ONE-SIDED HEEGAARD SPLITTINGS AND HOMEOTOPY GROUPS OF SOME 3-MANIFOLDS

J. H. RUBINSTEIN *and* J. S. BIRMAN

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1. *Introduction*

In this paper we compute the homeotopy groups of certain closed, orientable, irreducible 3-manifolds M which are non-Haken, i.e. do not contain any 2-sided incompressible surfaces. The homeotopy group $\mathscr{H}(M)$ is the quotient group of the group of all homeomorphisms from M to M modulo the normal subgroup of those homeomorphisms which are isotopic to the identity mapping of M.

The manifolds studied here all share a crucial property: each M contains a closed non-orientable and embedded surface K of non-orientable genus 3 *which is unique up to isotopy*, representing a fixed element of $H_2(M, Z_2)$. Also M admits a 1-sided Heegaard splitting $M = N(K) \cup Y$, where $N(K)$ is an orientable line bundle over K and Y is a handlebody of genus 2. The uniqueness of the surface K places a structure on M which is quite restrictive and it is this which allows the computation of $\mathscr{H}(M)$. Similar techniques have been used elsewhere [**1, 17**] in the case when K has non-orientable genus 2. However, we are only able to treat some cases when K has non-orientable genus 3, because we do not know whether the isotopy class of K is always unique (see [**18**]). The hypothesis 'K has genus 3' is essential to our techniques; it is quite unlikely that our methods generalize to genus greater than 3, although they probably can be applied to many other cases of genus 3.

The manifolds which we investigate include two classes.

(1) *The Seifert manifolds* $\{b; (o_1, 0); (2,1), (4k, 2k-1), (m,n)\}$ *where* $1 \leqslant k$, $0 < n < m$ *or* $(m,n) = (1,0)$. These manifolds are irreducible and non-Haken so long as the cases where $(m,n) = (4k, 1)$ and $b = -1$ are excluded. Our notation follows that on page 90 of [**13**]; however, we allow the case where $(m,n) = (1,0)$, understanding this to mean that M has two exceptional fibres. In that case M is a lens space $L(2,1)$ or $L(8pq+2\varepsilon,\ 4pq \pm 2p+\varepsilon)$ where $p, q \geqslant 1$ and $\varepsilon = \pm 1$ [**13**, pp. 99–100]. These are precisely the lens spaces which admit embeddings of K (see [**6**]). In terms of the Seifert invariants, $p = k$, $q = |b+1|$, and $\varepsilon = -\mathrm{sgn}(b+1)$. If $b = -1$ then $M = L(2,1)$. Note that there is a sign error in [**13**] on line 11 of page 100, i.e. $q = m\alpha_2 + n\beta_2$.

REMARK. The Seifert manifolds $\{b; (o_1, 0); (2,1), (4,1), (3,n)\}$, where $n = 1$ or 2, all have $\pi_1(M)$ equal to a finite group $O(48) \times Z_r$, where $O(48)$ is the binary octahedral group. These 3-manifolds are often called the binary octahedral spaces.

(2) *The* 3-*manifolds obtained by type*-$(2, 2N+1)$ *surgery on the complement of the figure*-8 *knot in* S^3. This knot complements fibres over S^1 with fibre a punctured torus. It is not difficult to see that type-$(2, 2N+1)$ surgery on any orientable once-punctured torus bundle over S^1 yields a 3-manifold M which admits a 1-sided

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Heegaard splitting of genus 3 (see [18]). Our method applies to all such 3-manifolds, and most of them are irreducible and non-Haken [17, 9]. In [21] it is proved that for $N \neq 0$, $(2, 2N+1)$-surgery on the figure-8 knot yields an irreducible, non-Haken and hyperbolic 3-manifold, whereas for $N = 0$ the surgery gives a Seifert manifold in Class (1) above.

The results of our work are summarized in:

MAIN THEOREM. I. *Let M be the lens space $L(8pq+2\varepsilon, 4pq \pm 2q+\varepsilon)$, where $\varepsilon = \pm 1$ and $p, q \geqslant 1$. Then $\mathscr{H}(M)$ is*

$$\begin{array}{ll} Z_2 & \text{if } p \neq q \text{ or } p = q = 1 \text{ and } \varepsilon = -1, \\ Z_2 \times Z_2 & \text{if } p = q \neq 1 \text{ and } \varepsilon = -1, \\ Z_4 & \text{if } p = q \text{ and } \varepsilon = 1. \end{array}$$

II. *Let M be a Seifert manifold of type*

$$\{b; (o_1, 0); (2, 1), (4k, 2k-1), (m, n)\}$$

where $1 \leqslant k$, $1 \leqslant n < m$, $b \in Z$, excluding the cases where $(m, n, b) = (4k, 1, -1)$. Then $\mathscr{H}(M)$ is

$$\begin{array}{ll} Z_2 \times Z_2 & \text{if } (m, n) = (4k, 2k-1), \text{ or } (m, n) = (2, 1) \text{ and } (k, b) \neq (1, -1), \\ Z_2 & \text{if } (m, n, k, b) = (2, 1, 1, -1), \text{ or} \\ & (m, n) \neq (2, 1), (4k, 2k-1) \text{ and } (m, n, k, b) \neq (3, 1, 1, -1), \\ \{1\} & \text{if } (m, n, k, b) = (3, 1, 1, -1). \end{array}$$

III. *Let M be the manifold obtained by type-$(2, 2N+1)$ surgery on the figure-8 knot, where $N \neq 0$. Then $\mathscr{H}(M)$ is*

$$Z_2 \times Z_2.$$

We now summarize our results as they relate to the connection between $\mathscr{H}(M)$ and $\operatorname{Out} \pi_1(M)$. There is a natural homomorphism θ from $\mathscr{H}(M)$ to $\operatorname{Out} \pi_1(M)$, given by assigning to a homeomorphism $h: M \to M$ the induced automorphism h_* of $\pi_1(M)$, and noting that if h is isotopic to the identity then h_* is an inner automorphism. For all our examples, we will find that θ is one-to-one, that is, if $h: M \to M$ is a homeomorphism such that h is homotopic to the identity, then h is isotopic to the identity. This result has been established by Waldhausen [22] whenever M is Haken.

Also we obtain that θ is surjective for those examples with $\pi_1(M)$ infinite. Equivalently, since then M is a $K(\pi, 1)$ space, if $f: M \to M$ is any homotopy equivalence then f is homotopic to a homeomorphism. Again Waldhausen has obtained this result when M is Haken [22]. Note that for a hyperbolic 3-manifold M, by Mostow's rigidity theorem [12] any homotopy equivalence is homotopic to an isometry. Also it is proved elsewhere (see [14, 15], and, especially, § 8 of [8]) that for a Seifert fibre space M, any homotopy equivalence $f: M \to M$ is homotopic to a fibre-preserving homeomorphism.

Our examples with $\pi_1(M)$ finite are all Seifert manifolds. For M Seifert fibred, both $\operatorname{Out} \pi_1(M)$ and the quotient group of fibre-preserving homeomorphisms of M, modulo the normal subgroup of those which are fibre-preserving isotopic to the identity, are computed in [8]. The latter quotient group is isomorphic to $\theta(\mathscr{H}(M))$.

We now summarize the method. Let G be the subgroup of $\mathscr{H}(M)$ containing isotopy classes of orientation-preserving homeomorphisms $h: M \to M$ such that the induced map $h_{\#}: H_1(M, Z_2) \to H_1(M, Z_2)$ is the identity in $\mathscr{H}(M)$. By § 7 of [**18**], if M is in Classes (1) or (2), then any isotopy class in G contains homeomorphisms h for which $h(K) = K$, where K is the embedded non-orientable surface of genus 3. Conversely, since the rank of $H_1(M, Z_2) \leqslant 2$ for M in Classes (1) or (2), it can be checked that any homeomorphism $h: M \to M$ which takes K to K is such that $h_{\#}$ is the identity. Let $\mathscr{H}(K)$ be the homeotopy group of K and let G_1 be the subgroup of $\mathscr{H}(K)$ consisting of homeomorphisms $f: K \to K$ which extend to orientation-preserving homeomorphisms $h: M \to M$. Then the map sending the isotopy class of f to that of h gives an epimorphism $\Psi: G_1 \to G$. The kernel G_2 of Ψ is the set of isotopy classes of homeomorphisms $f: K \to K$ which extend to homeomorphisms $h: M \to M$ such that h is isotopic to the identity $M \to M$.

Our main task is to compute G_1 and G_2, since it is easy to find $\mathscr{H}(M)$ knowing G, and $G \cong G_1/G_2$. By [3] (see Lemma 3 and Corollary 4 of [**18**]), there is a *unique* simple closed curve C in K up to isotopy with $K - C$ orientable, and $\mathscr{H}(K)$ is isomorphic to $\mathrm{GL}(2, Z)$, by the map which takes the isotopy class of f to $f_{\#}$. These two facts, namely that C is unique up to isotopy, and that $\mathscr{H}(K)$ is isomorphic to $\mathrm{GL}(2, Z)$, are the essential results that make our calculation of $\mathscr{H}(M)$ possible.

We now give an outline of the paper. In § 2 we give an explicit picture of the Seifert manifolds M in Class (1), in terms of 1-sided decompositions $M = N(K) \cup Y$. The idea of a 1-sided Heegaard diagram for a 3-manifold is introduced, which is the analogue of a Heegaard diagram, and explicit 1-sided Heegaard diagrams are determined for M in (1).

In § 3 Class (1) is subdivided into subclasses (1a) and (1b). The manifold M is in (1a) if $M - C$ is *not* fibred over S^1 with fibre the open punctured torus $K - C$, and in (1b) if $M - C$ is a fibre bundle with base S^1 and fibre $K - C$.

In § 4, $\mathscr{H}(M)$ is computed for M in (1a). The crucial step is to find a 'special' disk D properly embedded in the handlebody Y in the 1-sided Heegaard splitting $M = N(K) \cup Y$, and having the property that if $h: M \to M$ is a homeomorphism which respects the splitting, that is, takes $N(K)$ to $N(K)$ and Y to Y, then $h(D)$ and D are *always* isotopic in Y. Consequently, G_1 can be found by identifying the isotopy classes of homeomorphisms $f: K \to K$ which extend to homeomorphisms $\bar{f}: N(K) \to N(K)$ such that $\bar{f}(\partial D)$ and ∂D are homotopic loops on the surface $\partial N(K) = \partial Y$. It turns out that $G_2 = \{1\}$ or Z_2 and so G is isomorphic to either G_1 or G_1/Z_2 for M in (1a).

In § 5 $\mathscr{H}(M)$ is calculated for M in Classes (1b) and (2). The method is applicable to any $(2, 2N+1)$-surgery on a punctured torus bundle over S^1. By [3] (see Lemma 3 of [**18**]), a homeomorphism $h: M \to M$ with $h(K) = K$ can be isotoped until *in addition* $h(C) = C$. Since $M - C$ is fibred over S^1, the restriction of h to $M - C$ is a self-homeomorphism of a fibre bundle and by [**22**], $\mathscr{H}(M - C)$ can be easily computed. Let G' denote the subgroup of $\mathscr{H}(M - C)$ consisting of isotopy classes of homeomorphisms which extend to homeomorphisms of M. Let $\Psi': G' \to \mathscr{H}(M)$ be the map induced by extension of homeomorphisms. Then $\operatorname{Im} \Psi' = G$ and $\ker \Psi'$ can be readily calculated. So G and $\mathscr{H}(M)$ can be found. Note that G_2 is *infinite* in this case, contrary to the situation when M is in (1a).

The Appendix concerns the relationship between branched covering spaces and 1-sided Heegaard splittings of genus 3, and gives general ways to find all 3-manifolds which admit such splittings.

Recently, there have been two other independent calculations of the homeotopy groups $\mathscr{H}(M)$ for all the 3-dimensional lens spaces M (see [**5, 11**]). The methods involve two different usages of height functions on M, and are special to lens spaces. Another related paper which treats certain lens spaces is [**2**].

In [**10**], the problem of computing $\mathscr{H}(M)$ for 3-manifolds M with $\pi_1 M$ finite has been related to the existence of $\frac{1}{2}$ spin solutions for a quantum theory for gravity. We would like to thank J. Friedman and D. Witt for a helpful communication regarding $S^3/O(48)$. We would also like to thank the referee for extremely thorough and helpful suggestions.

2. *One-sided Heegaard diagrams*

Let M be a closed orientable 3-manifold with a 1-sided Heegaard splitting of genus g, $M = N(K) \cup Y$. Let L be a closed orientable surface, and let $j: L \to L$ be an orientation-reversing involution such that the orbit space $L/j \cong K$. Then $N(K)$ can be identified with the mapping cylinder of a free orientation-reversing involution j on L via the projection $\pi: L \times I \to N(K)$. Let $\{D_i: 1 \leqslant i \leqslant g\}$ be a complete system of meridian disks for the handlebody Y. Let j' be the involution on $L \times I$ given by $j'(x, t) = (jx, t)$. By an abuse of notation, we will denote $\partial N(K)$ by L and the involution $\pi \cdot j' \cdot \pi^{-1}|_L$ by j.

DEFINITION. A 1-*sided Heegaard diagram* for M (associated with the 1-sided splitting $M = N(K) \cup Y$) is given by the $(g+2)$-tuple $(L, j, \partial D_1 \dots \partial D_g)$. Note that given a $(g+2)$-tuple consisting of a closed, orientable surface L, a free orientation-reversing involution j of L, and g disjoint simple closed curves $C_1, \dots, C_g$ in L such that $L - \bigcup_i C_i$ is planar, a 3-manifold M with a 1-sided Heegaard decomposition $M = N(K) \cup Y$, which has $(L, j, C_1, \dots, C_g)$ as a 1-sided Heegaard diagram, can be uniquely constructed by gluing Y to $N(K)$ via a homeomorphism $\varphi: \partial Y \to L$ so that $\varphi(\partial D_i) = C_i$ for all i.

The purpose of this section is to describe convenient 1-sided Heegaard diagrams for the manifolds in Class (1).

If C is a loop in L, then its homology class in $H_1(L; Z)$ will be denoted by $[C]$. Let A, B, jA, jB be the curves illustrated in Fig. 1.

PROPOSITION 1. *Each Seifert manifold with invariants*

$$\{b; (o_1, 0), (2, 1), (4, 2k-1), (m, n)\},$$

where b is an arbitrary integer, $k \geqslant 1$, $0 < n < m$, *or* $b \neq -1$, $k \geqslant 1$ *and* $(m, n) = (1, 0)$, *may also be described by invariants* $\{-1; (o_1, 0), (2, 1), (4k, 2k-1), (m, n)\}$ *where* $k \geqslant 1$, $m \geqslant 1$ *and n is a non-zero integer. With the latter description, M has a 1-sided Heegaard diagram* $(L, j, \partial D_1, \partial D_2)$ *with*

$$[\partial D_1] = k[A] + [B] + k[jA],$$

$$[\partial D_2] = n[B] + m[jA] + n[jB].$$

The types of diagrams we will be using are illustrated in Fig. 2. Note that some of the later arguments will need diagrams as in Fig. 2, not just the homological data of Proposition 1.

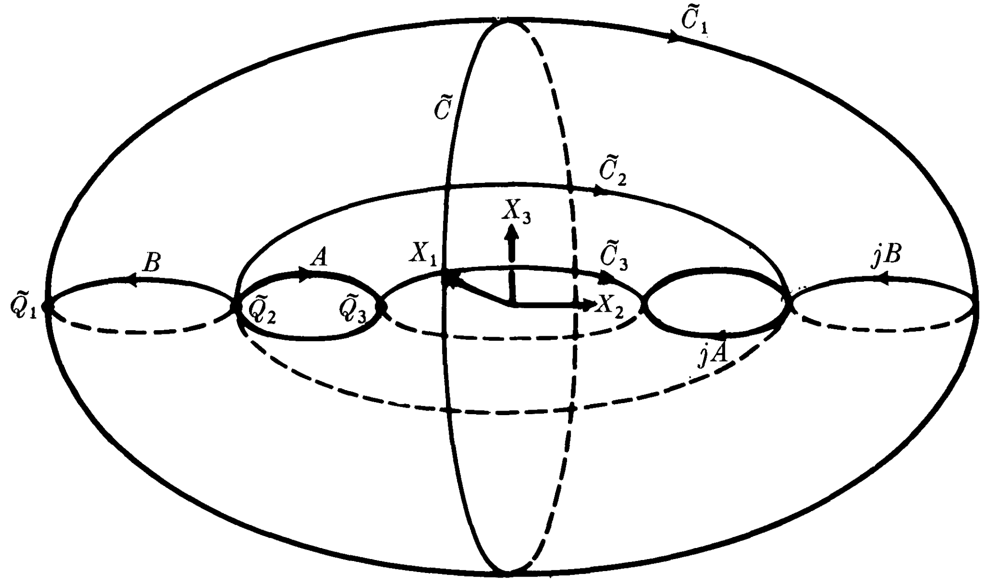
$\tilde{C}_1$
$\tilde{C}$
$\tilde{C}_2$
X_3
B
A
X_1
$\tilde{C}_3$
jB
$\tilde{Q}_1$
$\tilde{Q}_2$
$\tilde{Q}_3$
X_2
jA

FIG. 1

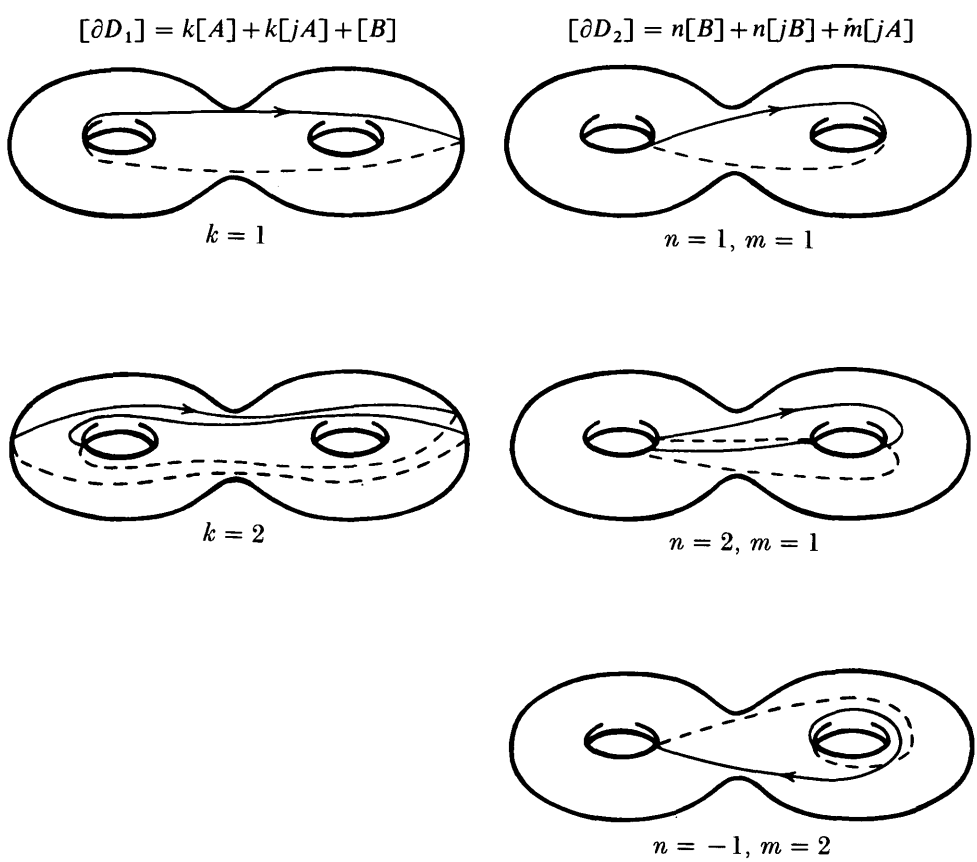
$[\partial D_1] = k[A]+k[jA]+[B]$
$[\partial D_2] = n[B]+n[jB]+m[jA]$
$k = 1$
$n = 1,\ m = 1$
$k = 2$
$n = 2,\ m = 1$
$n = -1,\ m = 2$

FIG. 2

Proof. Let N_1 and N_2 be small fibred neighbourhoods of the exceptional fibres of multiplicity 2 and $4k$ respectively. It can readily be seen that an ordinary fibre in ∂N_1 bounds a Mobius band J in N_1.

If a solid torus N is glued to N_2 so that a meridian curve for N is matched up with an ordinary fibre in ∂N_2, then the result is the lens space $L(4k, 2k-1)$. In [17] it is shown that $L(4k, 2k-1)$ has a 1-sided splitting of genus 2, $L(4k, 2k-1) = N(K_1) \cup Y_1$, and the Klein bottle K_1 can be chosen to meet N in a single meridian disk. Consequently, $K_1 \cap N_2$ is a punctured Klein bottle K_0.

An annulus A of ordinary fibres in $M - \operatorname{int} N_1 - \operatorname{int} N_2$ can be found so that $\partial A = \partial J \cup \partial K_0$. Then the non-orientable surface of genus 3 embedded in M is $K = J \cup A \cup K_0$.

We may identify $N(K_1)$ with the mapping cylinder of a free orientation-reversing involution j_1 on the torus $T = \partial N(K_1)$. Let C, C' be simple closed curves on T which intersect transversely at a single point such that $j_{1\#}[C] = -[C]$ and $j_{1\#}[C'] = [C']$. If D_1 is a meridian disk for Y_1, then $[\partial D_1] = m[C] + n[C']$ for relatively prime integers m, n. Also $\pi_1(K_1) = \langle x, y \colon y^{-1}xy = x^{-1} \rangle$, where C, C' have homotopy class x, y^2 respectively. Consequently,

$$\pi_1(N(K_1) \cup Y_1) = \langle x, y \colon y^{-1}xy = x^{-1}, y^{2n}x^m = 1 \rangle \cong Z_{4k}$$

if and only if $m = 1$ and $n = k$, for suitable orientation of C, C'.

In [17] it is proved that the solid torus N can be chosen to miss D_1. Consequently, D_1 lies in N_2 and can be selected as the first of a system of meridian disks $\{D_1, D_2\}$ for the handlebody Y in the splitting $M = N(K) \cup Y$.

The three curves $\tilde{C}_i$, with $1 \leqslant i \leqslant 3$, drawn in Fig. 1 are all j-invariant, and so if $\pi\colon L \to K$ is the covering projection with covering transformation j then the three curves $C_i = \pi(\tilde{C}_i)$ are the centres of three Mobius bands in K. We will assume that the punctured Klein bottle K_0 is $K - \operatorname{int} N(C_3)$, where $J = N(C_3)$ is a small Mobius band with centre C_3. Let $N(\tilde{C}_3)$ be the preimage of $N(C_3)$ in L. Then T can be regarded as $L - \operatorname{int} N(\tilde{C}_3)$ with two disks attached along the boundary curves. The curves C, C' in T can be chosen to be B and $\tilde{C}_1$ without loss of generality, so that

$$[\partial D_1] = [C] + k[C'] = [B] + k[\tilde{C}_1] = [B] + k[A] + k[jA] \subset L - \operatorname{int} N(\tilde{C}_3).$$

Note that ∂D_1 intersects $\tilde{C}_1$ and also $\tilde{C}_2$ transversely in a single point. (See Fig. 2.)

If D_2 is the second meridian disk for Y, then $\partial D_2 \cap \partial D_1 = \varnothing$, but $\partial D_2 \cap \tilde{C}_1$ is in general non-empty. However, by connecting D_2 with copies of D_1 by strips along arcs of $\tilde{C}_1$ we can obtain a different choice of meridian disk D_2 with $\partial D_2 \cap \tilde{C}_1 = \varnothing$. Consequently, ∂D_2 can be taken to be a curve in L as in the examples of Fig. 2, since $\partial D_2 \cap \partial D_1 = \varnothing$ and $\partial D_2 \cap \tilde{C}_1 = \varnothing$.

As C_3 is the centre of the Mobius band J in the complement of K_0 in K, C_3 can be regarded as the exceptional fibre of multiplicity 2 in M. Therefore $\tilde{C}_3$ is an ordinary fibre of M, without loss of generality. Let $N(K_0)$ be a regular neighbourhood of K_0 in M which is chosen so that $\partial N(K_0) \cap L = L - \operatorname{int} N(\tilde{C}_3)$. It is not difficult to check that if $N(D_1)$ is a small regular neighbourhood of D_1 in Y, then $N(D_1) \cup N(K_0)$ is a solid torus. Consequently, we may as well assume that $N_2 = N(D_1) \cup N(K_0)$. In addition, because C_3 is the exceptional fibre of multiplicity 2, the solid torus $N_3 = M - \operatorname{int}(N(D_1) \cup N(K)) = \operatorname{cl}(Y - N(D_1))$ is a regular neighbourhood of the other exceptional fibre of multiplicity m.

The disk D_2 is a meridian disk for N_3, as well as for Y. We will not be adopting the usual normalization of Seifert invariants (i.e. $0 < n < m$, as in [13]). First, the curve

jA in Fig. 1 can be chosen as a cross-section to the ordinary fibre $\tilde{C}_3$ in ∂N_3. With this choice, the 'invariants' of the exceptional fibre at the core of N_3 are given by integers (m, n), where $[\partial D_2] = m[jA] + n[\tilde{C}_3] = n[B] + m[jA] + n[jB]$. Note that k, m can be assumed to be positive integers, but n is an arbitrary (non-zero) integer.

Finally, we compute the value of the b invariant of the Seifert fibring, given the above choices of $\partial D_1, \partial D_2$ and k, m, n. A 3-manifold $M'' = N(K) \cup Y$ can be constructed so that Y has a set of meridian disks D'_1, D'_2 with boundary curves on L given by $\partial D'_1 = \partial D_1$ (as for M) and $\partial D'_2 = jA$. From the previous discussion, it follows immediately that M'' has Seifert invariants $\{b; (o_1, 0); (2, 1), (4k, 2k-1)\}$ where the value of b is the same for M'' as for M.

It is easy to check directly that $M'' = RP^3$. Alternatively, since $j(\partial D'_2) = A$ and $\partial D'_1$ cross transversely at a single point, it follows that the 2-fold cover $\tilde{M}''$ has a genus-2 Heegaard splitting with two pairs of cancelling meridian disks and so is S^3. A simple calculation shows that $b = -1$ is the only value for which $H_1(M'') = Z_2$. Consequently, M has Seifert invariants $\{-1; (o_1, 0); (2, 1), (4k, 2k-1), (m, n)\}$, where $k \geqslant 1$, $m \geqslant 1$, and $n \in Z$. This completes the proof of Proposition 1.

REMARK. If M has Seifert invariants $\{b; (o_1, 0); (2, 1), (4k, 2k-1), (m, n)\}$ where $k \geqslant 1$, $0 < n < m$, or $(m, n) = (1, 0)$, then M has 'unnormalized' Seifert invariants $\{-1; (o_1, 0); (2, 1), (4k, 2k-1), (m, m(b+1)+n)\}$. This can be easily checked by computing $|H_1(M, Z)|$ (see [13, p. 101]).

3. *Fibred knots in Seifert manifolds*

In this section, we determine which of the Seifert manifolds M in Class (1) have the property that C is a fibred knot in M.

LEMMA 2. *Let M be a Seifert manifold $\{-1; (o_1, 0); (2, 1), (4k, 2k-1), (m, n)\}$. The following are equivalent:*

(a) *$M - C$ is fibred over S^1 with fibre $K - C$;*

(b) *the meridian disks D_1, D_2 can be chosen so that ∂D_1 and ∂D_2 both cross $\tilde{C}$ in two points;*

(c) *$k = 1$ and $n = \pm 1$.*

Proof. It is clear that (c) implies (b). To see that (b) implies (a), suppose that ∂D_i intersects $\tilde{C}$ in two points, for $i = 1, 2$. If L_1, L_2 are the closures of the components of $L - \tilde{C}$, then $\partial D_i \cap L_j$ is a single arc for $i, j = 1, 2$. Hence L_1 and L_2 are parallel surfaces in Y, since ∂D_i gives an isotopy between an arc of L_1 and an arc of L_2, for $i = 1, 2$. Since $\operatorname{int} L_1$ and $\operatorname{int} L_2$ both project homeomorphically onto $K - C$, it follows that $M - C$ is a punctured torus bundle over S^1 with fibre $K - C$.

Finally, we show that (a) implies (c). If C is a fibred knot in M, then there is a system of meridian disks $\{\bar{D}_1, \bar{D}_2\}$ for Y with $\partial \bar{D}_1$ and $\partial \bar{D}_2$ both crossing $\tilde{C}$ in two points. Let $[\partial \bar{D}_1] = a[A] + b[B] + x[jA] + y[jB]$ and $[\partial \bar{D}_2] = c[A] + d[B] + u[jA] + v[jB]$. Assume that A, B are included in L_1. Since $L_1 - \partial \bar{D}_1 - \partial \bar{D}_2$ is simply connected and $L_1 \cap \partial \bar{D}_1$ is a single arc, for $i = 1, 2$, it follows that $\{a[A] + b[B], c[A] + d[B]\}$ and $\{[A], [B]\}$ generate the same subgroup of $H_1(L, Z)$. Therefore the subgroup G generated by $\{[\partial \bar{D}_1], [\partial \bar{D}_2]\}$ contains elements of the form

$$\alpha[A] + \beta[B] + \gamma[jA] + \delta[jB],$$

where (α, β) takes on all possible values in $Z \times Z$.

On the other hand, $\{[\partial D_1], [\partial D_2]\}$ also is a basis for G, and

$$[\partial D_1] = k[A]+[B]+k[jA], \quad [\partial D_2] = n[B]+m[jA]+n[jB].$$

So we must have

$$\det\begin{pmatrix} k & 0 \\ 1 & n \end{pmatrix} = \pm 1,$$

to obtain all possible integral values for the coefficients of $[A]$ and $[B]$. Consequently, $k = 1$ and $n = \pm 1$.

4. *Computation of $\mathcal{H}(M)$, when $M-C$ is not fibred*

Throughout this section, M will denote a Seifert manifold in Class (1a), that is, C is not a fibred knot in M. Therefore, by Lemma 2, M has Seifert invariants

$$\{-1;(o_1,0);(2,1),\ (4k,2k-1),\ (m,n)\},$$

where $1 \leqslant k$, $1 \leqslant m$, $n \neq 0$ with $(k,n) \neq (1, \pm 1)$. Let f be a homeomorphism of K. After an isotopy, it can be supposed that $f(C) = C$. Any extension of f to a homeomorphism $\bar{f}$: $N(K) \to N(K)$ can be chosen so that $\bar{f}(\tilde{C}) = \tilde{C}$ (there are two such extensions up to isotopy). If $\bar{f}$ extends to a homeomorphism h: $M \to M$, then h maps Y to Y and $\tilde{C}$ to $\tilde{C}$.

In Fig. 3, a separating simple closed curve C_0 in L is drawn which misses the loops ∂D_1 and ∂D_2 (see Fig. 2) but meets $\tilde{C}$ in exactly four points, for every choice of M in Class (1). It is easy to show (e.g. by Dehn's Lemma) that $C_0 = \partial D_0$ where D_0 is a separating meridian disk for Y, disjoint from D_1 and D_2. Note that $h(D_0)$ will be another separating meridian disk for Y with boundary curve meeting $\tilde{C}$ in four points.

Let $H(M) = \{h$: $M \to M$ such that h is a homeomorphism, $h(Y) = Y, h(\tilde{C}) = \tilde{C}\}$. As noted in § 1, the results in [**18**] show that for the manifolds studied in this paper $H(M)$ contains a representative from each isotopy class in G.

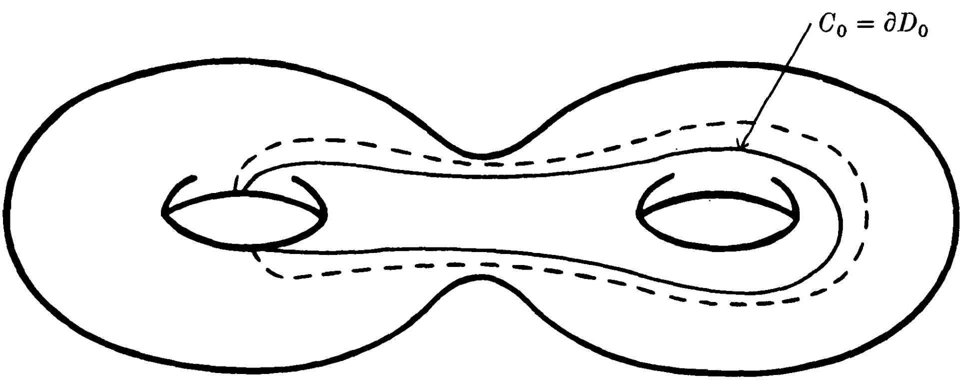

FIG. 3

LEMMA 3. *If M is in Class* (1a), *then one of the following mutually exclusive conditions holds:*

(a) *$h(D_0) \simeq D_0$ for all $h \in H(M)$;*

(b) *one of the meridian disks D_i for Y meets $\tilde{C}$ in two points and satisfies $h(D_i) \simeq D_i$ for all $h \in H(M)$.*

Proof. We assume that (a) does not hold, i.e. there exists some $h \in H(M)$ such that $h(D_0) \not\simeq D_0$. We first claim that in this situation there is a properly embedded disk in Y with its boundary not homotopically trivial, which is disjoint from D_0 and meets $\tilde{C}$ twice.

Let D' denote any disk which is isotopic to $h(D_0)$. We can assume that D' is chosen so that it intersects D_0 at most in arcs, and also that the number of such arcs is minimal for all $D' \simeq h(D_0)$. If $D' \cap D_0 = \varnothing$ then clearly D' and $h(D_0)$ are isotopic to D_0, because D_0 is separating and the genus of Y is 2. This contradicts the hypothesis $h(D_0) \not\simeq D_0$. Also the four points of $\tilde{C} \cap D_0$ can be selected to be h-invariant, since $h(\tilde{C}) = \tilde{C}$. Hence $h(D_0) \cap \tilde{C} = D_0 \cap \tilde{C}$. The isotopy which takes $h(D_0)$ to D', decreasing the number of arcs of intersection with D_0, can then be chosen so that $D' \cap \tilde{C} = D_0 \cap \tilde{C}$. Let λ be the interior of an arc of $D' \cap D_0$ which is outermost with respect to D', that is, there is a subdisk D of D' with $\partial D = \bar{\lambda} \cup \bar{\mu}$, where $\mu \subset \partial D'$ and $\mu \cap \partial D_0 = \varnothing$. (Notice that our arcs are all open; a bar denotes closure.)

Let N be a regular neighbourhood of $D_0 \cup D$ in Y. Then ∂N has three components, one of which is parallel to D_0. Let D_α, D_β be the other two. Note that D_α and D_β are both disjoint from D_0 by construction. Suppose that ∂D_α (or ∂D_β) is isotopic to a curve which does not meet $\tilde{C}$. Then ∂D_α projects one-to-one to a simple closed curve on K, and since K is incompressible this curve bounds a disk in L. The latter disk may be used to define an isotopy of $\partial D'$ in L, and hence an isotopy of D' in Y to remove the component λ of $D' \cap D_0$. However, by our hypothesis $|D' \cap D_0|$ is minimal, so this cannot occur. So both ∂D_α and ∂D_β intersect $\tilde{C}$ non-trivially. Since $\tilde{C}$ is separating, each intersects $\tilde{C}$ in an even number of points, and since $|\partial D_0 \cap \tilde{C}| = 4$ and $\mu \cap \tilde{C} = \varnothing$, each intersects $\tilde{C}$ twice. So either D_α or D_β, say D_α, can be chosen to be the disk we need.

We now claim that D_α is isotopic to D_1 or D_2 in Y. To see this, note that Y split along D_0 is a union of two solid tori, and D_α is in one of them. Since ∂D_α is not homotopically trivial, D_α must be a meridian disk for the solid torus. So D_α is non-separating in Y. Since a solid torus has unique meridian disks, and D_1 and D_2 are meridian disks for Y which are disjoint from D_0, it follows that D_α is isotopic to D_1 or D_2. Thus we have shown that if (a) does not hold, then either ∂D_1 or ∂D_2 can be assumed to meet $\tilde{C}$ in two points. Let $D_i = D_1$ or D_2, where $\partial D_i \cap \tilde{C}$ is two points.

It remains to prove that in this situation $h(D_i)$ is isotopic to D_i for every $h \in H(M)$. Suppose that this is not so. Then there exists some $h \in H(M)$ with $h(D_i)$ not isotopic to D_i. We may assume that D' meets D_i transversely in arcs only, with $|D' \cap D_i|$ minimal, for all disks D' isotopic to $h(D_i)$. By an outermost disk argument, it follows that D' is disjoint from D_i and hence that D' and $h(D_i)$ are isotopic to D_i, contrary to our hypothesis. This completes the proof of Lemma 3.

We are now ready to compute G_1, for M in Class (1a). Let $f: K \to K$ be a homeomorphism, and let $\tilde{f}, j\tilde{f}$ denote lifts to homeomorphisms of L. A basis $\mathcal{B}$ of the free abelian group $H_1(Z)$ is given by $\{[A], [B], [jA], [jB]\}$, where A, B are the curves in L as shown in Fig. 1. In terms of this basis, $j_\#$ has matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where I is a 2×2 identity matrix. We assume, without loss of generality, that the matrix of $\tilde{f}_\#$ relative to $\mathcal{B}$ is $\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$, where $F \in \mathrm{GL}(2, Z)$, because $\tilde{f}$ preserves $\tilde{C}$.

REMARK. The basis $\mathscr{B}$ is not a symplectic basis for $H_1(L, Z)$.

Case (A). Suppose that (a) but not (b) in Lemma 3 holds. So neither ∂D_1 nor ∂D_2 meets $\tilde{C}$ in two points. Clearly then $k \neq 1$ and $n \neq \pm 1$. There are two ways for a homeomorphism $f\colon K \to K$ to extend to a homeomorphism $h\colon M \to M$, where $\bar{f}$ denotes $\bar{f}$ restricted to $L = \partial N(K)$. (Note that $\bar{f}$ is an extension of f to $N(K)$.)

(i) Assume that $h(D_i)$ is isotopic to D_i for $i = 1, 2$, that is, h preserves the handles of Y on either side of D_0. If $\bar{f}_\sharp[\partial D_1] = \pm[\partial D_1]$, then

$$\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}(k \;\; 1 \;\; k \;\; 0)' = \pm(k \;\; 1 \;\; k \;\; 0)',$$

where $(a \;\; b \;\; c \;\; d)'$ denotes the transpose of $(a \;\; b \;\; c \;\; d)$. This gives $F = \pm I$, since $k \neq 0$. If $j_\sharp \bar{f}_\sharp[\partial D_1] = \pm[\partial D_1]$ then

$$\begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}(k \;\; 1 \;\; k \;\; 0)' = \pm(k \;\; 1 \;\; k \;\; 0)'.$$

This implies that $k = \pm 1$, contrary to our assumption, and there are no solutions of this type. Hence $f_\sharp = \pm I$ are the only solutions.

(ii) Suppose that $h(D_1)$ is isotopic to D_2 and $h(D_2)$ is isotopic to D_1, that is, h interchanges the handles of Y on either side of D_0. If $\bar{f}_\sharp[\partial D_1] = \pm[\partial D_2]$ then

$$\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}(k \;\; 1 \;\; k \;\; 0)' = \pm(0 \;\; n \;\; m \;\; n)'.$$

Again it follows that $k = \pm 1$ as m, n are relatively prime, which is a contradiction. So there are no solutions for f. If $j_\sharp \bar{f}_\sharp[\partial D_1] = \pm[\partial D_2]$ then

$$\begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}(k \;\; 1 \;\; k \;\; 0)' = \pm(0 \;\; n \;\; m \;\; n)'.$$

Let $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $a = d = 0$, $m = 1$, $k = \pm n$ and there are two solutions with $b = \pm 1$, $c = \pm 1$.

We conclude that G_1 has *at most* two elements unless $m = 1$ and $n = \pm k$, in which case G_1 has *at most* four elements if $n = +k$ and G_1 has *at most* two elements if $n = -k$ (G_1 has only orientation-preserving homeomorphisms).

We must realize the above as homeomorphisms. In Case (A(i)), the homeomorphism h which induces $-I$ on $H_1(L)$ is the covering transformation for the projection of M to S^3, which is a 2-fold covering branched over a link (see Theorem 4). Note that the involution gives a visible symmetry in the pictures of ∂D_1 and ∂D_2 in Fig. 2. If $m \neq 2$, then $\pi_1(M)$ has a cyclic centre with more than two elements and h sends a generator of the centre to its inverse. (See [**13**, Chapter 5].) Note that $\tilde{C}_3$ is an ordinary fibre of M and so gives an element of the centre. Hence h cannot be homotopic to the identity. If $m = 2$ then $H_1(M, Z)$ contains a cyclic subgroup with order greater than 2 and the same argument applies to show there is a no homotopy between h and the identity. In Case (A(ii)) M is a lens space with Seifert invariants $\{-1; (o_1, 0), (2, 1), (4k, 2k-1), (1, \pm k)\}$. Then $M = L(8k^2+2\varepsilon, 4k^2+2k+\varepsilon)$, where $\varepsilon = \pm 1 = \operatorname{sgn}(-n)$ (see [**13**, pp. 99–100] with the correct sign on p. 100). These are precisely the lens spaces $L(p, q)$ in Class (1a) for which $q^2 \equiv \pm 1 \bmod p$. As above, G_1 has at most four elements if $\varepsilon = -1$ and at most two elements if $\varepsilon = +1$. If $\varepsilon = -1$

then $q^2 \equiv +1 \bmod p$ and there are standard fibre-preserving homeomorphisms of such lens spaces giving $G_1 \cong Z_2 \times Z_2$. On the other hand, if $\varepsilon = +1$ then $q^2 \equiv -1 \bmod p$ and there is a standard orientation-reversing homeomorphism whose square represents the non-zero element of $G_1 \cong Z_2$. Since $q \neq 1$ ($L(6,1)$ is excluded) all these homeomorphisms give outer automorphisms of $\pi_1(M)$.

Note. For all our examples so far $G_2 = \{1\}$.

Case (B). Assume that (b) in Lemma 3 holds. There are two possibilities to consider.

(a) Suppose that D_1 meets $\tilde{C}$ in two points and is h-invariant up to isotopy. Then $k = 1$. If $f_\sharp[\partial D_1] = \pm[\partial D_1]$ then $f_\sharp = \pm I$. If $j_\sharp f_\sharp[\partial D_1] = \pm[\partial D_1]$ and $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $b = 0$ and $a = c = -d = \pm 1$. Therefore G_1 has at most four elements if $k = 1$.

(b) Assume that D_2 intersects $\tilde{C}$ in two points and $h(D_2)$ is isotopic to D_2. Then $n = \pm 1$. If $f_\sharp[\partial D_2] = \pm[\partial D_2]$ then $f_\sharp = \pm I$. Finally, if $j_\sharp f_\sharp[\partial D_2] = \pm[\partial D_2]$ and $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $c = 0$, $d = -a = \pm 1$, and $b = \pm mn$. Consequently, G_1 has at most four elements if $n = \pm 1$.

We show that all these elements occur as homeomorphisms, but in all cases two of them lie in G_2. So G is isomorphic to Z_2. The covering transformation for the standard 2-fold covering projection of M to S^3, branched over a link, gives a homeomorphism h such that $h_\sharp$ restricted to L is $-I$ (see the Appendix). The previous arguments show that h cannot be homotopic to the identity.

(a) Suppose that $k = 1$. We investigate the solution

$$F = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then $j_\sharp f_\sharp([B]+[jB]) = [B]+[jB]$, that is, $j_\sharp f_\sharp[\tilde{C}_3] = [\tilde{C}_3]$. So h preserves the homology class of an ordinary fibre of M (see Fig. 1 for $\tilde{C}_3$). Since $k = 1$, M has an exceptional fibre which has a regular neighbourhood N_2 which intersects K in a punctured Klein bottle K_0 (see the discussion at the beginning of § 2). In Fig. 4 the embedding of K_0 in N_2 is drawn explicitly; K_0 is a meridian disk for N_2 with two properly embedded strips attached, going from one side to the other of the meridian disk. Note that ∂K_0 is homotopic to 4 times a generator of $\pi_1(N_2)$ as desired, and is an ordinary fibre of the Seifert fibring of M. As usual we take C_1 and C_2 to be the centres of the cross caps of K_0, where C_i lifts to $\tilde{C}_i$ in L (Fig. 3) for $i = 1, 2$. It follows immediately that $j_\sharp f_\sharp[\tilde{C}_1] = [\tilde{C}_2]$ and $j_\sharp f_\sharp[\tilde{C}_2] = -[\tilde{C}_1]$.

Let $N_2 = B^2 \times S^1$, where B^2 (respectively S^1) is the unit disk (respectively circle) in C. The homeomorphism $h'\colon N_2 \to N_2$ given by $h'(x,t) = (-x,t)$ can be assumed to map K_0 to K_0 and to interchange the two cross caps of K_0. Clearly h' extends to a homeomorphism h of M which takes K to K, preserves C_3 but switches C_1 and C_2. Also h is obviously isotopic to the identity, but $h_\sharp$ restricted to L induces the desired homomorphism $j_\sharp f_\sharp$ of $H_1(L, Z)$. So we have shown that $G_1 \cong Z_2 \times Z_2$, $G_2 \cong Z_2$, and $G \cong Z_2$.

(b) Suppose that $n = \pm 1$. Consider the case when $F = \begin{pmatrix} 1 & -mn \\ 0 & -1 \end{pmatrix}$. Then $j_\sharp f_\sharp$ preserves $[\tilde{C}_1] = [A]+[jA]$. As in (a), let N_2 be a regular neighbourhood of the $(4k, 2k-1)$ exceptional fibre, so that $N_2 \cap K$ is a punctured Klein bottle K_0. Clearly

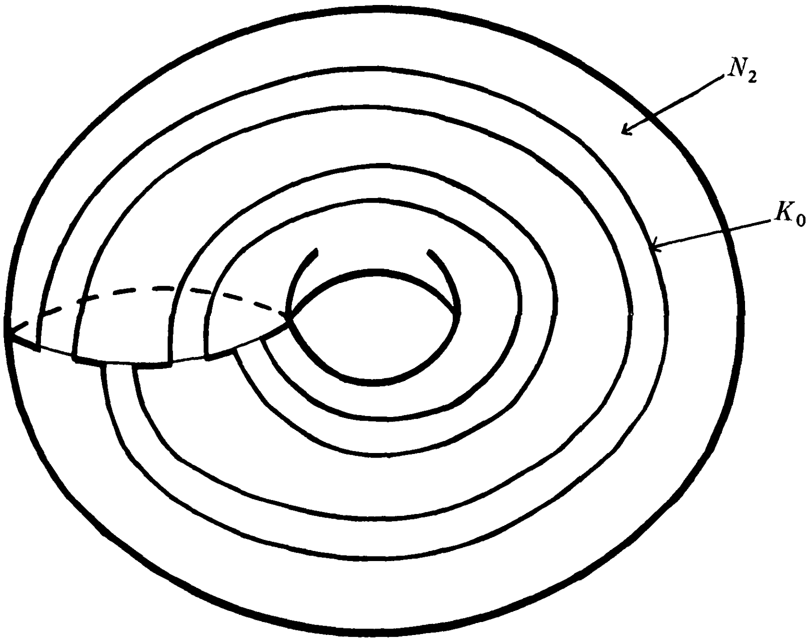

FIG. 4

the annulus in $N(K)$ which joins A to jA intersects the genus-2 handlebody $N(K_0) = N(K) - \operatorname{int} N(C_3)$ in a meridian disk D. Also the meridian disk D_1 for Y which lies inside N_2 has the property that ∂D_1 and ∂D cross transversely in one point (see Fig. 2). Hence adding $N(D_1)$ to $N(K_0)$ cancels one of the handles of $N(K_0)$ and yields the solid torus N_2. It is now easy to see that C_1 is a core circle of the remaining handle of $N(K_0)$, and hence of N_2. So we can isotopically shrink N_2 to a small regular neighbourhood N of C_1, which meets K in a Mobius-band neighbourhood $N(C_1)$ of C_1. Also the homeomorphism h that we are seeking can be chosen as the identity map on N.

Note that $M - \operatorname{int} N$ is a Seifert manifold with two exceptional fibres with multiplicity $(2, 1)$ and $(m, \pm 1)$. We can attach a solid torus N' to $M - \operatorname{int} N$ by an identification of $\partial N'$ and ∂N so that a meridian disk D' for N' satisfies $\partial D' = K \cap \partial N$. Then $D' \cup (K - \operatorname{int} N)$ is an embedded Klein bottle K' in $M' = N' \cup (M - \operatorname{int} N)$. So M' is a prism manifold with Seifert invariants $\{b; (o_1, 0), (2, 1), (2, 1), (m, \pm 1)\}$ [3, 17]. But then by [3] we see that M' has another Seifert fibring as an S^1 bundle over RP^2. The Klein bottle K' can be viewed as the set of fibres over a non-contractible simple closed curve C in the base RP^2. Also a core circle of the solid torus N' can be chosen to project one-to-one to a simple curve C' in RP^2 which meets C transversely in one point (so that N' and K' meet in the disk D').

We construct the desired homeomorphism h which is isotopic to the identity by 'inverting' the Klein bottle K. Precisely, there is an isotopy of RP^2 which reverses the orientation of C and can be chosen so that C' is invariant as a set throughout the isotopy. Lifting the isotopy to M', we obtain an isotopy inverting K' and leaving N' invariant. The homeomorphism h is given by the time-one mapping of the isotopy on $M - \operatorname{int} N'$ and by the identity map over N. Clearly h takes K to K and is isotopic to the identity.

Finally, $h_\sharp$ restricted to $H_1(L)$ takes $[A]+[jA]$ to $[A]+[jA]$, since

$$[\tilde{C}_1] = [A]+[jA]$$

is the homology class of a curve in ∂N. In addition, $h_\sharp$ maps $[B]$ to $[B]\pm[\partial D_2]$ (cf. the proof of Theorem 5 in [**17**]). Since $[\partial D_2] = n[B]+m[jA]+n[jB]$ and $n = \pm 1$, we can choose the direction of the isotopy so that $h_\sharp$ takes $[B]$ to $-mn[jA]-[jB]$. Consequently, h restricted to L must be of the form jf where $f_\sharp$ corresponds to the matrix $F = \begin{pmatrix} 1 & -mn \\ 0 & -1 \end{pmatrix}$ as desired. So we have proved that $G_1 \cong Z_2 \times Z_2$, $G_2 \cong Z_2$, and $G \cong Z_2$ as in (a).

We are now ready to prove the Main Theorem of § 1, for the manifolds in Class (1a). By Lemma 2, these are precisely the manifolds covered in Parts I and II of the Main Theorem, if we exclude the cases where $(k,n,b) = (1,\pm 1,-1)$ (cf. the Remark at the end of § 2). If m is odd then $H_1(M,Z_2) = Z_2$ and so $\mathscr{H}(M) = G$. So Part I follows immediately, since for M a lens space, $m = 1$ (see § 2 of [**18**]). Also Part II holds for all examples with m odd. If m is even then $H_1(M,Z_2) = Z_2 \times Z_2$. Suppose that there is a homeomorphism $h\colon M \to M$ with $h_\sharp\colon H_1(M,Z_2) \to H_1(M,Z_2)$ not equal to the identity. By [**8**], there must be a fibre-preserving homeomorphism h_0 of M with this property. This can only occur if $(m,n) = (4k, 2k-1)$ or $(2,1)$. Also there is just one automorphism of $H_1(M,Z_2)$ realizable by a homeomorphism and different from the identity in this case. Hence hh_0^{-1} or h belongs to G for all possible homeomorphisms h. We conclude that $\mathscr{H}(M) = Z_2$ or $Z_2 \times Z_2$.

5. *Calculation of $\mathscr{H}(M)$ for $M-C$ fibred over S^1*

We begin with M in Class (1b) or (2). Then $M-C$ is fibred over S^1 with fibre $K-C$. Let $\varphi\colon K-C \to K-C$ be the monodromy for a choice of orientation of the base S^1. Also let $\varphi_\sharp\colon H_1(K-C,Z) \to H_1(K-C,Z)$ be denoted by $\Phi \in \mathrm{SL}(2,Z)$. Then $M-C$ is determined up to bundle equivalence by the conjugacy class of Φ in $\mathrm{SL}(2,Z)$ (with a fixed orientation of the base S^1). Our first task is to find this conjugacy class.

As is well known (see, for example, [**16**]) the monodromy for the figure-8 knot complement corresponds to the conjugacy class of $\Phi = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$. Note that we will always use the basis $\{[\alpha],[\beta]\}$ for $H_1(K-C,Z)$, where α (respectively β) is the curve which is the projection of A (respectively B) to K. We now investigate the monodromy where M is in Class (1b).

The meridian disks, D_1, D_2 for Y (see Proposition 1 and Fig. 2), determine the fibring of $M-C$ since $\partial D_i \cap \tilde{C}$ has two points, for $i = 1,2$. Let L_1, L_2 be the components of $L-\tilde{C}$. We identify L_1 with $K-C$ by projection. The monodromy φ is then the composition of the map from L_1 to L_2 which takes the arc $L_1 \cap \partial D_i$ to $L_2 \cap \partial D_i$, for $i = 1,2$, and the covering transformation j restricted to L_2. Note that under the identification of L_1 and $K-C$, the curves A, B correspond to α, β. So

$$\varphi_\sharp\colon [A]+[B] \to -[jA] \to -[A],$$

$$\varphi_\sharp\colon [B] \to -mn[jA]-[jB] \to -mn[A]-[B], \quad \text{where } n = \pm 1,$$

because, by Proposition 1,

$$[\partial D_1] = [A]+[B]+[jA] \quad \text{and} \quad [\partial D_2] = n[B]+m[jA]+n[jB].$$

This implies that $\varphi_\sharp$ has matrix $X = \begin{pmatrix} mn-1 & -mn \\ 1 & -1 \end{pmatrix}$ relative to the basis $\{[\alpha], [\beta]\}$ for $H_1(K-C, Z)$. Replacing X by its conjugate $U^{-1}XU$, where $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we see that the conjugacy class of Φ is that of the matrix $\begin{pmatrix} mn-2 & -1 \\ 1 & 0 \end{pmatrix}$ in $\mathrm{SL}(2, Z)$. This gives a different conjugacy class for each example in Class (1b). Note that the conjugacy class is completely determined by $\operatorname{tr}\Phi = mn-2$, and all values of $\operatorname{tr}\Phi$ occur except ± 2.

EXAMPLES. (1) If $(m, n) = (1, 1)$, then

$$M = L(6,1) \quad \text{and} \quad \operatorname{tr}\Phi = -1.$$

If $(m, n) = (1, -1)$, then $M = L(10, 3)$ and $\operatorname{tr}\Phi = -3$. These are the only lens spaces in Class (1b).

(2) If $m = 2$ and $n = \pm 1$ then M is a prism manifold and $\operatorname{tr}\Phi = 0$ or -4. For these examples $\mathscr{H}(M)$ has been computed in [**1**] and [**17**].

(3) If $(m, n) = (3, \pm 1)$ then M is a binary octahedral space and $\operatorname{tr}\Phi = 1$ or -5.

(4) The case where $(m, n) = (4, 1)$ is excluded (M is Haken). If $(m, n) = (5, 1)$ then $\operatorname{tr}\Phi = 3$. This Seifert manifold $\{-1; (o_1, 0), (2, 1), (4, 1), (5, 1)\}$ is obtained by (2, 1)-surgery on the figure-8 knot (see [**21**]).

To find $\mathscr{H}(M)$, we start by calculating G', which is the subgroup of G consisting of isotopy classes of orientation-preserving homeomorphisms h which preserve the orientation of the base S^1 of the bundle $M-C$. Let G_1' denote the isotopy classes of homeomorphisms $f\colon K \to K$ which extend to orientation-preserving homeomorphisms $h\colon M \to M$ with isotopy class in G'. Let G_2' be the kernel of the homomorphism from $G_1' \to G'$, which sends the isotopy class of f to that of h. If f determines a class in G_1' then the restriction of f to $K-C$ induces a homomorphism of $H_1(K-C, Z)$ with matrix F and $F\Phi = \Phi F$. Conversely, given $F \in \mathrm{GL}(2, Z)$ such that F commutes with Φ, a homeomorphism $h\colon M-C \to M-C$, which preserves the orientation of the base of $M-C$, can be constructed so that h induces a homomorphism of $H_1(K-C, Z)$ with matrix F. We will say that h corresponds to F.

To find the centralizer of Φ in $\mathrm{GL}(2, Z)$, we replace Φ by the matrix

$$\hat{\Phi} = \begin{pmatrix} mn-2 & -1 \\ 1 & 0 \end{pmatrix}$$

by a change of basis and note that an abelian subgroup of $\mathrm{PSL}(2, Z) \simeq Z_2 * Z_3$ is cyclic. So if $\pm\hat{\Phi}$ has no roots in $\mathrm{GL}(2, Z)$, then the centralizer of $\hat{\Phi}$ is $\{\pm\hat{\Phi}^p\colon p \in Z\}$. On the other hand, if $\Psi^q = \pm\hat{\Phi}$, then since $\Psi^q = r\Psi + sI$, for integers r, s, and $\hat{\Phi} = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$ where $x = \operatorname{tr}\Phi$, we find that r and the off-diagonal terms of Ψ are all ± 1 and the only solutions are $q = 2$ and either $x = 3$ and $\Psi = \pm\begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$ or $x = -3$ and $\Psi = \pm\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$.

CONCLUSION. (1) If $x \neq \pm 3$, homeomorphisms $h\colon M-C \to M-C$ correspond to matrices $\pm\hat{\Phi}^p$.

(2) If $x = \pm 3$, the class of matrices is $\{\pm\Psi^p\}$, where $\Psi = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$ if $x = 3$ and $\Psi = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$ if $x = -3$.

Next we seek the matrices corresponding to homeomorphisms h of $M-C$ which extend over M. Obviously all the matrices $\pm\hat{\Phi}^p$ are of this type, since the homeomorphisms h can be chosen to be the identity map near C and so extend over M. Consider the matrix $\Psi = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$, in the case where $x = 3$. Then $\Psi^2 = \hat{\Phi}$ and so $M-C$ is a double cover of a $K-C$ bundle over S^1 with monodromy matrix Ψ. Let h' be the covering transformation for this covering. Then h' is isotopic to a homeomorphism $h\colon M-C \to M-C$ which leaves $K-C$ invariant. Clearly the restriction of h to $K-C$ can be assumed to correspond to the matrix Ψ. Let $N(C)$ be a small regular neighbourhood of C which is invariant under h and let f be the restriction of h to $\partial N(C)$. Since $\det\Psi = -1$, if γ is the homology class of $\partial(K - \operatorname{int} N(C))$ in $H_1(\partial N(C), Z)$, then $f_\sharp(\gamma) = -\gamma$. Also another member δ of $H_1(\partial N(C), Z)$ can be chosen so that $\{\delta, \gamma\}$ is a basis and $f_\sharp(\delta) = \delta$, because h is isotopic to h'. But then if D is a meridian disk for $N(C)$, we have $[\partial D] = \pm 2\delta + (2N+1)\gamma$ in $H_1(\partial N(C), Z)$ and so $f_\sharp[\partial D] \neq \pm[\partial D]$. Hence h does not extend over M. We conclude that only the matrices $\pm\hat{\Phi}^p$ correspond to homeomorphisms of M, if $x = 3$.

Finally, if $x = -3$, then $M = L(10, 3)$. Now $\pi_1(M-C)$ has a presentation

$$\{x, y, t\colon t^{-1}xt = x^{-4}yx,\ t^{-1}yt = x^{-1}\},$$

where $\{x, y\}$ generates $\pi_1(K-C)$. Note that adding the relation $t^2[x, y] = 1$ gives Z_{10}. Define an automorphism θ of $\pi_1(M-C)$ by $\theta(x) = y^{-1}xy^{-1}xy$, $\theta(y) = y^{-1}x$, and $\theta(t) = t[x, y]$. Then θ induces the matrix Ψ on $H_1(K-C)$. Let h be a homeomorphism of $M-C$ which induces θ on $\pi_1(M-C)$. Then h restricted to $\partial N(C)$ gives a homomorphism of $H_1(\partial N(C), Z)$ which takes $2\delta+\gamma$ to $2\delta+\gamma$, where δ is the homology class of t and γ is the homology class of $\partial(K - \operatorname{int} N(C))$. Since $2\delta+\gamma$ is the class of a meridian disk for $N(C)$, we see that h extends over M. Hence all the matrices $\pm\Psi^p$ come from homeomorphisms of M in the case that $x = -3$.

Our calculation of G_1' is now complete; $G_1' \cong Z \times Z_2$ in all cases. The next step is to compute G_2'. If h is a homeomorphism such that its restriction to $K-C$ has a homology action with matrix $\hat{\Phi}^p$, then clearly h is isotopic to the identity, because $\hat{\Phi}$ is the monodromy matrix for the bundle $M-C$. Hence the isotopy class of the restriction of h to K is in G_2'.

On the other hand, let h be the homeomorphism such that h restricted to $K-C$ induces the homomorphism $-I$ on $H_1(K-C, Z)$. If M is in Class (1b) and $\pi_1(M)$ is infinite, then a generator of the centre of $\pi_1(M)$ is mapped to its inverse by h_*. So h cannot be homotopic to the identity. The same argument works for the binary octahedral example M where $(m, n) = (3, -1)$ and $\pi_1(M) = O(48) \times Z_7$, as $\pi_1(M)$ has cyclic centre with order 14, and also for the two lens-space examples $L(6, 1)$ and $L(10, 3)$ (see [**13**, p. 101]). Finally, the prism manifold example M, where $(m, n) = (2, 1)$, has $|H_1(M, Z)| = 12$. Since $h_\sharp\colon H_1(M, Z) \to H_1(M, Z)$ is equal to $-I$, it follows that h cannot be homotopic to the identity. Note also that if $M = L(10, 3)$, the homeomorphism h corresponding to the matrix Ψ is not homotopic to the identity. In fact $h_*(\xi) = \xi^3$, where ξ generates $\pi_1(M)$. So we conclude that for all M in Class (1b), except for the cases where $(m, n) = (2, 1)$ or $(3, 1)$, G_2' corresponds to the set of matrices $\{\hat{\Phi}^p: p \in Z\}$. So $G' = G_1'/G_2' \cong Z_2$.

For the prism manifold $M = \{-1; (o_1, 0), (2, 1), (2, 1), (4, 1)\}$, by the proof of Theorem 5 in [17] it follows that the homeomorphism h of M corresponding to the matrix $-I$ is isotopic to the identity. Let M be the binary octahedral space $\{-1; (o_1, 0), (2, 1), (3, 1), (4, 1)\}$. If a small regular neighbourhood of the $(4, 1)$ exceptional fibre is removed from M, the result is the trefoil knot complement. So a non-orientable surface K can be formed by gluing a Seifert surface of genus 1 for the trefoil knot to a Mobius band which is properly embedded in the neighbourhood of the $(4, 1)$ fibre. Note that M is obtained by $(2, 1)$ surgery on the trefoil, so the boundary curves of these surfaces can be matched for appropriate choices of the Mobius band. This shows that the curve C in K is isotopic to the $(4, 1)$ exceptional fibre (any two embeddings of K in M are isotopic [18] and there is a unique curve C in K such that $K - C$ is orientable, up to isotopy).

The homeomorphism h of M which corresponds to $-I$ can be viewed as the covering transformation for the double covering projection of M to S^3, branched over a link. The fixed set of h has two components, one of which is C and the other is isotopic to the $(2, 1)$ exceptional fibre since $M - C$ has a Seifert fibration and is a fibred knot. The homeomorphism h on $M - C$ can be viewed as an involution with three fixed points on each punctured torus fibre. But h can also be constructed by a rotation through π along each of the ordinary fibres and on the $(3, 1)$ exceptional fibre for the Seifert fibring of $M - C$ and by fixing each point of C and the $(2, 1)$ exceptional fibre. Consequently, h is isotopic to the identity, and in the cases where $(m, n) = (2, 1)$ or $(3, 1)$, $G_1' = G_2'$ and $G' = \{1\}$.

Suppose now that M is in Class (2) and again let h correspond to the matrix $-I$. Then h is an isometry of the complete hyperbolic structure on M, so h cannot be homotopic to the identity by Mostow's theorem [12]. Alternatively, this can be checked by showing that $h_*: \pi_1(M) \to \pi_1(M)$ is not an inner automorphism. As M is hyperbolic, $\pi_1(M)$ has no centre, so this is easily proved by considering h^2.

The second step is to find G_1, G_2, and G. Let Y be the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Then $\hat{\Phi}^{-1} Y = Y\hat{\Phi}$ holds for all monodromy matrices $\hat{\Phi}$. Hence it easily follows that there is a homeomorphism $h'\colon M - \operatorname{int} N(C) \to M - \operatorname{int} N(C)$ such that h' corresponds to Y, and h' reverses the orientation of the base S^1 of the bundle $M - \operatorname{int} N(C)$. Let $\{\delta, \gamma\}$ be a basis for $H_1(\partial N(C), Z)$, where γ is the homology class of $\partial(K - \operatorname{int} N(C))$, and let f' denote the restriction of h' to $\partial N(C)$. Since $\det Y = -1$, it follows that $f'_\#(\gamma) = -\gamma$. Because h' is orientation-reversing on the base S^1, $f'_\#(\delta) = -\delta$ for any choice of δ. So h' extends to a homeomorphism of M, which we will again denote by h', in all cases. Hence G_1' is of index 2 in G_1.

To find G_2 suppose first that M is in Class (1b). If we change back to the basis $\{[\alpha], [\beta]\}$ for $H_1(K - C, Z)$, then Y is converted to the matrix $Z = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$. Since $\det Y = -1$ but h' is orientation-preserving, clearly h' restricted to L has the matrix $\begin{pmatrix} 0 & Z \\ Z & 0 \end{pmatrix}$ with respect to the basis $\{[A], [B], [jA], [jB]\}$. But in Case (B) of § 4, this matrix is shown to correspond to a homeomorphism which is isotopic to the identity.

(The desired isotopy is a rotation in a regular neighbourhood of the (4, 1) exceptional fibre.) Hence G_2' is of index 2 in G_2 and $G \cong G_1/G_2 \cong G_1'/G_2' \cong G'$ for M in Class (1b).

Finally, assume that M is in Class (2) and is hyperbolic. Then the homeomorphism h' can be chosen to be an isometry of the complete hyperbolic structure of M (see [**21**]) and so neither h' nor hh' is isotopic to the identity, where h corresponds to the matrix $-I$. This can also be checked by showing directly that h'_* and $h_* h'_*$ are both outer automorphisms of $\pi_1(M)$. So $G_2 = G_2'$.

The proof of the theorem is nearly complete. We need only check when there are homeomorphisms $h: M \to M$ such that $h_\#: H_1(M, Z_2) \to H_1(M, Z_2)$ is not the identity. This can only occur if $|H_1(M, Z_2)| > 2$, which completes the proof of Part III. For M in Class (1b), h can be assumed to be fibre-preserving by [**8**]. Hence $(m, n) = (2, \pm 1)$ are the only two cases. This finishes Parts I and II of the main theorem.

REMARK. It is interesting that (2, 1)-surgery on the figure-8 knot gives a Seifert manifold with a different homeotopy group from all the hyperbolic 3-manifolds obtained by $(2, 2N+1)$-surgery, for $N \neq 0$.

Appendix. Branched coverings and 1*-sided splittings of genus* 3

The manifolds considered in this paper all admit 1-sided Heegaard splittings of non-orientable genus 3; however, we only consider a subset of that class. Since the techniques used here undoubtedly apply to other manifolds in the class, we give in this section ways to construct all 3-manifolds which have 1-sided Heegaard splittings of non-orientable genus 3.

THEOREM 4. *Let M be a closed orientable 3-manifold.*

(1) *M has a 1-sided Heegaard splitting of genus 3 if and only if M is a 2-fold branched cover of S^3 branched over a link $\alpha \cup \mathcal{L}_1$, where $\mathcal{L}_1$ is a 6-plat and α is an unknotted circle which bounds a disk meeting $\mathcal{L}_1$ in exactly three points, as in Fig.* 5(a).

(2) *If M has such a splitting, then there is a commutative diagram of 2-fold covering spaces as in Fig.* 5(b), *all of which are branched except for p. The branch sets of p_1, p_2, p_3, p_4, p_5 are $\mathcal{L}_1$, $\mathcal{L}_2 = \alpha \cup \mathcal{L}_1$, $\mathcal{L}_3 = \alpha$, $\mathcal{L}_4 = p_3^{-1}(\mathcal{L}_1)$, $\mathcal{L}_5 = p_1^{-1}(\alpha)$. In particular, the link $\mathcal{L}_4$ has a symmetric 6-plat representation as in Fig.* 5(a), *where r is a rotation of* 180° *about an axis through the point O and perpendicular to the plane of the paper.*

(3) *Each 3-manifold M which admits a 1-sided Heegaard splitting of (non-orientable) genus 3 also admits a Heegaard splitting of (orientable) genus 3. Also the 3-manifold M' in Fig.* 5(b) *admits a Heegaard splitting of (orientable) genus 2.*

Proof. (1) Assume first that M has a genus-3 1-sided splitting $M = N(K) \cup Y$. Define $L, j, \pi: L \times I \to K$ as in §2. In Fig. 1, an embedding of L in R^3 is drawn which is symmetric relative to the coordinate axes. Then j is reflection in the origin, that is, $j(x) = -x$ for $x \in L$. Also the rotation through 180° about the x_2-axis gives an involution k on L which has six fixed points $\{\tilde{Q}_i, j\tilde{Q}_i: 1 \leqslant i \leqslant 3\}$ and commutes with j. Hence the involution on $L \times I$ defined by $(x, t) \to (kx, t)$ induces an involution k' on $N(K)$, via the projection π, and $\mathrm{Fix}(k')$ consists of three properly embedded arcs $\{Q_i \times I: 1 \leqslant i \leqslant 3\}$ in $N(K)$ together with $C = \pi(\tilde{C})$. Regard Y as the handlebody of genus 2 given by the compact region bounded by L in R^3 in Fig. 1. Let k'' be the involution on Y which is rotation through 180° about the x_2-axis. Exactly as in Theorem 5 in [**4**], the isotopy class of k is central in $\mathscr{H}(L)$. Consequently, the involutions k' on $N(K)$ and k'' on Y can be matched up (after isotopic adjustment) to produce an involution $\bar{k}$ on M. Since $M/\bar{k} = (N(K)/k') \cup (Y/k'') \approx B^3 \cup B^3 \approx S^3$, we have proved that there is a 2-fold branched covering $p_2\colon M \to S^3$ with $\bar{k}$ as covering translation.

The branch set of p_2 is a link $\mathscr{L}_2 = \alpha \cup \mathscr{L}_1$, where $\alpha = p_2(C)$ and $\mathscr{L}_1$ is the projection of the arcs of Fix(k') and Fix(k''). It is easy to check that K/k' is a disk, and so α bounds a disk $p_2(K)$ which meets $\mathscr{L}_1$ in three points. Since $M-K$ is an open handlebody of genus 2, it follows that $\mathscr{L}_2 \cap p_2(M-K)$ consists of three unknotted open arcs properly embedded in the open ball $S^3 - p_2(K)$ (cf. [**4**]). Hence $\mathscr{L}_1$ is a 3-bridge link determined by a 6-braid β as in Fig. 5.

Conversely, suppose that there is a 2-fold covering $p_2\colon M \to S^3$ with branch set a link $\mathscr{L}_2 = \alpha \cup \mathscr{L}_1$ as in Fig. 5. Let D be the disk bounded by α which meets $\mathscr{L}_1$ in three points. Then it can easily be shown that $K = {p_2}^{-1}(D)$ is a closed non-orientable surface of genus 3. Moreover, if B^3 is a small 3-ball neighbourhood of D in S^3, then ${p_2}^{-1}(B^3)$ is homeomorphic to $N(K)$. Finally, ${p_2}^{-1}(S^3 - \operatorname{int} B^3)$ is homeomorphic to a handlebody of genus 2 and so $M = N(K) \cup Y$.

(2) Assume that $M = N(K) \cup Y$ is a genus-3 1-sided Heegaard splitting and let $p\colon \tilde{M} \to M$ be the associated double covering. Also let $p_2\colon M \to S^3$ be the 2-fold

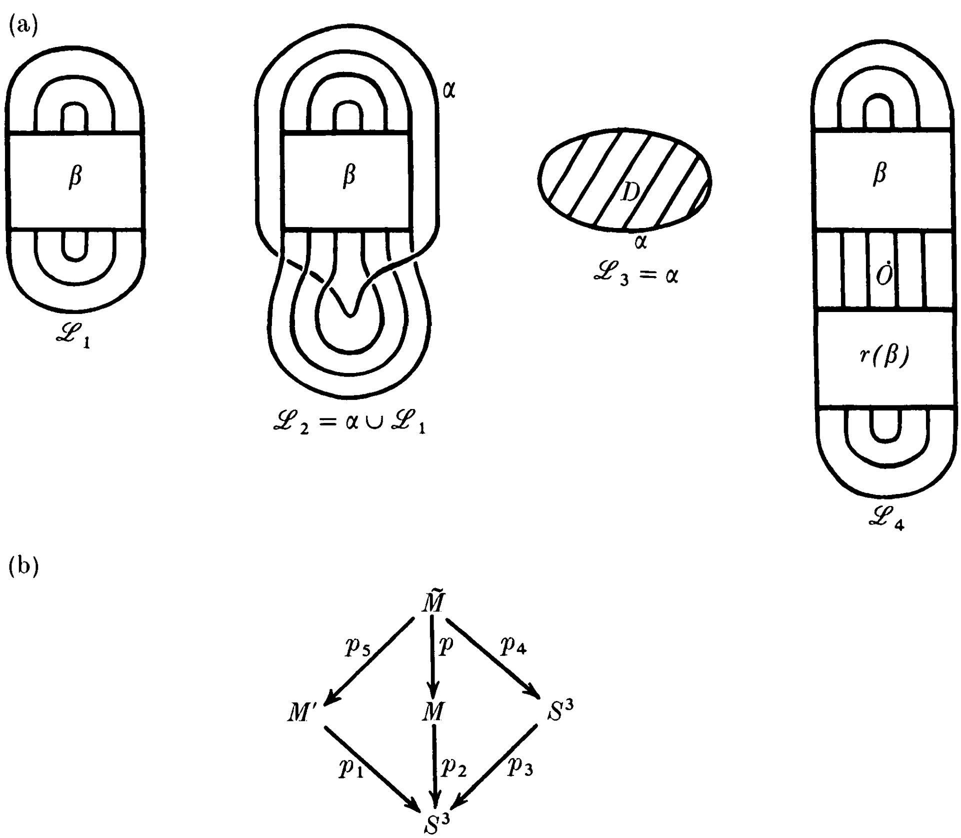

FIG. 5

branched covering as in (1). Finally, let p_1: $M' \to S^3$ and p_3: $S^3 \to S^3$ be the 2-fold coverings of S^3 branched over $\mathscr{L}_1$ and $\alpha = \mathscr{L}_3$ respectively, where $\mathscr{L}_2 = \alpha \cup \mathscr{L}_1$ is the branch set for p_2.

Now $q = p_2 \cdot p$ is a 4-fold covering from $\tilde{M}$ to S^3 with 2-fold branching over $\mathscr{L}_2$. To construct this covering, $H_1(S^3 - \mathscr{L}_2)$ can be split into a direct sum $Z_2 \oplus Z^n$, where the first summand is generated by a meridian of α and the second factor has meridians of the n components of $\mathscr{L}_1$ as generating set. Let η: $H_1(S^3 - \mathscr{L}_2) \to Z_2 \oplus Z_2$ be the direct sum of the epimorphisms $Z \to Z_2$ and $Z^n \to Z_2$. Then the composition of η with the epimorphism from $\pi_1(S^3 - \mathscr{L}_2)$ to $H_1(S^3 - \mathscr{L}_2)$ gives an epimorphism from $\pi_1(S^3 - \mathscr{L}_2)$ to $Z_2 \oplus Z_2$. The kernel of this latter map induces a regular 4-fold covering of $S^3 - \mathscr{L}_2$ which extends to give q: $\tilde{M} \to S^3$. However, the 4-fold covering of $S^3 - \mathscr{L}_2$ can be factorized into a composition of two 2-fold coverings in three distinct ways. This gives the diagram of coverings in Fig. 5(b), with $q = p_2 \cdot p = p_3 \cdot p_4 = p_1 \cdot p_5$.

Finally, the branch set $\mathscr{L}_4$ for p_4: $\tilde{M} \to S^3$ clearly satisfies $\mathscr{L}_4 = {p_3}^{-1}(\mathscr{L}_1)$. So $\mathscr{L}_4$ has a symmetric 6-plat representation as in Fig. 5 because p_3 is just the 2-fold covering of S^3.

(3) Since the branch set $\mathscr{L}_2$ (respectively $\mathscr{L}_1$) is represented in Fig. 5(a) as an 8-plat (respectively 6-plat) it follows from Theorem 5 of [**4**] that M (respectively M') admits a Heegaard splitting of genus 3 (respectively 2).

A partial converse to Theorem 4 will now be given.

Let M' be a closed orientable 3-manifold which admits a genus-2 Heegaard decomposition $M' = Y' \cup Y''$. Let f: $T_0 \times I \to Y'$ be any homeomorphism, where T_0 is a once-punctured torus, and let $C' = f(\partial T_0 \times \{\tfrac{1}{2}\})$. An epimorphism

$$\pi_1(M' - C') \to Z_2$$

can be constructed by taking intersection numbers modulo 2 of loops in $M' - C'$ with $f(T_0 \times \{\tfrac{1}{2}\})$. Let p_5: $\tilde{M} \to M'$ be the associated 2-fold covering of M', branched over C'.

THEOREM 5. *Let M', $\tilde{M}$, and p_5: $\tilde{M} \to M'$ be as above. Then there is a double covering p: $\tilde{M} \to M$, where M is a 3-manifold which has a 1-sided Heegaard splitting of genus 3. Also the projections p_5 and p can be included in a commutative diagram of coverings as in Fig.* 5(b).

Proof. By Theorem 5 of [**4**], there is a 2-fold covering projection p_1: $M' \to S^3$ with branch set a 3-bridge link $\mathscr{L}_1$. Let h: $M' \to M'$ be the covering translation for p_1. Then h can be chosen so that each surface $f(T_0 \times \{t\})$, for $0 \leqslant t \leqslant 1$, is invariant under h. Consequently, h restricted to $f(T_0 \times \{t\})$ is an involution with three fixed points and $p_1 f(T_0 \times \{t\})$ is a disk. So the simple closed curve $\alpha = p_1(C')$ bounds a disk $p_1 f(T_0 \times \{\tfrac{1}{2}\})$ in S^3 which intersects $\mathscr{L}_1$ in three points. Therefore the link $\mathscr{L}_2 = \alpha \cup \mathscr{L}_1$ is exactly as in Fig. 5(a).

As $\tilde{M}$ is the 2-fold covering of M' branched over $C' = {p_1}^{-1}(\alpha)$, it follows that the map $p_1 \cdot p_5$: $\tilde{M} \to S^3$ is a 4-fold covering with 2-fold branching at the link $\mathscr{L}_2 = \alpha \cup \mathscr{L}_1$. The 3 factorizations $p_1 \cdot p_5 = p_2 \cdot p = p_3 \cdot p_4$ are obtained exactly as in Theorem 4. Also by Theorem 4, M, which is the 2-fold covering of S^3 branched over $\mathscr{L}_2$, must have a 1-sided Heegaard splitting of genus 3.

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Department of Mathematics
University of Melbourne
Parkville, Victoria 3052
Australia

Department of Mathematics
Columbia University
New York
New York 10027
U.S.A.