PSEUDO-ISOTOPIES AND ISOTOPIES IN DIMENSION FOUR IN THE TOPOLOGICAL CATEGORY

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Abstract. This is a translation into English of 'Pseudo-isotopies et isotopies en dimension quatre dans la catégorie topologique' by B. Perron, published in Topology, Volume 25, No. 4, pp. 381–397, 1986. Translation by Mark Powell.

0. Introduction

Let M^4 be a compact topological manifold (not necessarily smoothable), of dimension four. The aim of this article is the study of the following conjecture. Let $I = [0, 1]$.

Conjecture 0.1 (Pseudo-Isotopy Conjecture)**.** Every homeomorphism F of $M \times I$, equal to the identity on $M \times \{0\} \cup \partial M \times I$ (a pseudo-isotopy of M) is isotopic, fixing $M \times \{0\} \cup \partial M \times I$, to the identity. In particular the restriction of F to $M \times \{1\}$ is isotopic, fixing $\partial M \times I$, to the identity.

This conjecture can evidently be posed in all dimensions and in each of the three manifold categories TOP, PL, and DIFF. In [Ce], J. Cerf proved that the conjecture holds in the C^∞ category for manifolds of dimension at least 5 and simply connected. The PL case is due to C. Morlet [Mor] (see also [R]). The topological case follows from the work of Kirby-Siebenmann [KS].

The main results of this article are given by the following theorem and its corollaries.

Theorem 0.2. *The Pseudo-Isotopy Conjecture is true for compact 4-manifolds M^4 obtained from the ball B^4 by attaching* (*topological*) *handles of index ≥ 2.*

Such manifolds are simply connected and smoothable (this last property follows from the theorem of Moise on the uniqueness of smooth structures on manifolds of dimension three [Moi].

The Milnor fibres of germs of holomorphic functions with an isolated singularity from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}, 0)$ and the smooth algebraic surfaces in $\mathbb{C}P^3$ admit such a decomposition (see [LP], [H] respectively; see also [AK]). On the other hand, Casson (unpublished; see [Man, Chapter 3]) gave an example of a smooth 4-manifold with nonempty boundary, that is 1-connected and such that every handle decomposition of it possesses handles of index one.

I owe the proof of the following results to L.C. Siebenmann.

Corollary 0.3. *The Pseudo-Isotopy Conjecture holds for closed, simply connected, topological 4-manifolds* (*not necessarily smoothable*)*.*

Corollary 0.4. *Let M^4 be a manifold given by Theorem 0.2 or Corollary 0.3 and let $h_1, h_2 \colon M \to M$ be two homeomorphisms that are homotopic rel. ∂M. Then they are isotopic rel. ∂M.*

Proof. It suffices to show that the homeomorphism $h_1 \circ h_2^{-1}$ is the extremity of a pseudo-isotopy. For that it suffices to show that the set of topological manifold structures up to homotopy on $M \times I$ is a single point. The proof is then identical to that of Theorem 6.1 of [Sh], using the fact that $\pi_{2i+1}(G/\mathrm{TOP}) = 0$ for $i = 0, 1, 2$, according to [KS] and [Su]. □

Remark 0.5. We do not know how to prove Theorem 0.2 in the piecewise linear category (and even less in the smooth category) due to the use of a theorem of Casson-Freedman [F] which only gives topological isotopies.

The idea of the proof of Theorem 0.2 is to improve the pseudo-isotopy handle by handle, starting from the boundary [R]. We can assume that $F|_{p_0 \times I} = \mathrm{Id}$, where p_0 is an interior point of M. We start by putting ourselves in a C^∞ framework, by showing that $F|_{(M \setminus \{p_0\}) \times I}$ is isotopic relative to $[\partial M \times I \cup (M \setminus \{p_0\}) \times \{0\}]$ to a pseudo-isotopy $G \colon (M \setminus \{p_0\}) \times I \to (M \setminus \{p_0\}) \times I$ such that the smooth structure Θ on the target, given by transporting by G a given smooth structure on the source, is “slice”, which means that the projection $p\colon [(M \setminus \{p_0\}) \times I]_\Theta \to I$ is a smooth submersion.

We suppose to begin with that M is obtained from the ball B^4 by attaching handles of index two only, and we set

$$V := \bigcup_i D_i^2$$

to be the union of the cocores of the 2-handles. We can assume, up to smooth isotopy, that

$$p \circ G| \colon V \times I \to [(M \setminus \{p_0\}) \times I]_\Theta \xrightarrow{p} I$$

is a smooth Morse function.

With the help of the topological isotopy extension theorem [EK], for every critical point of $p \circ G$, we can define a topological membrane [P]. Using the methods of [P], we show that $G|_{V\times I}$ is C^∞ isotopic to an embedding, $L\colon V \times I \to [(M \setminus \{p_0\}) \times I]_\Theta$, such that $p \circ L$ only has critical points of index 1 and 2.

In a middle level, using the results of Casson[Ca], Quinn [Q1], and Freedman [F], we can topologically separate the interiors of the membranes, and therefore topologically straighten the embedding $L|\colon V \times I \to M \times I$. Then we show that we can straighten a tubular neighbourhood of $V \times I$ (avoiding using the theorem of uniqueness of topological tubular neighbourhoods, which is unknown in this dimension.[1]) An application of the Alexander trick allows us to conclude. Analogous methods, but easier (because they do not use the results of Freedman) allow us to straighten the handles of index 3 and 4.

I thank L. Guillou and L. C. Siebenmann for the help they gave me to complete this work. L. C. Siebenmann helped me to greatly simplify some of the proofs, and moreover I owe him the proof of Corollary 0.3.

The organisation of the paper is as follows. (§1) sliced structures; (§2) membranes; (§3) removal of critical points of index 0 and 3; (§4) making membranes disjoint; (§5) the end of the proof of Theorem 0.2 in the case where M only has handles of index 2; (§6) proof of Theorem 0.2 in the presence of handles of index 3 and 4; (§7) proof of Corollary 0.3.

1. Sliced structures

Definition 1.1. Let Q^q be a topological manifold. A smooth structure Θ on $Q \times I$ is said to be *sliced* if the projection $p\colon [Q \times I]_\Theta \to I$ is a C^∞ submersion.

One of the essential tools for the proof of Theorem 0.2 is given by the following lemma.

Lemma 1.2. *Let M be a compact, connected, simply connected manifold of dimension four, let p_0 be an interior point of M, and let Σ be a C^∞ structure on $(M \setminus \{p_0\}) \times I$, which is a product on a neighbourhood of $\partial M \times I$. Then Σ is isotopic rel. $\partial[(M \setminus \{p_0\}) \times I]$ to a smooth structure Θ that is sliced, that is to say there is an isotopy $K_t\colon (M \setminus \{p_0\}) \times I \to (M \setminus \{p_0\}) \times I$, $t \in [0,1]$, such that:*

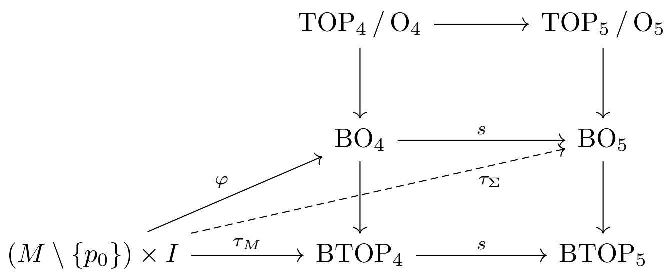
(1) $K_0 = \mathrm{Id}$; and $K_t|_{\partial[(M\setminus\{p_0\})\times I]} = \mathrm{Id}$ for all $t \in [0,1]$.
(2) K_1 is C^∞ for the structure Σ on the source and the structure Θ on the target.

Proof. Let $\tau = s \circ \tau_M \colon (M \setminus \{p_0\}) \times I \to \mathrm{BTOP}_4 \to \mathrm{BTOP}_5$ be the map classifyinf topological tangent microbundle of $(M\setminus\{p_0\})\times I$ and let τ_Σ be the lift of τ to BO_5 determined by the structure Σ [KS].

The hypothesis of simple connectivity implies that $(M\setminus\{p_0\})\times I$ is, up to homotopy equivalence, obtained from $\partial[(M \setminus \{p_0\}) \times I]$ cells of dimension 3. According to Quinn [Q1, Corollary 2.2.3], the stabilisation map $\mathrm{TOP}_4 / \mathrm{O}_4 \to \mathrm{TOP}_5 / \mathrm{O}_5$ is 3-connected. It follows that τ_Σ factorises up to homotopy rel. ∂ through a map $\varphi\colon (M \setminus \{p_0\}) \times I \to \mathrm{BO}_4$ (the arrows in the diagram below being

[1]MP: This is now known, and appeared in the book of Freedman and Quinn in 1990.

fibrations up to homotopy):



According to the theorem of Lees [Le], φ defines a sliced structure on $(M \setminus \{p_0\}) \times I$, isotopic rel. $\partial[(M \setminus \{p_0\}) \times I]$ to the structure Σ. ☐

Corollary 1.3. *Let $F\colon M \times I \to M \times I$ be a pseudo-isotopy such that $F|_{\{p_0\}\times I} = \mathrm{Id}$. Then $F|_{(M\setminus\{p_0\})\times I}$ is isotopic rel. $[\partial M \times I \cup (M \setminus \{p_0\}) \times \{0\}]$ to a pseudo-isotopy $H\colon (M \setminus \{p_0\}) \times I \to (M \setminus \{p_0\}) \times I$ that transports the $\mathcal{C}^\infty$ structure given on the domain to a slice structure θ on the codomain.*

Proof. Let Σ be the C^∞ structure on $(M\setminus\{p_0\})\times I$ transported by F from the given C^∞ structure. It suffices to set $H = K_1 \circ F$, where K_1 is given by Lemma 1.2. ☐

2. Membranes

In this section, and in sections 3 and 4, we consider the case where M is obtained from the 4-ball B^4 by attaching 2-handles only.

We set

$$V := \bigcup_i nD_i^2$$

to be the union of the cocores of the 2-handles of M, and we take $p_0 \in M \setminus V$.

Lemma 2.1. *We may assume that the function $p\circ H|\colon V \times I \to [(M\setminus\{p_0\})\times I]_\Theta \to I$ is a smooth Morse function. For each critical point c of $p \circ H|_{V\times I}$, there exists a C^∞ chart around c in $V \times I$, and a chart in $[(M \setminus \{p_0\}) \times I]_\Theta$ around $H(c)$, with respect to which $p\colon [(M \setminus \{p_0\}) \times I]_\Theta \to I$ and H have the form:*

- *$p(x, y, z, s, t) = t$;*
- *$H(x, y, z) = (x, y, z, 0, \varepsilon_1 x^2 + \varepsilon_2 y^2 + \varepsilon_3 z^2)$.*

Proof. This is immediate because p is C^∞. [2] ☐

We denote

$$D^{k+1} = \{x \in \mathbb{R}^{k+1} \mid \|x\| \le 1\};\;\; D^{k+1}_+ = \{x \in D^{k+1} \mid x_{k+1} \ge 0\};\;\; \partial_+ D^{k+1} = \partial D^{k+1} \cap D^{k+1}_+.$$

Definition 2.2. [See [P]] Let H be as in Lemma 2.1 and let c be a critical point of $p\circ H\colon V\times I \to I$ of index k. A *descending membrane of c until a level $M \times \{t_0\}$* is the image of a TOP embedding

$$\eta\colon (D^{k+1}_+, \partial_+ D^{k+1}, D^k) \to (M \times [t_0, p \circ H(c)], H(V \times I), M \times \{t_0\})$$

such that

(1) $\mathrm{Im}(\eta)$ and $H(V \times I)$ intersect transversely along $\eta(\partial_+ D)$;
(2) $\eta(0, \ldots, 0, 1) = H(c)$;
(3) $\rho|_{\eta(\mathring{D}^{k+1}_+)}$ and $p|_{\eta(\partial_+ D^{k+1})\setminus H(c)}$ are TOP submersions;
(4) $\eta(D^k) \subseteq M \times \{t_0\}$.

The disc $\eta(D^k)$ is called the *projection of the membrane* to the level $M \times \{t_0\}$. The disc $\eta(\partial_+ D^{k+1})$ is a descending submanifold of the critical point c [Ce]. We have a corresponding notion of ascending membrane.

[2] MP: there is a proof in Borodzik-Powell, J. Geom. Anal, 2016.

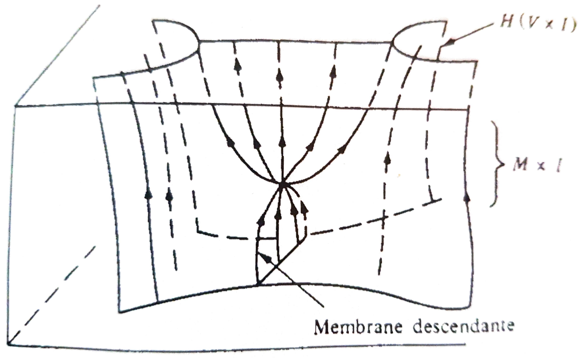


FIGURE 1.

Lemma 2.3 (Existence of Membranes). *Suppose that c is the unique critical point of $p \circ H$ between the levels $M \times \{t_0\}$ and $M \times \{t_1\}$ ($t_0 < t_1$). Then there exists a descending membrane (resp. ascending) for c until the level $M \times \{t_0\}$ (resp. $M \times \{t_1\}$) that is differentiable for the sliced structure θ in a neighbourhood of $H(V \times I)$.*

Proof. In a C^∞ chart given by Lemma 2.1, we have a C^∞ ascending membrane (resp. descending) by considering the trajectories of the field

$$\xi = (\varepsilon_1 x, \varepsilon_2 y, \varepsilon_3 z, s, 2(x^2 + y^2 + z^2 + s^2) + (t - \varepsilon_1 x^2 - \varepsilon_2 y^2 - \varepsilon_3 z^2)^2)$$

which leave (resp. converge to) c (Figure 1).

We remark that this field has c as a singular point, that it is tangent to the hypersurface $s = 0$, $t = \varepsilon_1 x^2 + \varepsilon_2 y^2 + \varepsilon_3 z^2$, and is transverse to $\{t = \text{Constant}\}$.

Thus we obtain a descending C^∞ membrane (resp. ascending) for θ, until the level $M \times (t_c - \varepsilon)$ (resp. $M \times (t_c + \varepsilon)$ (ε small), where $t_c := p \circ H(c)$. The image of the embedding

$$H^{-1}(M \times [t_0, t_c - \varepsilon]) \xrightarrow{H} [(M \setminus \{p_0\}) \times [t_0, t_c - \varepsilon]]_\theta \to M \times [t_0, t_c - \varepsilon]$$

is, by hypothesis, transverse to the horizontal foliation. The topological isotopy extension theorem [EK] implies we can smoothly extend the descending membrane until the level $M \times \{t_0\}$ along $H(V \times I)$. The same reasoning applies for the ascending membrane. □

Lemma 2.4. *We can assume that the descending (resp. ascending) membranes of the critical points of index 0 and 1 (resp. 2,3) are C^∞ with respect to the smooth structure θ.*

Proof. Let c be a critical point of index 0 or 1 of $p \circ H$, let $M \times \{t_0\}$ be a level below $H(c)$ such that $p \circ H$ has no critical point in $H^{-1}(M \times [t_0, p \circ H(c)])$. Let $\mathscr{D}$ be a C^∞ descending membrane of $H(c)$ up until a level $t'_0 = p \circ H(c) - \varepsilon > t_0$ (it exists by Lemma 2.3. Denote $W_{t_0,t'_0} = H(V \times I) \cap M \times [t_0, t'_0]$, $V_{t'_0} = H(V \times I) \cap M \times \{t'_0\}$ and let θ'_0 be the smooth structure on $(M \setminus \{p_0\}) \times \{t'_0\}$ induced by θ.

Then $((M \setminus \{p_0\}) \times [t_0, t'_0], W_{t_0,t'_0})_\theta$ is a smooth relative (δ, h)-cobordism that is $(\delta, 1)$-connected in the sense of [Q1, §2], for all map $M \setminus \{p_0\} \xrightarrow{\delta} (0, \infty)$. By the relative h-cobordism theorem [Q1, Theorem 2.1.1 and its addendum], there exists a topological product structure

$$h\colon ([(M \setminus \{p_0\}) \times \{t'_0\}]_{\theta'_0}, V_{t'_0}) \times [0,1] \to ((M \setminus \{p_0\}) \times [t_0, t'_0], W_{t_0,t'_0}). \quad (1)$$

such that:

(a) h is the identity on $[(M \setminus \{p_0\}) \times \{t'_0\}] \times \{0\}$
(b) h is smooth (for the structures indicated in (1)) outside a set $U \times I$, where U is a "standard ε-singular" subset of $(M \setminus \{p_0\}) \times \{t'_0\}$. This subset U is a smooth regular ε-neighbourhood of a locally finite smooth complex, of dimension 2, denoted $K \cup T$ (see the addendum of Theorem 2.1.1 of [Q1]). By smooth transversality, $\widetilde{\mathscr{D}} = \mathscr{D} \cap (M \times \{t'_0\})$ (of dimension 0 or 1) is disjoint from U. We can therefore extend the membrane $\mathscr{D}$ smoothly with the help of the smooth embedding $h(\widetilde{\mathscr{D}} \times I)$ up until the level $M \times \{t_0\}$. □

We recall some facts concerning membranes [P, Chap. 2, §3].

(1) One can lift up (respectively push down) a critical point along a membrane (since one can do it in the model), smoothly if the membrane is smooth.
(2) One can raise a critical point d above a critical point c if the projection onto an intermediate level of a descending membrane of c is disjoint from the projection of an ascending membrane of d.
(3) With the help of an embedding $\eta\colon (D^k, \partial) \to (M \times \{t\}, H(V \times I) \cup M \times \{t\})$ such that $\eta^{-1}(H(V \times I) \cap M \times \{t\}) = \partial$ we can introduce a critical point of index k (Lemma 2.10 of [P]).
(4) Let c and d be two consecutive Morse critical points of $p \circ H$ of indices k and $k+1$ respectively, and let $M \times \{t_0\}$ be an intermediate level. Then c and d cancel one another if and only if there exists an ascending membrane (respectively descending) of c (respectively d) until the level $M \times \{t_0\}$ such that the interiors of these two membranes are disjoint, and the boundaries of the projections intersect transversely in one point in $H(V \times I) \cap (M \times \{t_0\})$ (Lemma 2.9 of [P]) (Figure 2).

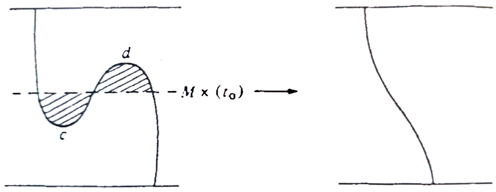


FIGURE 2.

(5) By the relative topological isotopy extension theorem [EK], every smooth descending submanifold (respectively ascending) of $p \circ H$ corresponds to a descending membrane (respectively ascending) satisfying the properties of Lemma 2.3.
(6) One can define the *dome* associated to a membrane ([P] Chapter 3, §1): in a chart with coordinates $(x_1, \ldots, x_n, x_{n+1}, x_{n+2})$ where H has the form $H(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, -(x_1^2 + \cdots x_p^2) + x_{p+1}^2 + \cdots x_n^2)$, the dome is defined in the hyperplane $x_{p+1} = \cdots = x_n = 0$ by Figure 3 (dome is coupole *en français*).

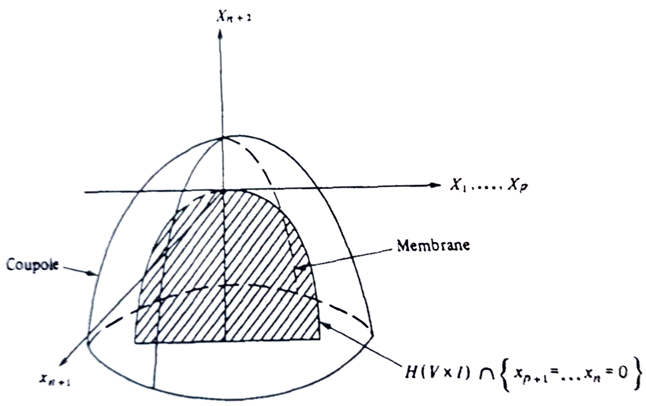


FIGURE 3.

(7) (a) One can take the connected sum of a membrane of a critical point d with the membrane of a critical point of the same index at a lower level ([P] Chapter 3, §2); see Figure 4.
(b) One can take the connected sum of a membrane of a critical point with the dome of a critical point of the same index situated at a lower level ([P] Chapter 3, §2); see Figure 5.

(8) Let $(V^n, V_0, V_1) \to M^{n+2} \times ([0,1], 0, 1)$ be an embedding with trivial normal bundle such that the projection $p|\colon V \to I$ has only one Morse critical point of index i, with descending membrane $\mathscr{D}$.

Then the complement $M \times I \setminus (V \times \mathring{D}^2)$ is obtained from $M \times \{0\} \setminus (V_0 \times \mathring{D}^2)$ by attaching a handle of index $i+1$ whose core is the dome associated to $\mathscr{D}$.

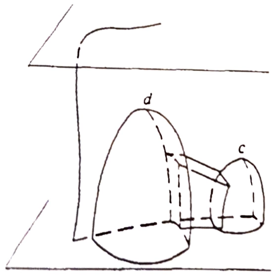


FIGURE 4.

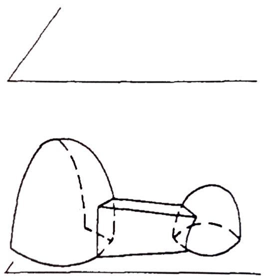


FIGURE 5.

Corollary 2.5. *The C^∞ embedding $H|\colon V \times I \to [(M \setminus \{p_0\}) \times I]_\theta$ from Lemma 2.1 is C^∞ isotopic to an embedding $K\colon V \times I \to [(M \setminus \{p_0\}) \times I]_\theta$ such that $p \circ K$ is a excellent ordered Morse function (the critical values are simple).*

Proof. Let a and b be two critical points of $p \circ H$ such that $p \circ H(a) < p \circ H(b)$ and $i = \text{index}(a) \geq \text{index}(b) = j$. According to Lemma 2.4, one of the descending membrane of b and the ascending membrane of a is C^∞. We perform a C^∞ slide (property (1) of membranes) of the corresponding point until a level where the other is C^∞. In this level, by C^∞ transversality, the membranes are disjoint (the dimension of the projection of the descending membrane of b (respectively ascending membrane of a) is j (respectively $3 - i$). One then applies property (2) of membranes to rearrange the order of the critical points. □

3. Cancellation of critical points of index 0 and 3 (for the map $p \circ K$ from Corollary 2.5)

Let $\mathcal{O}$ be the space of differentiable functions $f\colon V^n \times (I, 0, 1) \to (I, 0, 1)$ such that $f^{-1}(i) = V^n \times \{i\}$ for $i = 0, 1$. Let $\mathcal{O}^0$ be the subset of $\mathcal{O}$ consisting of the ordered, excellent Morse functions. $\mathcal{O}^1_\alpha$ denotes the subspace of ordered excellent functions except that they have a single critical point c of birth type $(-x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2 + x_{n+1}^3)$ and the level set of c separates the critical points of index i and $i+1$. Finally, $\mathcal{O}^1_\beta$ denotes the ordered excellent functions except that they have two critical points of the same index that are at the same level. We set $\mathcal{O}^1 := \mathcal{O}^1_\alpha \cup \mathcal{O}^1_\beta$ (see [Ce]).

Lemma 3.1. *Two functions $f_0, f_1 \in \mathcal{O}^0$ can be joined by a path $f_t \in \mathcal{O}^0 \cup \mathcal{O}^1$ whose graphic (in the sense of [Ce]) is of type (Fig. 6): that is the events in the graphic occur in the following order: birth of pairs of critical points, rearrangements of points of the same index, and finally death of pairs of critical points.*

Proof. This is done using the beak lemma ([Ce, chap. IV, §3]), the independent singularities lemma [Ce], and the bigon move shown in Figure 7. □

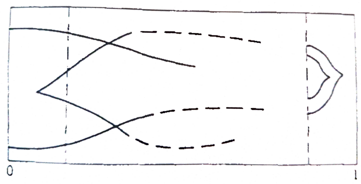


FIGURE 6.

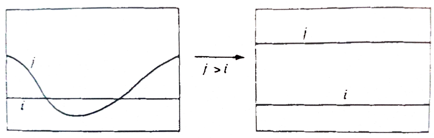


FIGURE 7.

Corollary 3.2. *Let $K\colon V^2 \times I \to [(M \setminus \{p_0\}) \times I]_\theta$ be the pseudo-isotopy given by Corollary 2.5 and let $f_t\colon V^2 \times I \to I$ be a path of functions in $\mathcal{O}^1$ joining $p \circ K = f_0$ to the projection p, given by Lemma 3.1, with graphic shown in Figure 8 (where t_1 denotes a value of the parameter such that on $[t_1, 1]$ the graphic only has deaths).*

Then there exists a C^∞ isotopy of embeddings, $K_t\colon V^2 \times I \to [(M \setminus \{p_0\}) \times I]_\theta$, $t \in [0, t_1]$, such that $K_0 = K$ and $p \circ K_t$ has, for $t \in [0, t_1]$, the same graphic as f_t.

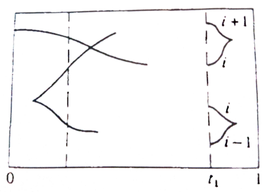


FIGURE 8.

Proof. It suffices to see that the events in the graphic can be realised by C^∞ ambient isotopies. This is clear for the births and for the rearrangements (by property (1) of membranes and Lemma 2.4). □

Let $\widetilde{K} = K_{t_1}: V \times I \to [(M \setminus \{p_0\}) \times I]_\theta$ be the C^∞ pseudo-isotopy given by Corollary 3.2. It satisfies that $p \circ \widetilde{K}$ is an ordered, excellent Morse function whose critical points are grouped into pairs that mutually cancel.

Let $M_{i+1} = M \times \{t_{i+1}\}$ (resp. $V_{i+1} = \widetilde{K}(V \times I) \cap M_{i+1}$) be an intermediate level lying between the critical points of index i and those of index $i+1$.

According to Corollary 3.2 and Smale's condition ([Ce, Prop. 3, §2 Chap. III]) one can find, for the critical points of index i ($i = 0, 1, 2, 3$), a system of ascending membranes $\mathscr{A}^i_j$ (resp. descending membranes $\mathscr{D}^i_j$) until the level M_{i+1} (resp. M_i) satisfying the properties of Lemmas 2.3 and 2.4, as well as the following property.

Property $(*)$. In the level V_{i+1}, the intersections A_j^i $(:= \mathscr{A}_j^i \cap \widetilde{K}(V \times I))$ are disjoint from the intersections D_jk^{i+1} $(:= \mathscr{D}_k^{i+1} \cap \widetilde{K}(V \times I))$ except for those which lead to the mutual cancellation of pairs of the critical points. These pairs intersect transversely in a single point that lies in V_{i+1}.

Lemma 3.3. *The pseudo-isotopy* $\widetilde{K}\colon V \times I \to [(M \setminus \{p_0\}) \times I]_\theta$ *is* C^∞*-isotopic to an embedding* L *such that* $p \circ L$ *only has critical points of index* 1 *and* 2*, grouped into pairs that mutually cancel. More precisely, one can assume that Property* $(*)$ *above is conserved.*

Proof. Using the ideas of Smale from the proof of the h-cobordism theorem, we will show that one can replace pairs of critical points with indices $0-1$ by pairs of indices $1-2$. By duality we will then also delete all pairs of indices $2-3$.

One can assume that the level M_1 and M_2 are such that the membranes of critical points of index 0 and 1 until these levels are C^∞. To achieve this, let M_1' be a level below all the critical points of index 0. We push these critical points down along their membranes (which are C^∞ by Lemma 2.4) until they are a bit below M_1'. According to Lemma 2.3 the ascending membranes until M_1 are C^∞. One proceeds in the same fashion for the critical points of index 1.

Let $(c_1, d_1), \ldots, (c_\ell, d_\ell)$ denote the pairs of critical points of index $0, 1$ respectively, which cancel in these pairs. Let $\{\mathscr{A}_i^0\}$ (resp. $\{\mathscr{D}_i^1\}$ be the ascending membranes (resp. descending membranes) of the points c_i (resp. d_i) until the level M_1, let $\{\widetilde{\mathscr{A}_i^0}\}$ (resp. $\{\widetilde{\mathscr{D}}_i^1\}$ denote their projections onto M_1. We also denote by $\mathscr{D}_k^1$ the descending membranes of the critical points of index 1 other than the $\{d_i\}$, that is those that are killed by the critical points of index 2.

Recall that $V_i = L(V \times I) \cap M_i$. Then $V_1 \subseteq M_1$ comprises a manifold V_1' isotopic to $V_0 = L(V \times I) \cap (M \times \{0\})$ and some 2-spheres that are the boundaries $\partial\widetilde{\mathscr{A}_i^0}$.

The submanifold $V_2 \subseteq M_2$ is obtained from V_1 by surgeries of index 1 corresponding to the handles of index 1 defined by the projections of the membranes $\widetilde{\mathscr{D}}_i^1, \widetilde{\mathscr{D}}_k^1$. By Property $(*)$ above, each projection $\partial\widetilde{\mathscr{A}_i^0}$ intersects V_1' in exactly one projection $\partial\widetilde{\mathscr{D}}_i^1$ (Figure 9). Let a_i (resp. b_i) be the endpoint of $\widetilde{\mathscr{D}}_i^1$ in $\partial\widetilde{\mathscr{A}_i^0}$ (resp. V_1'). Let α_i be a path in

$$M_1 \setminus V_1' \setminus \bigcup_i \widetilde{\mathscr{A}_i^0} \setminus \bigcup_i \widetilde{\mathscr{D}}_i^1 \setminus \bigcup_k \widetilde{\mathscr{D}}_k^1$$

joining a_i to b_i.[3]

By properties (3) and (4) of membranes, the path α_i allows us to introduce a critical point of index 1 cancelling with c_i. The path α_i being disjoint from the projections $\widetilde{\mathscr{D}}_i^1$, $\widetilde{\mathscr{D}}_k^1$, it survives to the level M_2. Let β_i be the loop $(\widetilde{\mathscr{D}}_i^1)' \cup \alpha_i$ in M_2, where $(\widetilde{\mathscr{D}}_i^1)'$ is a path parallel to $\widetilde{\mathscr{D}}_i^1$ (Fig. 9). Pushing β_i into $M_2 \setminus V_2$ along a section of the normal bundle of V_2, β_i represents an element of $\pi_1(M_2 \setminus V_2)$.

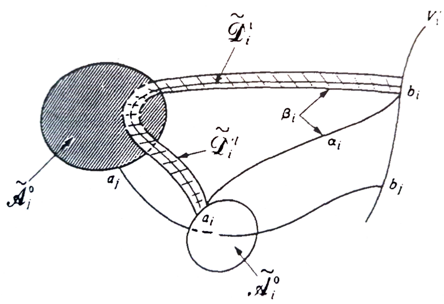


Figure 9.

[3]MP: Here and throughout, each $\setminus$ corresponds to removing an extra subset. So $X \setminus A \setminus B = (X \setminus A) \setminus B = X \setminus (A \cup B)$, and so on.

Using property (8) of membranes and remarking that $\pi_1(M \setminus V_0) \cong \pi_1(B^4) = 0$, we see easily that $\pi_1(M_2 \setminus V_2) = 0$. One can therefore find C^∞-immersions $\varphi_i \colon D^2 \to M_2^4$ such that:

(1) $\varphi_i|_{\partial D}$ is an embedding and $\varphi_i(\partial D) = \beta_i$;
(2) φ_i meets V_2 transversely and exactly along $\varphi_i(\partial D)$;
(3) φ_i has only double points $\{m_\ell^i\}$ where two branches intersect transversely;
(4) φ_i and φ_j intersect transversely.

Denote:

$$(p_\ell^i, q_\ell^i) := \varphi_i^{-1}(m_\ell^i);$$
$$\{r_h^i\} := \varphi_i^{-1}(\varphi_i(D^2)) \cap \Big(\bigcup_{i \neq j} \varphi_j(D^2)\Big);$$
$$\alpha_i' := \varphi_i^{-1}(\alpha_i) \subset \partial D^2;$$
$$\gamma_i := \varphi_i^{-1}((\widetilde{\mathscr{D}}_i^1)') \subset \partial D^2.$$

Consider disjoint paths v, joining the points p_ℓ^i, q_ℓ^i, r_h^i, to the points of α_i' such that $\varphi_i(v)$ are disjoint from the ascending membranes of the critical points of index 1, which means that the images of these paths appear at the level M_1. Consider the new path α_i'' obtained from α_i' by pushing one's finger along the paths v (Fig. 10) and let $(D_i^2)'$ be the disc D^2 after pushing by the finger moves. We set $\widetilde{\alpha}_i = \varphi(\alpha_i'')$ and $\Delta_i = \varphi((D_i^2)')$. We have the following properties.

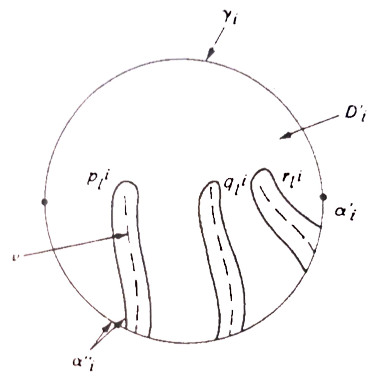


FIGURE 10.

(a) The curve $\widetilde{\alpha}_i$, considered in M_1, is a path in $M_1 \setminus V_1$ joining V_1' to $\partial\widetilde{\mathscr{A}_i^0}$, isotopic in $M_1 \setminus V_1' \setminus \widetilde{\mathscr{A}_i^0} \setminus \bigcup_i \widetilde{\mathscr{D}}_i^1 \setminus \bigcup_k \widetilde{\mathscr{D}}_k^1$ to the path α_i, the trace of the isotopy being given by the image of the fingers. The critical point of index 1 determined by $\widetilde{\alpha}_i$ (property 3 of membranes) cancels with the critical point of index 0 corresponding to $\widetilde{\mathscr{A}_i^0}$ (property 4 of membranes).
(b) The embedded 2-disc Δ_i bounded by $\widetilde{\beta}_i = \widetilde{\alpha}_i \cup \varphi_i(\gamma_i)$ determines a critical point of index 2 cancelling with the critical point of index 1 determined by $\widetilde{\alpha}_i$. The boundary $\partial\Delta_i$ intersects, transversely in exactly one point, the descending submanifold of the critical point of index 1 corresponding to the membrane $\widetilde{\mathscr{D}}_i^1$ (by construction) and does not intersect the descending submanifolds corresponding to the other critical points of index 1.

The lemma follows. □

4. SEPARATING MEMBRANES

Let $L \colon V \times I \to [(M \setminus \{p_0\}) \times I]_\theta$ be the C^∞ embedding given by Lemma 3.3 and let (a_i) (respectively (b_i) be the critical points of index 1 (respectively 2) of $p \circ L$. Proceeding as at the start of the proof of Lemma 3.3, we can suppose that the levels of (a_i) are close and just beneath a level $M \times \{t_2\}$, and that there exists a system of ascending membranes $\mathscr{A}_i$ (respectively descending $\mathscr{D}_j$) until the level $M \times \{t_2\}$ for $\{a_i\}$ (respectively $\{b_i\}$ such that:

(1) the membranes $\mathscr{A}_i$ are C^∞;
(2) the membranes $\mathscr{D}_j$ are C^∞ in a neighbourhood of $L(V \times I)$ and the boundaries of the projections $\partial\widetilde{\mathscr{A}_i}$, $\partial\widetilde{\mathscr{D}}_j$, which satisfy Property $(*)$.

The aim of this section is to prove the following lemma.

Lemma 4.1. *There exists a system of membranes satisfying properties (1) and (2) above such that the interiors of the projections are disjoint.*

Let N_2 denote a (small) C^∞ tubular neighbourhood (in Θ) of

$$V_{t_2} = (M \setminus \{p_0\} \times \{t_2\}) \cap L(V \times I),$$

let

$$\widetilde{M} := \overline{(M \times \{t_2\} \setminus N_2)} \setminus \{p_0\},$$

and write

$$\widetilde{\mathscr{A}_i'} = \overline{\widetilde{\mathscr{A}_i} \setminus N_2}, \quad \widetilde{\mathscr{D}}_j' = \overline{\widetilde{\mathscr{D}}_j \setminus N_2}.$$

To prove Lemma 4.1, we will apply the disjunction theorem of Casson-Freedman ([Ca], [F, Theorem 10.1], [Se2, Theorem 3]) to the ambient manifold $\widetilde{M}$ and to the submanifolds $\widetilde{\mathscr{A}_i'}$ and $\widetilde{\mathscr{D}}_j'$. This theorem requires a C^∞ structure in order to apply it: we will therefore prove (Lemma 4.2) that there exists a smooth structure on $M \setminus \{p_0\}$ rendering the submanifolds $\widetilde{\mathscr{A}_i'}$ and $\widetilde{\mathscr{D}}_j'$ smooth. It will then remain to check the other hypotheses of the disjunction theorem, namely π_1-negligibility of the families $\widetilde{\mathscr{A}_i'}$ and $\widetilde{\mathscr{D}}_j'$, and the existence of particular immersed dual 2-spheres.

4.1. Smooth structure on $M \setminus \{p_0\}$.

The aim of this subsection is to prove the following lemma.

Lemma 4.2. *There exists a smooth structure Γ on $M \setminus \{p_0\}$ that agrees with θ in a neighbourhood of V_{t_2}, such that V_{t_2}, $\widetilde{\mathscr{A}_i'}$, and $\widetilde{\mathscr{D}}_j'$, are all C^∞ submanifolds.*

Proof. Using the models for the membranes, for all j there exists a (topological) embedding $\varphi_j\colon (D^2, \partial) \times \mathbb{R}^2 \to (M \setminus \{p_0\}, V_{t_2})$, smooth (with respect to θ) on $\varphi_j^{-1}(N_2)$, such that $\varphi_j(D_2 \times \{0\}) = \widetilde{\mathscr{D}}_j$ (to see this it suffices to N_2 to be small and to use Lemma 2.3). We need the following lemma.

Lemma 4.3. *After a smooth isotopy of $\widetilde{\mathscr{A}_i}$, fixed on $\widetilde{\mathscr{A}_i} \cap N_2$, and a (topological) isotopy of $\widetilde{\mathscr{D}}_j$ with support in $M \setminus \{p_0\} \setminus N_2$, we can assume that the intersection points of $\widetilde{\mathscr{D}}_j$ and $\widetilde{\mathscr{A}_i}$ lie in N_2 (Figure 11) where everything is smooth (with respect to θ).*

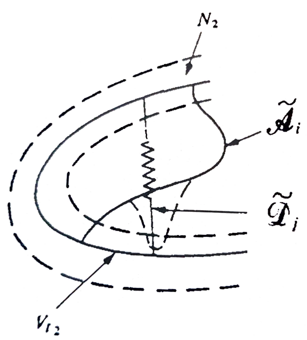


FIGURE 11.

Proof. We prove Lemma 4.3. By [Q1, Theorem 2.2.2], φ_j is C^∞ (with respect to θ) on a neighbourhood of $\widehat{D}$, where $\widehat{D} \subseteq D^2 \times \mathbb{R}^2$ is obtained from $D^2 \times \{0\}$ by introducing pairs of self-intersection

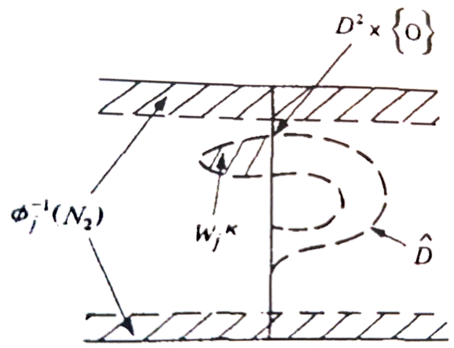


FIGURE 12.

points using finger moves (Casson's deformations: see [Ca] and Figure 12). These pairs of points appear with smooth[4] Whitney discs, denoted $\{W_j^k\}$, contained in $(D^2 \times \mathbb{R}^2) \setminus \varphi_i^{-1}(N_2)$.

We will denote $\widehat{\Delta}_j := \varphi_j(\widehat{D})$; we can assume that $\widehat{\Delta}_j \cap N_2 = \widetilde{\mathscr{D}}_j \cap N_2$. By [Q1, Proposition 2.2.4] (see also the proof of [Q1, Theorem 2.4.1]), in a small neighbourhood of $\varphi_j(W_k^j)$ (homeomorphic to $\varphi_j(W_k^j) \times \mathbb{R}^2$), we can put a Whitney tower[5] of five stages $\mathscr{C}_j^k$, smooth with respect to Θ, and with this we can put a smooth Casson handle CH_j^k (see [F, §2] for the definition), homeomorphic to $D^2 \times \mathbb{R}^2$ ([F, Theorem 1.1]), and weakly unknotted in the neighbourhood $\varphi_j(W_k^j) \times \mathbb{R}^2$ ([Q1, Proposition 2.2.4] and the proof of [Q1, Theorem 2.4.1]). This means that the core $D^2 \times \{0\}$ of CH_j^k is (topologically) isotopic rel. boundary to $\varphi_j(W_j^k) \times \{0\}$.

We can smoothly make $\widetilde{\mathscr{A}_i}$ disjoint from the towers $\mathscr{C}_j^k$ (proof of [Q1, Theorem 2.4.1]) at the cost of introducing additional points of intersection with $\widehat{\Delta}_j$. By a C^∞ isotopy of $\widetilde{\mathscr{A}_i}$, disjoint from $\{\mathscr{C}_j^k\}$, one can push the intersection points of $\widetilde{\mathscr{A}_i}$ and $\widehat{\Delta}_j$ into N_2: to do this it suffices to choose disjoint C^∞ paths in $\widehat{\Delta}_j$, disjoint from the towers $\{\mathscr{C}_j^k\}$, joining the points $\widehat{\Delta}_j \cap \widetilde{\mathscr{A}_i}$ to the points of N_2, and to then deform $\widetilde{\mathscr{A}_i}$ along these paths.

Next, we perform topological isotopies on $\widehat{\Delta}_j$ (proof of [Q1, Proposition 2.4.1]), within the Casson handles CH_j^k, to delete points of self-intersection of $\widehat{\Delta}_j$. The resulting family of discs $\{\Delta_j\}$ is topologically isotopic in $M \setminus N_2 \setminus \{p_0\}$ to the family $\{\widetilde{\mathscr{D}}_j\}$ (since CH_j^k is weakly unknotted). Thus $\{\Delta_j\}$ admits a topological normal bundle, C^∞ on N_2 (since there is a topological normal bundle for $\{\widetilde{\mathscr{D}}_j\}$. This completes the proof of Lemma 4.3. □

We continue with the proof of Lemma 4.2, as follows. Let $\mathcal{N}_1$ be a trivial C^∞ tubular neighbourhood of $\bigcup_i \widetilde{\mathscr{A}_i}$, let $\mathcal{N}_2$ be a (topological) tubular neighbourhood of $\bigcup_i \Delta_j$, that is C^∞ on N_2 (this exists by the previous proof). In a neighbourhood of a transverse point of intersection p between two of the three manifolds $\widetilde{\mathscr{A}} = \bigcup_i \widetilde{\mathscr{A}_i}$, $\Delta = \bigcup_j \Delta_j$, V_{t_2}, we can suppose that the model of the intersection of the tubular neighbourhoods is $B^2 \times B^2$ (where B^2 is the disc of dimension two), and the point p, the two submanifolds, and the fibres of the tubular neighbourhoods, correspond respectively to $\{0\} \times \{0\}$, $B^2 \cup \{0\} \cup \{0\} \times B^2$, and to the horizontal and vertical factors. This holds because in a neighbourhood of the points of intersection, everything is C^∞.

Each of the tubular neighbourhoods being trivial, the subspace $N_2 \cup \mathcal{N}_1 \cup \mathcal{N}_2$ has a natural C^∞ structure Γ_1 induced by the structure of V_{t_2}, $\widetilde{\mathscr{A}}$, and Δ. Let $W = M \setminus \{p_0\} \setminus \mathrm{int}(N_2 \cup \mathcal{N}_1 \cup \mathcal{N}_2)$ (where int means interior). The boundary ∂W, which is 3-dimensional, has a unique C^∞ structure

[4]MP: can they be smooth? If so, wouldn't the original embedding have been smooth? This doesn't seem to be required.

[5]MP: I belive this means what we would now call a Casson tower.

[Moi][6], which extends by [Q1, Corollary 2.2.3] to a smooth structure Γ_W on W. Then $\Gamma := \Gamma_1 \cup \Gamma_W$ defines a smooth structure on $(M \setminus \{p_0\}) \times \{t_2\}$ for which V_{t_2}, $\widetilde{\mathscr{A}}$, and Δ are C^∞ submanifolds.

In what follows we will continue to denote Δ_j by $\widetilde{\mathscr{D}}_j$. □

4.2. **π_1-negligibilty of the families $\{\widetilde{\mathscr{A}'_i}\}$, $\{\widetilde{\mathscr{D}}'_j\}$.** Recall that we set $\widetilde{\mathscr{A}'_i} = \overline{\widetilde{\mathscr{A}_i} \setminus N_2}$, $\widetilde{\mathscr{D}}'_j := \overline{\widetilde{\mathscr{D}}_j \setminus N_2}$, and $\widetilde{M} := \overline{M \setminus N_2} \setminus \{p_0\}$.

Lemma 4.4. *The two families $\{\widetilde{\mathscr{A}'_i}\}$ and $\{\widetilde{\mathscr{D}}'_j\}$ are π_1-negligible in $\widetilde{M}$, that is $\pi_1(\widetilde{M} \setminus \bigcup_i \widetilde{\mathscr{A}'_i}) = \pi_1(\widetilde{M} \setminus \bigcup_j \widetilde{\mathscr{D}}'_j) = 0$.*

Proof. We construct a topological foliation $\mathscr{F}$ into lines (field of trajectories in the sense of [KS, Essay III, §3]; see also [Se1]) on $M \times [0, t_2]$ whose only singularities are the critical points of $p \circ L$, such that:

(a) $\mathscr{F}$ is transverse to the level $M \times \{t\}$, for $t \in [0, t_2]$.
(b) $\mathscr{F}$ is tangent to $L(V \times I)$ and to the ascending and descending membranes of the critical points.

We construct such a field in the following way. In a compact chart $B^4 \times [-\varepsilon, \varepsilon]$ around a critical point given by Lemma 2.1 where p and L can be written: $p(x, y, z, s, t) = t$, $L(x, y, z) = (x, y, z, 0, \varepsilon_1 x^2 + \varepsilon_2 y^2 + \varepsilon_3 z^2)$, we can find a C^∞ field, χ, tangent to the part of the boundary $\partial B \times [-\varepsilon, \varepsilon]$, satisfying properties (a) and (b) above, and coinciding in a neighbourhood of 0 with the vector field ξ given in the proof of Lemma 2.3. We extend the field χ (topologically) to $M \times [0, t_2]$ using the topological isotopy extension theorem [EK].

We can assume, in addition, that $\{p_0\} \times I$ is a leaf of $\mathscr{F}$.

This foliation $\mathscr{F}$ determines a homeomorphism from $(M \setminus \{p_0\} \setminus L(V \times \{0\}) \setminus \cup \widetilde{d}_i) \times \{0\}$ to $(M \setminus \{p_0\} \setminus V_{t_2} \setminus \bigcup_i \widetilde{\mathscr{A}_i}) \times \{t_2\}$, where $\{\widetilde{d}_i\}$ denotes the projections of the descending membranes of the critical points of index 1 on $M \times \{0\}$. Since $\dim(\widetilde{d}_i) = 1$, we have isomorphisms

$$\pi_1(M \setminus \{p_0\} \setminus L(V \times \{0\}) \setminus \cup \widetilde{d}_i) \cong \pi_1(M \setminus \{p_0\} \setminus L(V \times \{0\})) \cong \pi_1(B^4 \setminus \{p_0\}) = 0.$$

On the other hand it is not difficult to see that

$$\pi_1((M \setminus \{p_0\} \setminus V_{t_2} \setminus \bigcup_i \widetilde{\mathscr{A}_i}) \times \{t_2\}) \cong \pi_1(\widetilde{M} \setminus \bigcup_i \widetilde{\mathscr{A}'_i}).$$

The lemma follows (to obtain the result for $\widetilde{\mathscr{D}}'_j$, it suffices to reverse the cobordism $M \times [0, 1]$. □

4.3. **Existence of dual 2-spheres.**

Lemma 4.5. *There exist C^∞ immersed 2-spheres*[7] *in $\widetilde{M}$ (with respect to the structure Γ of Lemma 4.2) $\mathscr{C}_i^a$, $\mathscr{C}_j^d$, such that*

(1) $\widetilde{\mathscr{A}'_i} \cdot \mathscr{C}_j^d = \widetilde{\mathscr{D}}'_j \cdot \mathscr{C}_i^a = \delta_{ij}$; and
(2) $\widetilde{\mathscr{D}}'_j \cdot \mathscr{C}_i^d = \widetilde{\mathscr{A}'_i} \cdot \mathscr{C}_j^a = 0$

for all pairs (i, j), where $(\cdot)$ denotes the algebraic intersection number.

Proof. It suffices to take for $\mathscr{C}_j^d$ (respectively $\mathscr{C}_i^a$) the boundary of the dome associated to the membrane $\widetilde{\mathscr{D}}_j$ (respectively $\widetilde{\mathscr{A}_i}$).

The calculation of algebraic intersection numbers follows from the Property (*) that was given after Corollary 3.2, and the fact that each point of intersection between $\widetilde{\mathscr{A}_i}$ and $\widetilde{\mathscr{D}}_i$ gives rise to two points of intersection with opposite signs between $\widetilde{\mathscr{A}_i}$ and $\mathscr{C}_j^d$ (respectively $\widetilde{\mathscr{D}}_j$ and $\mathscr{C}_i^a$) (Figure 13). □

[6]MP: it seems to me that this statement also uses work of Munkres – Moise only considered PL structures.

[7]MP: To apply the disc embedding theorem, these spheres need to be framed, i.e. to have trivial normal bundle. This is the case, so this does not create a problem with the proof. It was not discussed by Perron.

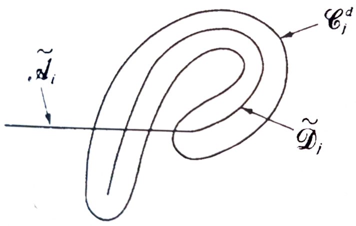


FIGURE 13.

4.4. The disjunction theorem of Casson-Freedman.

Theorem 4.6 ([F, Theorem 10.1], [Se2]). *Let X be a surface that is generically immersed in a C^∞ oriented 4-manifold $\widetilde{M}$ such that*

(a) $\pi_1(\widetilde{M} \setminus X) = 0$; and

(b) For each connected component X_i of X, there exists an immersed 2-sphere T_i, transverse to X, such that $|T_i \pitchfork X_j| = \delta_{ij}$ and $T_i \cdot T_i \in 2\mathbb{Z}$.[8]

Let $\overline{v}$ be a Whitney circle pairing two double points (p, q) of X with opposite intersection numbers. Then there exists a Whitney tower of 6 stages $(T_6, \partial^- T_6)$ (see [F, §2]) attached along $\overline{v} \subset \partial\eta(X)$, with $(T_6, \partial^- T_6) \subset (\widetilde{M} \setminus \eta(X), \partial\eta(X))$, where $\eta(X)$ is a tubular neighbourhood of X in $\widetilde{M}$. Moreover, $\pi_1(\widetilde{M} \setminus \eta(X) \setminus T_6) = 0$.

4.5. **Proof of Lemma 4.1.** By Lemma 4.4, the 2-spheres $\mathscr{C}_i^a$, $\mathscr{C}_j^d$ given by Lemma 4.5 are regularly homotopic in $\widetilde{M}$ to 2-spheres that are geometrically dual to $\widetilde{\mathscr{D}}'_j$ and $\widetilde{\mathscr{A}}'_i$ respectively (that is, $|\widetilde{\mathscr{A}}'_i \pitchfork \mathscr{C}_j^d| = |\widetilde{\mathscr{D}}'_j \pitchfork \mathscr{C}_i^a| = \delta_{ij}$ for all i, j). By the Casson lemma ([F, Lemma 10.1]), by introducing, in a smooth manner, pairs of intersection points of $\widetilde{\mathscr{A}}'_i$ with the $\widetilde{\mathscr{D}}'_j$ (Casson or finger moves), we can obtain a new family $\{\widetilde{\mathscr{A}}''_i\} \cup \{\widetilde{\mathscr{D}}'_j\}$ that is π_1-negligible in $\widetilde{M}$ (that is, $\pi_1(\widetilde{M} \setminus \bigcup_i \widetilde{\mathscr{A}}''_i \setminus \bigcup_j \widetilde{\mathscr{D}}'_j) = 0$).

By Lemmas 4.2 and 4.5, the family $\{\widetilde{\mathscr{A}}''_i, \widetilde{\mathscr{D}}'_j\}$ satisfies the hypothesis of the Disjunction Theorem 4.6. We can assume in addition that for each pair (i, j), the algebraic intersection numbers of the interiors of $\widetilde{\mathscr{A}}''_i$ and $\widetilde{\mathscr{D}}'_j$ is 0. This holds for $i = j$ by (possibly) spinning $\widetilde{\mathscr{D}}_j$ around the point $\partial\widetilde{\mathscr{A}}_i \cap \partial\widetilde{\mathscr{D}}_j$. For $i \neq j$, it is possibly necessary to add some domes (property 7 of §2), taking account of Lemma 4.5. We can therefore group the intersections of $\widetilde{\mathscr{A}}''_i$ and $\widetilde{\mathscr{D}}'_j$ into pairs of points with opposite intersection numbers. By Theorem 4.6 above, we can find disjoint Whitney towers T_6^k, for each pair of intersection points $\{p, q\}$.

By [F, Theorems 1.1 and 5.1], each tour T_6^k contains a topological Whitney model, which allows us to delete the intersections of $\widetilde{\mathscr{A}}''_i$ with $\widetilde{\mathscr{D}}'_j$ by a topological isotopy.

Corollary 4.7. *The embedding $L\colon V \times I \to [(M \setminus \{p_0\}) \times I]_\theta$ of Lemma 3.3. is topologically isotopic to the embedding $L_0 \times \mathrm{Id}_I$, where $L_0 = L|\colon V \times \{0\} \to (M \setminus \{p_0\}) \times \{0\}$.*

Proof. By the Cancellation Lemma (property 4 of membranes) and Lemma 4.1, L is topologically isotopic to an embedding L_1 such that $p \circ L_1$ has no critical points. The topological isotopy extension theorem then shows that L is isotopic to an embedding with vertical image. The corollary follows easily. ☐

5. END OF THE PROOF OF THEOREM 0.2 IN THE CASE THAT M ONLY HAS HANDLES OF INDEX 2

Recall that we began with a pseudo-isotopy $F\colon M \times I \to M \times I$ such that $F|_{\{p_0\} \times I} = \mathrm{Id}$, that we isotoped $F|_{(M \setminus \{p_0\}) \times I}$ to a C^∞ diffeomorphism $H\colon M \setminus \{p_0\}) \times I[(M \setminus \{p_0\}) \times I]_\theta$ (Corollary 1.3) and that $H|\colon V \times I \to [(M \setminus \{p_0\}) \times I]_\theta$ is C^∞ isotopic to an embedding $L\colon V \times I \to [(M \setminus \{p_0\}) \times I]_\theta$

[8] MP: in the application of the theorem, the hypothesis that $T_i \cdot T_i \in 2\mathbb{Z}$ is never checked, but follows from the dual spheres in the previous subsection being framed.

having only critical points of index 1 and 2 in cancellation position (Lemma 4.1). By the C^∞ isotopy extension theorem, there exists a smooth isotopy, $G_t\colon [(M\setminus\{p_0\})\times I]_\theta \to [(M\setminus\{p_0\})\times I]_\theta$ such that

(1) $G_0 = \mathrm{Id}$; G_t is the identity on $\partial M\times I\cup(M\setminus\{p_0\})\times\{0\}$ and outside a compact set;
(2) $G_1\circ H|_{V\times I} = L$.

We let φ denote the C^∞ diffeomorphism $G_1\circ H\colon M\setminus\{p_0\})\times I\to[(M\setminus\{p_0\})\times I]_\theta$.

Let $N\cong V\times D^2$ be a C^∞ tubular neighbourhood of V in $(M\setminus\{p_0\})$, $\mathcal{N}=N\times I$ the C^∞ neighbourhood corresponding to $V\times I$ in $[(M\setminus\{p_0\})\times I]_{\Sigma_0\times I}$, $\partial_+N = V\times\partial D^2$, and $\partial_+\mathcal{N} = \partial_+N\times I$.

Lemma 5.1. *$\varphi\colon M\setminus\{p_0\})\times I\to[(M\setminus\{p_0\})\times I]_\theta$ is topologically isotopic relative to $(\partial M\times I\cup (M\setminus\{p_0\})\times\{0\})$ to a homeomorphism $\widetilde{\varphi}$ such that $\widetilde{\varphi}|_{V\times I\cup\partial_+\mathcal{N}}$ is the inclusion.*

Proof. The C^∞ embedding $\varphi|\colon V\times I\to[(M\setminus\{p_0\})\times I]_\theta$ admits a C^∞ tubular neighbourhood $E\cong\varphi(V\times I)\times D^2$ such that

(1) $E\cap[(M\setminus\{p_0\})\times I]_\theta = N$; and
(2) the (Morse) critical points of $p|\colon\partial_+E\cong(V\times I)\times\partial D^2\to I$ are grouped into pairs (p_i,q_i), of indices $k,k+1$ respectively, each pair corresponding to a critical point of index k of $p\circ\varphi|_{V\times I}$. (Figure 14) (it suffices to do this in the model given by Lemma 2.1).

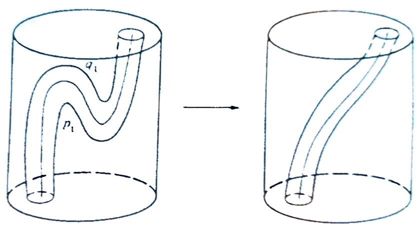


FIGURE 14.

By the uniqueness of C^∞ tubular neighbourhoods, we can assume that $\varphi(\mathcal{N}) = E$. By Lemma 4.1, the critical points of $\varphi|\colon V\times I[(M\setminus\{p_0\})\times I]_\theta$ cancel in pairs. We can arrange this so that in the isotopy the critical points of $p|_{\partial_+E}$ cancel (Figure 14) (again, it suffices to do this in the model). By the topological isotopy extension theorem [EK], $\varphi|_{V\times I\cup\partial_+\mathcal{N}}$ is isotopic to an embedding whose image is vertical. The lemma now follows easily. □

Lemma 5.2. *The pseudo-isotopy $\widetilde{\varphi}$ of Lemma 5.1 is isotopic relative to $\partial M\times I\cup(M\setminus\{p_0\})\times\{0\}\cup\partial_+\mathcal{N}\cup V\times I$ to a pseudo-isotopy $\overline{\widetilde{\varphi}}$ such that $\overline{\widetilde{\varphi}}|_{\mathcal{N}}$ is the inclusion.*

Proof. Recall that by the topological Alexander trick, $\mathrm{TOP}(D^n\times D^p, D^n\times\{0\}\cup\partial(D^{n+p}))$ is contractible, where TOP(,) denotes the space of homeomorphisms of $D^n\times D^p$ that are the identity on $D^n\times\{0\}\cup\partial(D^{n+p})$.

The homeomorphism $\widetilde{\varphi}_1=\widetilde{\varphi}|\colon(M\setminus\{p_0\})\times\{1\}\to(M\setminus\{p_0\})\times\{1\}$ sends $N\times\{1\}$ to itself, and is the identity on $\partial N\cup V\times\{1\}$. Identifying the pair $(N\times\{1\}, V\times\{1\}$ with $(D^2\times D^2, D^2\times\{0\})$, the Alexander trick shows that we can assume $\widetilde{\varphi}_1|_{N\times\{1\}}=\mathrm{Id}$.

Identifying $(\mathcal{N}, V\times I)$ with $(D^2\times D^2\times I, D^2\times\{0\}\times I)$, and remarking that $\widetilde{\varphi}$ send $\mathcal{N}$ to $\mathcal{N}$, the Alexander trick shows that we can assume $\widetilde{\varphi}|_{\mathcal{N}}=\mathrm{Id}$.

Remarking that $M\setminus\mathring{N}$ is homeomorphic to the ball B^4, yet one more application of the Alexander trick shows that the pseudo-isotopy F we started with is isotopic to the identity. □

6. Proof of Theorem 0.2 in the presence of handles of index 3 and 4

We assume the hypotheses of Section 0, and denote $V^{(i)}$, $i=3,4$, the union of the $(4-i)$-discs transverse to the handles of index i in M. We can assume that the map $p\circ H|\colon V^{(i)}\times I\to I$ is a C^∞ Morse function (where H is the pseudo-isotopy of Corollary 1.3). As in §2, we can associated an ascending and a descending membrane to each critical point of $p\circ H|_{V^{(i)}\times I}$.

6.1. **Isotopy of $H|_{V^{(4)}\times I}$ to the inclusion.** The map $p \circ H|_{V^{(4)}\times I}$ has only critical points of index 0 and 1; one can show as in Lemma 2.4 that all the membranes can be chosen to be C^∞. In an intermediate level between the critical points of index 0 and 1, for dimension reasons, we can smoothly isotope the membranes to be disjoint from the critical points of index 0 and 1. We then proceed as in Lemmas 5.1 and 5.2 to isotope the restriction of H to a tubular neighbourhood of $V^{(4)} \times I$ to the inclusion (here we can use the theorem of uniqueness of C^∞ tubular neighbourhoods in the proof of Lemma 5.1).

6.2. **Isotopy of $H|_{V^{(3)}\times I}$ to the inclusion.** One shows easily that on can kill (smoothly, with respect to θ) the critical points of index 0 (see Lemma 3.3). We choose a level a little below the critical points of index 2 such that the descending membranes (respectively, ascending membranes) of the critical points of index 2 (respectively 1) are C^∞ (the dimension of the projections of the ascending membranes of the critical points of index 1 is one). We can therefore perform a C^∞ isotopy in this level of the projections of the membranes, in order to make them disjoint. We then conclude as above.

7. Proof of Corollary 0.3

Lemma 7.1. *Let M^4 be a 4-manifold satisfying the hypotheses of Corollary 1. Then M^4 is homeomorphic to $W^4 \cup X^4$, where W^4 is obtained from the ball B^4 by attaching handles of index 2, and X^4 is a contractible manifold. In addition $W \cap X = \partial X = \partial W$ is a homology 3-sphere.*

Proof. It is easy to find a link $\mathscr{L}$ and a trivialisation τ of its normal bundle in $S^3 = \partial B^4$ such that the manifold W^4 obtained by attaching handles of index 2 along $(\mathscr{L}, \tau)$ has exactly the same intersection form as M^4. By [Ka, Theorem 4.2], we can choose $(\mathscr{L}, \tau)$ so that the Rochlin invariant of ∂W^4 in $\mathbb{Z}/2\mathbb{Z}$ (equal to $\sigma(Y)/8 \mod 2$, where Y is a parallelisable 4-manifold bounded by ∂W) is equal to the stabilised Kirby-Siebenmann obstruction to the existence of a PL triangulation of M^4.

The boundary ∂W^4 being a homology sphere (with coefficients in $\mathbb{Z}$), it bounds a contractible 4-manifold X^4 ([F, Theorem 1.4']). By the classification theorem for simply-connected closed 4-manifolds ([F, Theorem 1.5]) by the intersection form and the Kirby-Siebenmann invariant, M^4 is homeomorphic to $W^4 \cup_\partial X^4$ (here we need the condition on the Rochlin invariant when the intersection form of M^4 is odd). □

Lemma 7.2. *For the manifold $M^4 = W^4 \cup X^4$ described in Lemma 7.1, the contractible manifold X^4 is contained in the interior of a topological ball $B^4 \subset M^4$.*

Proof. Let X_1, X_1' be two copies of X; the manifold $X_1 \cup (-X_1')$ being a homotopy 4-sphere it is homeomorphic to S^4 ([F, Theorem 1.4']). Let $D^4 \subsetneq X_1'$ be a closed ball; the connected sum of X^4 with $X_1 \cup (-X_1') \cong S^4$ defined using D^4 is thus identified with X^4. We therefore have inclusions

$$X_1' \subset X_1 \cup (-X_1') \setminus \mathring{D} \subset X \subset M$$

where $X_1 \cup (-X_1') \setminus \mathring{D}$ is homeomorphic to a 4-ball. It is easy to see that there exists a homotopy equivalence, restricting to a diffeomorphism of the boundaries, between $W^4 = M^4 \setminus \mathring{X}$ and $W'^4 = M^4 \setminus \mathring{X}_1'$. The h-cobordism theorem of Freedman [F, Theorem 1.3] shows that W^4 and W'^4 are homeomorphic. □

End of the proof of Corollary 1. In §6, we saw that we can isotopy the entire pseudo-isotopy $F\colon M \times I \to M \times I$ to a pseudo-isotopy that is the identity on $B^4 \times I$, where B^4 is the ball given by Lemma 7.2. The pseudo-isotopy F is thus the identity on $X^4 \times I \subset B^4 \times I \subset M \times I$. We thus obtain a pseudo-isotopy of $W \times I$ ($W = \overline{M \setminus X}$), that is fixed on $\partial W \times I \cup W \times \{0\}$. Lemma 7.1 and Theorem 0.2 combine to allow us to conclude the proof. □

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Addendum. (February 1986). The results of this article were announced in a note to the *Compte rendu de l'Académie des sciences* **299** (1984). Since, in a preprint dated August 1985 and titled "Isotopy of 4-manifolds" [Q2], F. Quinn has proven, by completely different methods, Theorem 0.2 for all 1-connected compact topological 4-manifold.

He also proved Lemma 7.1 for all compact 1-connected 4-manifolds (not necessarily closed), which means that the methods of this current article can also prove Theorem 0.2 for these manifolds.[9]

[9] MP: i.e. prove that Conjecture 0.1 holds for all compact 1-connected 4-manifolds. However, Perron seems to be mistaken here: in the analogous statement in Quinn's article, M is indeed permitted to have nonempty boundary, but in the conclusion W can also have 1-handles. Nevertheless, Siebenmann's proof in §7 of this paper, can now be augmented with Boyer's 1993 homeomorphism classification to prove Corollary 0.3 for all compact, simply-connected 4-manifolds. Details are given in Gabai-Gay-Hartman-Krushkal-Powell. In addition, Quinn's independent proof establishes the PI conjecture for all compact, simply-connected 4-manifolds, albeit that Quinn's proof had a gap, that was fixed in Gabai-Gay-Hartman-Krushkal-Powell.