A Second Order Algebraic Knot Concordance Group

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6th December 2011

Setting the Scene

There is a surjection:

$$\mathcal{C} := \text{Concordance of } K\colon S^1 \subset S^3 \twoheadrightarrow$$

$$\text{Concordance of } K\colon S^{4k+1} \subset S^{4k+3}, (k \geq 1) \cong \mathcal{L}_k$$

High dimensional knots are determined by algebra, $\mathcal{L} = \mathcal{L}_k \cong \mathcal{L}_{k+4}$.

In low dimensions this is not the case: the map $\mathcal{C} \to \mathcal{L}$ has an interesting kernel.

High dimensional obstructions are then just the 1st order obstructions, out of an infinite sequence.

This talk is about 2nd order obstructions, which I try to understand using chain complexes and algebraic surgery theory.

Outline

Outline

Slice Knots

Definition

An oriented knot $K\colon S^1 \subset S^3$ is a *slice knot* if there is an embedded disk $D^2 \subset D^4$, $\partial D^4 = S^3$, with $\partial D^2 = K$ (Fox-Milnor 1959). This is called a slice disc.

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[d] & S^3 \arrow[d] \\ D^2 \arrow[r, hook] & D^4 \end{tikzcd}$$

All embeddings must be locally flat.

Slice Knots

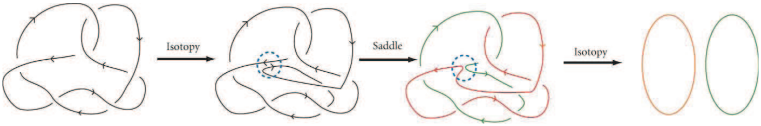
$$\begin{tikzcd}
S^1 \arrow[r, hook] \arrow[d] & S^3 \arrow[d] \\
D^2 \arrow[ur, hook, "A"] \arrow[r, hook, "B"] \arrow[dr, hook, "C"'] & D^4 \arrow[d] \\
& S^4
\end{tikzcd}$$

A exists only for the unknot. *C* exists for every knot.
Out of 2977 knots with 12 crossings or fewer, there are 158 slice knots.

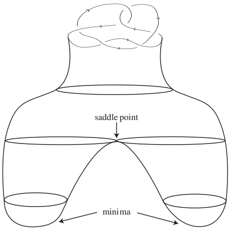
Source: http://www.indiana.edu/∼knotinfo/

Example - Twist Knot 6_1

Proving that a knot is slice: a Slice Movie:



Here is a schematic of the resulting disc in D^4:



Knot Concordance Group

Definition

The knot $-K$ is given by the mirror image knot.

Two knots K_1 and K_2 are *concordant* if $K_1 \sharp - K_2$ is slice.

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Two knots K_1 and K_2 are *concordant* if $K_1 \sharp -K_2$ is slice.

Forming the quotient of the monoid of knots by factoring out by slice knots,

$$\mathcal{C} := \frac{(\text{Knots}, \sharp)}{\text{Slice Knots}},$$

makes knots into a group under connected sum

Outline

Seifert Form

Definition

The Seifert form with respect to a Seifert Surface F is a pairing on $H_1(F;\mathbb{Z}) \cong \mathbb{Z}^{2g}$:

$$V\colon H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$$

which is defined by:

$$(x, y) \mapsto \mathrm{lk}(x^+, y)$$

where lk is the linking number in S^3 and x^+ is the push off of x along a positive normal direction to F.

Seifert Form - Example

With respect to the basis of curves shown the Seifert Form is given by:

$$V = \begin{pmatrix} n & 1 \\ 0 & -1 \end{pmatrix}$$

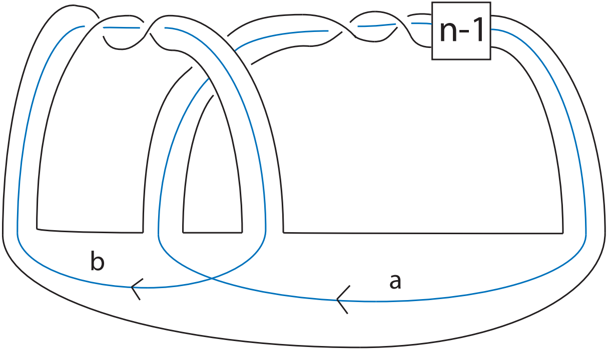

Figure: The *n*th twist knot

1st Order Algebraic Concordance

A knot is said to be *Algebraically Slice* if there is a basis for the 1st homology of a Seifert Surface such that the Seifert Form is:

$$\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$$

with block matrices A, B, C such that $C = C^T$ and $A - B^T$ is invertible.

We can add Seifert Forms over $\mathbb{Z}$ by $\oplus$.

Setting Seifert forms as above to be zero gives us a group $\mathcal{L}$, since there is a change of basis so that $A \oplus -A$ has the form above.

$$\mathcal{L} := \text{ the Witt group of Seifert forms.}$$

Proposition

A slice knot K is algebraically slice.

Proof: Push a Seifert surface F into D^4, and let D be a slice disk for K. $F \cup_K D = \partial M^3$ for some $M^3 \subset D^4 \setminus (F \cup_K D)$. Then

$$\ker(H_1(F \cup_K D) \to H_1(M))$$

gives a zero-linking half rank summand so the matrix looks like:

$$\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$$

as required.

Corollary

There is a surjective homomorphism:

$$Seifert \colon \mathcal{C} \twoheadrightarrow \mathcal{L}.$$

High Dimensional Knots

A high odd dimensional knot

$$K\colon S^{4n+1} \subset S^{4n+3},$$

for $n > 1$, is slice if and only if it is algebraically slice: the Whitney trick works.

$$\mathcal{C}_{4n+1} \xrightarrow{\simeq} \mathcal{L},$$

$n > 1$.

Outline

Knot Concordance

The 1st Order Algebraic Concordance Group $\mathcal{L}$

Cochran-Orr-Teichner Obstructions

A 2nd Order Algebraic Concordance Group

An Extension

Zero Surgery

We denote the zero-framed surgery on S^3 using a knot K as surgery data by

$$M_K = (S^3 \setminus \nu K) \cup_{S^1 \times S^1} D^2 \times S^1.$$

If K is a slice knot with slice disc D, and

$$W := D^4 \setminus \nu D,$$

then

$$\partial W = M_K.$$

Characterisation of Slice Knots

A knot K is topologically slice if and only if M_K is the boundary of a 4-manifold W which satisfies:

(i) $H_1(M_K;\mathbb{Z}) \xrightarrow{\simeq} H_1(W;\mathbb{Z}) \cong \mathbb{Z}$;

(ii) $H_2(W;\mathbb{Z}) = 0$;

(iii) $\pi_1(W) = \langle\langle \mu \rangle\rangle$.

Derived Series

Definition

For a group G, the derived series is defined inductively as iterated commutators:

$$G^{(0)} := G; \;\; G^{(i+1)} := [G^{(i)}, G^{(i)}].$$

Homology Surgery

Strategy (Cappell-Shaneson): Take a 4 manifold W with the right $H_1(W;\mathbb{Z})$ but $H_2(W;\mathbb{Z}) \neq 0$ and look for obstructions to being able to extirpate this H_2.

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Problem 1: in dimension 4, classes in $H_2(W;\mathbb{Z})$ are typically embedded surfaces N^2, but not embedded spheres. To detect these we must use twisted coefficients for the algebraic obstruction, the intersection form:

$$\lambda\colon H_2(W;\mathbb{Z}[\pi_1(W)]) \times H_2(W;\mathbb{Z}[\pi_1(W)]) \to \mathbb{Z}[\pi_1(W)].$$

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Problem 2: We don't know very much about $\pi_1(W)$; but we do know a lot about representations of $\pi_1(W)/\pi_1(W)^{(2)}$. Idea: If $\pi_1(N) \leq \pi_1(W)^{(2)}$, then the surface *looks like a sphere* to the 2nd level algebra.

Cochran-Orr-Teichner filtration of $\mathcal{C}$

Cochran-Orr-Teichner defined a filtration:

$$\cdots \subset \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(1)} \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C}$$

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$$\text{Arf}\colon \mathcal{C}/\mathcal{F}_{(0)} \xrightarrow{\simeq} \mathbb{Z}_2.$$

$$\text{Seifert}\colon \mathcal{C}/\mathcal{F}_{(0.5)} \xrightarrow{\simeq} \mathcal{L} \cong \bigoplus_{\infty} \mathbb{Z} \oplus \bigoplus_{\infty} \mathbb{Z}_2 \oplus \bigoplus_{\infty} \mathbb{Z}_4.$$

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$K \in \mathcal{F}_{(1.5)}$ implies that the Casson-Gordon obstructions vanish. Livingston, Jiang, Cochran-Orr-Teichner, T. Kim, Cochran-Harvey-Leidy:

$$\bigoplus_{\infty} \mathbb{Z} \oplus \bigoplus_{\infty} \mathbb{Z}_2 \hookrightarrow \mathcal{F}_{(1)}/\mathcal{F}_{(1.5)}$$

Cochran-Orr-Teichner filtration of $\mathcal{C}$

Definition

We say that knot K is (1.5)-solvable if the zero surgery M_K is the boundary of a spin 4-manifold W such that:

(i) $H_1(M_K;\mathbb{Z}) \xrightarrow{\simeq} H_1(W;\mathbb{Z}) \cong \mathbb{Z}$;

(ii) $H_2(W;\mathbb{Z})$ is generated by surfaces $L_1, \dots, L_n, D_1, \dots, D_n$ with geometric intersection numbers

$$L_i \cdot L_j = 0, L_i \cdot D_j = \delta_{ij},$$

which satisfy:

$$\pi_1(L_i) \leq \pi_1(W)^{(2)}, \pi_1(D_j) \leq \pi_1(W)^{(1)},$$

for all i, j.

Cochran-Orr-Teichner obstructions

Theorem (Cochran-Orr-Teichner)

Let K be a (1.5)-solvable knot. Then there exists a half-rank zero-self-linking summand P of $H_1(F)$ such that for all $p \in P$, there are defined representations

$$\phi_p \colon \pi_1(M_K) \to \pi_1(M_K)/\pi_1(M_K)^{(2)} \to \Gamma = \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}] \rtimes \mathbb{Z},$$

which depend on p, such that the corresponding Cheeger-Gromov Von Neumann ρ-invariant (an $L^{(2)}$-signature defect) satisfies.

$$\rho(M_K, \phi_p) = 0 \in \mathbb{R}.$$

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An Extension

Second Order Algebraic Concordance

Theorem (P.)

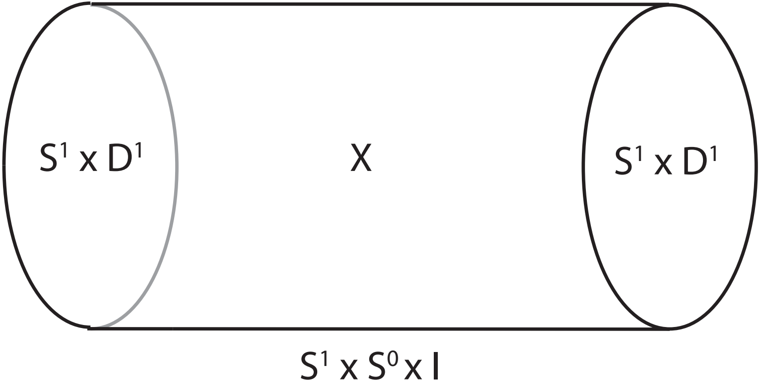
There exists an algebraically define group, $\mathcal{AC}_2$, which fits into the following diagram of groups:

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{AC}_2 \arrow[r, two heads] \arrow[d, dashed, "f"] & \mathcal{L} \\
\mathcal{C}/\mathcal{F}_{(1.5)} \arrow[ur] \arrow[r, dashed, "g"] & \mathcal{COT}_{(1.5)} &
\end{tikzcd}$$

where $\mathcal{COT}_{(1.5)}$ is a pointed set where the Cochran-Orr-Teichner obstructions live. f and g are only morphisms of pointed sets.

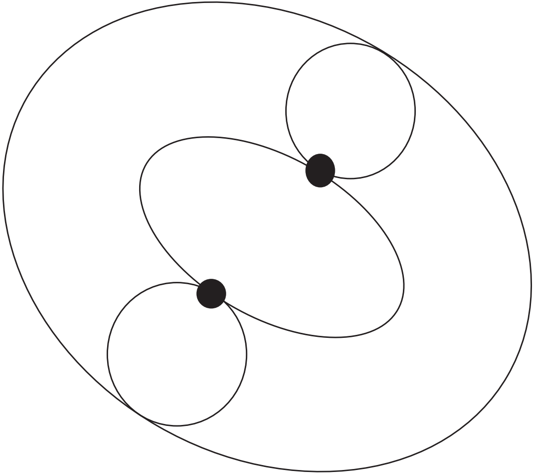
The Fundamental Cobordism

$$X := S^3 \setminus \nu K$$



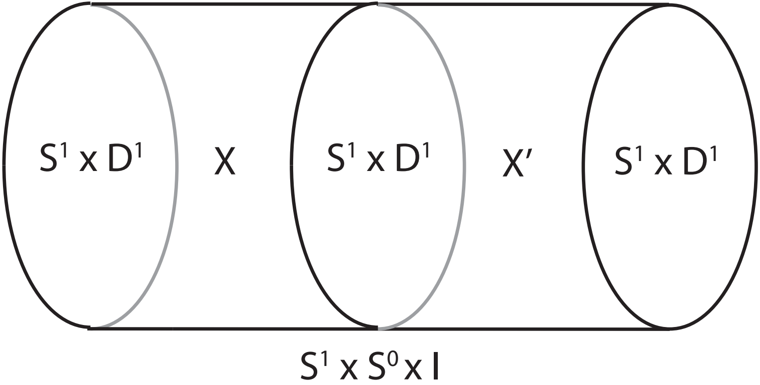
The Fundamental Cobordism

We consider the knot exterior as a $\mathbb{Z}$-homology cobordism from $S^1 \times D^1$ to itself, so we split the boundary torus $\partial X = S^1 \times S^1$, so the longitude is cut into two.



Adding Fundamental Cobordisms

This is very useful for adding knots together.



An Algebraically Defined Group

Pretend to forget about topology: we define a group of purely algebraic objects.

First, we define a monoid of chain complexes, and then take a quotient by an algebraic concordance relation.

A Monoid of Chain Complexes

Elements are chain equivalence classes of triples (H, C, ξ), where H is a $\mathbb{Z}[\mathbb{Z}]$-module, C is a 3-dimensional symmetric Poincaré triad:

$$\begin{tikzcd}
C_*(S^1 \times S^0; \mathbb{Z}[\mathbb{Z} \ltimes H]) \arrow[r, "i_-"] \arrow[d, "i_+"'] & C_*(S^1 \times D^1_-; \mathbb{Z}[\mathbb{Z} \ltimes H]) \arrow[d, "f_-"] \\
C_*(S^1 \times D^1_+; \mathbb{Z}[\mathbb{Z} \ltimes H]) \arrow[r, "f_+"] & Y.
\end{tikzcd}$$

such that

$$\mathrm{Id} \otimes f_\pm \colon C_*(S^1 \times D^1; \mathbb{Z}) \to \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z} \ltimes H]} Y$$

are isomorphisms of $\mathbb{Z}$-homology,
and

$$\xi \colon H \cong H_1(\mathbb{Z}[\mathbb{Z}] \otimes_{\mathbb{Z}[\mathbb{Z} \ltimes H]} Y)$$

is an isomorphism.

A Monoid of Chain Complexes

Given a knot K, with

$$X := S^3 \setminus \nu K,$$

define an element of our monoid by taking:

$$H := \frac{\pi_1(X)^{(1)}}{\pi_1(X)^{(2)}}$$

and

$$Y := C_*(X; \pi_1(X)/\pi_1(X)^{(2)}),$$

noting that:

$$\pi_1(X)/\pi_1(X)^{(2)} \cong \frac{\pi_1(X)}{\pi_1(X)^{(1)}} \ltimes \frac{\pi_1(X)^{(1)}}{\pi_1(X)^{(2)}} \cong \mathbb{Z} \ltimes H_1(X; \mathbb{Z}[\mathbb{Z}]).$$

Symmetric Structure

The Symmetric Structure is the chain level version of Poincaré duality; each chain complex C carries the extra structure of a chain map from $C^* \to C_*$:

$$\begin{tikzcd}
C^0 \arrow[r, "\delta_1"] \arrow[d, "\varphi_0"] & C^1 \arrow[r, "\delta_2"] \arrow[d, "\varphi_0"] & C^2 \arrow[r, "\delta_3"] \arrow[d, "\varphi_0"] & C^3 \arrow[d, "\varphi_0"] \\
C_3 \arrow[r, "\partial_3"'] & C_2 \arrow[r, "\partial_2"'] & C_1 \arrow[r, "\partial_1"'] & C_0
\end{tikzcd}$$

Chain level maps which induce the Poincaré duality isomorphisms.

The Symmetric Structure

Definition of cap product: the symmetric structure arises as the image of a fundamental class $[X, \partial X]$ under a chain level diagonal approximation map:

$$\Delta_0 \colon C_*(X) \to C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X})$$

$$\cong \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(C^{3-*}(\widetilde{X}), C_*(\widetilde{X})).$$

Adding Knots Algebraically

Tensor both triads with $\mathbb{Z}[\mathbb{Z} \ltimes (H \oplus H^\dagger)] = \mathbb{Z}[\mathbb{Z} \ltimes H^\ddagger]$ so all chain complexes are over the same ring.

$$\begin{tikzcd}
C(S^1 \times D^1_-) \arrow[d, "f_-"] & C(S^0 \times S^1) \arrow[l, "i_-"'] \arrow[r, "i_+^\dagger"] \arrow[d, "i_+"] & C(S^1 \times D^1_+)^\dagger \arrow[d, "f_+^\dagger"] \\
Y & C(S^1 \times D^1_+) \arrow[l, "f_+"'] \arrow[r, "f_-^\dagger"] & Y^\dagger
\end{tikzcd}$$

Glue by forming the mapping cone $\mathscr{C}(-f_+, f_-^\dagger)$.

Algebraic Concordance

An object is 2nd order algebraically null-concordant if there exists a $\mathbb{Z}[\mathbb{Z}]$-module H' with a homomorphism $H \to H'$ and a 4-dimensional chain complex V over $\mathbb{Z}[\mathbb{Z} \ltimes H']$ which fits into a 4-dimensional symmetric Poincaré triad:

$$\begin{tikzcd}
C(S^1 \times S^1; \mathbb{Z}[\mathbb{Z} \ltimes H']) \arrow[r, "f^U"] \arrow[d, "f"] & \mathbb{Z}[\mathbb{Z} \ltimes H'] \otimes Y^U \arrow[d, "j^U"] \\
\mathbb{Z}[\mathbb{Z} \ltimes H'] \otimes Y \arrow[r, "j"] & V
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\mathbb{Z}[\mathbb{Z} \ltimes H'] \otimes Y \arrow[r, "j"] & V
\end{tikzcd}$$

where Y^U corresponds to the unknot, satisfying:

1. $H_*(V; \mathbb{Z}) \cong H_*(S^1; \mathbb{Z})$; and
2. There exists an isomorphism $\xi' \colon H' \cong H_1(V; \mathbb{Z}[\mathbb{Z}])$.

Algebraic Concordance

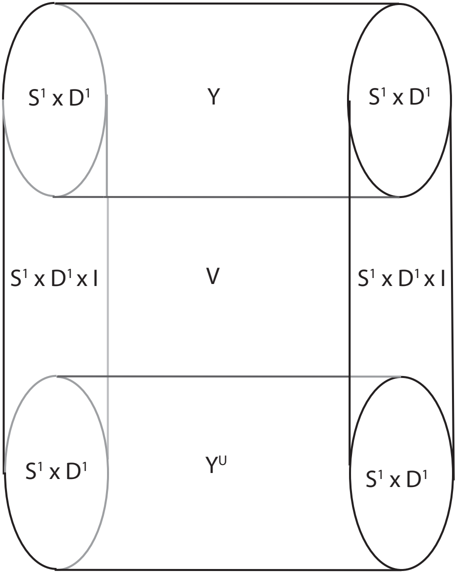
with a commutative diagram:

$$\begin{tikzcd}
H \arrow[rr] \arrow[d, "\cong"', "\xi"] & & H' \arrow[d, "\cong"', "\xi'"] \\
H_1(\mathbb{Z}[\mathbb{Z}] \otimes Y) \arrow[rr, "j_*"] & & H_1(\mathbb{Z}[\mathbb{Z}] \otimes V).
\end{tikzcd}$$

The duality information of the symmetric structure then limits the possible H' which can occur.

Algebraic Concordance

Think of V as $S^3 \times I$ minus a concordance $S^1 \times I$.



The Group $\mathcal{AC}_2$

Our *2nd order algebraic concordance group* $\mathcal{AC}_2$ is symmetric Poincaré triads modulo 2nd order algebraic concordance.

Concordance of knots modulo $\mathcal{F}_{(1.5)}$ is measured by $\mathbb{Z}$-homology cobordism of chain complexes.

Diagram of Obstructions

Recall our diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{AC}_2 \arrow[r, two heads] \arrow[d, dashed] & \mathcal{L} \\
\mathcal{C}/\mathcal{F}_{(1.5)} \arrow[ur] \arrow[r, dashed] & \mathcal{COT}_{(1.5)} &
\end{tikzcd}$$

Relation to Cochran-Orr-Teichner

Proposition

A (1.5)*-solvable knot is 2nd order algebraically concordant to the unknot.*

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Idea of proof:
The algebraic conditions on the intersection form of a (1.5)-solution 4-manifold W, with $\mathbb{Z}[\pi_1(W)/\pi_1(W)^{(2)}]$ coefficients, are that we have a basis of $H_2(W;\mathbb{Z})$ which lifts to a Lagrangian over

$$\mathbb{Z}[\pi_1(W)/\pi_1(W)^{(2)}] \cong \mathbb{Z}[\mathbb{Z} \ltimes H']$$

with a dual Lagrangian over

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$H_2(W;\mathbb{Z})$ looks spherical *algebraically*, so make V using algebraic surgery on the chain complex of W.

Relation to Levine

Proposition

There is a surjective homomorphism $\mathcal{AC}_2 \twoheadrightarrow \mathcal{L}$*.*

Idea of Proof:
Use representation $\mathbb{Z} \ltimes H \to \mathbb{Z}$: Chain complex over $\mathbb{Z}[\mathbb{Z}]$ with symmetric structure contains sufficient data to extract the Seifert Form.

Relation to Cochran-Orr-Teichner

Proposition

Let $\mathcal{Y} = 0 \in \mathcal{AC}_2$, *and let* $\Gamma = \mathbb{Z} \ltimes \mathbb{Q}(t)/\mathbb{Q}[t, t^{-1}]$ *be the Cochran-Orr-Teichner* (1)*-solvable group.*
Then there exists a set of zero linking curves $P \subset H_1(F; \mathbb{Z})$ *such that whenever we define a representation* $\phi_p \colon \mathbb{Z} \ltimes H \to \Gamma$ *using* $p \in P$, *the* ρ*-invariant (which can be defined algebraically)*

$$\rho(\mathcal{Y}, \phi_p) = 0.$$

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$$\rho(\mathcal{Y}, \phi_p) = 0.$$

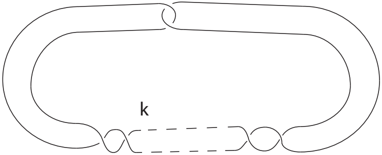
Idea of Proof:
Representation defined using $p \in P$ implies it extends over $\mathbb{Z} \ltimes H'$, so over the 4-dimensional complex V.

$$H_2(V; \mathbb{Z}) \cong 0,$$

so there is no intersection form, therefore $L^{(2)}$ and ordinary signatures vanish.

Example

Original Casson-Gordon example, can follow Cochran-Orr-Teichner proof of non-sliceness in the chain complex setting:
For example, $\mathcal{Y}$ = the symmetric Poincaré triad associated to the k-twist knot, $k \neq 0, 2$,



Then $\mathcal{Y} \neq 0 \in \mathcal{AC}_2$.

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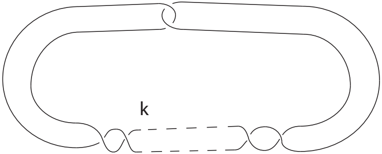

Then $\mathcal{Y} \neq 0 \in \mathcal{AC}_2$. Let p be zero linking curve on Seifert surface: a knot J.

$$\rho(\mathcal{Y}, \phi_p) = \rho(M_J, \psi \colon \pi_1(M_K) \to \mathbb{Z}).$$

For twist knots, J is a torus knot, and these have non-zero $L^{(2)}$-signatures:

$$\rho(M_J, \psi) = \int_{\omega \in S^1} \sigma_\omega.$$

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An Extension

Extension: work in progress with Kent Orr

Objects as symmetric Poincaré chain complexes C_* over $\mathbb{Z}[\pi]$, π finitely presented group.

Extra structure: isomorphisms

$$\xi_H \colon H_{ab} \xrightarrow{\simeq} H_1(C_* \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi/H])$$

for all $H \leq \pi$. A chain complex need not admit such isomorphisms in general.

Extension

(π, C_*, ξ) is null cobordant if there exists (Γ, D_*, ζ), Γ a finitely presented group, D_* a chain complex over $\mathbb{Z}[\Gamma]$,

$$\zeta_K \colon K_{ab} \xrightarrow{\simeq} H_1(D_* \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}[\Gamma/K])$$

for all $K \leq \Gamma$, with a homomorphism

$$\omega \colon \pi \to \Gamma$$

a symmetric Poincaré pair

$$j_* \colon C_* \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\Gamma] \to D_*$$

Extension

and a commutative square:

$$\begin{tikzcd}
\bigoplus_{\operatorname{im}\omega\backslash\Gamma/H}(\ker(\omega)\cap\phi^{-1}(H))_{ab} \arrow[r] \arrow[d] & H_{ab} \arrow[d] \\
\bigoplus_{\operatorname{im}\omega\backslash\Gamma/H} H_1(C_*\otimes\mathbb{Z}[\frac{\pi}{\ker(\omega)\cap\phi^{-1}(H)}]) \arrow[r] & H_1(D_*\otimes\mathbb{Z}[\Gamma/H])
\end{tikzcd}$$

for all subgroups $H \leq \Gamma$.

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We call ξ, ζ *Hurewicz structures*. They combine with duality structures to severely limit the possible Γ which can occur.

We can define corresponding surgery groups $L^n_H(G)$, which coincide with the usual L-groups in high dimensions but which have much more information in low dimensions.