The engulfing theorem for topological manifolds

By M. H. A. NEWMAN

A version of the Stallings engulfing theorem is proved by methods that do not depend on the theory of combinatorial manifolds and regular neighbourhoods, but are developed in detail within the paper.

THEOREM. *Every locally tame closed set X, of dimension $p \leqq n-3$ in a p-connected topological n-manifold-without-boundary M, can be engulfed by any $(p-1)$-connected open set V, such that $X \backslash V$ is compact.*[1]

This means that there is a self-homeomorphism h, of M such that $X \subseteqq hV$. There is an isotopy of h to the identity composed of a finite number of small *pushes*, each of which moves the points of a subset of a euclidean neighbourhood along parallel lines and leaves all else fixed.

The theorem is in fact proved for a somewhat larger class of sets X, the p-dominated sets $(p \leqq n-3)$ defined in ¶ 9.

By an argument due to E. H. Connell, based on Stallings' stretching process between dual skeletons, the Poincaré hypothesis for topological n-manifolds $(n \geqq 5)$ is derived from this form of the engulfing theorem.

1. Stretching theorems

1. In § 1 *space* means metric space, and *function* real function.[2]

If f is a bounded function on the space X, $f_{\cdot}(X)$, and $f^{\cdot}(x)$ denote lim inf f and lim sup f at x, respectively. If $A \subseteqq X$, $\Pi(f, g \mid A)$ is the set of points

$$\{(x, t) \mid x \in A, f_{\cdot}(x) \leqq t \leqq g^{\cdot}(x)\} \qquad \text{of } X \times R,$$

and $\Pi(f \mid A) = \Pi(f, f \mid A)$. $\Pi(f, g \mid A)$ is the (f, g)-*prism* on A, and if $f \leqq g$, $\Pi(f \mid A)$ and $\Pi(g \mid A)$ are the *base* and *top* of the prism.

2. LEMMA 1. *Let f be bounded and* u.s.c. (*upper semi-continuous*) *in X, and U be open in $X \times R$. If $\Pi(f \mid X) \subseteqq U$, there is a continuous function φ on X such that $f < \varphi$ and $\Pi(f, \varphi \mid X) \subseteqq U$.*

[1] The theorem remains true if the conditions "M p-connected, $V\,(p-1)$-connected" are replaced by "the pair (M, V) is p-connected."

[2] All the results of § 1 hold for sets in paracompact spaces, in view of Dowker's extension of the Baire insertion theorem to these spaces (see Dugundji, *Topology*, p. 170) and the proofs need only minor changes.

We may suppose that $U \subseteqq [t < 1 + \sup f]$. Then the set E in $X \times R$ defined by

$$E = [f_{\cdot}(x) \leqq t] \backslash U$$

is closed and meets each line $x \times R\,(x \in X)$. Therefore μ is well defined by

$$\mu(x) = \inf \{t \mid (x, t) \in E\}$$

and is l.s.c.. Since

$$[f_{\cdot}(x) \leqq t \leqq f^{\cdot}(x)] = \Pi(f \mid X) \subseteqq U\,,$$

E may also be defined as $[f(x) \leqq t] \backslash U$. Therefore for each x, $\mu(x) \geqq f(x)$. Since $(x, f(x)) \in U$, an open set, $\mu(x) \neq f(x)$, i.e., $\mu > f$.

By Baire's insertion theorem,[3] there is a continuous function φ, on X such that $f < \varphi < \mu$, and by the definition of μ, $\Pi(f, \varphi \mid X) \subseteqq U$. Lemma 1 is proved.

If $\alpha < \beta < \gamma$ and $\alpha < \beta' < \gamma$,

$$\theta(\alpha; \beta, \beta'; \gamma)$$

denotes the homeomorphism $\theta : R \doteq R$ which maps $[\alpha, \beta]$ and $[\beta, \gamma]$ linearly on to $[\alpha, \beta']$ and $[\beta', \gamma]$ respectively, leaving other points fixed. If α, β, β' and γ are continuous real functions on X, the same notation

$$\theta_x = \theta(\alpha; \beta, \beta'; \gamma)$$

may be used on the understanding that all the functions are to be evaluated at x. If then $h(x, t) = (x, \theta_x(t))$, h is a homeomorphism of $X \times R$.

If $\alpha \leqq \beta \leqq \gamma$, $\alpha \leqq \beta' \leqq \gamma$, h is still a homeomorphism provided that, for each x, the consistency conditions

$$\alpha(x) = \beta(x) \Leftrightarrow \alpha(x) = \beta'(x)$$

and

$$\beta(x) = \gamma(x) \Leftrightarrow \beta'(x) = \gamma(x)$$

hold. Any such equality makes $\theta_x = 1$ on the line $x \times R$.

3. It is convenient to use the word *engulfing* in the following sense. Let X be a space and A, U, a closed and an open set in X, respectively. Then A is *engulfed by* hU if h is a homeomorphism, $h : X \doteq X$ such that

(1) $A \subseteqq hU$, and

(2) h is isotopic to 1 (the identity) through h_τ. We say that h is *active only in* W if $h_\tau = 1$ outside W.

LEMMA 2. *Let $f_0\, f_1$ be bounded continuous functions on the closed set F in X, and U, W, open sets of $X \times R$. Let $f_0 \leqq f_1$; let $f_0 = f_1$ on* $\mathrm{fr}_X F$. *Sup-*

[3] For the theorem in metric spaces see, e. g., Aumann: *Reelle Funktionen*, p. 150.

pose that

$$\Pi(f_0, f_1 \mid F) \subseteqq W, \qquad \Pi(f_0 \mid F) \subseteqq U .$$

Then $\Pi(f_0, f_1 \mid F)$ *can be engulfed by* hU, *where* h *is active only in* W.

Put $\Pi(f, g \mid F) = \Pi(f, g)$ for any functions f, g. We make two applications of Lemma 1.

(1) There is a continuous function χ such that $f_1 < \chi$ and $\Pi(f_1, \chi) \subseteqq W$. Let $f_{1+} = \min(\chi, f_1 + (f_1 - f_0))$. Thus $\Pi(f_0, f_{1+}) \subseteqq W$.

(2) There is a continuous φ such that $f_0 < \varphi$ and $\Pi(f_0, \varphi) \subseteqq U$. Let

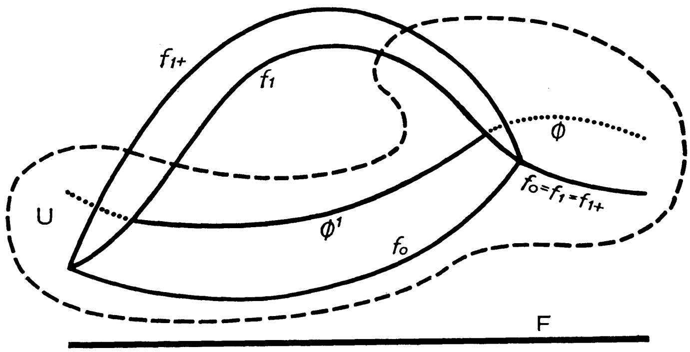

FIGURE 1.

$\varphi' = \min(\varphi, f_1)$. Then $f_0 \leqq \varphi' \leqq f_1 \leqq f_{1+}$. Let

$$\theta_x = \theta(f_0; \varphi', f_1; f_{1+}) , \qquad \text{for } x \in F.$$

The consistency conditions are satisfied, for since $f_0 < \varphi$,

$$f_0(x) = \varphi'(x) \Rightarrow \varphi'(x) = f_1(x);$$

hence $f_0(x) = f_1(x)$. Similarly

$$f_1(x) = f_{1+}(x) \Rightarrow f_{1+}(x) = \big(f_1 + (f_1 - f_0)\big)(x) .$$

Hence $f_1(x) = f_0(x)$, and therefore $\varphi'(x) = f_1(x)$. Thus $h(x, t) = (x, \theta_x(t))$ defines a homeomorphism of $X \times R$ which is the identity at (x, t) unless $x \in \operatorname{int}_X F$ and $f_0(x) < t < f_{1+}(x)$, and therefore $(x, t) \in W$. Since $\Pi(f_0, \varphi') \subseteqq U$, $\Pi(f_0, f_1) \subseteqq hU$, by the definition of θ_x.

An isotopy of h to the identity, active only in W, is defined by replacing f_1 by $(1 - \tau)\varphi' + \tau f_1$ in the definition of θ_x.

4. The following Theorem 1 shows that the stretching transformation of $X \times R$ in Lemma 2 can be extended into a product space $X' \times X \times R$, pulling the adjacent parts of the space with it. This is the theorem that replaces

Whitehead's theorem on the engulfing of a k-simplex ($k<n$) in a (combinatorial) n-manifold by its regular neighbourhood.

If $A \subseteqq X$ let

$$\Pi_0(f, g \mid A) = \Pi(f, g \mid A)\backslash\Pi(f \mid A)$$
$$= [x \in A, f^\cdot(x) < t \leqq g^\cdot(x)] .$$

Thus $\Pi_0(f, g) \cap \Pi(f) = \emptyset$.

Let Y, Z be spaces, and let a point o be chosen in Y as *origin*.

THEOREM 1. *Let g_0, g_1 be bounded functions on the closed set F in Z; let U, W, be open sets of $Y \times Z \times R$, and suppose that*

(4.1) *g_0 is* u.s.c., *g_1 continuous, $g_0 \leqq g_1$, and $g_0 = g_1$ on* $\mathrm{fr}_Z F$;

(4.2) $o \times \Pi(g_0 \mid F) \subseteqq U$;

(4.3) $o \times \Pi_0(g_0, g_1 \mid F) \subseteqq W$.

Then $o \times \Pi(g_0, g_1 \mid F)$ can be engulfed by hU, where h is active only in W.

Case 1. g_0 is continuous, and the stronger form

(4.3′) $o \times \Pi(g_0, g_1 \mid F) \subseteqq W$ of (4.3) holds.

The plan of the proof of Case 1 is first to costruct a *floor* V, an open set in $Y \times Z$ containing $o \times \mathrm{int}_Z F$, but narrow in the Y-dimension; then a continuous function φ on $\bar{V}$, equal to g_1 on $o \times F$ and to $g_0(z)$ at points (y, z) of $\mathrm{fr}_{Y\times Z} V$; and finally to use φ to taper off the h of Lemma 2, transferred to $o \times Z \times R$, to the identity on $(\mathrm{fr}\, V) \times R$.

Put $g_i^*(y, z) = g_i(z)$, for $i = 0, 1$ and $y \in Y$, $z \in F$. *There is an open set V in $Y \times Z$ such that*

(4.4.1) $o \times \mathrm{int}_Z F \subseteqq V \subseteqq Y \times \mathrm{int}_Z F$;

(4.4.2) $\Pi(g_0^*, g_1^* \mid \bar{V}) \subseteqq W$, $\Pi(g_0^* \mid \bar{V}) \subseteqq U$.

In the normal space $Y \times Z \times R$, there exist open sets U_1, W_1 satisfying (4.2) and (4.3′), and $\bar{U}_1 \subseteqq U$, $\bar{W}_1 \subseteqq W$. Let $z \in \mathrm{int}_Z F$. Then there exist real numbers $\lambda < g_0(z)$, $\mu > g_1(z)$ such that $o \times z \times [\lambda, \mu] \subseteqq W_1$. By the compactness of $[\lambda, \mu]$, there are open neighbourhoods $N_0(o)$ in Y and $N_1(z)$ in Z (both depending on z) such that $N_0 \times N_1 \times [\lambda, \mu] \subseteqq W_1$; and N_0 may be so chosen that, further,

$$z' \in N_0 \Rightarrow g_0(z') > \lambda \ \& \ g_1(z') < \mu .$$

Then

$$\Pi(g_0^*, g_1^* \mid N_0 \times N_1) \subseteqq N_0 \times N_1 \times [\lambda, \mu] \subseteqq W_1 .$$

Similarly it can be arranged that

$$\Pi(g_0^* \mid N_0 \times N_1) \subseteqq U_1 .$$

Then

$$V = \bigcup (N_0 \times N_1;\ z \in \operatorname{int}_Z F)$$

satisfies (4.4.1) and (4.2.2). (Recall that g_0^* is continuous in Case 1.)

Now let functions α and β be defined on $\bar{V}$:

$$\alpha \mid o \times F = g_1^* , \qquad \alpha \mid \bar{V} \backslash (o \times F) = g_0^* ,$$
$$\beta \mid V = g_1^* , \qquad \beta \mid \operatorname{fr}_{Y \times Z} V = g_0^* .$$

Then α is u.s.c., β is l.s.c., and $\alpha \leqq \beta$ on $\bar{V}$ since $g_0 = g_1$ on $\operatorname{fr}_Z F$. Let φ be a continuous function on $\bar{V}$ such that $\alpha \leqq \varphi \leqq \beta$, with strict inequalities where $\alpha < \beta$. It follows that

(4.5) $g_0^* \leqq \varphi \leqq g_1^*$ in $\bar{V}$, $g_0^* = \varphi$ on $\operatorname{fr} V$, $\varphi = g_1$ in $o \times F$.

By the last clause of (4.5),

$$o \times \Pi(g_0, g_1 \mid F) \subseteqq \Pi(g_0^*, \varphi \mid \bar{V})$$
$$\subseteqq \Pi(g_0^*, g_1^* \mid \bar{V}) \subseteqq W .$$

Therefore, $o \times \Pi(g_0, g_1 \mid F)$ can be engulfed by hU, h being active only in W (put $f_0 = g_0^*$, $f_1 = \varphi$, $F = \bar{V}$ in Lemma 2).

General case of Theorem 1. By (4.2) and Lemma 1, there is a continuous function ψ_0 on F such that $g_0 < \psi_0$ and $o \times \Pi(g_0, \psi_0 \mid F) \subseteqq U$. Let $\psi_1 = \min(\psi_0, g_1)$, and let H be the subset $[\psi_1(z) < g_1(z)]$ of F. In H,

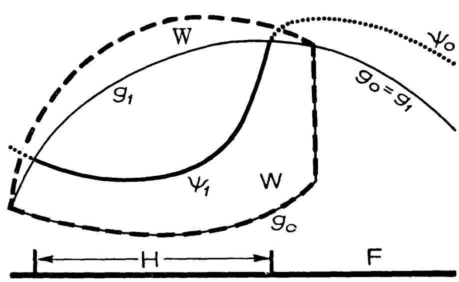

FIGURE 2

and therefore in $\bar{H}$, $\psi_1 = \psi_0$ and hence $\psi_1 > g_0$. The conditions of Case 1 are satisfied if ψ_1 and $\bar{H}$ replace g_0 and F, other sets and functions being unchanged: (4.2) because

$$\Pi(\psi_1 \mid \bar{H}) \subseteqq \Pi(g_0, \psi_0 \mid F);$$

and (4.3′) because $\psi_1 > g_0$ on $\bar{H}$ and so

$$o \times \Pi(\psi_1, g_1 \mid \bar{H}) \subseteqq o \times \Pi_0(g_0, g_1 \mid F) \subseteqq W .$$

Therefore hU engulfs $\Pi(\psi_1, g_1 \mid \bar{H})$, with h active only in W. Now

$$\Pi(\psi_1, g_1 \mid F \backslash \bar{H}) = \Pi(\psi_1 \mid F \backslash \bar{H})$$

by the definition of H, $\subseteqq \Pi(g_0, \psi_1 \mid F)$. Since ψ_1 replaced g_0 in the application of Case 1, and $g_0 < \psi_1$ in $\bar{H}$, $h = 1$ in $o \times \Pi(g_0, \psi_1 \mid F)$, and we infer that $o \times \Pi(\psi_1, g_1 \mid F) \subseteqq hU$. Hence finally the engulfing by h follows from

$$\Pi(g_0, g_1 \mid F) = \Pi(g_0, \psi_1 \mid F) \cup \Pi(\psi_1, g_1 \mid F) .$$

5. THEOREM 2. *With the notations of Theorem* 1, *let A be a closed set in $Y \times Z \times R$, and suppose in addition to the conditions of Theorem* 1, *that*

(5.1) $A \subseteqq U$

(5.2) $A \cap (o \times \Pi(g_0, g_1 \mid F)) \subseteqq o \times \Pi(g_0 \mid F)$.

Then the conclusion of Theorem 1 *can be strengthened to "$A \cup (o \times \Pi(g_0, g_1 \mid F))$ can be engulfed by hU", h being, as before, active only in W.*

Condition (5.2) implies that A does not meet $o \times \Pi_0(g_0, g_1 \mid F)$. Therefore if W is replaced by $W \backslash A$, the conditions of Theorem 1 are still satisfied. It follows that there is an engulfing map h such that $h = 1$ outside $W \backslash A$, and therefore $h = 1$ in A. Since $A \subseteqq U$ it follows that $A \subseteqq hU$.

2. Optimal maps

6. Polyhedral sets in the euclidean space R^n, i.e., loci of (rectilinear, locally finite, simplicial) complexes, are denoted by H, K, L, etc., and complexes with the locus H by H_0, H_1, etc.[4] If $L_0 \leqq K_0$ (i.e., if L_0 is a subcomplex of K_0) and K_1 is a further subdivision of K_0, L_1 is the induced subdivision of L_0.

All maps of polyhedral sets are understood to be locally finite (i.e., the set of image-simplexes is locally finite for some subdivision, and therefore for all).

A p.l. (piecewise linear) map $f: K \longrightarrow R^n$ is *optimal* if there is a subdivision K_0 such that

(1) f is a non-degenerate[5] map of K_0, and

(2) for each pair of *principal* simplexes σ_1, σ_2 of K_0,

(6.1) either $f\sigma_1 \cap f\sigma_2 = f(\sigma_1 \cap \sigma_2)$ or $\dim (f\sigma_1 \cap f\sigma_2) \leqq \dim \sigma_1 + \dim \sigma_2 - n$.

This is a geometrical property, i.e., it is preserved if K_0 is further subdivided. A non-degenerate linear map of K_0 may be called *optimal in all dimensions* if (6.1) holds for all $\sigma_1, \sigma_2 \in K_0$. This is not a geometrical property and will not be used in this paper.

LEMMA 3. *If*[6] $K_0 \cup [\sigma]$ *is a complex and* $\sigma \notin K_0$, *and if* $f: K \cup \sigma \longrightarrow R^n$ *is optimal, then*

[4] The notation $|H_0|$ for the locus is still required, e.g., for $|H_0^q|$, where H_0^q is the q-skeleton of H_0.

[5] $f \mid \sigma$ is injective if $\sigma \in K_0$.

[6] The complex $[\sigma]$ is σ and all its faces.

(6.2) $f\sigma \cap fK = f(\sigma \cap K) \cup A$,
where $A = \emptyset$ *or* $\dim A \leqq \dim \sigma + \dim K - n$.

This follows easily from (6.1) since σ is a principal simplex of $K_0 \cup [\sigma]$, and every simplex of $K_0 \cup [\sigma]$ is contained in a principal simplex.

It is an elementary result that the optimal p.l. maps of K lie dense in the space of continuous maps of K; but difficulties may arise if a subset $L \leqq K$, mapped optimally, but perhaps not in general position, is to be held fixed.

7. If K_0 is a complex, and $E \subseteqq K$, E_0 is the smallest subcomplex of K_0 containing E.

Two maps $f, g: K \to R^n$ *agree in* $X \subseteqq R^n$ if, for some subdivision K_0,

$$(f^{-1}X)_0 = (g^{-1}X)_0 = H_0$$

say, and $f \mid H_0 = g \mid H_0$.

Let C^n be the cube $[\mid x_i \mid \leqq 1,\ i = 1, \cdots, n]$ in R^n. If $\alpha > 0$, C^n_α is the concentric cube $[\mid x_i \mid \leqq \alpha]$. It is sometimes convenient to use also $C^n(c)$, the cube $[\mid x_i - c_i \mid \leqq 1]$, where $c = (c_i)$.

THEOREM 3. *Let K be a polyhedral set and $L \leqq K$. Let $f: K \to R^n$ be such that $f \mid L$ is a* p.l. *embedding. Then if $\varepsilon > 0$ and $0 < \alpha < 1$, there is a map $f': K \to R^n$ such that*

(7.1) *f' agrees with f in $R^n \backslash C^n$ and with a* p.l. *optimal map in C^n_α;*
(7.2) $f' \mid L = f \mid L$;
(7.3) $\mid f - f' \mid < \varepsilon$.

Let $\beta = (\alpha + 1)/2$. There are subdivisions C^β_1 of C^n_β and L_1 of L(which can be extended to K_1 of K) such that $fL \cap C^n_\beta$ is the locus of a common subcomplex of fL_1 and C^β_1.

For any set E of R^n let $\hat{E}$ be the convex cover. Let[7]

$$P_1 = (f^{-1}(R^n \backslash \mathring{C}^n))_1\,, \qquad Q_1 = [\sigma \mid \sigma \in K_1\ \&\ \widehat{f\sigma} \cap C^n_\beta \neq \emptyset]\,.$$

Suppose the division K_1 so fine that $P \cap Q = \emptyset$. A map f_1 agreeing with f in $R^n \backslash C^n$ and with a p.l. map in C^n_β is obtained by putting $f_1 = f$ at all points of P and at all vertices of K_1, making f_1 linear in each simplex of Q_1, and extending it to the transition zone $K \backslash (P \cup Q)$ by a *coning* process: if σ is a transitional simplex with centre c, and c' is the barycentre of the f_1-images of vertices of σ, let $f_1(c) = c'$; and supposing $f_1 \mid \dot{\sigma}$ defined extend f_1 linearly along radii. Then $f_1 \mid L = f \mid L$ and $f_1\sigma \subseteqq \widehat{f\sigma}$ for all $\sigma \in K$. From this the stated properties of f_1 follow.

Now let P_1', Q_1' be defined as P_1, Q_1 were, but with $\mathring{C}^n_\beta$, C^n_β in place of $\mathring{C}^n$,

[7] The square brackets in $Q_1 = [\cdot\cdot]$ denote the simplicial closure.

C^n_β respectively, and make the further assumption that $P' \cap Q' = \emptyset$. Let f_2 be obtained from f_1 by a δ-displacement of the images of vertices of Q'_1 (including those in L_1) into general position in $\overset{\circ}{C}{}^n$, other vertices remaining fixed, and $f_2 \mid P = f_1 \mid P$; and let f_2 then be extended linearly in Q'_1 and by *coning* in $K \backslash (P' \cup Q')$, as before. If δ is small enough, f_2 agrees with f in $R^n \backslash C^n$ and with a p.l. optimal map in C^n_α (indeed with a linear map optimal in all dimensions).

If $x = f(y)$, where y is a vertex of L_1, this vertex is unique, since $f \mid L$ is an embedding, and we put $\varphi(x) = f_2(y)$; if x is any other vertex of C^β_1 put $\varphi(x) = x$. If δ is small enough, φ, extended linearly into the simplexes of C^β_1, is a p.l. homeomorphism of C_β onto itself, and $\varphi \mid \dot{C}^n_\beta = 1$. Therefore φ may be extended to the rest of R^n by the identity.

Let $f'(x) = \varphi^{-1} f_2(x)$ for all $x \in K$. Then f' is p.l. in C^n_α, and since φ is a p.l. homeomorphism f' is optimal. (This uses the geometrical character of *optimal*, cf. ¶ 6.) If $y \in L$,

$$f'(y) = \varphi^{-1} f_2(y) = \varphi^{-1} \varphi f(y) = f(y).$$

That $\mid f - f' \mid$ can be made arbitrarily small is clear.

REMARK 1. Since $\varphi(x) = x$ at a vertex $x \notin fL$, such vertices are in general position.

REMARK 2. The following corollary, not used in this paper, but in fact proved above, may be found useful if the property *optimal in all dimensions* has to be used.

COROLLARY 3A. *Under the conditions of Theorem* 3 *there is a map* $f_2 : K \to R^n$, *a subdivision* K_1, *and a homeomorphism* $\varphi : R^n \doteq R^n$ *such that*

(7.3) f_2 *agrees with* f *in* $R^n \backslash \overset{\circ}{C}$ *and with a linear map of* K_1, *optimal in all dimensions in* C^n_α; *and if* $\beta = (\alpha + 1)/2$,

(7.4) $\varphi \mid C^n_\beta$ *is* p.l., $\varphi \mid R^n \backslash C^n_\beta = 1$, $\mid \varphi - 1 \mid < \varepsilon$, $\mid f - f_2 \mid < \varepsilon$;

(7.5) $\varphi^{-1} f_2 \mid L = f \mid L$.

8. A *manifold* is in this paper a topological manifold without boundary. A *euclidean* nbd in a manifold M^n is a pair (U, ψ) where U is an open set in M and $\psi : C^n \twoheadrightarrow \bar{U}$ a homeomorphism.[8] The symbol $\mid \psi \mid$ denotes the set $\psi(C^n)$ but also implies that ψ is the homeomorphism; $\mid \overset{\circ}{\psi} \mid$, $\mid \psi \mid_\alpha$, etc., will denote $\psi(\overset{\circ}{C}{}^n)$, $\psi(C^n_\alpha)$, etc.

A map $f : K \to M$ is *locally* p.l. if $f(K)$ has a covering by a set of euclidean nbds $\mid \overset{\circ}{\psi}_i \mid$, called a *smoothing system*, and for each i there is a subdivision[9]

[8] Or occasionally $\psi : C^n(c) \twoheadrightarrow \bar{U}$: see above.

[9] It follows that there is a single subdivision with this property.

K_0 such that

(8.1) $f^{-1}(|\psi_i|)$ *is a subcomplex*, H_0, *of* K_0 *and* $\psi^{-1}(f|H_0)$ *is linear*.

By a slight abuse of language (8.1) is sometimes abbreviated to f *is linear in* $|\psi_i|$, and the words "polyhedral", "subdivision", etc. are applied in a similar way to sets in $|\psi|$, (usually with quotation marks).

The following is a direct consequence of Theorem 3.

THEOREM 4. *Let* $f: K \to M$ *and* $L \leqq K$. *Let* $f|L$ *be a locally* p.l. *embedding and* $|\psi|$ *be one of the smoothing* nbds *for* $f|L$. *Let* $0 < \alpha < 1$ *and* $\varepsilon > 0$. *Then there is a map* $f': K \to M$ *such that* (*supposing* d *a metric on* M),

(8.2) f' *agrees with* f *in* $M \backslash |\mathring{\psi}|$ *and with an optimal* p.l. *map in* $|\psi|_\alpha$; *and* $d(f, f') < \varepsilon$;

(8.3) $f'|L = f|L$.

PROOF. Let $0 < \gamma < 1$ and let K_0 be a subdivision of K such that if

$$H_0 = (f^{-1}(|\psi|_\gamma))_0 ,$$

$f(H) \subseteqq |\mathring{\psi}|$. Apply Theorem 3 with H for K, $L \cap H$ for L and g for f, where

$$g = \varphi\psi^{-1}(f|H): H \to R^n ,$$

and φ is the radial θ-*map* of R^n onto itself:

$$\theta(\alpha; \gamma, 1; 2) .$$

If g' is the map provided by Theorem 3, then $f' = \psi\varphi^{-1}g'$, extended by the identity to $M^n \backslash |\psi|_\gamma$, is the required map.

There is a Corollary 4A corresponding to 3A.

3. The engulfing theorem

9. A set of points X in M^n is p-*dominated* if there is a covering of X by open nbds $|\mathring{\psi}|$ of M^n (the *smoothing* nbds) such that $X \cap |\psi|$ is contained in a "polyhedral set" in $|\psi|$ of dimension $\leqq p$.[10] Clearly every subset of a p-dominated set is p-dominated. If for each smoothing nbd $X \cap |\psi|$ itself is polyhedral, X is *locally tame*. Thus a p-dimensional locally tame set is p-dominated.

The main theorem of the paper is

THEOREM 5. *Let* X *be closed and* V *open in* M^n. *Suppose that* M *is* p-*connected*, V *is* $(p-1)$-*connected*, *and that* X *is* p-*dominated*, *where* $p \leqq$

[10] I. e., $\psi^{-1}(X \cap |\psi|)$ is contained in a p-dimensional polyhedral set in C^n. Note that $|\psi|$ is the closed set $\psi(C^n)$.

$n-3$. *Let* $X\backslash V$ *be compact.*

Then X *can be engulfed by* hV, *where* h *is active only in some compact set.*[11]

The proof shows that the isotopy is made up of small *pushes* along parallel lines in a euclidean nbd.

The main step in the argument, the use of induction to engulf the intersection of a simplex with the set already engulfed, is taken directly from Stallings' proof; but the logical structure of the argument is a good deal more complicated. To carry through the inductive arguments the following more general theorem will be proved.

THEOREM 6. *Let* X *be closed and* V *open in* M^n. *Let* Γ *be a polyhedral set in* R^N, $f:\Gamma\to M$ *a map. Let* $L\leqq\Gamma$, $H\leqq\Gamma$, $\varepsilon>0$. *Suppose that*

(9.1) M *is p-connected,* V *is* $(p-1)$*-connected,* $\dim\Gamma\leqq p$, $p\leqq n-3$;

(9.2) $(X\cup fH)\backslash V$ *is compact;*

(9.3) $f\,|\,L$ *is a locally* p.l. *embedding,* X *is p-dominated, and there is a smoothing system* Σ *for* $f\,|\,L$, *every member of which is a smoothing* nbd *for* X.

Then there is a map $g:\Gamma\to M$ *such that*

(9.4) $X\cup gh$ *can be engulfed by* hV, h *being active only in a compact subset of* M;

(9.5) $g\,|\,L=f\,|\,L$.

(9.6) $d(f,g)<\varepsilon$ *and* $f=g$ *outside a compact set.*

The inert complex Γ, which is not engulfed, is brought in to provide a common domain for the various maps g, g_i, etc., a technical convenience.

It will, we believe, be clear from the proof that the *nearness* conditions, (9.6) and the compact action of h_τ are satisfied. They will usually be referred to in such general terms as *$d(f,g)$ can be made sufficiently small.*

10. The first step is to show that it is sufficient to deal with the case where $X\subseteqq V$ and $H\backslash f^{-1}V\subseteqq\mathring{\sigma}^q$, where $\sigma^q\in H_0$, a subdivision of H.

LEMMA 5. *It is sufficient to prove Theorem* 6 *assuming that* $X\subseteqq V$.

(This does not mean that X disappears from the theorem. It must still be proved that $X\subseteqq hV$.)

For each point x of M choose

(i) if $x\in fL$, a member of the smoothing system Σ containing x;

(ii) if $x\in X\backslash fL$, a smoothing nbd of x for X, not meeting fL;

(iii) if $x\in M\backslash(X\cup fL)$ a euclidean nbd not meeting $X\cup fL$.

---

[11] See ¶ 3; and compare the footnote in ¶ 1.

We call the members of this covering *the* $|\mathring{\psi}_i|$ nbds.

Let $A(m)$ stand for the assertion of Theorem 6, with the added condition

(10.1) $X\backslash V$ is contained in the union of m $|\mathring{\psi}_i|$-nbds.

The fact that $X\backslash|\mathring{\psi}_1|$ is closed and p-dominated shows, by a simple induction on m, that $A(1)$ implies $A(m)$ for all $m \geqq 1$. Lemma 5 will therefore follow from: $A(0)$ *implies* $A(1)$. Suppose $X\backslash V \subseteqq |\mathring{\psi}_1|$.

It is given that $X \cap |\psi_1| \subseteqq \psi_1 P$, where P_0 is a p-complex in $\mathring{C}^n$; and we may suppose that $\Gamma \cap P = \varnothing$. Define $f' : \Gamma \cup P \to M$ by

$$f' \mid \Gamma = f\ , \qquad f' \mid P = \psi_1\ .$$

Let Γ^* be formed from $\Gamma \cup P$ by identifying points of $L \cup P$ that have the same f'-image. If p is the projection $p : \Gamma \cup P \to \Gamma^*$, f' can be factored through $p, f' = f^*p$, and, if $L^* = p(L \cup P)$,

$$\begin{tikzcd} & \Gamma^* \arrow[dr, "f^*"] & \\ \Gamma \cup P \arrow[ur, "p"] \arrow[rr, "f'"] & & M \end{tikzcd}$$

$f^* \mid L^*$ is an embedding. If Γ_0 and P_0 are suitable subdivisions, the identifying relation $f'(x) = f'(y)$ is a simplicial isomorphism between the parts identified, and therefore Γ_0^* can be regarded as a complex, p is simplicial, and $f^* \mid L^*$ a locally p.l. map with ψ_1 as one of its smoothing nbds. Let $X^* = X\backslash|\mathring{\psi}_1|$, $H^* = p(H \cup P)$.

The conditions of Theorem 6 are satisfied by Γ^*, H^*, X^*, f^*, and V; and we have $X^* \subseteqq V$. Therefore, assuming $A(0)$ true, it follows that h and g^* exist satisfying the smallness conditions, and such that

$$g^* \mid L^* = f^* \mid L^*\ , \qquad X^* \cup g^*H^* \subseteqq hV^*\ .$$

Let $g = g^*p$. Then if $x \in L$,

$$g(x) = g^*p(x) = f^*p(x) = f'(x) = f(x)\ ,$$

i.e., $g \mid L = f \mid L$. Similarly $g \mid P = \psi_1 \mid P$, and therefore

$$\begin{aligned} hV &\supseteqq X^* \cup g^*H^* = X^* \cup gH \cup \psi_1 P \\ &\supseteqq (X\backslash\psi_1^0) \cup gH \cup (X \cap |\psi_1|) = X \cup gH\ . \end{aligned}$$

The smallness condition for g follows easily, and Lemma 5 is proved.

11. Let $B(q)$ stand for $A(0)$ with the added condition:

$$\dim\,(H\backslash f^{-1}V) \leqq q\ .$$

Let $C(q, m)$ stand for $B(q)$ with the added conditions that, for some subdivision Γ_0,

$f(H_0^{q-1}) \subseteqq V$; there are at most m q-simplexes σ^q of H_0 such that $f\sigma^q$ meets $M\backslash V$; and none such that $f\sigma^q$ meets both X and $M\backslash V$.

LEMMA 6. *It is sufficient to prove that, for all $q \leqq p$, $B(q-1)$ implies $C(q, 1)$.*

By Lemma 5, it is sufficient to prove $B(q)$ for all q, and since $B(-1)$ is true we may assume $B(q-1)$ in doing so.

Since $fH\backslash V$ is compact, a simple induction argument shows that $C(q, 1)$ implies $C(q, m)$ for all positive m.

Therefore Lemma 6 will follow from

$$\big(B(q-1) \;\&\; \forall m\; C(q, m)\big) \Rightarrow B(q)\ .$$

Suppose that the conditions of $B(q)$ are satisfied. Let Γ_0 be so fine a division of Γ that if P_0 is the set of simplexes of H_0 mapped into V by f, $f(\overline{H\backslash P}) \cap X = \varnothing$. By $B(q-1)$ there exist h and g such that

$$X \cup g\big(P \cup [H_0\backslash P_0]^{q-1}\big) \subseteqq hV\,, \qquad g\,|\,L = f\,|\,L\ ,$$

and $d(f, g)$ is so small that $g(\overline{H\backslash P}) \cap X = \varnothing$. If there are $m_0(\geqq 1)$ q-simplexes[12] of $[H_0\backslash P_0]$ the conditions of $C(q, m_0)$ (which we may assume true) are satisfied, with g and hV in place of f and V. We may infer that $X \cup g'H$ can be engulfed, for some g' near g, with $g'\,|\,L = g\,|\,L$. Lemma 6 is proved.

It has now been shown that it is sufficient to prove Theorem 6 with the extra assumptions:

(11.1) $X \subseteqq V$

(11.2) $H\backslash f^{-1}V \subseteqq \mathring{\sigma}^q$

(11.3) $f\sigma^q \cap X = \varnothing$,

where $\sigma^q \in H_0$, a subdivision of H; and that in doing so we may assume the truth of $B(q-1)$.

12. Let a be a point of the euclidean space containing Γ, such that $a\sigma^q \cap \Gamma = \sigma^q$. Then $a\sigma^q \cup \Gamma$ is a polyhedral set, which we denote by G. By condition (9.1) of Theorem 6, $f\,|\dot{\sigma}^q$ can be extended to a map, which may also be called f, of $a\dot{\sigma}^q$ into V, and $f\,|\,(a\sigma^q)^{\boldsymbol{\cdot}}$, so defined, can be extended to a map $f: a\sigma^q \longrightarrow M$. It is convenient to add $a\dot{\sigma}^q$ to H. Condition (11.2) is then still satisfied, but now $H \cap a\sigma^q = (a\sigma^q)^{\boldsymbol{\cdot}}$.

In the rest of the proof the *membrane* $f(a\sigma^q)$ is used as a guide in engulfing σ^q.

It follows from (11.2) that σ^q is a principal simplex of H, which may therefore be called $K \cup \sigma^q$, where $\sigma^q \notin K$, $a\dot{\sigma}^q \subseteqq K$. Since the new simplex $a\sigma^q$ meets L at most in σ^q it follows from (11.3) that $f(L \cap a\sigma^q) \cap X = \varnothing$. It is

[12] If $m_0 = 0$ there is nothing to prove.

convenient to generalize this situation slightly by taking the new $(q+1)$-simplex of G to be any join, $\sigma^{q+1} = \sigma_1\sigma_2$, with $\sigma_1\dot{\sigma}_2 \subseteqq K$, and $\dot{\sigma}_1\sigma_2$ the part of H still to be engulfed. The whole proof is thus finally reduced to that of the following Lemma, and we may assume $B(q-1)$ in proving it.

LEMMA 7. *Let M^n, X be as in Theorem* 6, *and let M, V satisfy* (9.1). *Let G be a polyhedral set* (*in R^N*), $f: G \to M$ *a map*, $L \leqq G$, $K \cup \sigma^{q+1} \leqq G$, $\varepsilon > 0$. *Suppose that*

(12.1) $\sigma^{q+1} = \sigma_1\sigma_2$, $K \cap \sigma^{q+1} = \sigma_1\dot{\sigma}_2$, $\dim K \leqq p$;

(12.2) $X \cup fK \subseteqq V$;

(12.3) $X \cap f(L \cap \sigma^{q+1}) = \varnothing$;

(12.4) *is* (9.3) *of Theorem* 6.

Then the conclusions of Theorem 6 *hold if* $H = K \cup \dot{\sigma}_1\sigma_2$.

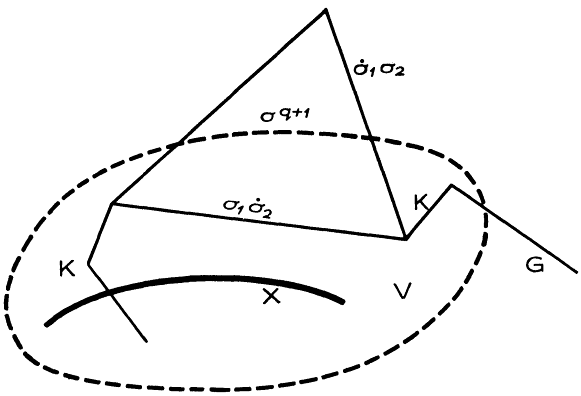

FIGURE 3

Case 1. Let a covering by $|\mathring{\psi}_i|$-nbds be chosen as in the proof of Lemma 5, and suppose that

(12.5) $f\sigma^{q+1} \subseteqq |\mathring{\psi}|_\alpha$ where $|\mathring{\psi}|$ is some ψ_i-nbd and $0 < \alpha < 1$.

For any set $E \subseteqq G$ let $E^\alpha = f^{-1}(|\psi|_\alpha \cap E)$.

By Theorem 4, there is a map $f': G \to M$ agreeing with f outside $|\mathring{\psi}|$, and with a p.l. and optimal map in $|\psi|_\alpha$; and $f' | L = f | L$. The ε of Theorem 4 is to be so chosen that $f'(K) \subseteqq V$. Let G_0 be a subdivision of G such that $f' | G_0^\alpha$ is linear[13] and let $Q_0 = (\sigma^{q+1})_0$, a $(q+1)$-complex.

From condition (12.3) of Lemma 7 it follows that if x is a vertex of a principal simplex, τ, of Q_0 such that f_τ meets X, $f'(x)$ may be moved into "general position" in $|\mathring{\psi}|$ relative to X, without infringing the condition $f' | L = f | L$; indeed we may suppose that f' itself already has the property (cf. Remark 1 of ¶ 7). If τ is a principal simplex of Q_0, then by Lemma 3,

(12.6) $f'(\tau) \cap (X \cup f'(K \cup |Q_0^q|)) = f'(\tau \cap (K \cup |Q_0^q|)) \cup T$, where Q_0^q is

[13] For the notation G_0^α see ¶ 10.

the q-skeleton of Q_0, T is a polyhedral subset of $f'(\tau)$, and

$$\begin{aligned}\dim T &\leqq \dim \tau + \dim\left(X \cup (K \cup |Q_0^q|)\right) - n \\ &\leqq q + 1 + p - n \leqq q - 2 ,\end{aligned}$$

since $p - n \leqq -3$.

13. A further subdivision, G_1, can be made so that K_1 expands to $K_1 \cup Q_1$ through $(q+1)$-simplexes*. This means that there exists a series of $(q+1)$-complexes ${}_iQ_1 (i = 0, \cdots, s)$ such that

$$\begin{aligned}&{}_0Q_1 = \varnothing , \quad {}_sQ_1 = Q_1 = (\sigma^{q+1})_1 \\ (13.1) \qquad &{}_{i+1}Q_1 = {}_iQ_1 \cup [{}_i\tau^{q+1}], \text{ where } {}_i\tau^{q+1} = \tau_1\tau_2 \\ &(K \cup {}_iQ) \cap ({}_i\tau^{q+1}) = \tau_1\dot{\tau}_2 .\end{aligned}$$

We make the inductive hypothesis that there is a map $g_i : G \to M$ such that

(13.2) $X \cup g_i(K \cup |{}_iQ_1^q|) \subseteqq h_i V$, for some engulfing h_i; and that

(13.3) $g_i | L \cup G^\alpha = f' | L \cup G^\alpha$

(13.4) $d(g_i, f')$ is sufficiently small.

The induction starts with $g_0 = f'$, $h_0 = 1$.

Let C^{n-p-1}, C^q, C^1 be the unit cubes in the $(x_1, \cdots, x_{n-q-1})$-subspace, the $(x_{n-q}, \cdots, x_{n-1})$-subspace, and the x_n-axis of R^n. Then $|\mathring{\psi}|$ may be regarded as $Y \times Z \times R$, where $Y = \psi(\mathring{C}^{n-q-1})$, $Z = \psi(\mathring{C}^q)$ and $R = \psi(\mathring{C}^1)$. By a linear change of the coordinate system ψ (without change of name) it can be arranged that the simplex $g_i({}_i\tau^{q+1}) = f'({}_i\tau^{q+1})$ is a prism in $Z \times I$:

$$g_i({}_i\tau^{q+1}) = \Pi(\gamma_0, \gamma_1 | F) , \qquad F \subseteqq Z,$$

with base $g_i(\tau_1\dot{\tau}_2) = \Pi(\gamma_0 | F)$ and top $g_i(\dot{\tau}_1\tau_2) = \Pi(\gamma_1 | F)$. The functions γ_0, γ_1 are continuous, $\gamma_0 \leqq \gamma_1$ and $\gamma_0 = \gamma_1$ on $\mathrm{fr}_Z F$.

Let J be the projection of the set T (above) into Z, so that $J \subseteqq F$. Let $\Pi J = \Pi(\gamma_0, \gamma_1 | J)$, $P = g_i^{-1}(\Pi J)$,

$$D = K \cup |{}_iQ_1^q| \cup P$$

a polyhedal subset of G.

The inductive hypothesis $B(q-1)$ will now be used to engulf $X \cup g_{i+1}D$, where g_{i+1} is to be specified. First G is replaced by G^*, formed by identifying points of $D \cap G^\alpha$ with the same g_i-image. As in Lemma 5, G^* is a polyhedral set since $g_i | G^\alpha = f' | G^\alpha$ is p.l.. If p is the projection $p : G \to G^*$, and f^* is defined by $g_i = f^* p$, L^* as $p(L \cup (D \cap G^\alpha))$, then $f^* | L^*$ is a locally p.l. embed-

*For example by using the theorem that every subdivision of a complex K_0 has a further subdivision which is a stellar subdivision of K_0.

ding. If the Γ, H, L, f, V, X of $B(q-1)$ are taken to be G^*, D^* $(= p(D))$, L^*, f^*, $h_i V$, X, the conditions of $B(q-1)$ are satisfied, since $X \subseteqq h_i V$, and "$H \backslash f^{-1} V$" is of dimension $\leqq q-1$ since

$$\text{“}H\backslash f^{-1}V\text{”} = D^*\backslash f^{*-1}(h_i V) = p(D\backslash g_i^{-1}(h_i V)) \subseteqq p(P) ,$$

by (13.2).

It follows that there is a map $g^* : G^* \to M$ such that $X \cup g^* D^*$ can be engulfed by $h'(h_i V)$, and $g^* \mid L^* = f^* \mid L^*$, $d(g^*, f^*)$ is sufficiently small. On putting $g_{i+1} = g^* p$, it follows, as in Lemma 5, that $g_{i+1} = f'$ in $L \cup (D \cap G^\alpha)$, that $X \cup g_{i+1} D \subseteqq h' h_i V$ and that the smallness conditions are satisfied.

14. What has so far been done is to engulf the subset ΠJ of $\Pi F = g_i({}_i\tau^{q+1}) = g_{i+1}({}_i\tau^{q+1})$. Theorem 2 is now used to engulf the rest of ΠF. The function "g_1" of Theorem 2 (which we hope will not be confused with the g_i of the preceding paragraphs) is to be γ_1; "g_0" is γ_0 in $F \backslash J$ and γ_1 in J, and so is u.s.c.; $U = V' \cap \mid \mathring{\psi} \mid$, where $V' = h' h_i V$; W is any open set containing ΠF and contained in $\mid \mathring{\psi} \mid$; and

$$A = \big((X \cup g_{i+1} D) \cap \mid \psi \mid_\alpha\big) \cup \Pi J .$$

The conditions of Theorem 2 are satisfied since ΠJ contains all the common points of A and ΠF except those in the base of ΠF.

It follows that $A \cup \Pi F$ can be engulfed by $h'' U$, where $h'' = 1$ in

$$\mid \mathring{\psi} \mid \backslash W \supseteqq \mid \mathring{\psi} \mid \backslash \mid \psi \mid_\alpha ,$$

and can therefore be extended to $M \backslash \mid \psi \mid_\alpha$ by the identity. Since the part of $X \cup g_{i+1}(K \cup \mid {}_{i+1}Q_1^q \mid)$ in $\mid \psi \mid_\alpha$ is $A \cup \Pi F \subseteqq h'' V'$, and the part in $M \backslash \mid \psi \mid_\alpha$ is in $X \cup g_{i+1} K \subseteqq V'$, it follows that $X \cup g_{i+1}(K \cup \mid {}_{i+1}Q_1^q \mid) \subseteqq h'' V'$.

On putting $h_{i+1} = h'' h' h_i$ the inductive step is completed and Case 1 proved.

15. *General Case.* Let a covering of M^n by a system of $\mid \mathring{\psi}_j \mid$-nbds be chosen, as in Case 1, and let Γ be so finely subdivided into Γ_0 that each simplex τ of $(\sigma^{q+1})_0$ is mapped by f into a set $\mid \mathring{\psi}_j \mid_\alpha$. Let Γ_0 be then further subdivided into Γ_1 so that K_1 expands through $(q+1)$-simplexes to $K_1 \cup (\sigma^{q+1})_1$. If now ${}_iQ_1$ and ${}_i\tau^{q+1}$ are defined as in Case 1, and the inductive assumption again made of maps $g_i : \Gamma \to M$ and $h_i : M \doteq M$ such that

$$X \cup g_i(K \cup \mid {}_iQ_1^q \mid) \subseteqq h_i V$$

$$g_i \mid L = f \mid L , \qquad d(g_i, f) \text{ is small,}$$

the conditions of Lemma 7 are satisfied if G, $K \cup \mid {}_iQ_1^q \mid$, ${}_i\tau^{q+1}$, $h_i V$, g_i, L replace G, K, σ^{q+1}, V, f, L; in particular

$$X \cap g_i\big(L \cap ({}_i\tau^{q+1})\big) = X \cap f(L \cap {}_i\tau^{q+1}) \subseteqq X \cap f(L \cap \sigma^{q+1}) = \varnothing ,$$

but more: ${}_i\tau^{q+1}$ (the new σ^{q+1}) is contained in a $\mid \mathring{\psi}_i \mid$-nbd. Therefore by Case 1,

$X \cup g_{i+1}(K \cup |{}_{i+1}Q_1^q|)$ can be engulfed for a suitable g_{i+1}.

The inductive step is complete and the proof of Theorem 6 is finished.

16. The following method of deducing the topological Poincaré hypothesis ($n \geqq 5$) from Theorem 5 is due to E. Connell.

THEOREM 7. *If $n \geqq 5$ every $[n/2]$-connected closed topological n-manifold M is a topological n-sphere.*

By a well-known argument, M is $(n-1)$-connected.

Suppose that M has a covering by k open topological n-cells, $|\mathring{\psi}_i|$, $i = 1, \cdots k$. It is known[14] that $k = 2$ implies that $M \doteq S^n$. It is therefore sufficient to show how to reduce k by 1 if $k \geqq 3$.

There is an $\alpha < 1$ such that $M \subseteqq \bigcup_1^k |\mathring{\psi}_i|_\alpha$. Let $\beta = (\alpha + 1)/2$, and let P_0 be a subdivision of $|\psi_3|_\beta$ into an "n-complex"[15] such that $|\psi_3|_\alpha$ is a subcomplex, and no simplex of P_0 meets both $|\dot{\psi}_1|_\alpha$ and $|\dot{\psi}_1|_\beta$.

If $p \leqq n-3$, $M_1 = M \backslash |\psi_1|_\beta$ is a p-connected open manifold, $V_1 = |\mathring{\psi}_1| \backslash |\psi_1|_\beta$ a $(p-1)$-connected open set in M_1, and $X_1 = |P_0^2| \backslash |\psi_1|_\beta$ a closed locally tame set in M_1 of dimension $2 \leqq n-3$; and $X_1 \backslash V_1 = |P_0^2| \backslash |\mathring{\psi}_1|$ is compact. By Theorem 5, $X_1 \subseteqq h_1 V_1$ for some $h_1 : M_1 \doteq M_1$ which is the identity outside a compact subset of M_1, and therefore near $|\psi_1|_\beta$. Hence h_1 can be extended to $|\psi_1|_\beta$ by the identity, giving

$$U_1 = h_1(|\psi_1|) = h_1 V_1 \cup |\mathring{\psi}_1|_\beta \supseteqq X_1 \cup |\psi_1|_\beta = |P_0^2| \cup |\psi_1|_\beta.$$

Similarly there is an open n-cell U_2 in M containing $|Q_0^{n-3}| \cup |\psi_2|_\beta$, where Q_0^{n-3} is the dual $(n-3)$-complex.[16]

Now[17] every point of $|P_0^n| \backslash (|P_0^2| \cup Q^{n-3})$ is on a unique segment joining a point of $|P_0^2|$ to a point of Q^{n-3}; and θ-homeomorphisms

$$\theta(0;\, \varepsilon,\, 1-\varepsilon;\, 1)$$

on these segments define a self-homeomorphism h_θ of $|\psi_0|_\beta$. The ε may vary continuously from segment to segment. We suppose that it is constant over the set of segments in $|\psi_3|_\alpha$, and tends to 1/2 as the segment tends to a position on $|\dot{\psi}_3|_\beta$. Then $h_\theta = 1$ on $|\dot{\psi}_3|_\beta$, and so can be extended to $M \backslash |\psi_3|_\beta$ by the identity. If the constant value of ε in $|\psi_3|_\alpha$ is small enough, $h_\theta U_1 \cup U_2 \supseteqq |\psi_3|_\alpha$. Therefore

[14] By the Mazur-Morton Brown collar theorem.

[15] Cf. ¶ 8. The quotation marks are omitted after this specimen.

[16] Q_0^{n-3} is the set of all simplexes of the derived complex $(P_0^n)'$ that do not meet $|P_0^2|$.

[17] Cf. Stallings' proof of the Poincaré hypothesis, Bull. Amer. Math. Soc., 66 (1960), 485.

$$h_\theta U_1 \cup U_2 \supseteqq h_\theta(|\psi_1|_\beta) \cup |\psi_2|_\beta \cup |\psi_3|_\alpha \,.$$

Now h_θ leaves x fixed if $x \notin |\psi_3|_\beta$ and moves it within a simplex of P_0 if $x \in |\psi_3|_\beta$. Since no simplex meets both $|\dot{\psi}_1|_\alpha$ and $|\dot{\psi}_1|_\beta$,

$$|\psi_1|_\alpha \subseteqq h_\theta(|\psi_1|_\beta) \,.$$

Therefore $|\mathring{\psi}_1|_\alpha \cup |\mathring{\psi}_2|_\alpha \cup |\mathring{\psi}_3|_\alpha$ is contained in two open n-cells.

UNIVERSITY OF WISCONSIN AND
RICE UNIVERSITY

(Received April 26, 1966)