MAPPING CLASS GROUPS OF 4-MANIFOLDS || PISA LECTURES

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Contents

1. Definitions of mapping class groups

These are the notes for my lecture series from the Living in Topology conference on 22nd, 23rd, and 24th September 2025, in Pisa, Italy. I am grateful to the organisers of LiTIII for the invitation to give these lectures, and the audience for the excellent atmosphere.

The aim of these lectures is to explain as much as possible within the time frame about the state of our knowledge on mapping class groups of 4-dimensional manifolds.

For a topological space X, we consider $\mathcal{C}^0(X)$, the space of continuous maps $f\colon X \to X$, with the compact-open topology. This topology is defined via a sub-basis, given by the subsets $V_{K,U}$, where $K \subseteq X$ is compact and $U \subseteq X$ is open. Let

$$V_{K,U} := \{f \in \mathcal{C}^0(X) \mid f(K) \subseteq U\}.$$

See the appendix of [Hat02] for details on this topology.

Now let X be a compact, smooth d-manifold. We define the space $\mathcal{C}^r(X)$, the space of C^r maps $f\colon X \to X$ with the Whitney topology, following Hirsch [Hir76]. This topology is defined via a sub-basis, given by the subsets $\mathcal{N}^r(f,(\varphi,U),(\psi,V),K,\varepsilon)$. Let $\varphi\colon U \to \mathbb{R}^d$ and $\psi\colon V \to \mathbb{R}^d$ be charts, let $K \subseteq U$ be compact, let $0 < \varepsilon \leq \infty$, and let $f\colon X \to X$ be a smooth map. Then $\mathcal{N}^r(f,(\varphi,U),(\psi,V),K,\varepsilon)$ is by definition

$$\{g\colon X \xrightarrow{C^r} X \mid g(K) \subseteq V,\ \|D^k(\psi f \varphi^{-1})(x) - D^k(\psi g \varphi^{-1})(x)\| < \varepsilon,\ \forall\, 0 \leq k \leq r,\ \forall\, x \in \varphi(K)\}.$$

This defines a topology on $C^r(X)$, for each r. For each finite r, we have the subset $\mathcal{C}^\infty(X) \subseteq \mathcal{C}^r(X)$ of smooth maps. We can use the topologies on $C^r(X)$ to define a topology $\mathcal{C}^\infty(X)$, the space of C^∞ maps $f\colon X \to X$, by taking the union of the topologies obtained from the inclusions $C^\infty(X) \to C^r(X)$, for all $r \geq 0$.

If X is non-compact we need a more refined definition of the topology (the *strong* version) which we will not go into here. See Hirsch [Hir76].

The forgetful map $\mathcal{C}^\infty(X) \to \mathcal{C}^0(X)$ is continuous (take $\varepsilon = \infty$), but the topology on $\mathcal{C}^\infty(X)$ is not the subspace topology.

Definition 1.1. Let X be a closed, topological 4-manifold. We write $\mathrm{Homeo}(X)$ for the homeomorphism group of X, the topological group of homeomorphisms $f\colon X \xrightarrow{\cong} X$ and we write $\mathrm{Homeo}^+(X)$ for the topological group of orientation preserving (o.p.) homeomorphisms $f\colon X \xrightarrow{\cong} X$. When X has (possibly) nonempty boundary, we write $\mathrm{Homeo}_\partial(X)$ for the topological group of homeomorphisms of X that fix ∂X pointwise.

Multiplication is via composition, and the topology is the compact-open topology, i.e. the subspace topology arising from $\mathrm{Homeo}(X) \subseteq \mathcal{C}^0(X)$. The connected components of this group, $\pi_0(\mathrm{Homeo}(X))$ is the *mapping class group* of X. The *o.p. mapping class group* of X is $\pi_0(\mathrm{Homeo}^+(X))$. The *boundary-fixing mapping class group* of X is $\pi_0(\mathrm{Homeo}_\partial(X))$.

If the boundary is nonempty, and X is connected, then every boundary-fixing homeomorphism is orientation-preserving, so both decorations are rarely needed simultaneously.

Definition 1.2. Let X be a closed, smooth 4-manifold. We write $\mathrm{Diff}(X)$ for the diffeomorphism group of X, the topological group of diffeomorphisms $f\colon X \xrightarrow{\cong} X$ and we write $\mathrm{Diff}^+(X)$ for the topological group of orientation preserving (o.p.) diffeomorphisms $f\colon X \xrightarrow{\cong} X$. When X has (possibly) nonempty boundary, we write $\mathrm{Diff}_\partial(X)$ for the topological group of diffeomorphisms of X that fix some neighbourhood of ∂X pointwise.

Multiplication is via composition, and the topology is the Whitney topology arising from $\mathrm{Diff}(X) \subseteq \mathcal{C}^\infty(X)$. The connected components of this group, $\pi_0(\mathrm{Diff}(X))$ is the *smooth mapping class group* of X. The *o.p. mapping class group* of X is $\pi_0(\mathrm{Diff}^+(X))$. The *boundary-fixing smooth mapping class group* of X is $\pi_0(\mathrm{Diff}_\partial(X))$.

We have a continuous forgetful map

$$\mathrm{Diff}(X) \to \mathrm{Homeo}(X)$$

and hence a map

$$\pi_0 \,\mathrm{Diff}(X) \to \pi_0 \,\mathrm{Homeo}(X).$$

2. Pseudo-isotopy

A key concept will be *pseudo-isotopy.* This notion enables us to break down the problem of understanding whether two homeomorphisms are isotopic into two distinct steps.

Definition 2.1 (Pseudo-Isotopy)**.** A pseudo-isotopy (PI) of X is a homeomorphism $F\colon X \times I \xrightarrow{\cong} X \times I$ with $F|_\sqsubset = \mathrm{Id}_\sqsubset$, where $\sqsubset := (X \times \{0\}) \cup (\partial X \times I)$. We say that $F|_{X\times\{1\}}$ is pseudo-isotopic to Id_X.

Let $\mathcal{P}(X)$ be the space of all pseudo-isotopies, again with the compact-open topology. We have a restriction map

$$\begin{aligned} r\colon \mathcal{P}(X) &\to \mathrm{Homeo}_\partial(X) \\ F &\mapsto F|_{X\times\{1\}}. \end{aligned}$$

Write

$$\operatorname{Im} r = \mathrm{Homeo}^{PI}_\partial(X).$$

We have a short exact sequence of groups

$$0 \to \pi_0 \mathrm{Homeo}^{PI}_\partial(X) \to \pi_0 \mathrm{Homeo}_\partial(X) \to \frac{\pi_0 \mathrm{Homeo}_\partial(X)}{\pi_0 \mathrm{Homeo}^{PI}_\partial(X)} \to 0. \tag{2.2}$$

Definition 2.3. We write

$$\widetilde{\pi}_0 \mathrm{Homeo}^+(X) := \frac{\pi_0 \mathrm{Homeo}_\partial(X)}{\pi_0 \mathrm{Homeo}^{PI}_\partial(X)} = \{f\colon X \xrightarrow{\cong} X\}/\text{pseudo-isotopy}.$$

This is called the *pseudo mapping class group of* X.

Here, we say that $f, g\colon X \xrightarrow{\cong} X$ with $f|_{\partial X} = g|_{\partial X} = \mathrm{Id}_{\partial X}$ are CAT *pseudo-isotopic* if and only if $g^{-1} \circ f\colon X \to X$ is pseudo-isotopic to Id_X. Note that if f and g are pseudo-isotopic then there is a homeomorphism $F\colon X\times I \to X \times I$ with $F|_{X\times\{0\}} = f$ and $F|_{X\times\{1\}} = g$.

Remark 2.4. For $f\colon X \xrightarrow{\cong} X$ a homeomorphism, we have that

$$\text{isotopic to Id} \Rightarrow \text{pseudo-isotopic to Id} \Rightarrow \text{homotopic to Id} \Rightarrow f_* = \mathrm{Id}_{H_*(X)}.$$

We will see that for X closed and 1-connected, all of these implications can be reversed.

We briefly justify the first two implications. Let $f_t\colon X \to X$ be an isotopy. Then $F\colon X\times I \to X\times I$ sending $(x,t) \mapsto (f_t(x), t)$ is a pseudo-isotopy. On the other hand, given a pseudo-isotopy $F\colon X\times I \to X \times I$ from $f = F|_{X\times\{1\}}$ to Id_X, we obtain a homotopy $\mathrm{pr}_1 \circ F\colon X\times I \to X$, which gives a homotopy from f to Id_X.

3. Closed, simply-connected 4-manifolds

The first theorem I want to discuss is a computation of the mapping class groups of closed, simply-connected 4-manifolds, in the sense of reducing it to algebra. For brevity I will sometimes write 1-connected for simply-connected. Note that this implies path connected as well. The right hand group of (2.2) can be computed as follows.

Theorem 3.1 (Freedman, Kreck, Quinn, Cochran-Habegger)**.** *Let X be a closed, 1-connected topological 4-manifold. Then*

$$\begin{aligned} \widetilde{\pi}_0(\mathrm{Homeo}^+(X) &\xrightarrow{\cong} \mathrm{Aut}(H_*(X), \lambda_X) \\ f &\mapsto f_* \end{aligned}$$

is an isomorphism of groups.

Here,

$$\begin{aligned}\lambda_X\colon H_2(X)\times H_2(X)\times H_4(X)&\to H_0(X)=\mathbb{Z}\\ (x,y)&\mapsto (\mathrm{PD}^{-1}(x)\cup \mathrm{PD}^{-1}(y))\cap z.\end{aligned}$$

is the intersection pairing of X, and an automorphism in $\mathrm{Aut}(H_2(X),\lambda_X)$ is an isomorphism $\varphi\colon H_2(X)\to H_2(X)$ and an isomorphism $\psi\colon H_4(X)\to H_4(X)$ such that $\lambda_X(\varphi(x),\varphi(y),\psi(z))=\lambda_X(x,y,z)\in\mathbb{Z}$ for all $x,y\in H_2(X)$ and all $z\in H_4(X)$.

The references for this theorem are [Fre82, Kre79, Qui86, CH90]. Freedman proved surjectivity, while Kreck and Quinn gave different proofs for injectivity. The homotopy theory in both directions needed some corrections by Cochran-Habegger.

This means we understand the right hand group in (2.2) well. Now for the left hand group.

Theorem 3.2 (Perron, Quinn, GGHKP)**.** *Let X be a compact, 1-connected topological 4-manifold. Then*

$$\pi_0\mathcal{P}(X)=\{[\mathrm{Id}_{X\times I}]\},$$

and hence $\pi_0\,\mathrm{Homeo}_\partial^{PI}(X)=\{[\mathrm{Id}_X]\}$*.*

The references for this theorem are [Per86, Qui86, GGH+23].

Remark 3.3. This result was inspired by Cerf's result that for $d\geq 5$, $\pi_0\mathcal{P}^{\mathrm{Diff}}(X^d)=0$ for $\pi_1(X^d)=\{1\}$.

It follows by combining the previous two theorems that for X^4 closed and 1-connected, we have the following theorem.

Theorem 3.4. *Let X be a closed, 1-connected topological 4-manifold. Then*

$$\begin{aligned}\pi_0(\mathrm{Homeo}^+(X)&\xrightarrow{\cong}\mathrm{Aut}(H_*(X),\lambda_X)\\ f&\mapsto f_*\end{aligned}$$

is an isomorphism of groups.

4. Examples

In the upcoming examples, we use Theorem 3.1 to compute $\pi_0(\mathrm{Homeo}^+(X))$, and then deduce $\pi_0(\mathrm{Homeo}(X))$ as a consequence. A unimodular, symmetric, bilinear form $(\mathbb{Z}^n,L)$ is called a lattice, and the groups of symmetries of a lattice is called its orthogonal group.

In general by considering the action of a homeomorphism on $H_4(X)\cong\mathbb{Z}$ to define a homomorphism to $\mathbb{Z}/2$, we have an exact sequence

$$0\to\pi_0\,\mathrm{Homeo}^+(X)\to\pi_0\,\mathrm{Homeo}(X)\to\mathbb{Z}/2.$$

If the signature of X is nonzero, there are no orientation-reversing homeomorphisms, and hence the map to $\mathbb{Z}/2$ is the zero map.

Example 4.1. Let $X=S^4$. Then since $H_2(S^4)=0$, we have that $\mathrm{Aut}(H_2(S^4),\lambda_X)=\{\mathrm{Id}_{H_2(S^4)}\}$, and so $\pi_0(\mathrm{Homeo}^+(S^4))=\{[\mathrm{Id}]\}$ by Theorem 3.1. Since S^4 admits an orientation-reversing homeomorphism, we have that $\pi_0(\mathrm{Homeo}(S^4))\cong\mathbb{Z}/2=\{[\mathrm{Id}_{S^4}],[R]\}$, where $R\colon S^4\to S^4$ is a reflection.

Example 4.2. Let $X = \mathbb{C}P^2$. Define

$$f\colon \mathbb{C}P^2 \to \mathbb{C}P^2$$
$$[z_0 : z_1 : z_2] \mapsto [\overline{z}_0 : \overline{z}_1 : \overline{z}_2].$$

When restricted to $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$ this gives complex conjugation, which has degree -1. Hence the action on $H_2(\mathbb{C}P^2) \cong \mathbb{Z}$ is multiplication by -1. Since $\mathbb{C}P^2$ has signature 1, every homeomorphism is o.p. So f generates the mapping class group

$$\pi_0 \operatorname{Homeo}(\mathbb{C}P^2) \cong \mathbb{Z}/2.$$

Example 4.3. Let $X = S^2 \times S^2$. Then $H_2(X) \cong \mathbb{Z}^2$, and the intersection form is hyperbolic, represented by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. So $\pi_0 \operatorname{Homeo}^+(X) \cong \operatorname{Aut}(\mathbb{Z}^2, H) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. The isometries here are straightforward to compute by hand. Generators are given by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $R\colon S^2 \to S^2$ be a reflection. Then $R \times \operatorname{Id}_{S^2}$ is an orientation reversing homeomorphism of order two. It follows that we have a split short exact sequence

$$0 \to \pi_0 \operatorname{Homeo}^+(X) \to \pi_0 \operatorname{Homeo}(X) \to \mathbb{Z}/2 \to 0.$$

The sequence splits, so we have that $\pi_0 \operatorname{Homeo}(X) \cong (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$. By computing the orders of elements using the action on H_2 and H_4, I computed that the action in the semi-direct product is such that this group is isomorphic to D_8, the dihedral group of order 8.

Example 4.4. Let $X = \#^n \mathbb{C}P^2$ be a connected sum of canonically oriented complex projective planes. This has $H_2(X) \cong \mathbb{Z}^n$ and λ_X represented by the size n identity matrix. The automorphism groups of this form, or in other words the orthogonal group of this lattice, which is isomorphic to $\pi_0 \operatorname{Homeo}^+(X)$, fits into a short exact sequence

$$\{0\} \to (\mathbb{Z}/2)^n \to \operatorname{Aut}(\mathbb{Z}^n, \operatorname{Id}) \to \Sigma_n \to \{1\}.$$

The sequence splits, so we have a semi-direct product, and the symmetric group Σ_n acts on $(\mathbb{Z}/2)^n$ by permuting the coordinates. It is known as the signed permutation group, or the Coxeter group of type B_n. Its order is $2^n \cdot n!$. Since the signature is nonzero, there is no orientation reversing homeomorphism, so $\pi_0 \operatorname{Homeo}(X) = \pi_0 \operatorname{Homeo}^+(X)$.

Example 4.5. Let $X = E_8$ be the E_8 manifold. It is built by plumbing D^2-bundles over S^2 with Euler number 2 together according to the E_8 Dynkin diagram, and then capping off the boundary with a contractible 4-manifold, who existence was proven by Freedman. The intersection form is the E_8 lattice. Its automorphism group is the Weyl or Coxeter group of type E_8. This is a famous group, whose order is $4! \cdot 6! \cdot 8!$.

In general, Wall gave explicit generators for automorphism groups of unimodular lattices $\operatorname{Aut} \lambda_X$ [Wal63].

5. Proof of Theorem 3.1

I will give an outline of the proof of the following result, which was Theorem 3.1.

Theorem 5.1 (Freedman, Kreck, Quinn, Cochran-Habegger)**.** *Let X be a closed, 1-connected topological 4-manifold. Then*

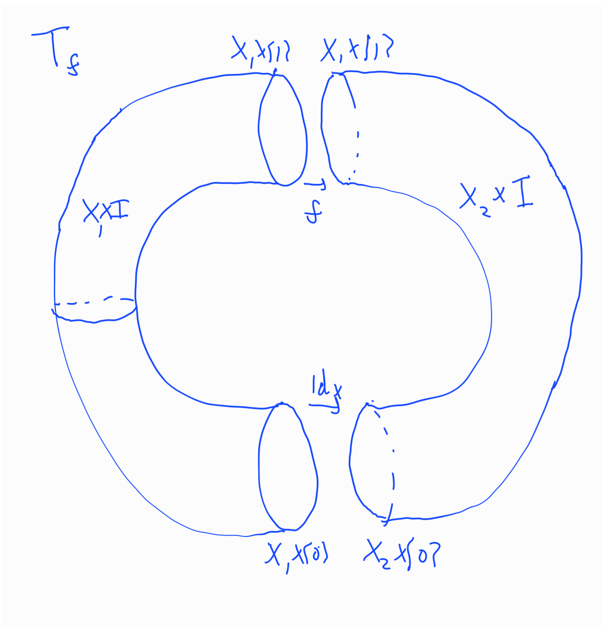
$$\widetilde{\pi}_0(\mathrm{Homeo}^+(X)) \xrightarrow{\cong} \mathrm{Aut}(H_*(X), \lambda_X)$$
$$f \mapsto f_*$$

is an isomorphism of groups.

As mentioned above, surjectivity follows from Freedman's famous classification. We discuss injectivity here.

Let $f\colon X \xrightarrow{\cong} X$ be a homeomorphism with $f_* = \mathrm{Id}_{H_2(X)}$. We consider the *mapping torus* $T_f := X \times [0,1]/(x,0) \sim (f(x),1)$. We think of a homeomorphic space to this, as follows. Let X_1 and X_2 be two copies of X, and consider f as a map $f\colon X_1 \to X_2$. Let $\mathrm{Id}\colon X_1 \to X_2$ be the identity map of X. Then

$$T_f \cong \frac{X_1 \times [0,1] \sqcup X_2 \times [0,1]}{(x_1,1) \sim (f(x_1),1), (x,0) \sim (\mathrm{Id}(x),0)}.$$



The aim is to find an h-cobordism $(V; X_1 \times I, X_2 \times I)$, relative to the boundary $X_1 \times \{0,1\}$, from $X_1 \times I$ to $X_2 \times I$, with the gluing indicated. That is V must be a 6-dimensional manifold with boundary $\partial V = T_f$, and being an h-cobordism means that the inclusion maps $X_1 \times I \to V$ and $X_2 \times I \to V$ are both homotopy equivalences. The h-cobordism theorem for 1-connected manifolds of dimension at least 5 is due to Smale and Kirby-Siebenmann [Sma62, KS77]; see also [Mil65]. It says that h-cobordism are homeomorphic to products, and we may use a given identification of one end of the two cobordisms.

Theorem 5.2 (h-cobordism theorem)**.** *There is a homeomorphism*

$$(G, \mathrm{Id}, g)\colon (V; X_1 \times I, X_2 \times I) \xrightarrow{\cong} (X_1 \times I \times I; X_1 \times I, X_1 \times I)$$

relative to the identity map on $X_1 \times I$ and some homeomorphism $g\colon X_2 \times I \to X_1 \times I$.

Then $g\colon X_2 \times I \xrightarrow{\cong} X_1 \times I$ is a pseudo-isotopy. Because of our initial choice of gluing, i.e. the fact that we took the mapping torus T_f, it is a pseudo-isotopy from f to Id_X. So if we can find the h-cobordism V, we will have proven that f is pseudo-isotopic to Id_X, as desired.

For the remainder of the sketch proof we assume that X is spin. The proof in the non-spin case is similar, replacing BSpin with BSO. Note that

$$\pi_2(X) \cong H_2(X) \cong \mathbb{Z}^n$$

Set

$$K := K(\pi_2(X), 2) = \prod^n \mathbb{C}P^\infty.$$

Since $f_* = \mathrm{Id}_{H_2(X)}$, and is orientation-preserving, and there is a unique spin structure on X, we obtain a lift of the stable normal bundle as follows:

$$\begin{array}{ccc} & \nearrow & K \times \mathrm{BSpin} \\ & & \downarrow \\ T_f & \xrightarrow{\nu_{T_f}} & \mathrm{BSO} \end{array}$$

Hence we obtain an element of the bordism group $\Omega_5^{\mathrm{Spin}}(K)$. An Atiyah-Hirzebruch spectral sequence computation, using that we understand well the homology of K and the bordism groups Ω_q^{Spin} for low values of q, shows the following.

Proposition 5.3. $\Omega_5^{Spin}(K) = 0$.

Hence there exists a null-bordism of T_f

$$\begin{array}{ccc} & \nearrow & K \times \mathrm{BSpin} \\ & & \downarrow \\ W^6 & \xrightarrow{\nu_W} & \mathrm{BSO}\,. \end{array}$$

In this simply-connected setting, it is a result of Kreck that any such null-bordism W is bordant rel. boundary to an h-cobordism V from $X_1 \times I$ to $X_2 \times I$, as above. This completes our outline of the proof of Theorem 3.1.

6. Open problems and the plan for the remainder of the lectures

Here are two salient open problems in the field.

Problem 6.1. Compute $\pi_0 \operatorname{Diff}_\partial X$ for some compact X^4.

Problem 6.2. Compute $\pi_0 \operatorname{Homeo}_\partial X$ for some compact X^4 with $\pi_1(X) \neq \{1\}$.

Here is the plan for the remainder of the lectures.

(1) Discuss the relationship between $\pi_0 \operatorname{Diff}(X)$ and $\pi_0 \operatorname{Homeo}(X)$ for $\pi_1(X) = \{1\}$, giving examples.
(2) Describe the computation of $\pi_0 \operatorname{Homeo}_\partial(X)$ for $\pi_1(X) = \{1\}$ and $\partial X \neq \emptyset$, giving examples.
(3) Introduce barbell diffeomorphisms, which are particularly important examples, and some of their applications.

(4) Discuss the Cerf theory approach to prove that $\pi_0 \operatorname{Homeo}^{PI}(X) = \{[\mathrm{Id}_X]\}$ for X compact and $\pi_1(X) = \{1\}$.
(5) Describe a potentially exotic element $[f] \in \pi_0 \operatorname{Diff}^+(S^4)$, and other interesting diffeomorphisms and open questions relating to the Cerf theory approach.

7. Relationship between smooth and topological mapping class groups

We restrict here to X simply-connected and closed. The following diagram may help us to understand the differences between the smooth and topological mapping class groups.

$$\begin{array}{ccccc}
\pi_0 \operatorname{Diff}^{PI}(X) & \rightarrowtail & \pi_0 \operatorname{Diff}(X) & \twoheadrightarrow & \widetilde{\pi}_0 \operatorname{Diff}(X) \\
\downarrow\!\!\!\downarrow & & \downarrow & & \downarrow \\
0 = \pi_0 \operatorname{Homeo}^{PI}(X) & \rightarrowtail & \pi_0 \operatorname{Homeo}(X) & \xrightarrow{\cong} & \widetilde{\pi}_0 \operatorname{Homeo}(X)
\end{array}$$

The proof of injectivity in Theorem 3.1 works just as well in the smooth category, explaining the right hand vertical injection. The left hand surjection and the isomorphism are just because $\pi_0 \operatorname{Homeo}^{PI}(X) = 0$.

Theorem 7.1 (Ruberman)**.** *There exists a closed, simply-connected 4-manifold X and a diffeomorphism $f\colon X \to X$ such that $[f] \neq 0 \in \pi_0 \operatorname{Diff}^{PI}(X)$. Moreover, there exists such an X with $\pi_0 \operatorname{Diff}^{PI}(X)$ infinitely generated.*

We describe Ruberman's example [Rub98, Rub99] next. Recall the K_3 surface, which is a famous smooth, close, 1-connected 4-manifold. It generates the 4-dimensional smooth spin cobordism group, for example. It is given by

$$\{[x, y, z, w] \in \mathbb{C}P^3 \mid x^4 + y^4 + z^4 + w^4 = 0\}.$$

There are several other descriptions.

Example 7.2. Let

$$X_0 := \#^3 \mathbb{C}P^2 \#^{20} \overline{\mathbb{C}P}^2$$

and

$$W := \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# \overline{\mathbb{C}P}^2.$$

Write $\xi_\pm = (1, \pm 1, 1) \in \mathbb{Z}^3 \cong H_2(W)$. Note that $\xi_\pm \cdot \xi_\pm = -1$. Hence a tubular neighbourhood is diffeomorphic to $\overline{\mathbb{C}P}^2 \smallsetminus \mathring{D}^4$.

Consider complex conjugation as a map $\overline{\mathbb{C}P}^2 \to \overline{\mathbb{C}P}^2$. Isotope it to fix a 4-ball, and remove that 4-ball, to obtain

$$g\colon \overline{\mathbb{C}P}^2 \smallsetminus \mathring{D}^4 \to \overline{\mathbb{C}P}^2 \smallsetminus \mathring{D}^4.$$

Implanting this using $\xi_\pm$, recalling that $\nu\xi_\pm \cong \overline{\mathbb{C}P}^2 \smallsetminus \mathring{D}^4$, we obtain

$$g_{\xi_\pm}\colon W \to W.$$

Next, for $p \geq 1$, let

$$X_p := E(2, p+1),$$

the result of a log transform on $E(2) = K_3$. There exists a diffeomorphism

$$\varphi_p \colon X_p \# W \xrightarrow{\cong} X_0 \# W.$$

We also define, for $p \geq 0$,

$$\rho_p = (\mathrm{Id}_{X_p} \# g_{\xi_+}) \circ (\mathrm{Id} \# g_{\xi_-}) \colon X_p \# W \to X_p \# W.$$

Then we set

$$f_p := \varphi_p \circ \rho_p \circ \varphi_p^{-1} \circ \rho_0^{-1} \colon X_0 \# W \xrightarrow{\cong} X_0 \# W = \#^4 \mathbb{C}P^2 \#^{22} \overline{\mathbb{C}P}^2.$$

Ruberman showed that f_p is not smoothly isotopic to the identity. He used Donaldson's invariants, but this can nowadays be done with *family Seiberg-Witten invariants*. Using this approach many new examples have been constructed; see in particular [BK20]. This acts trivially on $H_2(X_0 \# W)$, and so is pseudo-isotopic and topologically isotopic to the identity. In fact, by varying p we get that $\pi_0 \operatorname{Diff}^{PI}(X_0 \# W)$ is infinitely generated.

One can compare with the following theorem.

Theorem 7.3 (Baraglia [Bar23], Konno [Kon24])**.** *There exists a closed, simply-connected 4-manifold X with a summand*

$$(\mathbb{Z}/2)^\infty \subseteq \pi_0 \operatorname{Diff}(X)_{\mathrm{ab}}.$$

That is, the entire mapping class group, and not just the Torelli subgroup of diffeomorphisms acting trivially on homology, can be infinitely generated.

There is another more recent example of a nontrivial element of $\pi_0 \operatorname{Diff}^{PI}(X)$, due to Kronheimer-Mrowka, that was influential.

Example 7.4. We let $X := K_3 \# K_3$, and consider the connected sum sphere $S^3 \subseteq X$. Let $U \cong S^3 \times [0,1]$ be a neighbourhood of this S^3. Let $\rho_\theta \colon S^3 \to S^3$ be rotation of S^3 through an angle θ about a fixed axis. Define

$$\begin{aligned} \Phi \colon S^3 \times [0,1] &\to S^3 \times [0,1] \\ (x,t) &\mapsto (R_{2\pi t}(x), t). \end{aligned}$$

Define

$$\begin{aligned} f \colon X &\to X \\ x &\mapsto \begin{cases} \Phi(x) & x \in U \\ x & x \notin U. \end{cases} \end{aligned}$$

This is called Dehn twist on $S^3 \subseteq X$. Then $f_* = \mathrm{Id}_X \colon H_2(X) \to H_2(X)$, so f is topologically isotopic to the identity. However Kronheimer and Mrowka [KM20] proved that f is not smoothly isotopic to the identity of X. Moreover, Jianfeng Lin [Lin20] showed that f does not become smoothly isotopic to the identity even after connected summing with one $S^2 \times S^2$.

Several authors studied analogues of the Dehn twist for other Seifert fibred spaces and for 4-manifolds with boundary, too.

Next we address the fact that the right hand vertical map $\widetilde{\pi}_0 \operatorname{Diff}(X) \to \widetilde{\pi}_0 \operatorname{Homeo}(X)$ in the diagram above is not surjective in general.

Theorem 7.5 (Friedman-Morgan [FM88], Donaldson [Don90])**.** *There exist closed, simply-connected X^4, such that the map $\pi_0 \operatorname{Diff}(X) \to \pi_0 \operatorname{Homeo}(X)$ is not surjective.*

Examples include $K3$, homotopy $K3$s, and the Dolgachev surface. One can realise every self-isometry of the intersection pairing by a homeomorphism, by Freedman's theorem. The proof of the theorem above proceeds by showing that certain isometries cannot be realised smoothly, because of the way a diffeomorphism must act on the Seiberg-Witten basic classes, for example.

Here is a more recent result along the same lines, but for non-simply connected 4-manifolds.

Theorem 7.6 (Galvin [Gal24])**.** *Suppose X^4 is closed and $H_1(X;\mathbb{Z}/2) \neq 0$. Then $\pi_0 \operatorname{Diff}(X\#(S^2 \times S^2)) \to \pi_0 \operatorname{Homeo}(X\#(S^2 \times S^2))$ is not surjective, even after adding arbitrarily many copies of $S^2 \times S^2$.*

The proof uses a version of the Kirby-Siebenmann invariant. In this context work of Casson and Sullivan pre-dated Kirby-Siebenmann's work, and so the invariant that obstructs non-stable-smoothability of homeomorphisms of 4-manifolds is called the *Casson-Sullivan* invariant.

8. Mapping class groups of 1-connected 4-manifolds with boundary

The pseudo-isotopy theorem of Perron and Quinn works just as well for 1-connected compact 4-manifolds with boundary. However the pseudo-isotopy classification is more subtle. In joint work with Orson, we figured out the details of this Our work builds on important work of Saeki in this direction, who studied the analogous stable smooth mapping class group for smooth 1-connected 4-manifolds with connected nonempty boundary. We also rely on Boyer's work. The following description follows that in [OP22].

When $\partial X = \emptyset$, we have seen that if two o.p. homeomorphisms of X induce the same isometry of the intersection form then they are isotopic. When X has nonempty boundary, we need to consider a refinement of $\operatorname{Aut}(H_2(X), \lambda_X)$ to capture the algebraic data of a homeomorphism. A map $f \in \operatorname{Homeo}^+(X, \partial X)$ determines a homomorphism $\Delta_f \colon H_2(X, \partial X) \to H_2(X)$ called a *variation*, defined by $[x] \mapsto [x - f(x)]$. Using that X has Poincaré-Lefschetz duality, Saeki [Sae06] showed that Δ_f satisfies an additional condition, making it what we call a *Poincaré variation*. There is a binary operation on the set of Poincaré variations, together with which they form a group $\mathcal{V}(H_2(X), \lambda_X)$. The map $f \mapsto f_*$ factors through this group via homomorphisms:

$$\pi_0 \operatorname{Homeo}^+(X, \partial X) \xrightarrow{f \mapsto \Delta_f} \mathcal{V}(H_2(X), \lambda_X) \xrightarrow{\Delta \mapsto \operatorname{Id} - \Delta \circ j} \operatorname{Aut}(H_2(X), \lambda_X),$$

where $j \colon H_2(X) \to H_2(X, \partial X)$ is the quotient map. In general Δ_f contains more information than f_*, although if ∂X is a $\mathbb{Q}HS^3$ or a $\mathbb{Q}H(S^1 \times S^2)$ then the second map is an isomorphism. Saeki [Sae06] used $\mathcal{V}(H_2(X), \lambda_X)$ to describe the smooth stable mapping class group for simply connected 4-manifolds with nonempty, connected boundary.

Example 8.1. Let X be a 1-connected 4-manifold with boundary $T^3 = S^1_1 \times S^1_2 \times S^1_3$, the 3-torus. For example, such a 4-manifold arises by adding 0-framed 2-handles to D^4 along the Borromean rings. Rotating the S^1_1 direction yields a loop of diffeomorphism in $\pi_1 \operatorname{Diff}^+(T^3)$. We can apply this in a collar neighbourhood of ∂X to obtain a *generalised*

Dehn twist $f\colon X \xrightarrow{\cong} X$. Since f is supported in $\partial X \times I$, it acts trivially on $H_2(X)$. However, the curve S^1_2 bounds a nontrivial relative homology class $x_2 \in H_2(X, \partial X)$. The difference $x_2 - f(x_2)$ is the image under the injection $H_2(T^3) \rightarrowtail H_2(X)$ of the class $[S^1_1 \times S^1_2] \in H_2(T^3)$. Hence the variation Δ_f is nontrivial, and thus f is not isotopic rel. boundary to the identity. In contract, note that if the boundary is permitted to move in an isotopy, then f is isotopic to the identity.

When ∂X has more than one connected component and X admits a spin structure, there is a further invariant that does not appear in the closed case nor when the boundary is connected. For $f \in \mathrm{Homeo}^+(X, \partial X)$ we may compare a topological spin structure $\mathfrak{s}$ on X with the induced spin structure $f^*\mathfrak{s}$. The two agree on ∂X because f fixes the boundary pointwise. There is a free, transitive action of $H^1(X, \partial X; \mathbb{Z}/2)$ on the set of isomorphism classes of spin structures on X that agree on ∂X, and we denote by $\Theta(f) \in H^1(X, \partial X; \mathbb{Z}/2)$ the class representing the difference between $\mathfrak{s}$ and $f^*\mathfrak{s}$.

Example 8.2. Let $X := S^3 \times I$, and let $f\colon X \to X$ be the Dehn twist that we introduced earlier in the context of the connected sum sphere in $K3\#K3$. This diffeomorphism necessarily acts trivially on $H_2(X) = 0$, has trivial Poincaré variation for the same reason. However, f is not (pseudo-) isotopic rel. boundary to Id_X, because it acts nontrivially on the relative spin structures of X (of which there are two).

In joint work with Orson, we showed that these invariants describe the entire topological mapping class group.

Theorem 8.3. *Let $(X, \partial X)$ be a compact, simply connected, oriented, topological 4-manifold.*

(i) When X is spin, the map $f \mapsto (\Theta(f), \Delta_f)$ induces a group isomorphism

$$\pi_0 \mathrm{Homeo}^+(X, \partial X) \xrightarrow{\cong} H^1(X, \partial X; \mathbb{Z}/2) \times \mathcal{V}(H_2(X), \lambda_X).$$

(ii) When X is not spin, the map $f \mapsto \Delta_f$ induces a group isomorphism

$$\pi_0 \mathrm{Homeo}^+(X, \partial X) \xrightarrow{\cong} \mathcal{V}(H_2(X), \lambda_X).$$

If the boundary is nonempty, then the $+$ is superfluous: every homeomorphism that acts as the identity on the boundary is o.p. In order to state the result for the case of empty and nonempty boundary simultaneously, we leave the $+$ in the notation.

In Example 8.1 we gave an example of a nontrivial diffeomorphism that acts trivially on $H_2(X)$, i.e. that lies in the Torelli subgroup of the mapping class group of X. The question then arose whether all the elements of the Torelli group can be smoothly realised.

Theorem 8.4 (Galvin-Ladu [GL23])**.** *There exists a 1-connected, smooth, compact 4-manifold X (which they construct explicitly) together with a homeomorphism $f\colon X \xrightarrow{\cong} X$ in the Torelli subgroup of $\pi_0(\mathrm{Homeo}^+(X, \partial X))$ that is not topologically isotopic to any diffeomorphism of X.*

This is in contrast to the closed case, when every element of the Torelli subgroup is isotopic to the identity, hence is certainly isotopic to a diffeomorphism.

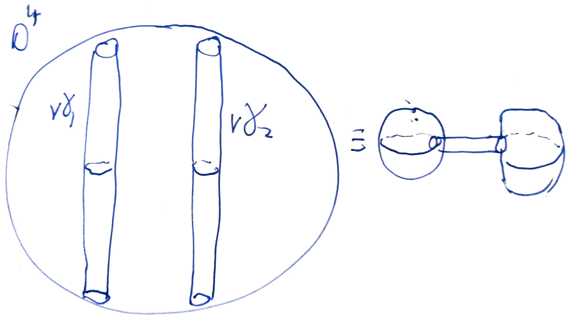


FIGURE 1. Schematic drawings of the model barbell $\mathcal{B}$, as $D^4 \setminus (\nu\gamma_1 \sqcup \nu\gamma_2)$ and as $(S^2 \times D^2) \natural (S^2 \times D^2)$.

9. Barbell diffeomorphisms

Some very interesting examples of diffeomorphisms are given by barbell diffeomorphisms. The *model barbell* is

$$\mathcal{B} := (S^2 \times D^2) \natural (S^2 \times D^2) \cong D^4 \setminus (D^1 \times \mathring{D}^3 \sqcup D^1 \times \mathring{D}^3),$$

where the two removed 4-balls are neighbourhoods of properly embedded arcs, which we denote $\nu\gamma_1$ and $\nu\gamma_2$. So

$$\mathcal{B} \cong D^4 \setminus (\nu\gamma_1 \sqcup \nu\gamma_2).$$

Note that

$$D^4 \setminus (\nu\gamma_1 \sqcup \nu\gamma_2) \subseteq D^4 \setminus \nu\gamma_2.$$

We consider a loop of embeddings of a thickened arc in $D^4 \setminus \nu\gamma_2$, starting and ending at $\nu\gamma_1$. That is, we consider an element of

$$\pi_1\big(\operatorname{Emb}_{S^0 \times D^3}(D^1 \times D^3, D^4 \setminus \nu\gamma_2); \nu\gamma_1\big).$$

The loop in question is illustrated in Figure 2. Isotopy extension gives give to a homomorphism

$$\pi_1\big(\operatorname{Emb}_{S^0 \times D^3}(D^1 \times D^3, D^4 \setminus \nu\gamma_2); \nu\gamma_1\big) \to \pi_0 \operatorname{Diff}_\partial(D^4 \setminus (\nu\gamma_1 \sqcup \nu\gamma_2)) \cong \pi_0 \operatorname{Diff}_\partial(\mathcal{B}).$$

The image of our loop of embeddings under this map is the *barbell diffeomorphism*, introduced by Budney and Gabai,

$$\phi \colon \mathcal{B} \xrightarrow{\cong} \mathcal{B}.$$

The diffemorphism ϕ has nontrivial Poincaré variation

$$e_1 \wedge e_2 \in \wedge^2 \mathbb{Z}^2 \cong \wedge^2 H^1(\partial X).$$

In fact by [OP22], or the last section of [KMPW24], this group is isomorphic to $\pi_0 \operatorname{Homeo}_\partial(\mathcal{B})$, and so $\pi_0 \operatorname{Homeo}_\partial(\mathcal{B}) \cong \wedge^2 \mathbb{Z}^2 \cong \mathbb{Z}$.

Budney and Gabai [BG19] used the barbell diffeomorphism to obtain nontrivial diffeomorphisms in $\pi_0 \operatorname{Diff}_\partial(S^1 \times D^3)$, and they showed in [BG23] that their diffeomorphisms are nontrivial in $\pi_0 \operatorname{Homeo}_\partial(S^1 \times D^3)$. To construct their diffeomorphisms, they considered an embedding

$$\psi \colon \mathcal{B} \to S^1 \times D^3 \cong S^1 \times D^2 \times I \cong ((I \times D^2)/\sim) \times I.$$

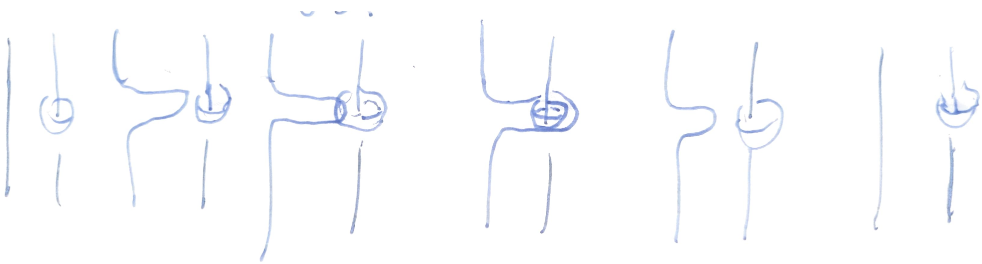


FIGURE 2. A loop of embeddings of $D^1 \times D^3$ in $D^4 \setminus \nu\gamma_2$. The arc $D^1 \times \{0\} \subseteq \nu\gamma_1 \subseteq D^4 \setminus \nu\gamma_2$ is pictured on the left. As time progressed, it swings around the meridional 2-sphere of $\nu\gamma_2$, and then returns to its starting point.

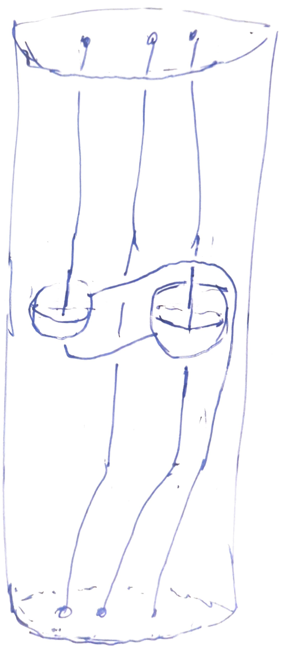


FIGURE 3. An embedding ψ of the model barbell $\mathcal{B}$ into $S^1 \times D^3$, such that the resulting diffeomorphism of $S^1 \times D^3$ is nontrivial in $\pi_0 \operatorname{Homeo}_\partial^{PL}(S^1 \times D^3)$.

In Figure 3, we draw the desired embedding as much as possible in $((I \times D^2)/\sim) \times \{1/2\}$. The 2-spheres intersect this 3-dimensional slice in circles that are capped with a disc in $((I \times D^2)/\sim) \times [0, 1/2]$ and a disc in $((I \times D^2)/\sim) \times [1/2, 1]$.

On the image of ψ, apply the barbell diffeomorphism ϕ. Extend it by the identity to obtain a diffeomorphism

$$\Phi \colon S^1 \times D^3 \xrightarrow{\cong} S^1 \times D^3.$$

FIGURE 4. An interesting barbell implantation in S^4.

We have described Budney-Gabai's construction, but the same result was proven at the same time by Watanabe.

Theorem 9.1 (Budney-Gabai, Watanabe)**.** $[\Phi] \neq 0 \in \pi_0 \operatorname{Homeo}_\partial^{PI}(S^1 \times D^3)$.

Let $U \subseteq S^4$ denote the trivial 2-knot. Note that $S^4 \setminus \nu U \cong S^1 \times D^3$, which contains $\{1\} \times D^3$. Budney-Gabai observed that $\Phi(\{1\} \times D^3)$ is a *knotted 3-ball*, that is isotopic to a standard one, but is not isotopic to a standard 3-ball embedded in S^4 rel. boundary.

Conjecture 9.2. *The barbell implantation shown in Figure 4 gives a nontrivial element of* $\pi_0 \operatorname{Diff}^+(S^4)$.

10. CERF THEORY

How can we hope to understand pseudo-isotopies? A key idea is to use *Cerf theory*, which is essentially 1-parameter Morse theory. I will use this theory to explain the idea of Quinn's proof of Theorem 3.2. It is a useful theory for analysing pseudo-isotopies in general, including in the smooth category. I will therefore close by explaining some applications to smooth mapping class groups, and describing some potential future applications.

I should mention that Perron's proof of Theorem 3.2 is also very nice, but so far the approach has found fewer uses in other work, so we focus on Quinn's approach. Note that I translated Perron's article into English; the result is available on my website.

Let X be a smooth, closed, 1-connected 4-manifold, and let $F\colon X \times I \to X \times I$ be a smooth pseudo-isotopy. We have two Morse functions on $X \times I$ without critical points. First,

$$g_0 := \operatorname{pr}_2\colon X \times I \to I$$

given by projection to the second coordinate. Next,

$$g_1 := \operatorname{pr}_2 \circ F\colon X \times I \to I$$

also has no critical points.

Recall that a Morse function $g\colon X \times I \to \mathbb{R}$ is a smooth function such that every critical point $p \in X \times I$, the Hessian matrix of second partial derivatives is nondegenerate (this condition is coordinate independent). Near a critical point we have coordinates $(x_1, \ldots, x_5)$ and $h \in \{0, \ldots, 5\}$ such that

$$g(\underline{x}) = g(p) - x_1^2 - \cdots - x_h^2 + x_{h+1}^2 + \cdots + x_5^2.$$

Here h is the *index* of the critical point.

Next, any two Morse functions on a manifold, so in particular on $X \times I$, can be connected by a 1-parameter path of *generalised Morse functions*

$$g_t \colon X \times I \to \mathbb{R}.$$

This is a smooth path of functions, such that for each $t \in I$, either g_t is a Morse function, or g_t is a Morse function everywhere except possibly at one critical point p, which is a *birth/death* type singularity. At p we have coordinates $(x_1, \ldots, x_5)$ such that in the coordinates

$$g_t(\underline{x}) = g(p) + x_1^3 - x_2^2 - \cdots - x_h^2 + x_{h+1}^2 + \cdots + x_5^2.$$

In a 1-parameter family near p, we can assume that g_{t+s} has the form

$$g_{t+s}(\underline{x}) = g(p) + x_1^3 \pm s x_1 - x_2^2 - \cdots - x_h^2 + x_{h+1}^2 + \cdots + x_5^2.$$

Then $\pm s$ is negative we see two Morse critical points, and when $\pm s$ is positive there are no critical points in the coordinate neighbourhood.

There is a corresponding 1-parameter family of *gradient-like vector fields* (glvf) ξ_t on $X \times I$, for $t \in [0, 1]$, such that ξ_t is a glvf for g_t. Using the glvf, we obtain, for each t where g_t is Morse, a handle decomposition of $X \times I$, where each critical point of index h gives rise to an h-handle. Trajectories of ξ_t between handles of index $(h + 1)$ and h correspond to attaching data, namely intersections of the attaching sphere of the index $(h+1)$-handle with the belt sphere of the index h handle. Trajectories of ξ_t between two index h handles can occur at isolated t-values, and correspond to handle slides.

Some of the data of g_t can be presented in a *Cerf graphic*, as shown in Figure 5. Here we plot with two axes, $t \in [0, 1]$, and the interval I in which our generalised Morse functions g_t take values. For each $t \in [0, 1]$, we consider all the critical points of g_t, namely $P_t := \{p_i \mid Dg_t(p_i) = 0\}$. Then we plot the critical values $g_t(P_t)$ at t. Doing this for every $t \in [0, 1]$ gives rise to the Cerf graphic. One cannot recover the Morse function from the Cerf graphic, but it turns out to contain some useful information that makes it easier to describe the qualitative features of our family g_t, and to describe the key features of the deformations that we wish g_t to undergo.

We can assume, generically, that two critical points have the same critical value at isolated value of t, and that in this case the critical lines in the Cerf graphic intersect transversely. Important data that is not shown in the graphic is the trajectories between critical points. We will not indicate this data in the Cerf graphic, although one is free to invent schema to do so.

Whenever we change (g_t, ξ_t), we speak of a *deformation* of the family. If we want to show that the pseudo-isotopy is isotopic to the identity, our aim will be to deform (g_t, ξ_t), without changing (g_i, ξ_i) for $i = 0, 1$, to a family with no critical points, i.e. one with empty Cerf graphic. Why is this our aim?

Given a glvf without critical points, we can integrate it in order to start with $X \times \{0\}$, and flow it along the integral curves to obtain a self-homeomorphism of $X \times I$. There is also a topological version of this, which we will not investigate.

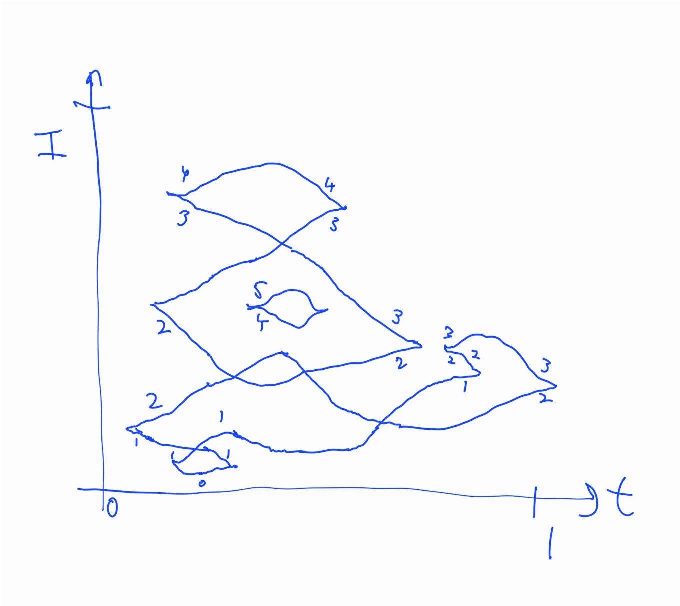


FIGURE 5. A generic Cerf graphic. I label the critical lines with the indices of the corresponding critical points.

If we integrate ξ_0, we obtain a homeomorphism of $X \times I$ isotopic to $\mathrm{Id}_{X\times I}$. If we integrate ξ_1, we obtain a homeomorphism of $X \times I$ isotopic to $F\colon X \times I \to X \times I$. Moreover, if ξ_t has no critical points, then for each t we obtain a homeomorphism $F_t\colon X \times I \to X \times I$, with $F_0 = \mathrm{Id}_X$ and $F_1 = F$. Since ξ_t depends continuously on t, we obtain an isotopy between $\mathrm{Id}_{X\times I}$ and F, as desired. It follows that it suffices to deform (g_t, ξ_t) rel. $t = 0, 1$ to a family with no critical points, as asserted.

Proposition 10.1 (Cerf, Hatcher-Wagoner)**.** *For X with $\pi_1(X) = \{1\}$, there exists a deformation of (g_t, ξ_t) to a family with a nested eye graphic.*

Here, a *nested eye graphic* is one of the form shown in the next figure. All births come first, one at a time, and they are all births of cancelling index 2 and 3 pairs of critical points. There are no rearrangements of critical values, and no handle slides, i.e. there are no 2/2 and no 3/3 trajectories. The births and deaths are independent, meaning that at each birth time and each death time, there are no trajectories that go either from or to the birth or death point, from or to another critical point. Each circle in the figure is called an *eye*.

The procedure to arrange the nested eye graphic is somewhat complicated, involving a careful use of codimension 2 singularities. These deformation are to generalised Morse functions as generalised Morse functions are to ordinary Morse functions. The procedure works in all dimensions at least 4. Here we are going to quote it and not attempt to justify it.

Figure 6 shows a nested eye Cerf graphic, and it indicates the times at which births and deaths appear. The additional labels will be explained carefully in the next section.

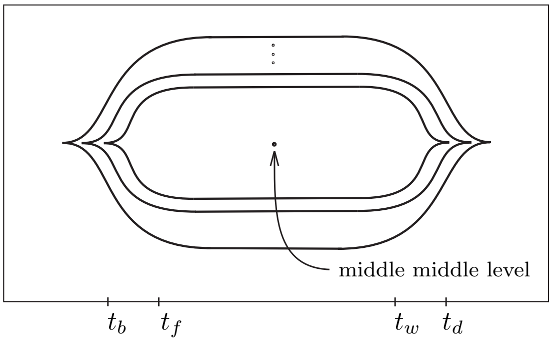


FIGURE 6. A nested eye Cerf graphic. Each loop is called an *eye*. The middle-middle level is also indicated, as well as the birth time t_b, the death time t_d, the finger move time t_f and the Whitney move time t_w. (Picture from [GGH+23].)

11. THE MIDDLE-MIDDLE LEVEL

In the proof of the h-cobordism theorem, for 5-dimensional h-cobordisms, we put a Morse function $f \colon W \to I$ on our h-cobordism W, and then perform handle trading to arrange that there are only 2- and 3-handles. We then consider the *middle level* $f^{-1}(1/2)$, in which we see two sets of mutually disjointly embedded 2-spheres. The first, $A_1, \dots, A_k$, are the ascending spheres of the index 2 critical points. In handle language, they are the belt spheres of the 2-handles. The second set of mutually disjointly embedded 2-spheres, $B_1, \dots, B_k$, are the descending spheres of the index 3 critical points, or the attaching spheres of the 3-handles. After some handle sliding we may assume that the intersection numbers of these spheres are $A_i \cdot B_j = \delta_{ij} \in \mathbb{Z}$, since W is an h-cobordism. The goal of the proof is to arrange by an isotopy that the geometric intersection numbers of these spheres agrees with the algebraic intersection numbers. Then we can cancel the critical points in 2-3 pairs, to obtain a cobordism with no critical points, which is hence a product. The proof of the pseudo-isotopy theorem is analogous to this, but one level of complexity higher.

We also consider the middle level, $g_t^{-1}(1/2)$, for each t. It turns out that the data of the pseudo-isotopy can be captured in the *middle-middle level*, which is the inverse image

$$M := g_{1/2}^{-1}(1/2).$$

This is the inverse image of the central point shown in Figure 6. This is a 4-manifold diffeomorphic to $X\#^k(S^2 \times S^2)$, where k is the number of 2-3 pairs.

Right after the birth time t_b, in the middle level $g_{t_b+\varepsilon}^{-1}(1/2) \cong X\#^k(S^2 \times S^2)$, we see the ascending 2-spheres $A_1, \dots, A_k$ of the index 2-critical points, of the form $\{\mathrm{pt}\} \times S^2$ in each of the $S^2 \times S^2$ summands. We also see the descending spheres $B_1, \dots, B_k$ of the index 3 critical points, of the form $S^2 \times \{\mathrm{pt}\}$ in each of the $S^2 \times S^2$ summands. Note that A_i and B_j intersect in exactly δ_{ij} points, so the critical points are in cancelling position. The situation is the same just before the death time, in $g_{t_d-\varepsilon}^{-1}(1/2) \cong X\#^k(S^2 \times S^2)$.

In between, looking at the ascending and descending spheres in $g_t^{-1}(1/2)$, for $t \in [t_b + \varepsilon, t_d - \varepsilon]$, we can assume that the $\{A_i\}$ stay fixed, and the $\{B_j\}$ move around by an isotopy,

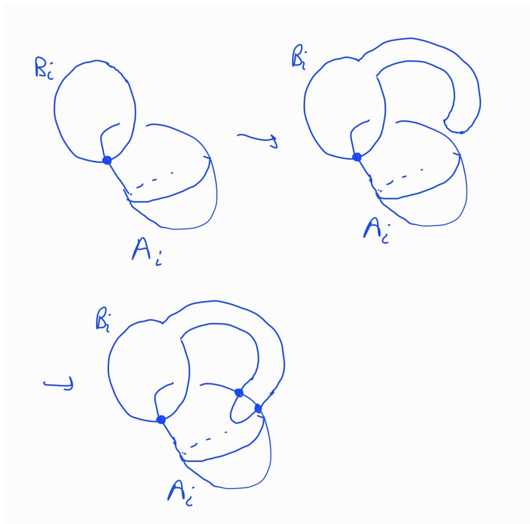


FIGURE 7. A finger move.

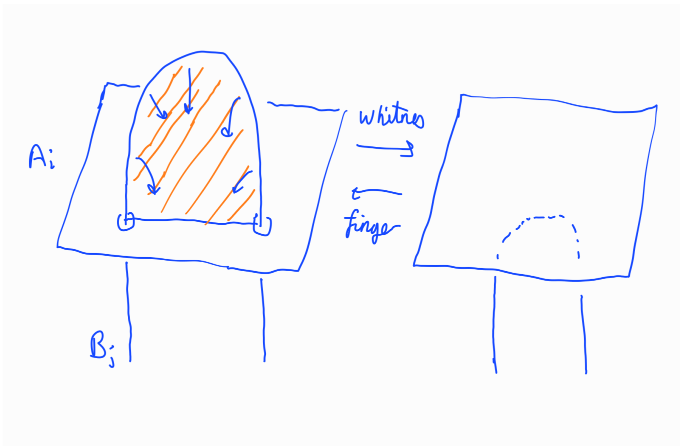


FIGURE 8. The Whitney move goes from left to right, and the finger move reverses it, going from right to left.

while staying pairwise disjoint and embedded. During this motion extra intersections between the $\{A_i\}$ and the $\{B_j\}$ can appear, and later disappear. At the start and end we know that the spheres are in cancelling position.

We can also assume that all the extra intersections are created first, and at the same time t_f, by *finger moves*; see Figure 7.

Then the extra intersections are all removed simultaneously by Whitney moves, at the time t_w. A Whitney move is guided by a Whitney disc W; see Figure 8.

Also, after a finger move there is a finger-move Whitney disc V, with the property that performing a Whitney move on that finger-move disc V undoes the finger move. In the middle-middle level, we see two collections of discs, the Whitney discs $\{W_\ell\}$, that guide

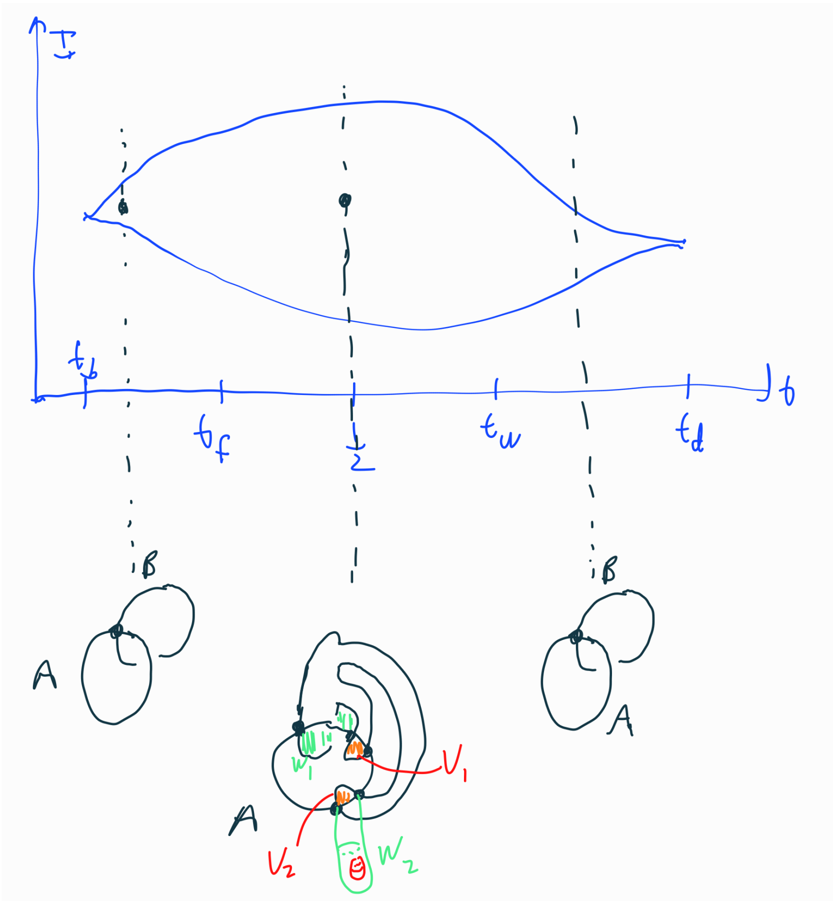


FIGURE 9. A Cerf graphic with one eye, together with schematics of the spheres in the middle levels at different time values. In the schematic, spheres are represented by circles. The key data is contained in the middle-middle level, where we see finger and Whitney discs that guide the pseudo-isotopy in the past and future respectively. The Whitney discs and the finger discs shown are distinct. First, W_1 and V_1 pair up the double points in a different way. Secondly, the Whitney discs V_2 and W_2 are assumed to have the same boundary (although this need not be the case, a priori), but thei union represent a nontrivial element in $\pi_2(M)$. This is supposed to be indicated by the small red sphere.

the Whitney moves that will happen at time $t = t_w$. After the Whitney move the $\{A_i\}$ and the $\{B_j\}$ are in cancelling position. We also see the finger-move discs $\{V_m\}$, which have the property that reversing time leads to Whitney moves using them. These Whitney moves also remove all excess intersections between the $\{A_i\}$ and $\{B_j\}$.

All of the important data about the pseudo-isotopy can now be captured by the spheres $\{A_i\}$ and $\{B_j\}$ in the middle-middle level, which intersect in δ_{ij} times algebraically, together with the two collections of Whitney discs $\{V_m\}$ and $\{W_\ell\}$. Each of these collections

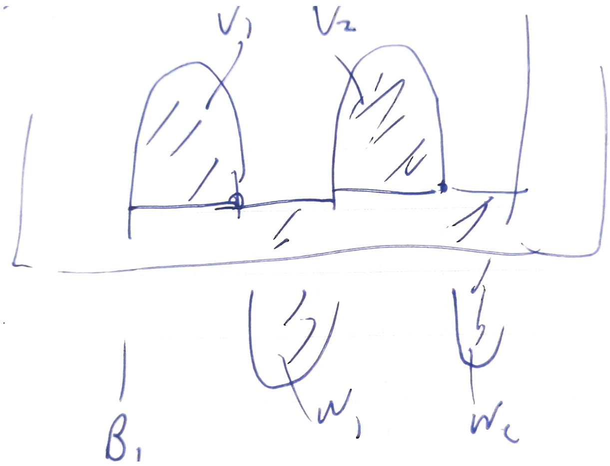


FIGURE 10. Finger and Whitney discs in standard position, with boundaries forming an arc in both A and B.

have mutually pairwise disjoint interiors, and has the property that using them for Whitney moves places the $\{A_i\}$ and $\{B_j\}$ in cancelling position.

In order to prove Theorem 3.2, Quinn showed the following (modulo a correction given in [GGH+23]).

Proposition 11.1. *We can perform a deformation to remove/cancel the innermost eye of the nested eye Cerf graphic.*

In order to be able to remove the innermost eye, it suffices that there is a unique trajectory between the index 3 and the index 2 critical point corresponding to that eye. That is, the critical points must be in cancelling position the entire time interval for which they exist. Then we can perform a 1-parameter families worth of cancellations, to entirely remove the eye. Note that the proposition implies Theorem 3.2, because we inductively close the eyes, always working on the innermost eye in the Cerf graphic.

To prove Proposition 11.1, Quinn arranged that the finger and Whitney discs can be arranged, using his 'sum square move' and the disc embedding theorem, into the position shown in Figure 10. In [GGH+23] we showed that one can only really do this modulo homeomorphisms supported in D^4, and so one also needs an application of the Alexander trick to conclude that $\pi_0 \operatorname{Homeo}_\partial^{PI}(X) = 0$.

12. SMOOTH PSEUDO-ISOTOPIES

I will close by describing some past, and potential future, applications of Cerf theory to the analysis of smooth pseudo-isotopies, in particular highlighting some interesting examples.

Without the disc embedding theorem (which can only be applied in the topological category), it is not possible to simplify the situation as much as in Proposition 11.1. A key difficulty is that the V and W discs can intersect each other, and it is unclear how to remove these intersections, particularly if they are algebraically nontrivial.

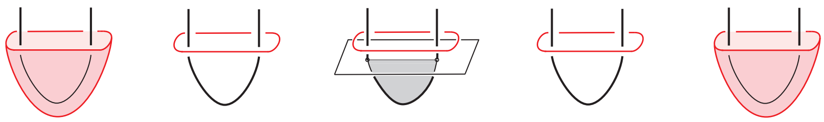


FIGURE 11. A Whitney sphere. Such a sphere lives in a neighbourhood of any Whitney disc. (Picture from [GGH+23].)

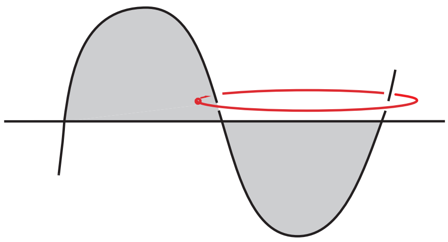


FIGURE 12. A middle-middle level, with a portion of A and B shown, together with one finger disc V (left) and one Whitney disc W' (right). The red circle is the image in the present of the Whitney sphere S_W of W'. Tube W' into S_W to obtain a new Whitney disc W. Note that the interior of V intersects the interior of W in a single point. (Picture from [GGH+23].)

However, by analysing the (A, B, V, W) data in the case of a single eye, we were able to prove the following.

Theorem 12.1 ([KMPW24]). *Let $f\colon X \to X$ be a diffeomorphism of a 1-connected, compact smooth 4-manifold X, and suppose that f is 1-stably isotopic to Id_X. Then there exists a contracible submanifold $U \subseteq X$ such that f is smoothly isotopic to f', where f' is supported on U.*

Using this theorem we constructed new nontrivial diffeomorphisms on contractible 4-manifolds.

Next, we give two particularly interesting examples of pseudo-isotopies. First note that given a Whitney disc pairing two double points of $A \cap B$, there is a *Whitney sphere*, shown in Figure 11. This sphere is disjoint from A, B, and the Whitney disc.

Example 12.2. Let $X = S^4$. Build a Cerf family with finger-Whitney data (A, B, V, W) as described in Figure 12. This is a family with a single eye, and hence a single A-B pair. Integrating ξ_1 yields a pseudo-isotopy $F\colon S^4 \times I \to S^4 \times I$.

Question 12.3. Is $f := F|_{S^4 \times \{1\}}$ smoothly isotopic to the identity?

This turns out to be the same, up to isotopy, as the barbell implantation from Figure 4. This pseudo-isotopy formulation first appeared in [GGH+23].

Gabai-Gay-Hartman claim they can prove Conjecture 9.2, that this is in fact nontrivial. Their proposed proof, involves a combinatorial obstruction that counts intersections between V and W discs, and showing that this is a robust invariant of the diffeomorphism, and not just an invariant of the choice of pseudo-isotopy and the auxiliary Cerf data. At the time of writing only half of their claimed proof is available.

Question 12.4. The diffeomorphism from the previous example can be arranged to be supported on D^4, and hence can be implanted in any 4-manifold X. For which X is it nontrivial?

Example 12.5. In this example $X = S^1 \times S^2 \times I$. Again we can build a pseudo-isotopy, by building a family of Cerf data, which in turn is (sufficiently) determined by finger and Whitney data. It will be a family with a single eye. Take the A and B spheres in standard position. Perform a single finger move on B, through A, in such a way that the double point loop is homotopic to $S^1 \times \{p\} \times \{1/4\}$, for some $p \in S^2$. We obtain a local finger disc V. Tube V into the 2-sphere $\{q\} \times S^2 \times \{3/4\}$, for some $q \in S^1$. Call the resulting disc W, and use this to perform a Whitney move. This describes a pseudo-isotopy, and hence a diffeomorphism $f\colon X \to X$. Singh [Sin21] showed, using the Hatcher-Wagoner [HW73] obstructions, that f is nontrivial in $\pi_0 \operatorname{Diff}^{PI}_\partial(X)$.

I mentioned the Hatcher-Wagoner obstructions in the last example. I do not have time to define them here, but let me mention the following question.

Question 12.6. What are the Hatcher-Wagoner invariants of some pseudo-isotopy from the Budney-Gabai barbell implantations in $S^1 \times D^3$ to $\mathrm{Id}_{S^1 \times D^3}$?

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