TOPOLOGICAL MANIFOLDS || PROBLEM SHEET 8

Problem 1. Prove that every homeomorphism $h\colon T^n \to T^n$ is stable, where T^n denotes the n-torus $S^1 \times \cdots \times S^1$.

Hints:

- Easy mode: Apply SH_n.
- Expert mode: The result can be proved independently of SH_n, and was the key step in Kirby's proof of SH_n. (We sidestepped it by using a slightly stronger result about PL homotopy tori.) First prove the case where the induced map on fundamental groups is the identity. Then show that for any $n \times n$ matrix A with integer entries and determinant one, there exists a diffeomorphism $h : T^n \to T^n$ such that $h_* = A$ where $h_*\colon \pi_1(T^n, x) \to \pi_1(T^n, x)$. Prove that diffeomorphisms of T^n are stable.

Problem 2. Use the torus trick to show that a homeomorphism of $\mathbb{R}^n$ is stable if and only if it is isotopic to the identity.

Hints:

(1) It suffices to show that the space of stable homeomorphisms of $\mathbb{R}^n$, denoted $\mathrm{SHomeo}(\mathbb{R}^n)$, is both open and closed in $\mathrm{Homeo}(\mathbb{R}^n)$.
(2) Use the torus trick from our proof of local contractibility of $\mathrm{Homeo}(\mathbb{R}^n)$ to show that an open neighbourhood of the identity in $\mathrm{Homeo}(\mathbb{R}^n)$ consists of stable homeomorphisms. Conclude that every stable homeomorphism of $\mathbb{R}^n$ has an open neighbourhood consisting of stable homeomorphisms.
(3) Every coset of $\mathrm{SHomeo}(\mathbb{R}^n)$ in $\mathrm{Homeo}(\mathbb{R}^n)$ is open since $\mathrm{Homeo}(\mathbb{R}^n)$ is a topological group. Conclude that $\mathrm{SHomeo}(\mathbb{R}^n)$ is closed in $\mathrm{Homeo}(\mathbb{R}^n)$.