\|\frac{\nabla u_r - \nabla u_s}{2}\right\|^n \leqslant \frac{1}{2}(\|\nabla u_r\|^n + \|\nabla u_s\|^n). \tag{2}$$

Since $\frac{1}{2}(u_r+u_s)$ is an admissible function for D, it follows from (2) that

$$\lim_{r,s\to\infty}\int_D \left|\nabla\left(\frac{u_r+u_s}{2}\right)\right|^n dD = \mathscr{C}(D)$$

and that

$$\lim_{r,s\to\infty} \|\nabla u_r - \nabla u_s\| = 0.$$

Hence the i-th component function of ∇u_r converges in L^n to a function f_i $(i=1, \ldots, n)$. We may suppose, by the remark above, that in fact the sequence $u_1, u_2, \ldots$ converges uniformly on Moebius space to a limit function u. We show that u is admissible for D, $\nabla u = (f_1, \ldots, f_n)$ a.e., and $\int_D |\nabla u|^n dD = \mathscr{C}(D)$.

Let $I = [p, q]$ be a finite line segment in $\mathbf{R}^n$ parallel to the x_1-axis. Let J_ε denote the cube $|x_2|<\varepsilon, \ldots |x_n|<\varepsilon$ in the $(n-1)$-plane $x_1=0$. Since $\frac{\partial u_r}{\partial x_1}$ converges in L^n to f_1, it converges, by Hölder's inequality, in L^1 to f_1 on any bounded region and thus

$$\begin{aligned}\int_{I\times J_\varepsilon} f_1(x+y)\,dx_1\,dy &= \lim_{r\to\infty}\int_{I\times J_\varepsilon}\frac{\partial u_r}{\partial x_1}dx_1\,dy \\ &= \lim_{r\to\infty}\left(\int_{J_\varepsilon} u_r(q+y)\,dy - \int_{J_\varepsilon} u_r(p+y)\,dy\right) \\ &= \int_{J_\varepsilon}(u(q+y)-u(p+y))\,dy.\end{aligned}$$

Dividing by ε^{n-1} and letting $\varepsilon\to 0$, the left side approaches $\int_I f_1(x)\,dx_1$ for almost all $(x_2, \ldots, x_n)$, whereas the right side approaches $u(q)-u(p)$, since u is continuous. Thus $\int_p^q f_1\,dx_1 = u(q)-u(p)$, u is absolutely continuous on almost all line segments $[p, q]$ parallel to the x_1-axis, and $\frac{\partial u}{\partial x_1} = f_1$ a.e. This argument applies to lines parallel to any of the coordinate axes and thus u is ACL and $\nabla u = f = (f_1, \ldots, f_n)$ a.e. Consequently,

$$\int_D |\nabla u|^n dD = \|f\|^n = \lim_{r\to\infty}\|\nabla u_r\|^n = \mathscr{C}(D).$$

Given any admissible function v on D such that $\int_D |\nabla u|^n dD = \mathscr{C}(D)$, we obtain by Clarkson's inequality

$$\left\|\nabla\left(\frac{u+v}{2}\right)\right\|^n + \left\|\frac{\nabla u - \nabla v}{2}\right\|^n \leqslant \mathscr{C}(D)$$

and, since $\frac{u+v}{2}$ is admissible for D, $\nabla(u-v)=0$ a.e. It follows, since u and v are continuous, that $u-v=0$. This establishes the uniqueness of u.

The fact that u is monotone follows at once from the observation that Lebesgue's straightening process would yield an admissible function u_0 with $\int_D |\nabla u_0|^n dD < \int_D |\nabla u|^n dS$ if u were not monotone.

Definition. — The function u of Theorem 5.1 *is called the extremal function for* D. (It depends of course on which of the components of the complement of D is labelled C_0.)

Note. — The extremal function satisfies the variational condition

$$\int_D |\nabla u|^{n-2} \nabla u . \nabla w \, dD = 0$$

for any C^1 function w having compact support in D, or equivalently, in the weak sense

$$\int_D (\operatorname{div} |\nabla u|^{n-2} \nabla u) w \, dD = 0.$$

Thus $\operatorname{div}|\nabla u|^{n-2}\nabla u = 0$ in the weak sense.

Using in succession results of E. di Giorgi, C. B. Morrey, and E. Hopf on elliptic partial differential equations, F. W. Gehring proves for $n=3$ in " Rings and quasi-conformal mappings in space ", *Trans. Amer. Math. Soc.*, v. 103 (196), that if $\frac{1}{M} < \nabla u < M$ a.e., then u is real analytic. Gehring's proof applies with obvious modifications for $n \geqslant 2$ as well.

We shall not require the analyticity theorem in this account.

§ 6. CONTINUITY OF $\mathscr{C}(D)$ IN D

We introduce a metric topology on the closed subsets of Moebius space $\mathbf{R}^n \cup \infty$, defining the distance between two closed subsets E and F:

$$p(E, F) = \sup_{x \in E - F} p(x, F) + \sup_{x \in F - E} p(x, E)$$

$p(x, y)$ denoting the standard metric on the sphere S^n which is identified with Moebius space via some fixed stereographic projection. This metric topology is equivalent to the topology obtained from the standard euclidean distance $d(x, y)$ when we restrict attention to the compact subsets of $\mathbf{R}^n$. Without this restriction, the latter topology is finer.

For arbitrary shells D and D′, we define the distance between D and D′

as $p(C_0 \cup C_1, C'_0 \cup C'_1)$ where $C_0 \cup C_1$ and $C'_0 \cup C'_1$ are the complements of D and D' respectively.

Theorem **(6.1)**. — *$\mathscr{C}(D)$ depends continuously on* D.

Proof. — We must show that given any $\varepsilon > 0$, we have $|\mathscr{C}(D) - \mathscr{C}(D')| < \varepsilon$ for all shells D' in some neighborhood of D. Let x_1 denote a fixed point in C_1, and let I denote the set of all shells not containing x_1. For any neighborhood U of D in the space of all shells, the orbit under the Moebius group of $U \cap I$ is a neighborhood of D. For, since the metric as well as conformal capacity is preserved by a transitive subgroup of the Moebius group, we can assume $\infty \in D$. Let U_δ denote the set of all shells D' such that $d(C'_0 \cup C'_1, C_0 \cup C_1) < \delta$. Given $D' \in U_\delta$, we can by a translation t of length at most δ carry C'_1 into a subset containing the point x_1 of C_1. Since

$$d(C_0 \cup C_1, t(C'_0 \cup C'_1)) \leqslant d(C_0 \cup C_1, C'_0 \cup C'_1) + d(C'_0 \cup C'_1, t(C'_0 \cup C'_1)) < 3\delta$$

we have $T(U_{3\delta} \cap I) \supset U_\delta$, where T denotes the group of translations on $\mathbf{R}^n \cup \infty$. Since $C_0 \cup C_1 \subset \mathbf{R}^n$, the subsets U_δ form a base of neighborhoods of D, and thus our assertion is established.

Since $\mathscr{C}(D)$ is invariant under the Moebius group, it suffices to show that $\mathscr{C}$ is continuous on I. We may clearly take x_1 to be any point in Moebius space. We now fix $x_1 = \infty$. That is, it remains only to prove that $\mathscr{C}(D)$ depends continuously on D for shells $D \subset \mathbf{R}^n$.

Let D be a shell in $\mathbf{R}^n$, let $\varepsilon > 0$, and let u be a smoothly admissible function for D such that $\int_D |\nabla u|^n dD < \mathscr{C}(D) + \varepsilon$. Let K denote the support of ∇u. By definition of u, K is a compact subset of D. For any shell D' sufficiently near D, we have $K \subset D'$. Hence u is admissible for D' and

$$\mathscr{C}(D') < \mathscr{C}(D) + \varepsilon. \tag{1}$$

The opposite inequality will be deduced effectively from Lemma (4.5) which provides uniform estimates for the variation of monotone admissible functions. Let C_0, C'_0 denote the bounded component of the complement of D and D' respectively. Let a' denote the diameter of C'_0 in the euclidean metric, let $b' = d(C'_0, C'_1)$, and $c'(x, y) = \inf(d(x, C'_0), d(y, C'_0))$. For each shell D' in $\mathbf{R}^n$, select a monotone admissible function u' such that

$$\int_{D'} |\nabla u'|^n dD' \leqslant \mathscr{C}(D') + \varepsilon.$$

(We could for example take u' to be the extremal function.) Define u' to be constant on C'_0 and C'_1.

Define the function v' on $\mathbf{R}^n$:

$$v'(x) = \begin{cases} 0 & \text{if } u'(x) < \varepsilon \\ \dfrac{u'(x) - \varepsilon}{1 - 2\varepsilon} & \text{if } \varepsilon \leqslant u'(x) \leqslant 1 - \varepsilon \\ 1 & \text{if } 1 - \varepsilon < u'(x) \leqslant 1. \end{cases}$$

Then v' is ACL in $\mathbf{R}^n$. Let K′ denote the support of $\nabla v'$. Then K′ is a compact subset of D′. The inequality

$$|u'(x)-u'(y)|^n \leqslant \mathrm{AM}\left(\log \frac{a'}{d(x,y)}\right)^{-1} \quad \text{if } d(x,y)<a'$$

implies for any $x \in \mathrm{K}'$,

$$d(x, \mathrm{C}_i') \geqslant \frac{a'}{\exp(\mathrm{AM}'\varepsilon^{-n})} \geqslant \frac{a-\varepsilon}{\exp(\mathrm{A}(\mathrm{M}+2\varepsilon)\varepsilon^{-n})} = \delta(\varepsilon)$$

$(i=0, 1)$, provided D′ is sufficiently close to D.

The inequality

$$|u'(x)-u'(y)|^n \leqslant \mathrm{AM}'\left(\log \frac{c'(x,y)}{b}\right)^{-1} \quad \text{if } c'(x,y)<a'$$

implies for any $x \in \mathrm{K}'$ (when we take $y=\infty$)

$$d(x, \mathrm{C}_0') \leqslant b' \exp(\mathrm{AM}'\varepsilon^{-n}) < (b+\varepsilon)\exp(\mathrm{A}(\mathrm{M}+2\varepsilon)\varepsilon^{-n}) = r(\varepsilon)$$

provided D′ is sufficiently close to D.

Thus for any shell D′ satisfying

$$(2) \qquad d((\mathrm{C}_0' \cup \mathrm{C}_1') \cap \mathrm{B}, (\mathrm{C}_0 \cup \mathrm{C}_1) \cap \mathrm{B}) < \delta(\varepsilon)$$

where B is the closed n-ball with center at some fixed point of C_0 and radius $r(\varepsilon)+b(\varepsilon)+\delta(\varepsilon)$, and for any $x \in \mathrm{K}'$, we have $d(x, \mathrm{C}_0 \cup \mathrm{C}_1) > 0$; that is, $\mathrm{K}' \subset \mathrm{D}$.

Consequently, v' is admissible for D and therefore

$$(3) \qquad \mathscr{C}(\mathrm{D}) \leqslant \int_{\mathrm{D}} (\nabla v)^n d\mathrm{D} \leqslant (1-2\varepsilon)^{-n}(\mathscr{C}(\mathrm{D}')+\varepsilon)$$

for all shells D′ satisfying the condition (2). The set of such shells is a neighborhood of D. From assertions (1) and (3) the theorem follows.

Remark. — For closed subsets of $\mathbf{R}^n \cup \infty$ containing ∞, the metric topology introduced above is equivalent to the inductive topology with respect to compact sets of $\mathbf{R}^n$. The same is true therefore for *closed* subsets of $\mathbf{R}^n \cup \infty$. Let D and D_m denote shells in $\mathbf{R}^n \cup \infty$ with complementary components C_0, C_1 and $\mathrm{C}_{m,0}$, $\mathrm{C}_{m,1}$ respectively such that

$$\lim_{m\to\infty} \mathrm{C}_{m,0} \cap \mathrm{K} = \mathrm{C}_0 \cap \mathrm{K}$$
$$\lim_{m\to\infty} \mathrm{C}_{m,1} \cap \mathrm{K} = \mathrm{C}_1 \cap \mathrm{K}$$

for each compact set $\mathrm{K} \subset \mathbf{R}^n$. Then D_m converges to D in the topology of our space of shells.

§ 7. THE CONFORMAL CAPACITY AND MODULUS OF SOME SPECIAL SHELLS

Let $D_{a,b}$ denote the spherical shell consisting of all x in $\mathbf{R}^n$ with $a<|x|<b$, where $0<a<b$.

Lemma (7.1). — $\mathscr{C}(D_{a,b})=c_{n-1}\left(\log\frac{b}{a}\right)^{-(n-1)}$, *where* c_{n-1} *denotes the* $(n-1)$ *measure of the surface* S *of the unit n-ball.*

Proof. — Let u be an admissible C^1 function for $D=D_{a,b}$. Then integrating along a ray, we get

$$1\leqslant\left(\int_a^b|\nabla u|\,dr\right)^n=\left(\int_a^b|\nabla u|\,r^{\frac{n-1}{n}}r^{-\frac{n-1}{n}}\,dr\right)^n$$
$$\leqslant\left(\int_a^b|\nabla u|^n r^{n-1}\,dr\right)\left(\int_a^b r^{-1}\,dr\right)^{n-1}$$

by Hölder's inequality. Integrating over all rays, we get

$$c_{n-1}\leqslant\int_D|\nabla u|^n\,dD.\left(\log\frac{b}{a}\right)^{n-1}.$$

Hence $\mathscr{C}(D)\geqslant c_{n-1}\left(\log\frac{b}{a}\right)^{-(n-1)}$. On the other hand, taking $u(x)=\left(\log\frac{|x|}{a}\right)\left(\log\frac{b}{a}\right)^{-1}$, we find

$$\int_D|\nabla u|^n\,dD=c_{n-1}\left(\log\frac{b}{a}\right)^{-n}\int_a^b\left(\frac{1}{r}\right)^n r^{n-1}\,dr=c_{n-1}\left(\log\frac{b}{a}\right)^{-(n-1)}.$$

Hence $\mathscr{C}(D)=c_{n-1}\left(\log\frac{b}{a}\right)^{-(n-1)}$.

It is convenient to introduce another conformal invariant, equivalent to conformal capacity, but rendering somewhat simpler the formulas to be encountered below.

Definition. — *The modulus of a shell* D *in Moebius n-space is* $\left(\frac{c_{n-1}}{\mathscr{C}(D)}\right)^{\frac{1}{n-1}}$. *It is denoted* mod D.

Thus we have $\operatorname{mod} D_{a,b}=\log\frac{b}{a}$, generalizing the familiar formula for $n=2$.

Given two shells D and D′ such that $C_0'\supset C_0$ and $C_1'\supset C_1$, where C_0 and C_1 are two components of the complement of D and C_0' and C_1' are the components of the complement of D′, we say that D′ *separates* the boundaries of D. It is clear that $D'\subset D$ and that any admissible function for D′ is admissible for D. Hence

$$\mathscr{C}(D)\leqslant\mathscr{C}(D')\quad\text{and}\quad\operatorname{mod}D\geqslant\operatorname{mod}D'$$

if D′ separates the boundaries of D.

It follows from this and Lemma (7.1) that $\mathscr{C}(D)=0$ *if one of the components of the complement of* D *reduces to a single point.* For without loss of generality, one can assume that the origin is the unique point of the component C_0. Then for suitably small positive b, the spherical shell $D_{a,b}$ is contained in D and it separates the boundaries of D. Hence

$$\operatorname{mod} D \geqslant \log \frac{b}{a}$$

for arbitrarily small a. It follows that $\operatorname{mod} D = \infty$ and $\mathscr{C}(D) = 0$.

Note. — The condition that the component C_0 of the complement of the shell D consist of a single point is equivalent to the condition that the boundary component $C_0 \cap \overline{D}$ consist of a single point p. For the complement of a single point in S^n is connected. On the other hand $C \cup D = S^n - C_0$ is open, and $C_0 - p = S^n - (C_1 \cup D) - p = S^n - (C_1 \cup \overline{D})$ is open. Since $S^n - p = (C_1 \cup D) \cup (C_0 - p)$, we conclude that $C_0 - p$ is empty.

Referring to Lemma (4.2), we can now assert that the *conformal capacity of a shell is zero if and only if one of its boundary components reduces to a point.*

Lemma **(7.2)**. — *Let* D *be a shell and let* $D_1, \ldots, D_m$ *be mutually disjoint shells each separating the boundaries of* D. *Then*

$$\operatorname{mod} D \geqslant \operatorname{mod} D_1 + \ldots + \operatorname{mod} D_m.$$

Proof. — By hypothesis we have $C_0 \subset C_{0,i}$ and $C_1 \subset C_{1,i}$ for the connected components of the complements. Let u_i be a smoothly admissible function for D_i taking the value 0 on $C_{0,i}$ and 1 on $C_{1,i}$ $(i = 1, \ldots, m)$. Set $u = \sum_{i=1}^{m} a_i u_i$, where $a_i \geqslant 0$ and $\sum_{i=1}^{m} a_i = 1$. Then

$$\int_D |\nabla u|^n \, dx = \sum_{i=1}^{m} a_i^n \int_{D_i} |\nabla u_i|^n \, dx.$$

Hence $\mathscr{C}(D) \leqslant \sum_{i=1}^{m} a_i^n \mathscr{C}(D_i)$. We can assume $\mathscr{C}(D) > 0$, otherwise there is nothing to prove. Hence $\mathscr{C}(D_i) > 0$ and we may take $a_i = \mathscr{C}(D_i)^{-\frac{1}{n-1}} \left(\sum_{j=1}^{m} \mathscr{C}(D_j)^{-\frac{1}{n-1}} \right)^{-1}$. Then we get

$$\mathscr{C}(D) \leqslant \frac{\sum_{i=1}^{m} \mathscr{C}(D_i)^{-\frac{1}{n-1}}}{\left(\sum_{i=1}^{m} \mathscr{C}(D_i)^{-\frac{1}{n-1}} \right)^n} = \left(\sum_{i=1}^{m} \mathscr{C}(D_i)^{\frac{1}{n-1}} \right)^{-(n-1)}$$

$$\mathscr{C}(D)^{-\frac{1}{n-1}} \geqslant \sum_{i=1}^{m} \mathscr{C}(D_i)^{-\frac{1}{n-1}}$$

$$\operatorname{mod} D \geqslant \sum_{i=1}^{m} \operatorname{mod} D_i.$$

Definition. — *The Grötzsch shell* $D_G D_G = (a)$ *is the shell in* $\mathbf{R}^n$ *whose complementary components consist of the sphere* $|x| < 1$ *and the ray* $a \leqslant x_1 < \infty$, $x_2 = x_3 = \ldots = x_n = 0$, *where* $a > 1$.

The Teichmüller shell $D_T = D_T(b)$ *is the shell in* $\mathbf{R}^n$ *whose complementary components consist of the segment* $-1 \leqslant x_1 \leqslant 0$, $x_2 = \ldots = x_n = 0$ *and the ray* $b < x_1 < \infty$, $x_2 = x_3 = \ldots = x_n = 0$, *where* $b > 0$.

Teichmüller has introduced the functions Φ_n and Ψ_n (in the case $n=2$) defined by:

$$\text{mod } D_G = \log \Phi_n(a), \qquad \text{mod } D_T = \log \Psi_n(b).$$

Lemma **(7.3)**. — $\dfrac{\Phi_n(a)}{a}$ *is a monotonically increasing function of* a, $a>1$. *For any* $b>0$,

$$\Psi_n(b) = (\Phi_n((b+1))^{1/2})^2.$$

Proof. — Suppose $a'>a>1$. Let D denote the shell obtained on shrinking $D_G(a')$ by the factor $\dfrac{a}{a'}$. Then, by Lemma (7.2), we get

$$\text{mod } D_G(a') = \text{mod } D \geqslant \text{mod } D_{\frac{a}{a'},1} + \text{mod } D_G(a).$$

Thus

$$\log \Phi_n(a') \geqslant \log \frac{a'}{a} + \log \Phi_n(a)$$

and this gives the first assertion.

To prove the second assertion, let D denote the shell obtained from $D_T(b)$ by translating $x \to (x_1+1, x_2, \ldots, x_n)$ and then shrinking by the factor $(b+1)^{-1/2}$.

Let $D' = D_G((b+1)^{1/2})$. Let σ denote the Moebius inversion in the unit sphere and let u denote an extremal function for the shell D. Then σ interchanges the components of the complement of D and $\sigma D = D$. Since σ preserves $\int_D |\nabla u|^n dD$, the function $u \circ \sigma$ is extremal and thus $u \circ \sigma = 1 - u$. Since $\sigma(x) = x$ for $|x| = 1$, we find $u(x) = \frac{1}{2}$ for $|x| = 1$. From this it follows that the restriction to D' of $2\left(u - \frac{1}{2}\right)$ is extremal for D'.

Hence

$$\begin{aligned}\mathscr{C}(D_T(b)) &= \int_D |\nabla u|^n dx = \int_{D'} |\nabla u|^n dx + \int_{\sigma(D')} |\nabla u|^n dx \\ &= 2\int_{D'} |\nabla u|^n dx = \frac{2}{2^n}\int_{D'} |\nabla 2((u-1)/2)|^n dx \\ &= 2^{-(n-1)}\mathscr{C}(D').\end{aligned}$$

Thus

$$\text{mod } D_T(b) = 2 \text{ mod } D_G((b+1)^{1/2})$$

or

$$\Psi_n(b) = (\Phi_n((b+1)^{1/2}))^2.$$

§ 8. SPHERICAL SYMMETRIZATION

The idea we are about to describe goes back to Jacob Steiner who defined for any plane region R and any line L in the plane, the symmetrization R^* as follows: At each point $x \in L$ such that the perpendicular line S_x to L at x meets R, one places a line segment along S_x centered at x whose length is the length of $S_x \cap R$; the union of these segments is R^*. Schwarz introduced the symmetrization of a solid region in 3-space with respect to a line L. If one thinks of Schwarz's planes perpendicular to L as a family of concentric spheres of infinite radius, then one is led to the definition of the *spherical symmetrization* of a set R in $\mathbf{R}^n$ with respect to the line $L = \{(t, 0, \ldots, 0); -\infty < t \leq 0\}$.

For each sphere $S_r : |x| = r$ meeting the set R, place along S_r, centered at $(-r, 0, \ldots, 0)$ a spherical cap (of dimension $n-1$) of spherical measure equal to the measure of $S_r \cap R$; the union of these caps is denoted R^*.

To complete this definition, we have to specify whether the spherical caps are open or closed. One does the expected: If R is open, the spherical cap is taken open and includes the point $(r, 0, \ldots, 0)$ if and only if $S_r \subset R$. If R is closed, the spherical cap is taken closed.

It is clear that for any connected set E, open or closed, the set E^* is connected, and that E^* is open if E is open. Since the complement of E^* is clearly the spherical symmetrization of the complement of E with respect to the *positive* x_1-axis, it follows that E^* is closed if E is closed, and the complement of E^* is connected if the complement of E is connected. Also, $E_1^* \subset E_2^*$ if $E_1 \subset E_2$.

Definition. — Let D *be a shell in* $\mathbf{R}^n$ *and let* C_0 *denote the bounded component of its complement. The sphere-symmetrization of* D *is the set* D^0 *defined by*

$$D^0 = (D \cup C_0)^* - C_0^*.$$

It follows from the remarks above that D^0 is a shell if D is a shell. For the Grötzsch and Teichmüller shells, we have $D_G^0 = D_G$, $D_T^0 = D_T$.

This section is devoted to the proof that $\mathscr{C}(D^0) \leq \mathscr{C}(D)$ for any shell D, a fact which is suggested by isoperimetric considerations. In the course of comparing a shell with its symmetrization, we shall use the following notion.

Let u be a real-valued continuous function on $\mathbf{R}^n$. For any real number a, set

$$\begin{aligned} F_a &= u^{-1}(-\infty, a] \\ G_a &= u^{-1}(-\infty, a) \\ \Delta_a &= u^{-1}(a) = F_a - G_a \\ \Delta_a^* &= F_a^* - G_a^*. \end{aligned}$$

Define the real valued function u^* on $\mathbf{R}^n$ by:

$$u^*(x) = a \text{ if and only if } x \in \Delta_a^*.$$

Clearly $\bigcap_{b>a} F_b = F_a$, and $\bigcup_{b<a} F_b = G_a$. Hence $\bigcap_{b>a} F_b^* = F_a^*$ and $\bigcup_{b<a} F_b^* = G_a^*$. It follows that $u^{*-1}(-\infty, a) = G_a^*$ and is open. In addition, $u^{*-1}(-\infty, a] = F_a^*$ is closed. Thus $u^{*-1}(b, a)$ is open for $b < a$ and therefore u^* is continuous on $\mathbf{R}^n$. The function u^* is called the *spherical symmetrization* of u. It has the property:

For any interval $I = (a, b)$, $[a, b]$, or $(a, b]$, $u^{-1}(I) \cap S_r$ and $u^{*-1}(I) \cap S_r$ have the same measure.

We wish to compare $\mathscr{C}(D)$ and $\mathscr{C}(D^0)$. This we do by comparing $\int_D |\nabla u|^n dx$

and $\int_{D^0} |\nabla u^*|^n dx$ for piecewise linear approximations u to smoothly admissible functions. In the proof below we require the following two results:

The first result is an isoperimetric inequality:

Theorem A. — *Let* E *be a subset of the simply connected space* X *of constant curvature, and let* $T_s(E)$ *be a tubular neighborhood of radius* s (that is, the union of all balls of radius s and center in E). *Then for fixed volume* $m(E)$, *the volume* $m(T_s(E))$ *is minimized for* E *a ball.*

This theorem, for convex bodies in euclidean space is known as the " Brunn-Minkowski inequality ". It was first proved for arbitrary measurable sets in euclidean space by Lusternik, in *C.R. Acad. Sci. U.R.S.S.*, v. 3 (1935), pp. 55-58. Theorem A is proved by E. Schmidt, in *Math. Nachr.*, v. 1 (1948), pp. 81-157. We shall require this theorem only for the cases $X = S^n$ and $X = \mathbf{R}^n$.

The usual form of the isoperimetric theorem concerning the boundary $\partial E (= E - \text{int } E)$ is a consequence of Theorem A. Namely, for any subset E of a metric space, one can define the k-dimensional *Hausdorff measure*

$$m_k(E) = \lim_{s \to 0} \inf \sum_U b_k (\text{radius } U)^k$$

where $\{U\}$ is a countable covering of E of balls U having radius less than s, and

$$b_k = 2^k \Gamma\left(\frac{1}{2}\right)^{k-1} \Gamma\left(\frac{k+1}{2}\right) \Gamma(k+1)^{-1}$$

that is, the volume of the unit ball in $\mathbf{R}^k$. In the highest dimension, Hausdorff measure coincides with Lebesgue measure on S^n and $\mathbf{R}^n$. Moreover if E is a *closed region* (i.e. $E = \overline{\text{int } E}$) of either S^n of $\mathbf{R}^n$, *with a piecewise smooth boundary*, then

$$m_{n-1}(\partial E) = \lim_{s \to 0} \frac{m(T_s(E)) - m(E)}{s}.$$

Let $\hat{E}$ denote the ball such that $m_n(E) = m_n(\hat{E})$. (In the case of S^n, the ball $\hat{E}$ is a spherical cap of course). Then Theorem A implies

Theorem A'. — $m_{n-1}(\partial E) \geq m_{n-1}(\partial \hat{E})$.

Notation. — Given $E \subset S_r = \{x, |x| = r, x \in \mathbf{R}^n\}$ we shall be considering below the tubular neighborhoods $T_s(E)$ in S_r and in $\mathbf{R}^n$. To avoid ambiguity, we shall denote by $T_s(E)$ the tubular neighborhood in $\mathbf{R}^n$ and by $\hat{T}_s(E)$ the tubular neighborhood in S_r.

The second theorem we shall refer to is:

Theorem B. — *Let u be a real valued function on the domain* $D \subset \mathbf{R}^n$ *satisfying the uniform Lipschitz condition*

$$|u(x) - u(y)| < M|x - y| \qquad \textit{for } x, y \in D.$$

Let f *be a continuous function on* D. *Then*

$$\int_D f |\nabla u| dx = \int_{-\infty}^{\infty} \left(\int_{u^{-1}(t)} f d\sigma \right) dt$$

where $d\sigma$ *denotes Hausdorff* $(n-1)$*-measure on* $u^{-1}(t)$.

This theorem was first proved by E. di Giorgi (*Ann. Mat. Pur. Appl.*, ser. 4, v. 36 (1954), pp. 191-213) under the hypothesis that u is differentiable and was generalized independently by H. Federer and L. C. Young to Lipschitz maps u of an n-manifold in to an m-manifold, $n \geqslant m$ (cf. H. Federer, Curvature measures, *Trans. Amer. Math. Soc.*, v. 93 (1959), pp. 418-431).

Lemma (**8.1**). — *Let* F *be a closed region in* $\mathbf{R}^n$. *Then*

$$T_s(F)^* \supset T_s(F^*) \quad \textit{for any } s>0$$

where T_s *denotes a tubular neighborhood of radius* s.

Proof. — It suffices to show that for each $r>0$,

$$S_r \cap T_s(F)^* \supset S_r \cap T_s(F^*);$$

that is

$$m_{n-1}(S_r \cap T_s(F)) \geqslant m_{n-1}(S_r \cap T_s(F^*)).$$

We have

$$S_r \cap T_s(F) = \bigcup_{-s \leqslant t \leqslant s} S_r \cap T_s(S_{r-t} \cap F).$$

For any radius r_1, r_2, we have

$$S_{r_2} \cap T_s(S_{r_1} \cap F) = \hat{T}_{\theta r_2}\left(\frac{r_2}{r_1}(S_{r_1} \cap F)\right)$$

where θ is the angle defined by $s^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta$.

Hence, by Theorem A

$$m_{n-1}(S_r \cap T_s(S_{r-t} \cap F)) \geqslant m_{n-1}(S_r \cap T_s(S_{r-t} \cap F^*)).$$

Consequently,

$$\begin{aligned} m_{n-1}(S_r \cap T_s(F)) &\geqslant \sup_t m_{n-1}(S_r \cap T_s(S_{r-t} \cap F)) \\ &\geqslant \sup_t m_{n-1}(S_r \cap T_s(S_{r-t} \cap F^*)) \\ &= m_{n-1}\Big(\bigcup_{-s \leqslant t \leqslant s} S_r \cap T_s(S_{r-t} \cap F^*)\Big) \end{aligned}$$

since $T_s(F^*)$ is a solid of revolution. Hence

$$m_{n-1}(S_r \cap T_s(F)) \geqslant m_{n-1}(S_r \cap T_s(F^*)).$$

Lemma (**8.2**). — *Let* F *be a closed region in* $\mathbf{R}^n$ *with a piecewise smooth boundary. Then*

$$m_n(F^*) = m_n(F)$$

and

$$m_{n-1}(\partial F^*) \leqslant m_{n-1}(\partial F).$$

Proof. — Set $V(r) = m_n(F \cap B_r)$, $V^*(r) = m_n(F^* \cap B_r)$.

Then $\dfrac{dV}{dr} = m_{n-1}(F \cap S_r) = m_{n-1}(F^* \cap S_r) = \dfrac{dV^*}{dr}$. It follows that $V(r) = V^*(r)$ and $m_n(F^*) = m_n(F)$.

To prove the second assertion, we observe

$$m_{n-1}(\partial F) = \lim_{s\to 0} \frac{m(T_s(F)) - m(F)}{s}$$
$$= \lim_{s\to 0} \frac{m(T_s(F)^*) - m(F^*)}{s}$$
$$\geqslant \lim_{s\to 0} \frac{m(T_s(F^*)) - m(F^*)}{s}, \qquad \text{by Lemma (8.1).}$$

Hence $m_{n-1}(\partial F) \geqslant m_{n-1}(\partial F^*)$.

Lemma **(8.3)**. — *Let u be a real-valued function on R^n satisfying the uniform Lipschtiz condition*

$$|u(x) - u(y)| \leqslant M|x-y| \quad \textit{for} \quad x, y \in \mathbf{R}^n.$$

Let u^* denote the spherical symmetrization of u. Then

$$|u^*(x) - u^*(y)| \leqslant M|x-y| \quad \textit{for} \quad x, y \in \mathbf{R}^n.$$

Proof. — Let F_t denote the set in $\mathbf{R}^n$ on which $u(x) \leqslant t$. It suffices to prove that if $x \in T_s(F_t^*)$, then $u^*(x) \leqslant t + Ms$; that is, $T_s(F_t^*) \subset F_{t+Ms}^*$. By Lemma (8.1) and our hypothesis

$$T_s(F_t^*) \subset T_s(F_t)^* \subset F_{t+Ms}^*.$$

Lemma **(8.4)**. — *Let F be a finite union of simplexes in $\mathbf{R}^n$, and let u be a continuous piecewise linear function on $\mathbf{R}^n$, linear on each simplex of F and whose gradient vanishes outside F. Let u^* denote the spherical symmetrization of u. Let φ be a piecewise continuous function on $\mathbf{R}^2$. Set $\Delta_t = u^{-1}(t)$, $\Delta_t^* = u^{*-1}(t)$. Then*

$$\int_{\mathbf{R}^n} \varphi(|x|, u(x))\,dx = \int_{\mathbf{R}^n} \varphi(|x|, u^*(x))\,dx$$

and for all but a finite number of t,

$$\int_{\Delta_t} \varphi(|x|, t)\,d\Delta_t \geqslant \int_{\Delta_t^*} \varphi(|x|, t)\,d\Delta_t^*.$$

Proof. — Set $W(r) = \int_{B_r} \varphi(|x|, u(x))\,dx$, $W^*(r) = \int_{B_r} \varphi(|x|, u^*(x))\,dx$.
Then $\dfrac{dW}{dr} = \int_{S_r} \varphi(r, u(x))\,dS_r$ and a similar formula holds for $\dfrac{dW^*}{dr}$.

From the definition of u^* it follows at once that the set of points on S_r for which $a < u \leqslant b$ and $a < u^* \leqslant b$ respectively have the same measure, a and b being arbitrary. It follows at once that $\dfrac{dW}{dr} = \dfrac{dW^*}{dr}$ and $W(r) = W^*(r)$. This establishes the first assertion.

Turning to the second assertion, let $t_1, \ldots, t_p$ be the values of u on the vertices of F. Then for t distinct from these values, $\Delta_t = u^{-1}(t)$ is the boundary of $F_t = u^{-1}[-\infty, t]$; that is, $F_t = \overline{\operatorname{Int} F_t}$. Moreover Δ_t is piecewise smooth and $\Delta_t \cap S_r$ is the boundary of $F_t \cap S_r$ for each $r > 0$, provided $t \neq t_1, \ldots, t_p$.

Let $F_t; r_1, r_2 = (F_t \cap B_{r_2}) - (F_t \cap \operatorname{Int} B_{r_1})$, where $0 < r_1 < r_2$, and let

$$\Delta_t; r_1, r_2 = \Delta_t \cap (F_t; r_1, r_2).$$

Similarly define $F_t^*; r_1, r_2$ and $\Delta_t^*; r_1, r_2$. Then

$$\partial F_t; r_1, r_2 = (\Delta_t; r_1, r_2) \cup (F_t \cap S_{r_1}) \cup (F_t \cap S_{r_2})$$
$$\partial F_t^*; r_1, r_2 = (\Delta_t; r_1, r_2) \cup (F_t^* \cap S_{r_1}) \cup (F_t^* \cap S_{r_2}).$$

By definition $m_{n-1}(F_t \cap S_{ri}) = m_{n-1}(F_t^* \cap S_{ri})$ $(i=1, 2)$. Hence by Lemma (8.2)

$$m_{n-1}(\Delta_t; r_1, r_2) \geqslant m_{n-1}(\Delta_t^*; r_1, r_2)$$

for $0 < r_1 < r_2$. Let M denote the set function on the positive r-axis given by $M(r_1, r_2) = m_{n-1}(\Delta_t; r_1, r_2)$, and similarly set $M^*(r_1, r_2) = m_{n-1}(\Delta_t^*; r_1, r_2)$. Then

$$\int_{\Delta_t} \varphi(|x|, t)\, d\Delta_t = \int_{r>0} \varphi(r, t)\, dM \geqslant \int_{r>0} \varphi(r, t)\, dM^* = \int_{\Delta_t^*} \varphi(|x|, t)\, d\Delta_t^*.$$

Theorem **(8.1)**. — *Let* D *be a shell in* R^n *and let* D^0 *be its sphere-symmetrization. Then*

$$\mathscr{C}(D) \geqslant \mathscr{C}(D^0).$$

Proof. — Let u be a smoothly admissible function for D. By the method of § 3, one can approximate D by a piecewise linear function u' and the function u' can be so chosen that

1. u' is admissible for D and $\nabla u'$ has as support a finite union of simplexes.
2. $|\nabla u - \nabla u'|$ is arbitrarily small.

Since the spherical symmetrization of u' will then be admissible for D^0, it suffices, to prove Theorem (8.1), to show

$$\int_{\mathbf{R}^n} |\nabla u|^n\, dx \geqslant \int_{\mathbf{R}^n} |\nabla u^*|^n\, dx$$

for any piecewise linear function which " lives " on a finite union of simplexes. Let $t_1 < t_2 < \ldots < t_h$ denote the values of u on the vertices of these simplexes.

Set $f^* = |\nabla u^*|^{n-1}$. We claim $f^*(x) = \varphi(|x|, u^*(x))$ with φ a piecewise continuous function on $\mathbf{R}^2$. For by the axial symmetry of u^*, ∇u is determined by its restriction to the upper half (x_1, x_2)-coordinate plane P; let G_i denote the subset of points in $\mathbf{R}^n$ for which $t_i < u^*(x) < t_{i+1}$. Then the map of $\mathbf{R}^n$, $\pi : x \to (|x|, u^*(x))$ gives a homeomorphism π_i of $G \cap P_i$. For by definition of spherical symmetrization, u^* is a monotone function on the semi-circle $S_r \cap P$ and it is not constant on any open subset of $S_r \cap G_i \cap P$. Since in addition u^* is piecewise differentiable, it is strictly monotone on the arc $S_r \cap G_i \cap P$ and provides a homeomorphism of the arc for each r. It follows that π_i, the restriction of π to $G_i \cap P$, is a homeomorphism. Therefore the restriction of f^* to G_i can be expressed by $f^*(x) = (f^* \circ \pi_i^{-1})(\pi(x))$. Set $\varphi = f^* \circ \pi_i^{-1}$ on $\pi(G_i)$ $(i=1, \ldots, h-1)$. Define φ to be zero on the complement of $\pi(\mathbf{R}^n)$. It is clear that φ is piecewise continuous on $\mathbf{R}^2$.

We have

$$\int_{\mathbf{R}^n}|\nabla u^*|^n\,dx=\int_{\mathbf{R}^n}|\nabla u^*|^{n-1}|\nabla u^*|\,dx=\int f^*|\nabla u^*|\,dx$$

$$=\int_0^1\Big(\int_{u^{*-1}(t)} f^*\,d\sigma\Big)dt, \qquad \text{by Theorem B}$$

$$\leqslant\int_0^1\Big(\int_{u^{-1}(t)} f\,d\sigma\Big)dt, \qquad \text{where } f(x)=\varphi(|x|,u(x)), \qquad \text{by Lemma (8.4)}$$

$$=\int_{\mathbf{R}^n} f|\nabla u|\,dx, \qquad \text{by Theorem B}$$

$$\leqslant\Big(\int_{\mathbf{R}^n} f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}\Big(\int_{\mathbf{R}^n}|\nabla u|^n\Big)^{\frac{1}{n}}, \qquad \text{by Hölder's inequality}$$

$$=\Big(\int_{\mathbf{R}^n}|\nabla u^*|^n\Big)^{\frac{n-1}{n}}\Big(\int_{\mathbf{R}^n}|\nabla u|^n\Big)^{\frac{1}{n}}, \qquad \text{by Lemma (8.4).}$$

Hence

$$\int_{\mathbf{R}^n}|\nabla u|^*\,dx\leqslant\int_{\mathbf{R}^n}|\nabla u|^n.$$

Theorem **(8.2)**. — *Let* D *be a shell in* $\mathbf{R}^n$, *and let* C_0, C_1 *denote the bounded and unbounded components of its complement. Let* $x_0\in C_0$, *and set* $a=\sup_{y\in C_0} d(x_0,y)$, $b=\inf_{y\in C_1} d(x_0,y)$.
Then

$$\operatorname{mod} D\leqslant\log\Psi_n(b/a).$$

Proof. — Let D^0 denote the sphere symmetrization of D based on x_0 as the origin and the line L as axis. Then clearly $C_0^*\supset[-a,0]$, and $C_1^*\supset[b,\infty]$. Hence D^0 separates the boundaries of the shell $D'=\mathbf{R}^n-[-a,0]-[b,\infty]$ and we have

$$\operatorname{mod} D\leqslant\operatorname{mod} D^0\leqslant\operatorname{mod} D'=\operatorname{mod} D_T(b/a)=\log\Psi_n(b/a).$$

§ 9. QUASI CONFORMAL MAPPINGS

In this section we shall define quasi-conformal mappings and establish the characteristic properties that will be needed in § 10 to prove:

A quasiconformal mapping of the open n-ball onto itself extends to a homeomorphism of the boundary, and this boundary homeomorphism is quasi-conformal.

Let $\varphi: D\to D'$ be a homeomorphism of a Riemannian manifold D onto the Riemannian manifold D'. We associate with φ the functions H_φ, I_φ, J_φ, L_φ, and l_φ defined as follows:

$$L_\varphi(p,r)=\sup_{d(q,p)=r} d(\varphi(q),\varphi(p))$$

$$l_\varphi(p,r)=\inf_{d(q,p)=r} d(\varphi(q),\varphi(p))$$

$$H_\varphi(p)=\overline{\lim_{r\to 0}}\frac{L(p,r)}{l(p,r)}$$

$$I_\varphi(p)=\overline{\lim_{r\to 0}}\frac{L(p,r)}{r}$$

$$J_\varphi(p)=\overline{\lim_{r\to 0}}\frac{m(\varphi(T_r(p)))}{m(T_r(p))}.$$

Here we have written $L=L_\varphi$, $l=l_\varphi$, and similarly we write H, I, J when the homeomorphism φ is unambiguous in the context. We note that the tubular neighborhood $T_r(p)$ of the point p is merely an open ball.

Some elementary observations should be noted.

(1) $s \leqslant l(p, t)$ if and only if $T_s(\varphi(p)) \subset \varphi(T_t(p))$

(2) $L(p, t) \leqslant s$ if and only if $\varphi(T_t(p)) \subset T_s(\varphi(p))$

(3) $L_\varphi(p, t)=s$ if and only if $l_{\varphi^{-1}}(\varphi(p), s) = t$

and $l_\varphi(p, t)=s$ if and only if $L_{\varphi^{-1}}(\varphi(p), s)=t$

(4) $I_\varphi(p)^n \leqslant H_\varphi(p)^n J_\varphi(p)$ and $I_{\varphi^{-1}}(\varphi(p))^n \leqslant H_\varphi(p)^n J_{\varphi^{-1}}(\varphi(p))$.

Properties (1), (2), and (3) are obvious. The first property in (4) comes from

$$\left(\frac{L_\varphi(p, t)}{t}\right)^n = \left(\frac{L_\varphi(p, t)}{l_\varphi(p, t)}\right)^n \cdot \left(\frac{l_\varphi(p, t)}{t}\right)^n.$$

The second property in (4) comes from setting $t=L_{\varphi^{-1}}(\varphi(p), s)$ and observing, by (3), that

$$\left(\frac{t}{s}\right)^n = \left(\frac{t}{l_\varphi(p, t)}\right)^n = \left(\frac{L_\varphi(p, t)}{l_\varphi(p, t)}\right)^n \cdot \left(\frac{t}{L_\varphi(p, t)}\right)$$

and

$$\frac{t}{L_\varphi(p, t)} = \frac{l_{\varphi^{-1}}(p, r)}{r}$$

where $r=L_\varphi(p, t)$.

We say that a mapping φ is *differentiable* at a point p if it has a *differential* at p. If the domain and image of φ are in $\mathbf{R}^n$, the differential $\dot\varphi$ of φ at p is a linear mapping such that

$$\varphi(x)=\varphi(p)+\dot\varphi(x-p)+\varepsilon(x, p)$$

where $\lim_{x\to p} \frac{|\varepsilon(x, p)|}{|x-p|}=0$. If φ is differentiable at p, then the above functions satisfy $H_\varphi(p)=H_{\dot\varphi}(0)$ if $\dot\varphi \neq 0$, $I_\varphi(p)=I_{\dot\varphi}(0)$, $J_\varphi(p)=J_{\dot\varphi}(0)$.

For a *linear* mapping $\varphi : \mathbf{R}^n \to \mathbf{R}^n$, we have

$$L_\varphi(0, r)^2 = \sup_{|x|=r} \langle \varphi x, \varphi x \rangle = \sup_{|x|=r} \langle {}^t\varphi\varphi x, x \rangle = \lambda_1^2 r^2$$

where λ_1^2 is the maximum eigenvalue of the positive definite self-adjoint operator ${}^t\varphi\varphi$. Similarly $l_\varphi(0, r)^2=\lambda_n^2 r^2$ where λ_n^2 is the smallest eigenvalue of ${}^t\varphi\varphi$. Since

$$J_\varphi(0)=|\det \varphi| = (\det {}^t\varphi\varphi)^{1/2} = \lambda_1\lambda_2\ldots\lambda_n,$$

where $\lambda_1^2 \geqslant 0$ and $\lambda_1^2 \geqslant \lambda_2^2 \geqslant \ldots \geqslant \lambda_n^2$ are the eigenvalues of ${}^t\varphi\varphi$, we have

$$\lambda_1^n \leqslant (\lambda_1/\lambda_n)^{n-1}\lambda_1\lambda_2\ldots\lambda_n.$$

It follows that

(5) If φ is differentiable at p, then $I_\varphi(p)^n \leqslant H_\varphi(p)^{n-1} J_\varphi(p)$ since $I_\varphi(p) = J_\varphi(p) = 0$ if $\dot{\varphi} = 0$.

We shall require the following result, known as the Rademacher-Stepanoff Theorem.

Let f be a measurable function on a measurable set E *such that*

$$\varlimsup_{\Delta x \to 0} \left| \frac{f(x+\Delta x) - f(x)}{\Delta x} \right| < \infty \qquad a.e.$$

Then f is differentiable a.e. in E.

This result was proved by H. Rademacher (*Math. Annalen*, v. 79, pp. 340-359 (1919)) under the stronger hypothesis that $I_f < \infty$ everywhere in the domain E. The result quoted here is given by W. Stepanoff, *Rec. Math. Soc. Math.*, Moscou, v. 32, pp. 511-526 (1925), cf. also Saks, p. 310.

Definition. — *A homeomorphism φ of a Riemannian manifold* D *onto a Riemannian manifold* D' *is called quasi-conformal if* H_φ *is bounded on* D. *The mapping φ is called* K-*quasi-conformal if* H_φ *is bounded on* D *and* $H_\varphi(x) \leqslant K$ *almost everywhere on* D.

For example, a conformal mapping is 1-quasi-conformal.

Theorem **(9.1)**. — *A quasi-conformal mapping is differentiable almost everywhere.*

Proof. — By definition, $J_\varphi(p)$ is the upper derivative of the set function $E \to m(\varphi(E))$. Hence by Lebesgue's Theorem, $J_\varphi < \infty$ a.e. (Saks, p. 115). Hence by (4), $I_\varphi < \infty$ a.e. Theorem (9.1) now follows from the Rademacher-Stepanoff Theorem.

Let $E \subset \mathbf{R}^n$ and let a be a positive real number. Let $\{U\}$ be a denumerable cover of E by balls of radius not exceeding a. We write

$$\Lambda(E, a) = \inf_{\{U\}} (\sum_U 2 \text{ rad } U).$$

Then $\lim_{a \to 0} \Lambda(E, a) = m_1(E)$, the 1-dimensional Hausdorff measure of the set E.

Lemma **(9.1)**. — *Let φ be a topological mapping of a domain* $D \subset \mathbf{R}^n$ *onto a domain in* $\mathbf{R}^n$, $n > 1$, *such that* $H_\varphi(x) < K$ *for all* $x \in D - P$, *where* P *is a hyperplane in* $\mathbf{R}^n$. *Then φ is* ACL *in* D *and* φ^{-1} *is* ACL *in* $\varphi(D)$.

Proof. — The lemma is equivalent to the assertion that φ is ACL in D if either (i) H_φ is bounded in $D - P$ or (ii) $H_{\varphi^{-1}}$ is bounded in $\varphi(D-P)$.

To prove that φ is ACL in D, it suffices to prove that for almost all closed line segments X interior to D and parallel to the x_1 axis, φ is absolutely continuous on X; that is, $m_1(\varphi(E)) = 0$ if E is a subset of X with $m_1(E) = 0$. Set

$$\tau(X) = \varlimsup_{t \to 0} \frac{m(\varphi(T_t(X)))}{t^{n-1}}.$$

Then by Lebesgue's Theorem on differentiation of set functions (Saks, *Theory of the Integral*, p. 115), $\tau(X) < \infty$ for almost all line segments X. The lemma will be established once we have proved for any compact set $E \subset X - P$ (which we can assume to be X minus one point)

(6) $$m_1(\varphi(E))^n \leqslant A\tau(X) m_1(E)^{n-1}$$

where A is a constant depending on the bound for H_φ in case (i) and the bound for $H_{\varphi^{-1}}$ in case (ii).

First we take up case (i); that is we assume $H(x)<K$ for all $x \in D-P$. We can assume without loss of generality that there is a positive number b satisfying for all $p \in E$

$$L(p, t) \leqslant Kl(p, t) \qquad \text{whenever} \quad 0<t<b. \tag{7}$$

For let E_b be the set of $p \in E$ such that (7) is satisfied. Since E_b is compact and expands to E as $b \to 0$, the validity of (6) for E follows from its validity for E_b. Next we fix a positive number a (which ultimately will tend to zero) and we select a positive c such that $L(p, c)<a$ for all $p \in E$; this we can do by the uniform continuity of φ on E. We can find a real number t, $0<t<\inf(b, c)$ and points $p_1, p_2, \ldots, p_N$ in the linear set E such that the balls $T_t(p_k)$ cover E, $|p_i-p_j| \geqslant (|i-j|-1)t$ for all $i, j=1, \ldots, N$, and $Nt \leqslant m_1(E)+a$. Namely, choose $t<\min(b, c)$ so that $m_1(T_t(E) \cap X)<m_1(E)+a$, divide the line containing E into equal disjoint half-open intervals of length t, select in each interval meeting E a point of E, and arrange these points $p_1, p_2, \ldots, p_N$ in order. We note in passing that for any real number K, no point of the ball $|x-p_k|<Kt$ lies in more than 3K of the balls with center at $p_1, \ldots, p_N$ and radius Kt. Set

$$s_k=L(p_k, t) \qquad (k=1, \ldots, N).$$

Clearly $s_k<a$.

From (7) we get

$$s_k/K \leqslant l(p_k, t)$$

and thus by (1) and (2)

$$T_{s_k/K}(\varphi(p_k)) \subset \varphi(T_t(p_k)) \subset T_{s_k}(\varphi(p_k)).$$

Set $B_k=T_{s_k/K}(\varphi(p_k))$, $B^k=T_{s_k}(\varphi(p_k))$.

Since the B^k form a cover of $\varphi(E)$ by balls of radius less than a, we have

$$\Lambda(\varphi(E), a) \leqslant \sum_k 2s_k.$$

By Hölder's inequality,

$$\Lambda(\varphi(E), a)^n \leqslant 2^n N^{n-1} \sum_k s_k = b_n^{-1} 2^n N^{n-1} \sum_k m(B^k) \tag{8}$$

where b_n denotes the volume of the unit n-ball. Since $B_k \subset \varphi(T_t(p_k))$, each point of B_k lies in at most three of $B_1, \ldots, B_N$. Hence

$$\sum_k m(B_k) \leqslant 3m(\bigcup_k B_k)$$

and

$$\sum_k m(B^k)=K^n \sum_k m(B_k) \leqslant 3K^n m(\bigcup_k B_k) \leqslant 3K^n m(\varphi(T_t(E))).$$

Consequently

$$\Lambda(\varphi(E), a)^n \leqslant b_n^{-1} 2^n . 3K^n (Nt)^{n-1} \frac{m(\varphi(T_t(E)))}{t^{n-1}} \leqslant A(m_1(E)+a)^{n-1} \frac{m(\varphi(T_t(X)))}{t^{n-1}}.$$

Letting $a \to 0$, we find formula (6).

In the case (ii), we can assume instead of (7), the condition

$$L_{\varphi^{-1}}(\varphi(p), t) \leqslant K l_{\varphi^{-1}}(\varphi(p), t) \qquad 0 < t < b \tag{7'}$$

for all $p \in E$. Again we fix a, and select c such that $L(p, c) < a$. Again we choose $t < \inf(b, c)$ and the points $p_1, \ldots, p_N$ so that the balls $T_t(p_k)$ cover E,

$$|p_i - p_j| \geqslant (|i-j| - 1)t$$

for all i, j, and $Nt \leqslant m_1(E) + a$.

Again we set $s_k = L_\varphi(p_k, t)$ $(k = 1, \ldots, N)$. Then

$$l_{\varphi^{-1}}(\varphi(p_k), s_k) = t, \qquad \text{and} \qquad L_{\varphi^{-1}}(p_k, s_k) < Kt$$

by (3) and (7'). Hence

$$\varphi(T_t(p_k)) \subset T_{s_k}(\varphi(p_k)) \subset \varphi(T_{Kt}(p_k))$$

As before, formula (8) applies. But in this case we estimate $\sum_k m(B^k)$ by observing that each point of B^k meets at most 3K of $\varphi(T_{Kt}(p_k))$, $k = 1, \ldots, N$.

Consequently, $\sum_k m(T_{s_k}(\varphi(p_k))) \leqslant 3Km(\bigcup_k T_{s_k}(\varphi(p_k)))$, and

$$\Lambda(\varphi(E), a)^n \leqslant b_n^{-2} 2^n 3KN^{n-1} m(\varphi(T_{Kt}(X))).$$

Therefore

$$m_1(\varphi(E)) \leqslant b_n^{-1} 2^n 3K^n (m_1(E))^{n-1} \tau(X) = Am_1(E)^{n-1} \tau(X).$$

The proof of Lemma (9.1) is now complete.

Lemma **(9.2)**. — *Let* $\varphi : R \to R'$ *be a homeomorphism of domains in* $\mathbf{R}^n$. *Assume* φ *is* ACL *in* R *and* $I_\varphi^n \leqslant K^{n-1} J_\varphi$ *a.e. in* R, K *being a constant. Then*

$$\text{mod } \varphi(D) \leqslant K \text{ mod } D$$

for all shells D *in* R.

Proof. — Let u' be a differentiable admissible function on the shell $\varphi(D)$ and set $u = u' \circ \varphi$. Then u is ACL on D and hence admissible on D.

Now

$$|\nabla u|(p) = \overline{\lim_{q \to p}} \frac{|u(q) - u(p)|}{|q-p|}$$

$$\leqslant \overline{\lim_{q \to p}} \frac{|u(q) - u(p)|}{|\varphi(q) - \varphi(p)|} \cdot \overline{\lim_{q \to p}} \frac{|\varphi(q) - \varphi(p)|}{|q-p|}$$

$$|\nabla u|(p) \leqslant |\nabla u'|(\varphi(p)) \cdot I(p). \tag{9}$$

Hence

$$|\nabla u|^n \leqslant (|\nabla u'| \circ \varphi) \cdot I^n \leqslant K^{n-1}(|\nabla u'|^n \circ \varphi) \cdot J$$

and

$$\mathscr{C}(D) \leqslant K^{n-1} \int_D (|\nabla u'|^n \circ \varphi) \cdot J \, dx \leqslant K^{n-1} \int_{\varphi(D)} |\nabla u'|^n \, dx$$

for all differentiable admissible functions u' on $\varphi(D)$. It follows that $\mathscr{C}(D) \leqslant K^{n-1}\mathscr{C}(\varphi(D))$ and $\bmod \varphi(D) \leqslant K \bmod D$.

Note. — It follows from (5), Theorem (9.1), and Lemma (9.1) that $\bmod \varphi(D) \leqslant K \bmod D$ if φ is K-quasi-conformal. It follows from (4) and Lemma (9.1) that $\bmod \varphi^{-1}(D) \leqslant K^{\frac{n}{n-1}} \bmod D$ if φ is K-quasi-conformal. Following Lemma (9.3) below, we will know that φ^{-1} is quasi-conformal, and being differentiable with non-zero differential almost everywhere, we will be able to assert that φ^{-1} is also K-quasi-conformal. It will therefore follow that $\bmod \varphi^{-1}(D) \leqslant K \bmod D$.

Theorem **(9.2)**. — *Let φ be a homeomorphism of a domain* R *in* $\mathbf{R}^n$ *onto a domain in* $\mathbf{R}^n$. *Assume* $n>1$ *and*

$$\bmod \varphi(D) \leqslant K \bmod D$$

for all shells D *in* R. *Then φ is quasi-conformal and* $H_\varphi < \Psi_n(1)^K$ *on* R.

Proof. — We must prove that H_φ is bounded on R.

Let $p \in R$, and let r be any positive number such that the ball $T_r(p) \subset R$. Then

$$\varphi^{-1}(T_{l(p,r)}(\varphi(p))) \subset T_r(p) \subset \varphi^{-1}(T_{L(p,r)}(\varphi(p))).$$

Let D denote the shell whose complement consists of

$$C_0 = \text{closure of } \varphi^{-1}(T_{l(p,r)}(\varphi(p)))$$
$$C_1 = \text{exterior of } \varphi^{-1}(T_{L(p,r)}(\varphi(p))).$$

Then the spherical symmetrization D^0 separates the boundaries of the Teichmüller-like shell D' whose complement consists of the intervals $C'_0 = [-r, 0]$, $C'_1 = [r, \infty]$ along the axis of the symmetrization. Hence

$$\bmod \varphi(D) \leqslant K \bmod D \leqslant K \bmod D^0 \leqslant K \bmod D_T(1) = K \log \Psi_n(1).$$

But $\varphi(D)$ is the spherical shell $D_{l,L}$ so that we get

$$\log \frac{L(p, r)}{l(p, r)} \leqslant K \log \Psi_n(1).$$

It follows at once that $H_\varphi(p) \leqslant \Psi_n(1)^K$ for all $p \in R$. Hence φ is quasi-conformal.

We may summarize our conclusions in the next theorem :

Theorem **(9.3)**. — *The following assertions are equivalent for a homeomorphism φ of a domain* R *in* $\mathbf{R}^n$ *onto a domain of* $\mathbf{R}^n$, $n>1$:

a) H_φ *is bounded on* R; *that is φ is quasi-conformal.*

b) φ *is* ACL *in* R *and* $I^n < KJ$ *a.e. for some constant* K.

c) $\bmod \varphi(D) \leqslant K \bmod D$ *for all shells* D *in* R *for some constant* K.

d) φ^{-1} *is quasi-conformal.*

e) $K^{-1} \bmod D \leqslant \bmod \varphi(D)$ *for all shells* D *in* R *for some constant* K.

Proof. — *a)* ⇒ *b)* ⇒ *c)* ⇒ *a)* has already been proved. *a)* ⇒ *d)* follows from (4), Lemma (9.1), and *b)*, *d)* ⇒ *a)* is obviously equivalent to *a)* ⇒ *d)*, and *d)* ⇔ *e)* by *a)* and *c)*.

Theorem **(9.4)**. — *Let* φ *be a quasi-conformal mapping of a domain* R *of* $\mathbf{R}^n$ *onto a domain of* $\mathbf{R}^n$, $n>1$. *Then* $J_\varphi>0$ *a.e. in* R, *the set function* $E\to m(\varphi(E))$ *is absolutely continuous in* R *and*

$$m(\varphi(E))=\int_E J\,dx$$

for any measurable set E *in* R.

Proof. — Let Z be the set of points in R at which J_φ is zero. To prove that the set Z has measure zero, it suffices to prove that Z has no points of density; that is, a point p such that $\overline{\lim_{r\to 0}}\dfrac{m(T_r(p)\cap Z)}{m(T_r(p))}=1$. (Saks, p. 129).

Let D denote the spherical shell $a<|x-p|<2a$ centered at p and let u' be a differentiable admissible function on $\varphi(D)$. Set $u=u'\circ\varphi$. Then u is an admissible function on D. Let y denote a variable point in the coordinate plane $x=x_1(p)$, and let X_y denote the intersection with D of the line through D parallel to the x_1-axis. Then we have

$$\int_D|\nabla u|\geqslant\int_{|q-p|<a}\int_{X_y}|\nabla u|\,dx_1\,dy\geqslant 2b_{n-1}a^{n-1}$$

where b_{n-1} is the volume of the unit $(n-1)$-ball.

On the other hand, since $|\nabla u|(p)\leqslant|\nabla u'|(\varphi(p)).I(p)$,

$$2b_{n-1}a^{n-1}\leqslant\int_D(|\nabla u'|\circ\varphi).I\,dx.$$

Assume φ to be K-quasi-conformal, we have $I^n\leqslant K^{n-1}J$, and applying Hölder's inequality,

$$(2b_{n-1}a^{n-1})^n\leqslant\left(\int_{D-Z}(|\nabla u'|\circ\varphi)I\,dx\right)^n\leqslant K^{n-1}m(D-Z)^{n-1}\int_{D-Z}(|\nabla u'|^n\circ\varphi).J\,dx$$
$$\leqslant K^{n-1}m(D-Z)^{n-1}\int_{\varphi(D)}|\nabla u'|^n\,dx.$$

It follows that $(2b_{n-1}a^{n-1})^n\leqslant K^{n-1}m(D-Z)^{n-1}\mathscr{C}(\varphi(D))$ or

$$m(D-Z)\geqslant K^{-1}Aa^n\bmod\varphi(D)$$

where A is a constant depending only on the dimension n. Now

$$\bmod\varphi(D)\geqslant K^{-n/(n-1)}\bmod D$$

by Lemma (9.2) applied to φ^{-1}. Since $\bmod D=\bmod D_{a,2a}=\log 2$, we find

$$\frac{m(D-Z)}{m(T_{2a}(p))}>A'>0$$

where A′ is a constant depending only on K and n. It follows that p is not a point of density of Z, and that Z has measure zero.

Let P denote the set of points p in R at which $J_\varphi(p)=\infty$. Then any point of $\varphi(P)$ at which φ^{-1} is differentiable must lie in $Z_{\varphi^{-1}}$, the set of zeros of $J_{\varphi^{-1}}$. Since φ^{-1} is quasi-conformal, we find that $\varphi(P)$ is a set of measure zero. It follows from the De la

Vallée Poussin Decomposition Theorem (Saks, p. 127) that $E \to m(\varphi(E))$ is absolutely continuous and $m(\varphi(E)) = \int_E J\, dx$.

Corollary. — *If* φ *is* K-*quasi-conformal, then* φ^{-1} *is* K-*quasi-conformal.*

Proof. — If $n=1$, the assertion is obvious. We can assume therefore that $n>1$. We can assert now that $\dot{\varphi} \neq 0$ a.e. and thus $H_\varphi(p) = H_{\dot{\varphi}_p}$ for almost all p if φ is quasi-conformal, and similarly for φ^{-1}. The Corollary follows from the fact that $H_\varphi = H_{\varphi^{-1}}$ for a non-singular linear mapping φ.

Remark. — As a consequence of (5) and Lemma (9.2), we have for any shell D

$$K^{-1} \operatorname{mod} D \leqslant \operatorname{mod} \varphi(D) \leqslant K \operatorname{mod} D$$

if φ is K-quasi-conformal on D.

§ 10. THE BOUNDARY MAP OF THE n-BALL

Theorem (**10.1**). — *Let* φ *be a quasi-conformal mapping of an open ball in* $\mathbf{R}^n$ *onto itself. Then* φ *extends to a homeomorphism of the closed n-ball.*

Proof. — Mapping the domain of φ onto the upper half space

$$X = \{(x_1, \ldots, x_n);\, x_n > 0\}$$

via a Moebius transformation, the theorem is seen to be equivalent to the assertion: a quasi-conformal mapping $\varphi: X \to Y = \{y;\, |y| < 1\}$ extends to a continuous mapping at any point x of the boundary of X. For convenience, we take $x=0$.

The proof is by contradiction. If $\lim_{p \to 0} \varphi(p)$ $(p \in X)$ does not exist, we can find two sequences $\{p_k\}$ and $\{q_k\}$ in X approaching 0 with $\lim_{k\to\infty} \varphi(p_k) = p'$, $\lim_{k\to\infty} \varphi(q_k) = q'$ and $|q'-p'| = a > 0$. Denoting by $\overline{pq}$ the line segment joining two points p and q, we select points p_0' and q_0' in Y such that $d(\overline{p_0' p_k'}, \overline{q_0' q_k}) > a$ for all large k, where $p_k' = \varphi(p_k)$, $q_k' = \varphi(q_k)$. Set $p_0 = \varphi^{-1}(p_0')$, $q_0 = \varphi^{-1}(q_0')$. Then for $\sup(|p_k|, |q_k|) < r < \inf(|p_0|, |q_0|)$, the hemisphere $S_r^+ = \{x;\, |x| = r,\, x_n > 0\}$ meets the curves $\varphi^{-1}(\overline{p_0' p_k'})$ and $\varphi^{-1}(\overline{q_0' q_k'})$. For each such r at least one of the coordinate functions of $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$ satisfies

$$\operatorname*{osc}_{S_r^+} \varphi_i > a/\sqrt{n}.$$

Hence

$$\sum_i \int_0^\infty (\operatorname*{osc}_{S_r^+} \varphi_i)^n \frac{dr}{r} = \infty.$$

By Lemma (9.1), φ_i is ACL in X. Applying Lemma (4.4), we get for each $i = 1, \ldots, n$

$$\int_0^\infty (\operatorname*{osc}_{S_r^+} \varphi_i)^n \frac{dr}{r} \leqslant 2A \int_X |\nabla \varphi_i|^n dx \leqslant 2A \int_X I_\varphi^n dx \leqslant 2A \int_X K^{n-1} J_\varphi dx \leqslant 2AK^{n-1} \int_Y dy.$$
$$< \infty$$

This yields a contradiction.

Theorem **(10.2)**. — *Let* φ *be a* K*-quasi-conformal mapping of an open ball in* $\mathbf{R}^n$ *onto itself,* $n \geqslant 2$, *and let* $\overline{\varphi}$ *denote the boundary homeomorphism induced by* φ. *Then* $\overline{\varphi}$ *is* $\Psi_n(1)^{\mathrm{K}}$*-quasi-conformal.*

Proof. — Denote the extended homeomorphism to the closed ball by φ and set $\mathrm{K}' = \Psi_n(1)^{\mathrm{K}}$. We wish to show that $\mathrm{H}_{\overline{\varphi}} < \mathrm{K}'$ on the boundary $(n-1)$-sphere. Given any point p on the boundary, no generality is lost in assuming that $\overline{\varphi}(p_\infty) = p_\infty$ for some point p_∞ on the boundary distinct from p : we need merely compose φ with some rotations of the closed ball to achieve this. Let θ be a Moebius transformation of $\mathbf{R}^n \cup \infty$ carrying the closed ball onto the upper half space $x_n \geqslant 0$ with p_∞ going into ∞. Set $\psi = \theta\varphi\theta^{-1}$. Then $\mathrm{H}_\psi(x) = \mathrm{H}_\varphi(\theta^{-1}(x))$ for all x in the open upper half space and $\mathrm{H}_{\overline{\psi}}(x) = \mathrm{H}_{\overline{\varphi}}(\theta^{-1}x)$ for all x in the coordinate plane $\mathrm{P} : x_n = 0$.

We extend ψ to a homeomorphism of $\mathbf{R}^n$ to $\mathbf{R}^n$, defining

$$\psi(x_1, \ldots, x_{n-1}, -x_n) = -\psi(x_1, \ldots, x_n).$$

Then $\mathrm{H}_\psi(x) < \mathrm{K}$ for all $x \in \mathbf{R}^n - \mathrm{P}$. It follows from Lemma (9.1), that φ is ACL in $\mathbf{R}^n$ and it follows from § 9, formula (5), that $\mathrm{I}^n \leqslant \mathrm{K}^{n-1}\mathrm{J}$ a.e. in $\mathbf{R}^n$. Hence by Lemma (9.2),

$$\operatorname{mod} \psi(\mathrm{D}) \leqslant \mathrm{K} \operatorname{mod} \mathrm{D}$$

for all shells D in $\mathbf{R}^n$. Applying Theorem (9.2), we get $\mathrm{H}_\psi(x) \leqslant \mathrm{K}'$ for all $x \in \mathbf{R}^n$. In particular

$$\mathrm{H}_{\overline{\varphi}}(p) = \mathrm{H}_{\overline{\psi}}(\theta^{-1}(p)) \leqslant \mathrm{H}_\psi(\theta(p)) \leqslant \mathrm{K}'$$

and thus $\overline{\varphi}$ is K′-quasi-conformal.

§ 11. AN ERGODIC THEOREM

Lemma **(11.1)**. — *Let* S *be a non-abelian group of the form* B.A *where* B *is a normal subgroup isomorphic to the additive group of real or complex numbers and* A *is a subgroup isomorphic to the multiplicative group of positive real numbers, and with multiplication*

$$(b.a) \circ (b'.a') = (b + ab').aa'.$$

Let ρ *be a continuous unitary representation of* S *on a hilbert space* V *and let* v *be a vector in* V *such that* $\rho(\mathrm{A})v = v$. *Then* $\rho(\mathrm{S})v = v$.

Proof. — For any $a \in \mathrm{A}$ and $b \in \mathrm{B}$ we have $a \circ b \circ a^{-1} = ab$ and thus

$$\begin{aligned}\langle \rho(b)v, v \rangle &= \langle \rho(a^n)\rho(b)v, \rho(a^n)v \rangle \\ &= \langle \rho(a^n \circ b \circ a^{-n})\rho(a^n)v, \rho(a^n)v \rangle \\ &= \langle \rho(a^n b)v, v \rangle\end{aligned}$$

for all $n > 0$. Choosing $|a| < 1$, we get as $n \to \infty$

$$\langle \rho(b)v, v \rangle = \langle \rho(\text{identity})v, v \rangle = \langle v, v \rangle.$$

By Schwarz's inequality, $\rho(b)v = v$ for all $b \in \mathrm{B}$ and hence $\rho(\mathrm{S})v = v$.

Note. — The foregoing lemma was first proved by Mautner by expressing any unitary representation as a direct integral of irreducible ones and verifying the assertion for the irreducible representations of S. The proof given here is taken from an article of L. Auslander and L. Green in *Amer. J. Math.*, v. 88 (1966), p. 58, who in turn attribute the underlying idea to J. von Neumann and I. E. Segal. It may be further noted that $\rho(B)v = v$ if we make the weaker hypothesis that v is an eigenvector for A.

Lemma (**11.2**). — *Let* W *be a measure space with finite total measure* μ. *Let* A *be a group of measurable measure-preserving transformations of* W *such that the only* A*-invariant functions in* $L^2(W, \mu)$ *are the constant functions. Let* A^+ *be a semi-group in* A *such that* A^+ *and its inverse generate* A. *Assume that there is a separable topology on* A *with respect to which the representation of* A *on* $L^2(W, \mu)$ *is continuous. Let* $\{W_n\}$ *be a denumerable family of measurable sets in* W. *Then for almost all* $x \in W$, A^+x *meets* W_n.

Proof. — Set $A^- = (A^+)^{-1}$. Consider the characteristic function f of the measurable set DW_n, where D is the semi-group generated by a denumerable dense subset of A^-. For any $d \in D$ we have $dDW_n \subset DW_n$ and since the action of d is measure-preserving, $\mu(dDW_n) = \mu(DW_n)$. Hence $df = f$ for all $d \in D$. Since D is dense in A^-, we find $A^- f = f$ and therefore $Af = f$. By hypothesis f is constant almost everywhere and therefore $\mu(DW_n) = \mu(W)$. Consequently $A^- W_n$ differs from W in merely a set of measure zero and $\mu(\bigcap_n A^- W) = \mu(W)$. For any $x \in \bigcap_n A^- W_n$, we find $A^+ x \cap W_n$ non-empty for all n.

Theorem (**11.1**). — *Let* G *be a separable locally compact group and* Γ *a discrete subgroup such that* G/Γ *has finite Haar measure. Let* A *be a subgroup of* G *isomorphic to the multiplicative group of positive real numbers and let* A^+ *be the semi-group in* A *given by* $\{a; a \leq 1\}$. *Assume that* G *is generated by* A *and the family of subgroups isomorphic to the real numbers* $\{B\}$ *such that* B.A *satisfies the multiplication rule:* $(b.a)\circ(b'.a') = (b + ab').aa'$. *Then for almost all cosets* $x\Gamma$, $A^+x\Gamma$ *is topologically dense in* G.

Proof. — Consider the canonical representation of G on $L^2(G/\Gamma)$ and let f be a function fixed under A. By Lemma (11.1), f is fixed under B.A for all subgroups in the family $\{B\}$. By hypothesis, $\{BA\}$ generates G. Hence f is fixed by G; that is, f is a constant function on G/Γ. The hypotheses of Lemma (11.2) are satisfied by A and the conclusion of Theorem (11.1) follows by taking for $\{W_n\}$ a denumerable base of open sets in G/Γ.

Note. — The subset $A\Gamma$ may well be closed.

§ 12. PROOF OF THE MAIN THEOREM

Theorem (**12.1**). — *Let* G *be the group of isometries of hyperbolic n-space* X. *Let* Γ *and* Γ' *be subgroups of* G *such that* $\Gamma\backslash G$ *and* $\Gamma'\backslash G$ *have finite Haar measure. Let* $\theta : \Gamma \to \Gamma'$ *be an isomorphism and* $\varphi : X \to X$ *a quasi-conformal homeomorphism such that*

$$\varphi(\gamma x) = \theta(\gamma)\varphi(x)$$

for all $\gamma \in \Gamma$, $x \in X$. *Then* θ *extends to an inner automorphism of* G *provided* $n > 2$.

The proof is divided into several parts. We begin with some remarks on the algebraic structure of G.

1. We have defined hyperbolic n-space at the end of Section 1 as the unit ball $B^n : |x|<1$ in $\mathbf{R}^n$ with metric $(1-|x|^2)^{-2}\sum_{i=1}^{n} dx_i^2$ and we have seen in Theorem (1.2) that the subgroup G′ of the Moebius group GM(n) which stabilizes a hemisphere in S^{n+1} operates on hyperbolic n-space transitively with isotropy subgroup O(n). It follows at once that any isometry of hyperbolic space differs from an isometry in G′ by an isometry T which keeps a point fixed and induces on the tangent space at some point p the identity transformation. In such circumstances T leaves fixed all the geodesics through p and is therefore the identity transformation. Hence the group G of all isometries of X is precisely the group $G'=GM(n-1)=O(1, n)/(\pm 1)$. In other words, when we take $B^n=\{x; |x|<1\}$ as our model of hyperbolic n-space X, each isometry of X induces a Moebius transformation on the boundary sphere $S^{n-1}=\{x; |x|=1\}$ and conversely, each Moebius transformation of S^{n-1} extends to a unique isometry of X.

Let p_0 and p_∞ be antipodal points on the $n-1$ sphere S^{n-1} in $\mathbf{R}^n$. Let π_∞ denote stereographic projection of S^{n-1} from p_∞ onto the tangent plane E_0 to S^{n-1} at p_0 and let T_0 denote the $(n-1)$-dimensional vector group of translations in the plane E_0. Set $N_+=\pi_\infty^{-1}T_0\pi_\infty$. Then $N_+\subset G$. Similarly, performing stereographic projection from p_0, we define the vector subgroup N_- of G_0. Let A denote the one-parameter subgroup of G corresponding to the homotheties $x\to tx$ where $t\in\mathbf{R}$ and $x\in E_0$ which is regarded as a vector space with origin p_0. It is clear that A corresponds equally to the homotheties from p_∞ in the plane E_∞. Then AN_+ is a solvable subgroup of G and similarly AN_- is a solvable subgroup.

We have seen in Section 1 that GM$(n-1)$ is generated by the reflection σ in the equatorial plane bisecting p_0p_∞ together with AN_+. Clearly $\sigma^2=1$, $\sigma A\sigma=A$ and $\sigma N_+\sigma=N_-$. Hence $G=GM(n-1)$ has exactly two connected components and G_0, the connected component of the identity element in G, is generated by the solvable subgroups AN_+ and AN_-. Let B denote a one parameter vector subgroup of either N_+ or N_-. Then BA is a solvable subgroup of G_0; if we identify B with the additive group of real numbers and A with the multiplicative group of positive real numbers, we have for $b, b'\in B$ and $a, a'\in A$:

$$(b.a)\circ(b'.a')=(b+ab').aa'.$$

2. The fixed point set of the subgroup A is the set of points $\{p_0, p_\infty\}$ and therefore the stabilizer of $\{p_0, p_\infty\}$ in G is N(A), the normalizer of A. Hence we may identify the space of cosets G/N(A) with the orbit under G of the set $\{p_0, p_\infty\}$, that is, the set of unordered pairs of distinct points $\{p, q\}$ of S^{n-1}. On the other hand each coset of G/N(A) determines a unique conjugate of A; namely $xN(A)\leftrightarrow xAx'$. Given the distinct points p, q in S^{n-1}, we denote by $A_{p,q}$ the conjugate of A corresponding to $\{p, q\}$; $A_{p,q}$ is the unique conjugate of A leaving fixed p and q.

The hypotheses of Theorem (11.1) are satisfied by the triple G_0, $\Gamma \cap G_0$, A. It follows that for almost all cosets Γx in the group ΓG_0 (which may be either G_0 or G), the subset $\Gamma x A^+$ is topologically dense in ΓG_0, where A^+ denotes the semi-group in A corresponding to $\{a; a<1\}$. Hence for almost all $x \in G_0$, $\Gamma x A^+ x^{-1}$ is topologically dense in ΓG_0. In other words $\Gamma A^+_{p,q}$ is topologically dense in ΓG_0 for almost all $\{p, q\}$.

3. By hypothesis we have a quasi-conformal mapping $\varphi : X \to X$ of hyperbolic n-spaces such that $\varphi(\gamma x) = \theta(\gamma)\varphi(x)$, for all $\gamma \in \Gamma$, $x \in X$. The mapping φ is thus a homeomorphism of the unit n-ball B^n onto itself which is quasi-conformal with respect to the hyperbolic metric $(1-|x|^2)^{-2} \sum_{i=1}^{n} dx_i^2$. It follows at once that φ is quasi-conformal with respect to the euclidean metric $\sum_{i=1}^{n} dx_i^2$. By Theorems (10.1) and (10.2) we may assert that φ extends to a homeomorphism ψ of the boundary S^{n-1} and ψ is quasi-conformal. Clearly

$$\psi(\gamma p) = \theta(\gamma)\psi(p)$$

for all $\gamma \in \Gamma$, $p \in S^{n-1}$. We shall prove that ψ *is a Moebius transformation* if $n>2$.

4. Since $\psi : S^{n-1} \to S^{n-1}$ is quasi-conformal, its differential $\dot{\psi}_p$ exists at almost all points $p \in S^{n-1}$, by Theorem (9.1). In addition $\Gamma A^+_{p,q}$ is topologically dense in ΓG_0 for almost all $\{p, q\}$, when we identify the unordered pair of distinct point $\{p, q\}$ with an element of $G/N(A)$ and introduce a quotient measure on $G/N(A)$. The centralizer $Z(A)$ is the fixer of p_0 and p_∞. It is easy to verify that, up to a constant factor, the invariant quotient measure of $G/Z(A) = (S^{n-1} \times S^{n-1}$—diagonal) correspond to the product of the standard measures on $S^{n-1} \times S^{n-1}$. Applying Theorem (11.1), we find for almost all points $p \in S^{n-1}$, there is a point q distinct from p such that $\Gamma A^+_{p,q}$ is topologically dense in ΓG_0. Finally, if $n>1$, $\dot{\psi}_p$ is invertible for almost all p, by Theorem (9.4).

5. Let p be a point of S^{n-1} such that

a) the differential $\dot{\psi}_p$ exists and is invertible.

b) $\Gamma_0 A^+_{p,q}$ is topologically dense in G_0 for some q.

We prove *under these hypotheses* that $\dot{\psi}_p$ is a conformal linear mapping of the tangent space to S^{n-1} at p. To reach this conclusion about $\dot{\psi}_p$, we may compose ψ with a Moebius transformation of the closed unit ball and thus no generality is lost in assuming $\psi(p)=p$, and $\psi(q)=q$. Next we carry the pair (p, q) into the antipodal points (p_0, p_∞) by an element $\tau \in G$ and we replace Γ by $\tau\Gamma\tau^{-1}$, φ by $\varphi \circ \tau^{-1}$, $\theta(\gamma)$ by $\theta(\tau^{-1}\gamma\tau)$. Thereby we arrive at the situation : $(p, q) = (p_0, p_\infty)$, $\psi(p_0) = p_0$, and $\Gamma_0 A^+_{p_0, p_\infty}$ is topologically dense in G_0.

We identify the tangent space to S^{n-1} at p_0 with the euclidean $(n-1)$-plane E_0 in $\mathbf{R}^n$ and via stereographic projection from p_∞ onto E_0, we identify S^{n-1} with the Moebius space $E_0 \cup \infty$. By abuse of notation, we regard ψ as a mapping of $E_0 \cup \infty$ to itself, and similarly for the action of the Moebius group of S^{n-1}.

Consider now the real-valued function f on G_0 defined by

$$f(g) = \mathscr{C}(\dot{\psi}_{p_0}(gD))$$

where D is a shell in E_0 and $\mathscr{C}$ denotes the conformal capacity. We regard E_0 as a vector space with origin at p_0. By definition of the differential, $\dot{\psi}_{p_0}(x) = \lim_{t \to 0} t^{-1}\psi(tx)$ for all $x \in E_0$ and the convergence is uniform on compact subsets of E_0. It follows from the last remark of Section 6, that the shell $\dot{\psi}_p(gD)$ is the limit in the space of shells $\lim_{t \to 0} t^{-1}\psi(tgD)$. By Theorem (6.1), conformal capacity depends continuously on shells, so that

$$f(g) = \lim_{t \to 0} \mathscr{C}(t^{-1}\psi(tgD)).$$

The homotheties $a_t : x \to tx$, where $x \in E_0$ and $0 < t \leqslant 1$ make up the semi-group A^+ $(= A^+_{p_0, p_\infty})$. Since $\Gamma_0 A^+$ is topologically dense in G_0, we may select a sequence of elements $\gamma_n \in \Gamma$ and elements t_n such that $\gamma_n a_{t_n} \to g^{-1}$ and $t_n \to 0$. Set $a_n = a_{t_n}$ $(n = 1, 2, \ldots)$. Then

$$\begin{aligned} f(g) &= \lim_{t \to \infty} \mathscr{C}(t^{-1}\psi(tgD)) \\ &= \lim_{t \to \infty} \mathscr{C}(\psi(tgD)) \\ &= \lim_{t \to \infty} \mathscr{C}(\theta(\gamma)\psi(tgD)) \end{aligned}$$

for any $\gamma \in \Gamma$, since conformal capacity is invariant under Moebius transformations. We have $\theta(\gamma) \circ \psi = \psi \circ \gamma$ for all $\gamma \in \Gamma$ and therefore we may write

$$\begin{aligned} f(g) &= \lim_{n \to \infty} \mathscr{C}(\psi(\gamma_n t_n gD)) \\ &= \lim_{n \to \infty} \mathscr{C}(\psi(\gamma_n a_n gD)) \\ &= \mathscr{C}(\psi(D)). \end{aligned}$$

In particular

$$\mathscr{C}(\dot{\psi}_{p_0}(kD)) = \mathscr{C}(\dot{\psi}_{p_0}(D))$$

for all elements k in the rotation subgroup $SO(n-1, \mathbf{R})$ of the group G_0 which keeps fixed the point p_0.

Lemma (**12.1**). — *Let* T *be a linear transformation of* $\mathbf{R}^{n-1}$ *such* $\mathscr{C}(T(kD)) = \mathscr{C}(T(D))$ *for every* $k \in SO(n-1, \mathbf{R})$ *and for every shell* D *in* $\mathbf{R}^{n-1}$. *Then* T *is conformal.*

Proof. — We can write $T = T_1 T_2$ where T_1 is orthogonal and T_2 is a positive definite self-adjoint automorphism of $\mathbf{R}^{n-1}$. Since T_1 is conformal, it suffices to prove the lemma for $T = T_2$. Introducing an orthonormal base of eigenvectors for T_2, we can assume that in fact T is the transformation

$$(x_1, \ldots, x_{n-1}) \to (\lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})$$

with $\lambda_1 = \sup\{\lambda_1, \ldots, \lambda_{n-1}\}$ and $\lambda_2 = \inf\{\lambda_1, \ldots, \lambda_{n-1}\}$ if $n-1 \geqslant 2$. Composing T with λ_1^{-1} times the identity transformation, we can assume moreover that $\lambda_1 = 1$. It remains only to show, in case $n-1 \geqslant 2$, that $\lambda_2 = 1$. We set $\lambda = \lambda_2$ and we know $\lambda \leqslant 1$.

Consider now the shell $D(a)$ whose complement consists of the line segment $0 \leqslant x_2 \leqslant a$, $0 = x_1 = x_3 = \ldots = x_n$, and the infinite line segment $1 \leqslant x_1 < \infty$, $0 = x_2 = \ldots = x_n$. Let k be the rotation by 90^0 in the x_1, x_2 plane which leaves fixed the $x_3, \ldots, x_n$ axes. Then for all $0 \leqslant a < \infty$, $T(kD(a))$ is similar to $D(\lambda^{-1}a)$, whereas $T(D(a))$ is similar to $D(\lambda a)$. By hypothesis, $\mathscr{C}(D(\lambda^{-1}a)) = \mathscr{C}(D(\lambda a))$, that is $\mathscr{C}(D(a)) = \mathscr{C}(D(\lambda^2 a))$ for all $a > 0$. Hence $\mathscr{C}(D(1)) = \mathscr{C}(D(\lambda^{2n}))$. If $\lambda < 1$, we find $\mathscr{C}(D(1)) = \lim_{n \to \infty} \mathscr{C}(D(\lambda^{2n})) = \mathscr{C}(D(0)) = 0$, since $D(0)$ has a point as one of its complementary components, by Theorem (6.1) and the remark following Lemma (7.1). But $\mathscr{C}(D(1)) \neq 0$ by Lemma (4.2). This contradiction shows that $\lambda = 1$. Hence T is conformal.

It follows from Lemma (12.1) that $\dot{\psi}_{p_0}$ is conformal and thus $\dot{\psi}_p$ is conformal for almost all $p \in S^{n-1}$. The quasi-conformal mapping ψ is therefore 1-quasi-conformal in the sense of Section 9.

6. *Lemma* **(12.2)**. — *Let* $\psi : S^{n-1} \to S^{n-1}$ *be a* 1-*quasi-conformal mapping. Then* ψ *is a Moebius transformation if* $n-1 \geqslant 2$.

Proof. — Composing ψ with a Moebius transformation, we may assume that for some point S^{n-1} denoted p_∞, we have $\psi(p_\infty) = p_\infty$, ψ is differentiable at p_∞, and the differential of ψ at p_∞ is the identity transformation. We shall prove after this normalization that ψ is an isometry of S^{n-1}.

Let $\pi : S^{n-1} \to \mathbf{R}^{n-1} \cup \infty$ denote stereographic projection from p_∞. Set

$$\xi = \pi \psi \pi^{-1}.$$

Then ξ is a 1-quasi-conformal mapping of Moebius space with $\xi(\infty) = \infty$. For any point $p \in \mathbf{R}^{n-1}$ and positive real number a, set

$$l(p, a) = \inf \{ |\xi(q) - \xi(p)| ; |q-p| = a \}$$
$$L(p, a) = \sup \{ |\xi(q) - \xi(p)| ; |q-p| = a \}$$
$$H(p) = \overline{\lim_{a \to 0}} L(p, a) / l(p, a)$$
$$I(p) = \overline{\lim_{a \to 0}} L(p, a) / a.$$

For the point ∞, we define for any positive b

$$l(\infty, b) = \sup \{ |\xi(q)| ; |q| = b \}$$
$$L(\infty, b) = \inf \{ |\xi(q)| ; |q| = b \}.$$

Since ξ is 1-quasi-conformal, we know that $H(p) = 1$ almost everywhere. The fact that $\dot{\psi}_{p_\infty}$ is the identity implies that

$$\textbf{(12.1)} \qquad \lim_{b \to \infty} l(\infty, b) / b = 1 = \lim_{b \to \infty} L(\infty, b) / b.$$

Set $B_r(p) = \{ q ; |q-p| \leqslant r \}$ for any point $p \in \mathbf{R}^{n-1}$ and $B_r(\infty) = \{ q ; |q| \geqslant r \}$ for any positive real number r.

Let p be any point of $\mathbf{R}^{n-1}$ at which $H(p)=1$. Let $D_{a,b}(p)$ denote the spherical shell $\{q; a<|q-p|<b\}$. We have

$$B_{l(p,a)}(\xi(p)) \subset \xi(B_a(p)) \subset B_{L(p,a)}(\xi(p))$$

and

$$B_{L(\infty,b)}(\infty) \subset \xi(B_b(\infty)) \subset B_{l(\infty,b)}(\infty).$$

Since $B_r(p)$ and $B_r(\infty)$ are the complementary components of $D_{a,b}(p)$, we have

$$D_{L(p,a),L(\infty,b)}(\xi(p)) \subset \xi(D_{a,b}(p)) \subset D_{l(p,a),l(\infty,b)}(\xi(p))$$

and therefore by Lemmas (7.2) and (7.1), since $n-1>1$

(12.2) $$\log L(\infty, b)/L(p, a) \leqslant \log b/a \leqslant \log l(\infty, b)/l(p, a).$$

Hence

$$\frac{L(\infty, b)/b}{L(p, a)/a} \leqslant 1 \leqslant \frac{l(\infty, b)/b}{l(p, a)/a}.$$

Now let $a\to 0$ and $b\to\infty$. We get by (12.1) and the fact that $H(p)=1$,

$$\frac{1}{I(p)} \leqslant 1 \leqslant \frac{1}{I(p)}$$

that is, $I(p)=1$ for almost all points $p\in\mathbf{R}^{n-1}$. The mapping ξ thus preserves the length of all lines on which it is absolutely continuous. Since ξ is absolutely continuous on almost all lines parallel to the coordinate axes, it is clearly an isometry. It follows at once that ψ is a Moebius transformation.

7. We may regard ψ as an element of the Moebius group $G=GM(n-1)$. Since

$$\psi(\gamma p)=\theta(\gamma)\psi(p)$$

for all $\gamma\in\Gamma$ and $p\in S^{n-1}$, we have

$$\psi\gamma=\theta(\gamma)\psi$$

and thus

$$\theta(\gamma)=\psi\gamma\psi^{-1}$$

with $\psi\in G$. The proof of Theorem (12.1) is complete.

Remark. — The hypothesis $n>2$ was employed only twice in our proof, namely in the assertion that the modulus of the spherical shell $D_{a,b}: a<|x|<b$ is $\log b/a$ and in Theorem (9.4). In dimension 1 the conformal capacity of such a domain has the constant value 2. And of course, Theorem (12.1) is false in case $n=2$. A counter example is provided by two compact Riemann surfaces which have the same topological genus but not the same conformal type. In fact, with that example in mind, we state the following corollaries of Theorem (12.1).

Corollary **(12.2)**. — *Let* Y *and* Y′ *be complete Riemannian manifolds having constant negative curvature and finite volume. Assume that there is a quasi-conformal homeomorphism of* Y *onto* Y′. *Then there is a unique conformal mapping of* Y *onto* Y′ *inducing the same isomorphism* *of fundamental groups, provided the dimension is greater than two. The conformal mapping may be taken as an isometry if* Y *and* Y′ *have the same curvature.*

Proof. — Let X and X′ denote the simply connected covering manifolds of Y and Y′ respectively. Then, as is well-known, X and X′ are hyperbolic n-spaces and up to isometries are completely characterized by a single invariant, namely the curvature. By modifying the metric via a multiplicative constant at all points, the curvatures can be made equal. In other words, X and X′ are conformally equivalent. We may assume without loss of generality that $X=X'$. Let G denote the group of isometries of X, and let Γ and Γ' denote the fundamental groups of Y and Y′ respectively. A quasi-conformal mapping of Y onto Y′ induces an isomorphism $\theta:\Gamma\to\Gamma'$ and a quasi-conformal mapping $\varphi: X\to X$ such that

$$\varphi(\gamma x)=\theta(\gamma)\varphi(x)$$

for all $\gamma\in\Gamma$, $x\in X$. The result now follows from Theorem (12.1). Namely, if

$$\psi\gamma\psi^{-1}=\theta(\gamma),\qquad \psi\in G$$

then $\psi(\gamma x)=\theta(\gamma)\psi(x)$ for all $\gamma\in\Gamma$, $x\in X$ and ψ induces a conformal mapping of Y to Y′ — in fact an isometry.

To prove that ψ is unique, suppose that ψ and ψ' are elements of G satisfying

$$\psi\gamma=\theta(\gamma)\psi \qquad\text{and}\qquad \psi'\gamma=\theta(\gamma)\psi'$$

for all $\gamma\in\Gamma$. Set $\zeta=\psi^{-1}\psi'$. Then

$$\zeta\gamma=\gamma\zeta$$

for all $\gamma\in\Gamma$. Since $\Gamma\backslash G$ has finite measure, it follows that ζ is central in G (cf. A. Borel [2], or G. D. Mostow [14]). Since the Moebius group has trivial center, $\zeta=1$ and $\psi=\psi'$.

Corollary **(12.3)**. — *Any two compact Riemannian manifolds of the same constant negative curvature are isometric if they are diffeomorphic, provided the dimension is greater than two.*

Proof. — This follows immediately from Corollary (12.1) once we remark that a diffeomorphism of compact Riemannian manifolds is quasi-conformal.

Remark. — Let φ_0 and φ_1 be continuous mappings of hyperbolic n-space X into itself such that for all $x\in X$ and $\gamma\in\Gamma$

$$\varphi_0(\gamma x)=\theta(\gamma)\varphi_0(x)$$
$$\varphi_1(\gamma x)=\theta(\gamma)\varphi_1(x).$$

It is not hard to see that there is a unique geodesic joining any two points in X. Let $\varphi_t(x)$ denote the point on the geodesic segment $[\varphi_0(x),\varphi_1(x)]$ joining $\varphi_0(x)$ to $\varphi_1(x)$ that divides it in the ratio $t/(1-t)$ where $0\leqslant t\leqslant 1$. Then $\varphi_t: x\to\varphi_t(x)$ is continuous and we have for all $0\leqslant t\leqslant 1$, $\gamma\in\Gamma$, $x\in X$

$$(*)\qquad \varphi_t(\gamma x)=\theta(\gamma)\varphi_t(x)$$

since $\theta(\gamma)$ is an isometry of X sending $[\varphi_0(x), \varphi_1(x)]$ to $[\varphi_0(\gamma x), \varphi_1(\gamma x)]$. We apply this observation to two continuous mapping $\Phi_0 : \Gamma\backslash X \to \Gamma'\backslash X$ and $\Phi_1 : \Gamma\backslash X \to \Gamma'\backslash X$ which induce the same homomorphism $\theta : \Gamma \to \Gamma'$ of the fundamental groups. We obtain from condition (*) a deformation $\Phi_t : \Gamma\backslash X \to \Gamma'\backslash X$ $(0 \leqslant t \leqslant 1)$ of Φ_0 to Φ_1. Thus the homotopy class of a continuous mapping of hyperbolic space forms is uniquely determined by the induced homomorphism of the fundamental groups. In particular, the conformal mapping ψ of Corollary (12.2) is homotopic to the quasi-conformal mapping φ.

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Yale Univ. and I.H.E.S.

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