TOPOLOGICAL MANIFOLDS || PROBLEM SHEET 7

Problem 1. Let M and N be manifolds. Let d be a metric on N. Show that the collection of sets of the form

$$W(f, K, \varepsilon) := \{f \in \mathcal{C}(M, N) \mid d(f(x), g(x)) < \varepsilon \text{ for all } x \in K\}$$

where $K \subseteq M$ is compact and $\varepsilon > 0$ is a basis for the compact open topology on $\mathcal{C}(M, N)$.

Problem 2. Let M be a manifold. Show that $\mathrm{Homeo}(M)$ is locally contractible at each $f \in \mathrm{Homeo}(M)$ if and only if it is locally contractible at Id.

Problem 3. For $i \in \mathbb{N}$, let B_i denote the ball of radius $\frac{1}{3}$ centred at $(i, 0) \in \mathbb{R}^2$. Define $M := \mathbb{R}^2 \setminus \bigcup_i B_i$.

Let $h_i \in \mathrm{Homeo}(M)$ be a homeomorphism which is the identity outside the disc of radius 1 centred at $(i + \frac{1}{2}, 0)$, and which maps B_i to B_{i+1} and vice versa. Why does such a homeomorphism exist?

Show that h_i is not homotopic to the identity for any i, but $\{h_i\}$ converges to the identity in the compact open topology on $\mathrm{Homeo}(M)$.

Conclude that $\mathrm{Homeo}(M)$ is not locally contractible, nor locally path connected.