TOPOLOGICAL MANIFOLDS || PROBLEM SHEET 6

Problem 1. Every connected topological manifold with empty boundary is *homogeneous*. That is, for any two points $a, b \in M$, there exists a homeomorphism $h\colon M \to M$ with $h(a) = b$.

Hint: show that for any two points a, b in $\mathrm{Int}(D^n)$, there is a homeomorphism of D^n mapping a to b and fixed on the boundary. Next show that the orbit of any given point in M under the action of $\mathrm{Homeo}(M)$ is both open and closed in M.

Remark: this problem does not require the use of anything recently covered in the course. It will help us this week when we prove that connected sum of manifolds is well defined.

Problem 2.

(i) The space of homeomorphisms of $\mathbb{R}^2$ is not contractible.
(ii) The space of orientation preserving homeomorphisms of $\mathbb{R}^2$ is not contractible.

Hint: Feel free to use the fact that for a manifold M, the map $\mathrm{Homeo}(M) \times M \to M$ given by $(f, x) \mapsto f(x)$ is continuous. Recall also that $\mathrm{Homeo}_0(\mathbb{R}^2) \hookrightarrow \mathrm{Homeo}(\mathbb{R}^2)$ is a homotopy equivalence. Construct a loop of homeomorphisms that does not contract to a point.

Remark: Kneser (1926) showed that $\mathrm{Homeo}(\mathbb{R}^2) \simeq O(2)$. Further, we know that $O(2) \cong S^1 \sqcup S^1$.