4-MANIFOLDS WITH BOUNDARY AND FUNDAMENTAL GROUP $\mathbb{Z}$

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Abstract. We classify topological 4-manifolds with boundary and fundamental group $\mathbb{Z}$, under some assumptions on the boundary. We apply this to classify surfaces in simply-connected 4-manifolds with S^3 boundary, where the fundamental group of the surface complement is $\mathbb{Z}$. We then compare these homeomorphism classifications with the smooth setting. For manifolds, we show that every Hermitian form over $\mathbb{Z}[t^{\pm 1}]$ arises as the equivariant intersection form of a pair of exotic smooth 4-manifolds with boundary and fundamental group $\mathbb{Z}$. For surfaces we have a similar result, and in particular we show that every 2-handlebody with S^3 boundary contains a pair of exotic discs.

In what follows a 4-manifold is understood to mean a compact, connected, oriented, topological 4-manifold. Freedman classified closed simply-connected 4-manifolds up to orientation-preserving homeomorphisms. Groups π for which classifications of closed 4-manifolds with fundamental group π are known include $\pi \cong \mathbb{Z}$, [FQ90, SW00], π a finite cyclic group [HK88], and π a solvable Baumslag-Solitar group [HKT09]. Complete classification results for manifolds with boundary essentially only include the simply-connected case [Boy86, Boy93]; see also [Sto93].

This paper classifies 4-manifolds with boundary and fundamental group $\mathbb{Z}$, under some extra assumptions on the boundary. We give an informal statement now: for a nondegenerate Hermitian form λ over $\mathbb{Z}[t^{\pm 1}]$ and a 3-manifold Y, we define $\mathscr{V}^0_\lambda(Y)$ to be the set of 4-manifolds M with a homeomorphism $\partial M \cong Y$, fundamental group $\pi_1(M) \cong \mathbb{Z}$, equivariant intersection form λ, and $\pi_1(Y) \twoheadrightarrow \pi_1(M)$ surjective, considered up to orientation-preserving homeomorphism rel. boundary; see Definition 5 for a precise definition of $\mathscr{V}^0_\lambda(Y)$. The fact that λ is nondegenerate implies that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is torsion, with the coefficient system determined by the homomorphism $\pi_1(Y) \to \pi_1(M) \cong \mathbb{Z} = \langle t \rangle$.

Our main result, Theorem 1.1, provides a bijection

$$b \colon \mathscr{V}^0_\lambda(Y) \to \mathscr{A}(\lambda, Y)$$

where $\mathscr{A}(\lambda, Y)$ is a set defined algebraically in terms of λ and Y; the description of $\mathscr{A}(\lambda, Y)$ can be found in Theorem 1.1 (and Definition 1) and the construction of the map b can be found in Subsection 1.4. Injectivity of b is a consequence of [CP20, Theorem 1.10]. Surjectivity of b is the main technical result of this paper, Theorem 1.15. We also give a similar classification of such M up to homeomorphism *not* rel. boundary, Theorem 1.2. A feature of our classification, which we shall demonstrate in Section 7, is the existence of arbitrarily large sets of homeomorphism classes of such 4-manifolds, all of which have the same boundary Y and the same form λ.

We apply these results to study compact, oriented, locally flat, embedded surfaces in simply-connected 4-manifolds where the fundamental group of the exterior is infinite cyclic; we call these $\mathbb{Z}$-*surfaces*. The classification of closed surfaces in 4-manifolds whose exterior is simply-connected was carried out by Boyer [Boy93]; see also [Sun15]. Literature on the classification of discs in D^4 where the complement has fixed fundamental group includes [FT05, CP21, Con22]. For surfaces in more general 4-manifolds, [CP20] gave necessary and sufficient conditions for a pair of $\mathbb{Z}$-surfaces to be equivalent. In this work, for a 4-manifold N with boundary S^3 and a knot $K \subset S^3$, we classify $\mathbb{Z}$-surfaces in N with boundary K in terms of the equivariant intersection form of the surface exterior; see Theorem 1.6. An application to H-sliceness can be found in Corollary 1.8, while Theorem 1.10 classifies closed $\mathbb{Z}$-surfaces.

Finally, we compare these homeomorphism classifications with the smooth setting. We demonstrate that for every Hermitian form λ over $\mathbb{Z}[t^{\pm 1}]$ there are pairs of smooth 4-manifolds with boundary, $\pi_1 \cong \mathbb{Z}$, and equivariant intersection form λ which are homeomorphic rel. boundary but not diffeomorphic; see Theorem 1.12. We also show in Theorem 1.13 that for every Hermitian form λ satisfying conditions which are conjecturally necessary, there is a smooth 4-manifold N with S^3 boundary containing a pair of smoothly embedded $\mathbb{Z}$-surfaces whose exteriors have equivariant intersection form λ and which are topologically but not smoothly isotopic rel. boundary.

1. Statement of results

Before stating our main result, we need to introduce some terminology. A 4-manifold M is said to have *ribbon boundary* if the inclusion induced map $\pi_1(\partial M) \to \pi_1(M)$ is surjective. Our oriented 3-manifolds Y will always be equipped with an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$. When we say that a 4-manifold M with an identification $\pi_1(M) \cong \mathbb{Z}$ has ribbon boundary Y, we require that M comes equipped with a homeomorphism $\partial M \xrightarrow{\cong} Y$ such that the composition $\pi_1(Y) \twoheadrightarrow \pi_1(M) \xrightarrow{\cong} \mathbb{Z}$ agrees with φ. We will always assume that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion; here $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ refers to the first homology group of the infinite cyclic cover $Y^\infty \to Y$ corresponding to $\ker(\varphi)$.

1.1. The classification result.

Our goal is to classify 4-manifolds M with $\pi_1(M) \cong \mathbb{Z}$ whose boundary $\partial M \cong Y$ is ribbon with $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ torsion, up to orientation-preserving homeomorphism. The isometry class of the *equivariant intersection form* λ_M on $H_2(M;\mathbb{Z}[t^{\pm 1}])$ is an invariant of such M (whose definition is recalled in Subsection 2.1) and so, to classify such M, it is natural to first let λ be a nondegenerate Hermitian form over $\mathbb{Z}[t^{\pm 1}]$ and classify 4-manifolds M with $\pi_1(M) \cong \mathbb{Z}$, ribbon boundary $\partial M \cong Y$, and fixed equivariant intersection form λ. As mentioned above, the fact that λ is nondegenerate implies that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is torsion.

For such a 4-manifold M there is a relationship between the equivariant intersection form λ_M on $H_2(M;\mathbb{Z}[t^{\pm 1}])$ and the *Blanchfield form*

$$\mathrm{Bl}_Y\colon H_1(Y;\mathbb{Z}[t^{\pm 1}]) \times H_1(Y;\mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

whose definition is recalled in Subsection 2.2. Thus we can restrict attention to forms λ which *present* $(H_1(Y;\mathbb{Z}[t^{\pm 1}]), \mathrm{Bl}_Y)$, an algebraic notion which we make precise now.

If $\lambda\colon H \times H \to \mathbb{Z}[t^{\pm 1}]$ is a nondegenerate Hermitian form on a finitely generated free $\mathbb{Z}[t^{\pm 1}]$-module (for short, a *form*), then we write $\widehat{\lambda}\colon H \to H^*$ for the linear map $z \mapsto \lambda(-,z)$, and there is a short exact sequence

$$0 \to H \xrightarrow{\widehat{\lambda}} H^* \to \mathrm{coker}(\widehat{\lambda}) \to 0.$$

Such a presentation induces a *boundary linking form* $\partial\lambda$ on $\mathrm{coker}(\widehat{\lambda})$ in the following manner. For $[x] \in \mathrm{coker}(\widehat{\lambda})$ with $x \in H^*$, one can show that there exist elements $z \in H$ and $p \in \mathbb{Z}[t^{\pm 1}] \setminus \{0\}$ such that $\lambda(-,z) = px \in H^*$. Then for $[x],[y] \in \mathrm{coker}(\widehat{\lambda})$ with $x, y \in H^*$, we define

$$\partial\lambda([x],[y]) := \frac{y(z)}{p} \in \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}].$$

One can check that $\partial\lambda$ is independent of the choices of p and z.

Definition 1. For T a torsion $\mathbb{Z}[t^{\pm 1}]$-module with a linking form $\ell\colon T \times T \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$, a nondegenerate Hermitian form (H,λ) *presents* (T,ℓ) if there is an isomorphism $h\colon \mathrm{coker}(\widehat{\lambda}) \to T$ such that $\ell(h(x),h(y)) = \partial\lambda(x,y)$. Such an isomorphism h is called an *isometry* of the forms, the set of isometries is denoted $\mathrm{Iso}(\partial\lambda,\ell)$. If (H,λ) presents $(H_1(Y;\mathbb{Z}[t^{\pm 1}]), -\mathrm{Bl}_Y)$ then we say (H,λ) *presents* Y.

This notion of a presentation is well known to high dimensional topologists (see e.g. [Ran81, CS11]) but also appeared in the classification of simply-connected 4-manifolds in [Boy86, Boy93] and in [CP20] for 4-manifolds with $\pi_1 \cong \mathbb{Z}$, as well as in e.g. [BF15, FL19]. Presentations capture the geometric relationship between the linking form of a 3-manifold and the intersection form of a 4-manifold filling. More precisely, as explained in the following paragraph, the form $(H_2(M;\mathbb{Z}[t^{\pm 1}]),\lambda_M)$ presents ∂M.

The long exact sequence of the pair $(M,\partial M)$ with coefficients in $\mathbb{Z}[t^{\pm 1}]$ reduces to the short exact sequence

$$0 \longrightarrow H_2(M;\mathbb{Z}[t^{\pm 1}]) \longrightarrow H_2(M,\partial M;\mathbb{Z}[t^{\pm 1}]) \longrightarrow H_1(\partial M;\mathbb{Z}[t^{\pm 1}]) \longrightarrow 0,$$

where $H_2(M;\mathbb{Z}[t^{\pm 1}])$ and $H_2(M,\partial M;\mathbb{Z}[t^{\pm 1}])$ are f.g. free $\mathbb{Z}[t^{\pm 1}]$-modules [CP20, Lemma 3.2]. The left term of the short exact sequence supports the equivariant intersection form λ_M and the right supports $\mathrm{Bl}_{\partial M}$. As explained in detail in [CP20, Remark 3.3], some algebraic topology gives the following commutative diagram of short exact sequences, where the isomorphism D_M is defined so that the right-most square commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(M;\mathbb{Z}[t^{\pm 1}]) & \xrightarrow{\widehat{\lambda}_M} & H_2(M;\mathbb{Z}[t^{\pm 1}])^* & \longrightarrow & \operatorname{coker}(\widehat{\lambda}_M) & \longrightarrow & 0 \\ & & =\big\downarrow \mathrm{Id} & & \cong\big\downarrow \mathrm{ev}^{-1}\circ \mathrm{PD} & & \cong\big\downarrow D_M & & \\ 0 & \longrightarrow & H_2(M;\mathbb{Z}[t^{\pm 1}]) & \longrightarrow & H_2(M,\partial M;\mathbb{Z}[t^{\pm 1}]) & \longrightarrow & H_1(\partial M;\mathbb{Z}[t^{\pm 1}]) & \longrightarrow & 0. \end{array} \tag{1}$$

It then follows that $(H_2(M;\mathbb{Z}[t^{\pm 1}]),\lambda_M)$ presents ∂M, where the isometry $\partial\lambda_M \cong -\mathrm{Bl}_{\partial M}$ is given by D_M. For details see [CP20, Proposition 3.5].

Thus, as mentioned above, to classify the 4-manifolds M with $\pi_1(M)\cong\mathbb{Z}$ and ribbon boundary $\partial M\cong Y$, it suffices to consider forms (H,λ) which present Y. The set of self isometries of (H,λ) is denoted $\mathrm{Aut}(\lambda)$. We will describe the action of $\mathrm{Aut}(\lambda)$ on $\mathrm{Iso}(\partial\lambda,-\mathrm{Bl}_Y)$ in Equation (2) of Construction 1, in Section 1.4. Additionally, recall that a Hermitian form (H,λ) is *even* if $\lambda(x,x)=q(x)+\overline{q(x)}$ for some $q\colon H\to\mathbb{Z}[t^{\pm 1}]$ and is *odd* otherwise. Our first classification now reads as follows; its proof modulo our main technical theorem can be found in Subsection 1.4.

Theorem 1.1. *Let Y be a 3-manifold with an epimorphism $\varphi\colon \pi_1(Y)\twoheadrightarrow\mathbb{Z}$ whose Alexander module is torsion, and let (H,λ) be a nondegenerate Hermitian form over $\mathbb{Z}[t^{\pm 1}]$. Consider the set $\mathscr{V}^0_\lambda(Y)$ of 4-manifolds with $\pi_1(M)\cong\mathbb{Z}$, ribbon boundary $\partial M\cong Y$, and $\lambda_M\cong\lambda$, considered up to homeomorphism rel. boundary.*

If the form (H,λ) presents Y, then $\mathscr{V}^0_\lambda(Y)$ is nonempty and corresponds bijectively to

(1) $\mathrm{Iso}(\partial\lambda,-\mathrm{Bl}_Y)/\mathrm{Aut}(\lambda)$, *if λ is an even form;*
(2) $(\mathrm{Iso}(\partial\lambda,-\mathrm{Bl}_Y)/\mathrm{Aut}(\lambda))\times\mathbb{Z}_2$ *if λ is an odd form. The map to $\mathbb{Z}_2$ is given by the Kirby-Siebenmann invariant.*

Use $\mathrm{Homeo}^+_\varphi(Y)$ to denote the orientation-preserving homeomorphisms of Y such that the induced map on π_1 commutes with $\varphi\colon\pi_1(Y)\twoheadrightarrow\mathbb{Z}$. As we show in Subsection 1.4 below, Theorem 1.1 leads to a description of $\mathscr{V}_\lambda(Y)$, the analogous set of fillings of Y with the rel. boundary condition omitted. We will describe the action of $\mathrm{Aut}(\lambda)\times\mathrm{Homeo}^+_\varphi(Y)$ on $\mathrm{Iso}(\partial\lambda,-\mathrm{Bl}_Y)$ in detail in (4), just below Definition 6. The classification statement for $\mathscr{V}_\lambda(Y)$ reads as follows.

Theorem 1.2. *Let Y be a 3-manifold with an epimorphism $\pi_1(Y)\twoheadrightarrow\mathbb{Z}$ whose Alexander module is torsion, let (H,λ) be a nondegenerate Hermitian form over $\mathbb{Z}[t^{\pm 1}]$. Consider the set $\mathscr{V}_\lambda(Y)$ of 4-manifolds with $\pi_1(M)\cong\mathbb{Z}$, ribbon boundary $\partial M\cong Y$, and $\lambda_M\cong\lambda$, considered up to orientation-preserving homeomorphism.*

If the form (H,λ) presents Y, then $\mathscr{V}_\lambda(Y)$ is nonempty and corresponds bijectively to

(1) $\mathrm{Iso}(\partial\lambda,-\mathrm{Bl}_Y)/(\mathrm{Aut}(\lambda)\times\mathrm{Homeo}^+_\varphi(Y))$, *if λ is an even form;*
(2) $\big(\mathrm{Iso}(\partial\lambda,-\mathrm{Bl}_Y)/(\mathrm{Aut}(\lambda)\times\mathrm{Homeo}^+_\varphi(Y))\big)\times\mathbb{Z}_2$ *if λ is an odd form. The map to $\mathbb{Z}_2$ is given by the Kirby-Siebenmann invariant.*

Remark 1.3. We collect a couple of remarks about these results.

- In both of these theorems, the bijection is explicit, and we describe it in Subsection 1.4. Additionally, note that since (H,λ) is assumed to present Y, there is an isometry $\partial\lambda\cong$

$-\mathrm{Bl}_Y$ and this leads to a bijection

$$\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda) \approx \operatorname{Aut}(\partial\lambda)/\operatorname{Aut}(\lambda),$$

where $\operatorname{Aut}(\partial\lambda)$ denotes the group of self-isometries of $\partial\lambda$. Note however that this bijection is not canonical as it depends on the choice of the isometry $\partial\lambda \cong -\mathrm{Bl}_Y$. There are pairs (Y, λ) for which the set $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/(\operatorname{Aut}(\lambda) \times \mathrm{Homeo}^+_\varphi(Y))$ is arbitrarily large, as we will outline in Example 1.5 and show in detail in Section 7.

- The *automorphism invariant* that distinguishes $\mathbb{Z}$-manifolds with the same equivariant form is nontrivial to calculate in practice, as its definition typically involves choosing identifications of the boundary 3-manifolds; see Subsection 1.4.
- These theorems should be thought of as extensions of the work of Boyer [Boy86, Boy93] that classifies simply-connected 4-manifolds with boundary and fixed intersection form and of the classification of closed 4-manifolds with $\pi_1 = \mathbb{Z}$ [FQ90, SW00]. Boyer's main statements are formulated using presentations instead of isometries of linking forms, but both approaches can be shown to agree when the 3-manifold is a rational homology sphere [Boy93, Corollary E]. Rational homology spheres should be thought of as the analogue of pairs (Y, φ) with torsion Alexander module.
- We have focused on the case of pairs (Y, φ) where the Alexander module $H_1(Y; \mathbb{Z}[t^{\pm 1}])$ is torsion because it is sufficient to treat our main application (surfaces in 4-manifolds) and because the general case leads to additional phenomena involving spin structures. In addition, in general $H_1(Y; \mathbb{Z}[t^{\pm 1}])$ need not have projective dimension one.

Example 1.4. If $Y \cong \Sigma_g \times S^1$ and $\varphi\colon \pi_1(\Sigma_g \times S^1) \to \pi_1(S^1) \to \mathbb{Z}$ is induced by projection onto the second factor, then as shown in [CP20, Proposition 5.6], the isometries of the Blanchfield form of Y exactly coincide with the symplectic group of isometries of the intersection form of Σ_g. Since every such isometry is realised by a self-homeomorphism of Σ_g [FM12, Section 2.1], it follows that the action of $\mathrm{Homeo}^+_\varphi(Y)$ has one orbit, and therefore the quotients in Theorem 1.2 consist of a single orbit. In other words, for a fixed non-degenerate Hermitian form λ that presents Y, if λ is even there is a unique homeomorphism class of 4-manifolds with $\pi_1 \cong \mathbb{Z}$, boundary Y and equivariant intersection form λ, and if λ is odd there are two such homeomorphism classes.

Example 1.5. We will show in Proposition 7.5 that there are examples of pairs (Y, φ) for which the set of 4-manifolds with fixed boundary Y and fixed equivariant intersection form, up to homeomorphism, can have arbitrarily large cardinality. Here is an outline of the argument. Start with an integer P that is a product of n distinct primes $P = p_1 \cdot p_2 \cdots p_n$ and let W be a 4-manifold with fundamental group $\mathbb{Z}$, ribbon boundary, and equivariant intersection form λ_W represented by the integer matrix $(2P)$. We will show in Section 7 that the quotient $\operatorname{Aut}(\mathrm{Bl}_{\partial W})/\big(\operatorname{Aut}(\lambda_W) \times \mathrm{Homeo}^+_\varphi(\partial W)\big)$ contains at least 2^{n-1} elements. These correspond to the different ways of factoring P as a product of two unordered coprime integers.

Then it suffices to produce such a W, which we can do by attaching a single 2-handle to $S^1 \times D^3$ using the process described in the proof of Theorem 6.5. By carefully modifying the attaching maps we can also arrange that the mapping class group of ∂W is trivial. Define $Y := \partial W$ and let $\varphi\colon \pi_1(Y) \to \pi_1(W) \cong \mathbb{Z}$ be the inclusion induced map.

Then by Theorem 1.2 there are at least 2^{n-1} homeomorphism classes of 4-manifolds with the same boundary and equivariant intersection form $2P$, detected by their pairwise distinct automorphism invariants. This is similar to the idea used in [CCPS21a] and [CCPS21b] to construct closed manifolds of dimension $4k \geq 8$ with nontrivial homotopy stable classes.

It is not necessary for this phenomenon that P be an integer, rather we are describing the simplest instance of the phenomenon. A more general criterion for P is given in Section 7. We note that this phenomenon also exists for simply connected 4-manifolds bounding rational homology spheres, which can be deduced from Boyer's work [Boy86, Boy93] with a similar proof.

In Subsection 1.4 we describe the bijections in Theorems 1.1 and 1.2 explicitly, give the main technical statement, outline the proof of Theorem 1.1 and explain how Theorem 1.1 implies Theorem 1.2. But first, in Subsections 1.2 and 1.3, we discuss some applications.

1.2. **Classification of $\mathbb{Z}$-surfaces in simply connected 4-manifolds with S^3 boundary.** For a fixed simply-connected 4-manifold N with boundary S^3 and a fixed knot $K \subset \partial N = S^3$, we call two locally flat embedded compact surfaces $\Sigma, \Sigma' \subset N$ with boundary $K \subset S^3$ *equivalent rel. boundary* if there is an orientation-preserving homeomorphism $(N, \Sigma) \cong (N, \Sigma')$ that is pointwise the identity on $S^3 \cong \partial N$. We are interested in classifying the $\mathbb{Z}$-surfaces in N with boundary K up to equivalence rel. boundary.

As for manifolds, we first observe that the classification naturally decomposes into more accessible classification problems once we fix the appropriate invariants. Indeed the genus of Σ and the equivariant intersection form λ_{N_Σ} on $H_2(N_\Sigma; \mathbb{Z}[t^{\pm 1}])$ are invariants of such a surface Σ, where N_Σ denotes the exterior $N \smallsetminus \nu(\Sigma)$. Thus it is natural to split $\mathbb{Z}$-surfaces for K in N into the subsets

$$\mathrm{Surf(g)}^0_\lambda(N, K) := \{\text{genus } g\ \mathbb{Z}\text{−surfaces } \Sigma \subset N \text{ for } K \text{ with } \lambda_{N_\Sigma} \cong \lambda\}/ \text{ equivalence rel. } \partial.$$

Again as for manifolds, we now describe some necessary conditions for the set $\mathrm{Surf(g)}^0_\lambda(N, K)$ to be nonempty. Write $E_K := S^3 \setminus \nu(K)$ for the exterior of K and recall that the boundary of N_Σ has a natural identification

$$\partial N_\Sigma \cong E_K \cup_\partial (\Sigma_{g,1} \times S^1) =: M_{K,g}.$$

As discussed in Subsection 1.1, there is a relationship between the equivariant intersection form λ_{N_Σ} on $H_2(N_\Sigma; \mathbb{Z}[t^{\pm 1}])$ and the Blanchfield form $\mathrm{Bl}_{M_{K,g}}$ on $H_1(M_{K,g}; \mathbb{Z}[t^{\pm 1}])$: the Hermitian form $(H_2(N_\Sigma; \mathbb{Z}[t^{\pm 1}]), \lambda_{N_\Sigma})$ presents $M_{K,g}$. Thus it suffices to restrict our attention to the subsets $\mathrm{Surf(g)}^0_\lambda(N, K)$, where (H, λ) is a nondegenerate Hermitian form over $\mathbb{Z}[t^{\pm 1}]$ that presents $M_{K,g}$.

There is one additional necessary condition for a given form (H, λ) to be isometric to the intersection pairing $(H_2(N_\Sigma; \mathbb{Z}[t^{\pm 1}]), \lambda_{N_\Sigma})$ for some surface Σ. Observe that we can reglue the neighborhood of Σ to N_Σ to recover N. This is reflected in the intersection form, as follows. We write $\lambda(1) := \lambda \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}_\varepsilon$, where $\mathbb{Z}_\varepsilon$ denotes $\mathbb{Z}$ with the trivial $\mathbb{Z}[t^{\pm 1}]$-module structure. If W is a $\mathbb{Z}$-manifold, then $\lambda_W(1) \cong Q_W$, where Q_W denotes the standard intersection form of W; see e.g. [CP20, Lemma 5.10]. Therefore, if $\lambda \cong \lambda_{N_\Sigma}$, then we have the isometries

$$\lambda(1) \cong \lambda_{N_\Sigma}(1) = Q_{N_\Sigma} \cong Q_N \oplus (0)^{\oplus 2g},$$

where the last isometry follows from a Mayer-Vietoris argument. Thus, for the set $\mathrm{Surf(g)}^0_\lambda(N, K)$ to be nonempty, it is necessary both that λ presents $M_{K,g}$ and that $\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}$. The following theorem (which is stated slightly more generally in Theorem 5.1 below) shows that these two necessary conditions are in fact also sufficient and lead to a description of $\mathrm{Surf(g)}^0_\lambda(N, K)$.

Theorem 1.6. *Let N be a simply-connected 4-manifold with boundary S^3 and let $K \subset S^3$ be a knot. If a nondegenerate Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ presents $M_{K,g}$ and satisfies $\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}$, then the set $\mathrm{Surf(g)}^0_\lambda(N, K)$ is nonempty and there is a bijective correspondence*

$$\mathrm{Surf(g)}^0_\lambda(N, K) \cong \mathrm{Aut}(\mathrm{Bl}_K)/ \mathrm{Aut}(\lambda).$$

The action of the group $\mathrm{Aut}(\lambda)$ on the set $\mathrm{Aut}(\mathrm{Bl}_K)$ arises by restricting the action of $\mathrm{Aut}(\lambda)$ on $\mathrm{Aut}(\partial\lambda) \cong \mathrm{Aut}(\mathrm{Bl}_{M_{K,g}}) \cong \mathrm{Aut}(\mathrm{Bl}_K) \oplus \mathrm{Sp}_{2g}(\mathbb{Z})$ to the first summand. Here the (non-canonical) isomorphism $\mathrm{Aut}(\partial\lambda) \cong \mathrm{Aut}(\mathrm{Bl}_{M_{K,g}})$ holds because the form λ presents $M_{K,g}$, while the isomorphism $\mathrm{Aut}(\mathrm{Bl}_{M_{K,g}}) \cong \mathrm{Aut}(\mathrm{Bl}_K) \oplus \mathrm{Sp}_{2g}(\mathbb{Z})$ is a consequence of [CP20, Propositions 5.6 and 5.7].

Again, we give an explicit description of the bijection. The idea is as follows. In Section 5 we relate $\mathrm{Surf(g)}^0_\lambda(N, K)$ with a particular set of 4-manifolds $\mathscr{V}^0_\lambda(M_{K,g})$, and then we relate this to $\mathrm{Aut}(\mathrm{Bl}_K)/ \mathrm{Aut}(\lambda)$ via Theorem 1.1. Finally, we note that when $N = D^4$, equivalence rel. boundary can be upgraded to isotopy rel. boundary via the Alexander trick.

Remark 1.7. Previous classification results of locally flat discs in 4-manifolds include $\mathbb{Z}$-discs in D^4 [FQ90, CP21], $BS(1, 2)$-discs in D^4 [FT05, CP21] and G-discs in D^4 (under some assumptions on the group G) [FT05, Con22]. In the latter case it is not known whether there are groups satisfying the assumptions other than $\mathbb{Z}$ and $BS(1, 2)$. Our result is the first classification of discs with non simply-connected exteriors in 4-manifolds other than D^4.

Before continuing with $\mathbb{Z}$-surfaces, we mention an application of Theorem 1.6 to H-sliceness. A knot K in ∂N is said to be (topologically) *H-slice* if K bounds a locally flat, embedded disc D in N that represents the trivial class in $H_2(N, \partial N)$. The study of H-slice knots has garnered some interest recently because of its potential applications towards producing small closed exotic 4-manifolds [CN20, MMSW19, MMP20, IMT21, MP21, KMRS21]. Since $\mathbb{Z}$-slice knots are H-slice (see e.g. [CP20, Lemma 5.1]), Theorem 1.6 therefore gives a new criterion for topological H-sliceness. Our results also apply in higher genus. When $N = D^4$, this is reminiscent of the combination of [FL18, Theorems 2 and 3] and [BF14, Theorem 1.1] (and for $g = 0$ it is Freedman's theorem that Alexander polynomial one knots bound $\mathbb{Z}$-discs [Fre84, FQ90]). In connected sums of copies of $\mathbb{C}P^2$, this is closely related to [KMRS21, Theorem 1.3]. Compare also [FL19, Theorem 1.10], which applies in connected sums of copies of $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ and $S^2 \times S^2$.

Corollary 1.8. *Let N be a simply-connected 4-manifold with boundary S^3 and let $K \subset S^3$ be a knot. If $\mathrm{Bl}_{M_{K,g}}$ is presented by a nondegenerate Hermitian matrix $A(t)$ such that $A(1)$ is congruent to $Q_N \oplus (0)^{\oplus 2g}$, then K bounds a genus g $\mathbb{Z}$-surface in N. In particular, when $g = 0$, K is H-slice in N.*

We also study $\mathbb{Z}$-surfaces up to equivalence (instead of equivalence rel. boundary). Here an additional technical requirement is needed on the knot exterior $E_K := S^3 \setminus \nu(K)$.

Theorem 1.9. *Let K be a knot in S^3 such that every isometry of Bl_K is realised by an orientation-preserving homeomorphism $E_K \to E_K$. If a nondegenerate Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ presents $M_{K,g}$ and satisfies $\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}$, then up to equivalence, there exists a unique genus g surface $\Sigma \subset N$ with boundary K and whose exterior has equivariant intersection form λ.*

The classification of closed $\mathbb{Z}$-surfaces then follows from Theorem 1.9. To state the result, given a closed simply-connected 4-manifold X, we use X_Σ to denote the exterior of a surface $\Sigma \subset X$ and write

$$\mathrm{Surf(g)}_\lambda(X) = \{\text{genus } g\ \mathbb{Z}\text{−surfaces } \Sigma \subset X \text{ with } \lambda_{X_\Sigma} \cong \lambda\}/ \text{ equivalence}$$

as well as $N := X \setminus \mathring{D}^4$ for the manifold obtained by puncturing X. The details are presented in Section 5.3. The idea behind the proof is that for $K = U$ the unknot, the sets $\mathrm{Surf(g)}_\lambda(N, U)$ and $\mathrm{Surf(g)}_\lambda(X)$ are in bijective correspondence, so we can apply Theorem 1.9.

Theorem 1.10. *Let X be a closed simply-connected 4-manifold. If a nondegenerate Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ presents $\Sigma_g \times S^1$ and satisfies $\lambda(1) \cong Q_X \oplus (0)^{\oplus 2g}$, then there exists a unique (up to equivalence) genus g surface $\Sigma \subset X$ whose exterior has equivariant intersection form λ.*

Note that the boundary 3-manifold in question here, $\Sigma_g \times S^1$, is the same one that appeared in Example 1.4. We conclude with a couple of remarks on Theorems 1.6, 1.9, and 1.10. Firstly, we note that for the bijection in each theorem, the injectivity (i.e. uniqueness) statements follow from [CP20]. Our contributions in this work are the surjectivity (i.e. existence) statements. Secondly, we note that similar results were obtained for closed surfaces with simply-connected complements by Boyer [Boy93]. Some open questions concerning $\mathbb{Z}$-surfaces are discussed in Subsection 5.4.

1.3. **Exotica for all equivariant intersection forms.** So far, we have seen that the equivariant intersection form and the automorphism invariant from Theorems 1.1 and 1.2 determine the topological type of our 4-manifolds (and $\mathbb{Z}$-surfaces). In what follows, we investigate the smooth failure of this statement.

One of the driving questions in smooth 4-manifold topology is whether every smoothable simply-connected closed 4-manifold admits multiple smooth structures. This question has natural generalisations to 4-manifolds with boundary and with other fundamental groups; we set up these generalisations with the following definition.

Definition 2. For a 3-manifold Y, a (possibly degenerate) symmetric form Q over $\mathbb{Z}$ (resp. Hermitian form λ over $\mathbb{Z}[t^{\pm 1}]$) is *exotically realisable rel.* Y if there exists a pair of smooth simply-connected (resp. $\pi_1 \cong \mathbb{Z}$) 4-manifolds M and M' with boundary Y (resp. ribbon boundary Y) and intersection form Q (resp. equivariant intersection form λ) such that there is an orientation-preserving homeomorphism $F\colon M \to M'$ (for $\pi_1 \cong \mathbb{Z}$, we additionally require that F respects the identifications of $\pi_1(M)$ and $\pi_1(M')$ with $\mathbb{Z}$) but no diffeomorphism $G\colon M \to M'$.

In this language, the driving question above becomes (a subquestion of) the following: which symmetric bilinear forms over $\mathbb{Z}$ are exotically realisable rel. S^3? There is substantial literature demonstrating that some forms are exotically realisable rel. S^3 (we refer to [AP08, AP10] both for the state of the art and for a survey of results on the topic) but there remain many forms, such as definite forms or forms with $b_2 < 3$, for which determining exotic realisability rel. S^3 remains out of reach. For more general 3-manifolds, the situation is worse; in fact it is an open question whether for every integer homology sphere Y there exists *some* symmetric form Q that is exotically realisable rel. Y [EMM19].

Presently there only seems to be traction on exotic realisability of intersection forms if one relinquishes control of the homeomorphism type of the boundary.

Definition 3. A symmetric form Q over $\mathbb{Z}$ (resp. a Hermitian form λ over $\mathbb{Z}[t^{\pm 1}]$) is *exotically realisable* if there exists pair of smooth simply-connected (resp. $\pi_1 \cong \mathbb{Z}$) 4-manifolds M and M' with intersection form Q (resp. equivariant intersection form λ and with ribbon boundary) such that there is an orientation-preserving homeomorphism $F\colon M \to M'$ (for $\pi_1 \cong \mathbb{Z}$, we additionally require that F respects the identifications of $\pi_1(M)$ and $\pi_1(M')$ with $\mathbb{Z}$) but no diffeomorphism $G\colon M \to M'$.

The following theorem does not appear in the literature but is known to experts; it can be proven using techniques developed in [Fre82] and, for example, [Akb91, AM97, AR16]. It shows that contrarily to the closed setting, *every* symmetric bilinear form over $\mathbb{Z}$ is exotically realisable.

Theorem 1.11 (Folklore)**.** *For every symmetric bilinear form $(\mathbb{Z}^n, Q)$ over $\mathbb{Z}$, there exists a pair of simply-connected smooth 4-manifolds M and M' with the same boundary such that:*

(1) there is a homeomorphism $F\colon M \to M'$;
(2) the equivariant intersection forms Q_M and $Q_{M'}$ are isometric to Q;
(3) there is no diffeomorphism from M to M'.

In other words, every symmetric bilinear form $(\mathbb{Z}^n, Q)$ over $\mathbb{Z}$ is exotically realisable.

Theorem 1.11 can be proved using a simplification of the argument we use to prove Theorem 1.12, so we do not include a separate proof.

Following our classification of 4-manifolds with fixed ribbon boundary, fixed equivariant intersection form λ and $\pi_1 \cong \mathbb{Z}$, it is natural to ask which Hermitian forms λ are exotically realisable, with or without fixing a parametrisation of the boundary 3-manifold. We resolve the latter.

Theorem 1.12. *For every Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ there exists a pair of smooth 4-manifolds M and M' with ribbon boundary and fundamental group $\mathbb{Z}$, such that:*

(1) there is a homeomorphism $F\colon M \to M'$;
(2) F induces an isometry $\lambda_M \cong \lambda_{M'}$, and both forms are isometric to λ;
(3) there is no diffeomorphism from M to M'.

In other words, every Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ is exotically realisable.

Smooth 4-manifold topologists are also interested in finding smooth surfaces which are topologically but not smoothly isotopic. While literature in the closed case includes [FKV88, FS97, Kim06, KR08a, KR08b, Mar13, HS20] there has been a recent surge of interest in the relative setting on which we now focus [JMZ21, Hay20, HKK+21, HS21, DMS22]. Most relevant to us are the exotic ribbon discs from [Hay20]. In order to prove that his discs in D^4 are topologically isotopic, Hayden showed that their exteriors have group $\mathbb{Z}$ and appealed to [CP21]. From the perspective of this paper and [CP20], any two $\mathbb{Z}$-ribbon discs are isotopic rel. boundary because their exteriors are aspherical and therefore have trivial equivariant intersection form. To generalise Hayden's result to other forms than the trivial one, we introduce some terminology.

Definition 4. For a fixed smooth simply-connected 4-manifold N, with boundary S^3, a form λ over $\mathbb{Z}[t^{\pm 1}]$ is *realised by exotic $\mathbb{Z}$-surfaces in N* if there exists a pair of smooth properly embedded $\mathbb{Z}$-surfaces Σ and Σ' in N, with the same boundary, whose exteriors have equivariant intersection forms isometric to λ, and which are topologically but not smoothly isotopic rel. boundary.

Using this terminology, Hayden's result states that the trivial form is realised by exotic $\mathbb{Z}$-discs (in D^4). The next result shows that in fact *every* form is realised by exotic $\mathbb{Z}$-discs.

Theorem 1.13. *For any Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ such that $\lambda(1)$ is realised as the intersection form of a smooth simply-connected 4-dimensional 2-handlebody N with boundary S^3, there exists a pair of smooth $\mathbb{Z}$-discs D and D' in N with the same boundary and the following properties:*

(1) the equivariant intersection forms λ_{N_D} and $\lambda_{N_{D'}}$ are isometric to λ;
(2) D is topologically isotopic to D' rel. boundary;
(3) D is not smoothly equivalent to D' rel. boundary.

In other words, for every λ, N satisfying the hypothesis, λ can be realised by exotic $\mathbb{Z}$-discs in N.

Remark 1.14. We make a couple of remarks on Theorems 1.12 and 1.13.

- The 11/8 conjecture predicts that every integer intersection form which is realisable by a smooth 4-manifold with S^3 boundary is realisable by a smooth 4-dimensional 2-handlebody with S^3 boundary, thus our hypothesis on the realisability of $\lambda(1)$ by 2-handlebodies is likely not an additional restriction (a nice exposition on why this follows from the 11/8 conjecture is given in [HLSX19, page 24]). It is likely that with a little more care the proof of Theorem 1.13 could be upgraded to prove the result under the milder assumption that $\lambda(1)$ be realisable by a 4-manifold with S^3 boundary and no 3-handles. Since the 11/8 conjecture again predicts that this is the same set of forms included in the present hypothesis, we do not pursue this upgrade.
- The handlebody N is very explicit: it can be built from D^4 by attaching 2-handles according to $\lambda(1)$. In particular, when λ is the trivial form, then $N = D^4$ and so Theorem 1.13 demonstrates that there are exotic discs in D^4. This was originally proved in [Hay20], and we note that our proof relies on techniques developed there.
- The proof of Theorem 1.13 also shows that every smooth 2-handlebody with S^3 boundary contains a pair of exotic $\mathbb{Z}$-discs. We expand on this above the statement of Theorem 6.6.
- Similarly to Theorem 1.11, Theorem 1.12 shows that contrarily to the topological setting, the equivariant intersection form and the automorphism invariant (unsurprisingly) do not determine the diffeomorphism type of a smooth manifold. It must be noted however that during the proof of Theorem 1.12 and (resp. Theorem 1.13), we show by direct means that the manifolds (resp. surfaces) are homeomorphic (resp. isotopic) rel. boundary and deduce that their automorphism invariants agree. In other words, we do *not* use the uniqueness statement of Theorem 1.1 to establish the topological rigidity and are unable to control which automorphism invariant we realise.
- It would be interesting to find a form λ presenting a 3-manifold Y such that for every element of $b \in \operatorname{Isom}(\partial\lambda, -\mathrm{Bl}_Y)$ there exists a pair of exotic manifolds with equivariant intersection form λ and automorphism invariant b. This also provokes a perhaps more primary question; assuming a form λ over $\mathbb{Z}[t^{\pm 1}]$ arises as the equivariant intersection form of a smooth 4-manifold with boundary Y, it is intriguing to wonder whether every automorphism invariant in $\operatorname{Iso}(\partial\lambda, -\mathrm{Bl}_Y)$ can be realised by a *smooth* manifold.

We briefly mention the idea of the proof of Theorem 1.12. For a given Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$, we construct a Stein 4-manifold M with $\pi_1(M) \cong \mathbb{Z}$ and $\lambda_M \cong \lambda$ that contains a cork. Twisting along this cork produces the 4-manifold M' and the homeomorphism $F\colon M \cong M'$. We show that if $F|_\partial$ extended to a diffeomorphism $M \cong M'$, two auxiliary 4-manifolds W and W' (obtained from M and M' by adding a single 2-handle) would be diffeomorphic. We show this is not the case by proving that W is Stein whereas W' is not using work of Lisca-Matic [LM97]. This proves that M and M' are non-diffeomorphic rel. $F|_\partial$. We then use a result of [AR16] to show that there exists a pair of smooth manifolds V and V', which are homotopy equivalent to M and M' respectively, and which are homeomorphic but not diffeomorphic to each other. The proof of Theorem 1.13 uses similar ideas.

1.4. **The main technical realisation statement.** In this section we begin by describing a map $b : \mathscr{V}^0_\lambda(Y) \to \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$. Theorem 1.1 then reduces to the statement that b is a bijection. As we will explain, the injectivity of b follows from [CP20, Theorem 1.10]. The main technical result of this paper is Theorem 1.15, which gives the surjectivity of b (and thus implies Theorem 1.1). We also prove in this section that Theorem 1.2 follows from Theorem 1.1, and we finish the section with an outline of the proof of Theorem 1.15.

We start by describing the set $\mathscr{V}^0_\lambda(Y)$ from Theorem 1.1 more carefully.

Definition 5. Let Y be a 3-manifold with an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module is torsion, and let (H, λ) be a Hermitian form presenting Y. Consider the set $S_\lambda(Y)$ of pairs (M, g), where

- M is a 4-manifold with a fixed identification $\pi_1(M) \xrightarrow{\cong} \mathbb{Z}$, equivariant intersection form isometric to λ, and ribbon boundary Y;
- $g\colon \partial M \xrightarrow{\cong} Y$ is an orientation-preserving homeomorphism such that $Y \xrightarrow{g^{-1},\cong} \partial M \to M$ induces φ on fundamental groups.

Define $\mathscr{V}^0_\lambda(Y)$ as the quotient of $S_\lambda(Y)$ in which two pairs $(M_1, g_1), (M_2, g_2)$ are deemed equal if and only if there is a homeomorphism $\Phi\colon M_1 \cong M_2$ such that $\Phi|_{\partial M_1} = g_2^{-1} \circ g_1$. Note that such a homeomorphism is necessarily orientation-preserving because g_1 and g_2 are. For conciseness, we will say that (M_1, g_1) and (M_2, g_2) are *homeomorphic rel. boundary* to indicate the existence of such a homeomorphism Φ.

Construction 1. [Constructing the map $b\colon \mathscr{V}^0_\lambda(Y) \to \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$.] Let Y be a 3-manifold with an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose corresponding Alexander module is torsion, and let (H, λ) be a form presenting Y. Let (M, g) be an element of $\mathscr{V}^0_\lambda(Y)$, i.e. M is a 4-manifold with $\pi_1(M) \cong \mathbb{Z}$, equivariant intersection form λ and ribbon boundary, and $g\colon \partial M \cong Y$ is a homeomorphism as in Definition 5.

In the text preceding Theorem 1.1, we showed how M leads to an isometry $D_M \in \operatorname{Iso}(\partial\lambda_M, -\operatorname{Bl}_{\partial M})$. Morally, one should think that this isometry D_M is the invariant we associate to M. For this to be meaningful however, we instead need an isometry that takes value in a set defined in terms of just the 3-manifold Y and the form (H, λ), without referring to M itself. We resolve this by composing D_M with other isometries, so that our invariant is ultimately an element of $\operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)$. Once we have built the invariant, we will show it is well defined up to an action by $\operatorname{Aut}(\lambda)$.

We first use g to describe an isometry $\operatorname{Bl}_{\partial M} \cong \operatorname{Bl}_Y$. Since on the level of fundamental groups g intertwines the maps to $\mathbb{Z}$, [CP20, Proposition 3.7] implies that g induces an isometry

$$g_*\colon \operatorname{Bl}_{\partial M} \cong \operatorname{Bl}_Y .$$

Next we describe an isometry $\partial\lambda \cong \partial\lambda_M$. The assumption that M has equivariant intersection form λ means by definition that there is an isometry $F\colon \lambda \cong \lambda_M$, i.e. an isomorphism $F\colon H \to H_2(M; \mathbb{Z}[t^{\pm 1}])$ that intertwines the forms λ and λ_M. Note that there is no preferred choice of F. Any such F induces an isometry $\partial F \in \operatorname{Aut}(\partial\lambda, \partial\lambda_M)$ as follows: $F\colon H \to H_2(M; \mathbb{Z}[t^{\pm 1}])$ gives an isomorphism $(F^*)^{-1}\colon H^* \to H_2(M; \mathbb{Z}[t^{\pm 1}])^*$ that descends to an isomorphism $\operatorname{coker}(\widehat{\lambda}) \cong \operatorname{coker}(\widehat{\lambda}_M)$ and is in fact an isometry; this is by definition

$$\partial F := (F^*)^{-1}\colon \partial\lambda \cong \partial\lambda_M.$$

This construction is described in greater generality in [CP20, Subsection 2.2]. We shall henceforth abbreviate $(F^*)^{-1}$ to F^{-*}.

We are now prepared to associate an isometry in $\operatorname{Iso}(\partial\lambda, -\mathrm{Bl}_Y)$ to $(M,g) \in \mathcal{V}^0_\lambda(Y)$ as follows: choose an isometry $F\colon \lambda_M \cong \lambda$ and consider the isometry

$$b_{(M,g,F)} := g_* \circ D_M \circ \partial F \in \operatorname{Iso}(\partial\lambda, -\mathrm{Bl}_Y).$$

We are not quite done, because we need to ensure that our invariant is independent of the choice of F and that b defines a map on $\mathcal{V}^0_\lambda(Y)$.

First, we will make our invariant independent of the choice of F. We require the following observation. Given a Hermitian form (H,λ) and linking form (T,ℓ), there is a natural left action $\operatorname{Aut}(\lambda) \curvearrowright \operatorname{Iso}(\partial\lambda,\ell)$ defined via

$$G \cdot h := h \circ \partial G^{-1} \text{ for } G \in \operatorname{Aut}(\lambda) \text{ and } h \in \operatorname{Iso}(\partial\lambda,\ell). \tag{2}$$

In particular, we can consider

$$b_{(M,g)} := g_* \circ D_M \circ \partial F \in \operatorname{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda).$$

It is now not difficult to check that $b_{(M,g)}$ is independent of the choice of F.

The fact that if (M_0,g_0) and (M_1,g_1) are homeomorphic rel. boundary (recall Definition 5), then $b_{(M_0,g_0)} = b_{(M_1,g_1)}$ follows fairly quickly. From now on we omit the boundary identification $g\colon \partial M \cong Y$ from the notation, writing b_M instead of $b_{(M,g)}$. This concludes the construction of our automorphism invariant.

We are now ready to state our main technical theorem.

Theorem 1.15. *Let Y be a 3-manifold with an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module is torsion, and let (H,λ) be a nondegenerate Hermitian form presenting Y. If $b \in \operatorname{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda)$ is an isometry, then there is a 4-manifold M with $\pi_1(M) \cong \mathbb{Z}$, equivariant intersection form $\lambda_M \cong \lambda$, ribbon boundary Y and $b_M = b$. If the form is odd, then M can be chosen to have either* $\operatorname{ks}(M) = 0$ *or* $\operatorname{ks}(M) = 1$.

We now describe how to obtain Theorem 1.1 by combining this result with [CP20].

Proof of Theorem 1.1 assuming Theorem 1.15. First, notice that Theorem 1.15 implies the surjectivity portion of the statement in Theorem 1.1. It therefore suffices to prove that the assignment $\mathcal{V}^0_\lambda(Y) \to \operatorname{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda)$ which sends M to b_M is injective for λ even, and that the assignment $\mathcal{V}^0_\lambda(Y) \to (\operatorname{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda)) \times \mathbb{Z}_2$ which sends M to $(b_M, \operatorname{ks}(M))$ is injective for λ odd.

Let $(M_0,g_0),(M_1,g_1)$ be two pairs representing elements in $\mathcal{V}^0_\lambda(Y)$. Each 4-manifold M_i comes with an isometry $F_i\colon (H,\lambda) \to (H_2(M_i;\mathbb{Z}[t^{\pm1}]),\lambda_{M_i})$ and for $i = 0,1$, the homeomorphisms $g_i\colon \partial M_i \to Y$ are as in Definition 5. We then get epimorphisms

$$(g_i)_* \circ D_{M_i} \circ \partial F_i \circ \pi\colon H^* \twoheadrightarrow H_1(Y;\mathbb{Z}[t^{\pm1}]).$$

Here $\pi\colon H^* \to \operatorname{coker}(\widehat{\lambda})$ denotes the canonical projection. We assume that $b_{M_0} = b_{M_1}$ and, if λ is odd, then we additionally assume that $\operatorname{ks}(M_0) = \operatorname{ks}(M_1)$. The fact that $b_{M_0} = b_{M_1}$ implies that there is an isometry $F\colon (H,\lambda) \cong (H,\lambda)$ that makes the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \xrightarrow{\widehat{\lambda}} & H^* & \xrightarrow{(g_0)_*\circ D_{M_0}\circ\partial F_0\circ\pi} & H_1(Y;\mathbb{Z}[t^{\pm1}]) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F^{-*}} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & H & \xrightarrow{\widehat{\lambda}} & H^* & \xrightarrow{(g_1)_*\circ D_{M_1}\circ\partial F_1\circ\pi} & H_1(Y;\mathbb{Z}[t^{\pm1}]) & \longrightarrow & 0.
\end{array}$$

But now, by considering the isometry $G\colon \lambda_{M_0} \cong \lambda_{M_1}$ defined by $G := F_1 \circ F \circ F_0^{-1}$, a quick verification shows that (G, Id_Y) is a compatible pair in the sense of [CP20]. Consequently [CP20, Theorem 1.10] shows that there is a homeomorphism $M_0 \cong M_1$ extending Id_Y and inducing G; in particular M_0 and M_1 are homeomorphic rel. boundary. □

Next, we describe how the classification in the case where the homeomorphisms need not fix the boundary pointwise (i.e. Theorem 1.2) follows from Theorem 1.1. To this effect, we describe the set $\mathscr{V}_\lambda(Y)$ from Theorem 1.2 more precisely, similarly to the way we defined $\mathscr{V}^0_\lambda(Y)$ in Definition 5.

Definition 6. For Y and (H,λ) as in Definition 5, define $\mathscr{V}_\lambda(Y)$ as the quotient of $S_\lambda(Y)$ in which two pairs $(M_1,g_1),(M_2,g_2)$ are deemed equal if and only if there is a homeomorphism $\Phi\colon M_1 \cong M_2$ such that $\Phi|_{\partial M_1} = g_2^{-1}\circ f\circ g_1$ for some $f \in \operatorname{Homeo}^+_\varphi(Y)$; note that such a homeomorphism is necessarily orientation-preserving. Here, recall that $\operatorname{Homeo}^+_\varphi(Y)$ denotes the set of those orientation-preserving homeomorphisms $f\colon Y\cong Y$ such that $\varphi\circ f_* = \varphi\colon \pi_1(Y)\twoheadrightarrow \mathbb{Z}$.

We continue to set up notation to describe how Theorem 1.2 follows from Theorem 1.1. Observe that the group $\operatorname{Homeo}^+_\varphi(Y)$ acts on $\mathscr{V}^0_\lambda(Y)$ by setting $f\cdot(M,g) := (M, f\circ g)$ for $f\in \operatorname{Homeo}^+_\varphi(Y)$. Further, observe that

$$\mathscr{V}_\lambda(Y) = \mathscr{V}^0_\lambda(Y)/\operatorname{Homeo}^+_\varphi(Y). \tag{3}$$

Recall that any $f \in \operatorname{Homeo}^+_\varphi(Y)$ induces an isometry f_* of the Blanchfield form Bl_Y. Thus the group $\operatorname{Homeo}^+_\varphi(Y)$ acts on $\operatorname{Iso}(\partial\lambda, -\mathrm{Bl}_Y)$ by $f\cdot h := f_*\circ h$. Finally, there is a natural left action $\operatorname{Aut}(\lambda)\times \operatorname{Homeo}^+_\varphi(Y)$ on $\operatorname{Iso}(\partial\lambda,-\mathrm{Bl}_Y)$ defined via

$$(F,f)\cdot h := f_*\circ h\circ \partial F^{-1}. \tag{4}$$

We now explain how Theorem 1.2, which gives a description of $\mathscr{V}_\lambda(Y)$ in terms of the orbit set $\operatorname{Iso}(\partial\lambda,-\mathrm{Bl}_Y)/(\operatorname{Aut}(\lambda)\times\operatorname{Homeo}^+_\varphi(Y))$, follows from Theorem 1.1 which described the set $\mathscr{V}^0_\lambda(Y)$ in terms of the orbit set $\operatorname{Iso}(\partial\lambda,-\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda)$.

Proof of Theorem 1.2 assuming Theorem 1.1. Thanks to Theorem 1.1 and (3), it suffices to prove that the map b respects the $\operatorname{Homeo}^+_\varphi(Y)$ actions, i.e. that $b_{f\cdot(M,g)} = f\cdot b_{(M,g)}$, where $g\colon \partial M\cong Y$ is a homeomorphism as in Definition 5 and $f\in\operatorname{Homeo}^+_\varphi(Y)$. This now follows from the following formal calculation: $b_{f\cdot(M,g)} = b_{(M,f\circ g)} = f_*\circ g_*\circ D_M\circ\partial F = f\cdot b_{(M,g)}$, where $F\colon \lambda_M\cong\lambda$ is an isometry and we used the definitions of the $\operatorname{Homeo}^+_\varphi(Y)$ actions and of the map b. □

We conclude the introduction by outlining the strategy of the proof of Theorem 1.15.

Outline of the proof of Theorem 1.15. The idea is to perform surgeries on Y along generators of $H_1(Y;\mathbb{Z}[t^{\pm1}])$ to obtain a 3-manifold Y' with $H_1(Y';\mathbb{Z}[t^{\pm1}]) = 0$. The verification that $H_1(Y';\mathbb{Z}[t^{\pm1}])=0$ uses Reidemeister torsion. We then use surgery theory to show that this Y' bounds a 4-manifold B with $B\simeq S^1$; this step relies on Freedman's work in the topological category [Fre82, FQ90, BKK+21]. The 4-manifold M is then obtained as the union of the trace of these surgeries with B. To show that in the odd case both values of the Kirby-Siebenmann invariant are realised, we use the star construction [FQ90, Sto93]. The main difficulty of the proof is to describe the correct surgeries on Y to obtain Y'; this is where the fact that λ presents Bl_Y comes into play: we show that generators of $H_1(Y;\mathbb{Z}[t^{\pm1}])$ can be represented by a link $\widetilde{L}$ with equivariant linking matrix equal to minus the transposed inverse of a matrix representing λ. □

This is a strategy similar to the one employed in Boyer's classification of simply-connected 4-manifolds with a given boundary [Boy93]. The argument is also reminiscent of [BF15, Theorem 4.1], where Borodzik and Friedl obtain bounds (in terms of a presentation matrix for Bl_K) on the number of crossing changes required to turn K into an Alexander polynomial one knot: they perform surgeries on $Y = M_K$ to obtain $Y' = M_{K'}$, where K' is an Alexander polynomial one knot.

Remark 1.16. As we mentioned in Construction 1, if M_0 and M_1 are homeomorphic rel. boundary, then $b_{M_0} = b_{M_1}$ in $\operatorname{Iso}(\partial\lambda,-\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda)$. In fact the same proof shows more. If two 4-manifolds M_0 and M_1 that represent elements of $\mathscr{V}^0_\lambda(Y)$ are *homotopy equivalent* rel. boundary, then $b_{M_0} = b_{M_1}$ in $\operatorname{Iso}(\partial\lambda,-\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda)$.

Organisation. In Section 2, we recall and further develop the theory of equivariant linking numbers. In Section 3 we review the facts we will need on Reidemeister torsion. Section 4, we prove Theorem 1.15. Section 5 is concerned with our applications to surfaces and in particular, we prove Theorems 1.6, 1.9 and 1.10. Our results in the smooth category, namely Theorems 1.12 and 1.13, are proved in Section 6. Finally, Section 7 shows that the sets $\mathscr{V}^0_\lambda(Y)$ and $\mathscr{V}^0_\lambda(Y)$ can be arbitrarily large.

Conventions. In Sections 2-5 and 7, we work in the topological category with locally flat embeddings unless otherwise stated. In Section 6, we work in the smooth category.

From now on, all manifolds are assumed to be compact, connected, based and oriented; if a manifold has a nonempty boundary, then the basepoint is assumed to be in the boundary.

If P is manifold and $Q \subseteq P$ is a submanifold with closed tubular neighborhood $\overline{\nu}(Q) \subseteq P$, then $P_Q := P \setminus \nu(Q)$ will always denote the exterior of Q in P, that is the complement of the open tubular neighborhood. The only exception to this use of notation is that the exterior of a knot K in S^3 will be denoted E_K instead of S^3_K.

We write $p \mapsto \overline{p}$ for the involution on $\mathbb{Z}[t^{\pm 1}]$ induced by $t \mapsto t^{-1}$. Given a $\mathbb{Z}[t^{\pm 1}]$-module H, we write $\overline{H}$ for the $\mathbb{Z}[t^{\pm 1}]$-module whose underlying abelian group is H but with module structure given by $p \cdot h = \overline{p}h$ for $h \in H$ and $p \in \mathbb{Z}[t^{\pm 1}]$. We write $H^* := \overline{\operatorname{Hom}_{\mathbb{Z}[t^{\pm 1}]}(H, \mathbb{Z}[t^{\pm 1}])}$.

If a pullback map F^* is invertible we shall abbreviate $(F^*)^{-1}$ to F^{-*}. Similarly, for an invertible square matrix A we write $A^{-T} := (A^T)^{-1}$.

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2. Equivariant linking and longitudes

We collect some preliminary notions that we will need later on. In Subsection 2.1 we fix our notation for twisted homology and equivariant intersections. In Subsection 2.2, we collect some facts about linking numbers in infinite cyclic covers, while in Subsection 2.3, we define an analogue of integer framings of a knot in S^3 for knots in infinite cyclic covers.

2.1. Covering spaces and twisted homology.

We fix our conventions on twisted homology and recall some facts about equivariant intersection numbers. We refer the reader interested in the intricacies of transversality in the topological category to [FNOP19, Section 10].

We first introduce some notation for infinite cyclic covers. Given a CW complex X together with an epimorphism $\varphi\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$, we write $p\colon X^\infty \to X$ for the infinite cyclic cover corresponding to $\ker(\varphi)$. If $A \subset X$ is a subcomplex, then we set $A^\infty := p^{-1}(A)$ and often write $H_*(X, A; \mathbb{Z}[t^{\pm 1}])$ instead of $H_*(X^\infty, A^\infty)$. Similarly, since $\mathbb{Q}(t)$ is flat over $\mathbb{Z}[t^{\pm 1}]$, we often write $H_*(X, A; \mathbb{Q}(t))$ or $H_*(X, A; \mathbb{Z}[t^{\pm 1}]) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t)$ instead of $H_*(X^\infty, A^\infty) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t)$.

Remark 2.1. Assume $H_1(X; \mathbb{Z}[t^{\pm 1}])$ is finitely generated. In this case, the *Alexander polynomial* of X, denoted Δ_X is the order of the *Alexander module* $H_1(X; \mathbb{Z}[t^{\pm 1}])$. While we refer to Remark 3.2 below for some recollections on orders of modules, here we simply note that Δ_X is a Laurent polynomial that is well defined up to multiplication by $\pm t^k$ with $k \in \mathbb{Z}$ and that if $X = M_K$ is the 0-framed surgery along a knot K, then Δ_X is the Alexander polynomial of K.

Next, we move on to equivariant intersections in covering spaces.

Definition 7. Let M be an n-manifold (with possibly nonempty boundary) with an epimorphism $\pi_1(M) \twoheadrightarrow \mathbb{Z}$. For a k-dimensional closed submanifold $A \subset M^\infty$ and an $(n-k)$-dimensional closed submanifold $A' \subset M^\infty$ such that A and $t^j A'$ intersect transversely for all $j \in \mathbb{Z}$, we define the *equivariant intersection* $A \cdot_{\infty,M} A' \in \mathbb{Z}[t^{\pm 1}]$ as

$$A \cdot_{\infty,M} A' = \sum_{j\in\mathbb{Z}} (A \cdot_{M^\infty} (t^j A'))t^{-j},$$

where $\cdot_{M^\infty}$ denotes the usual (algebraic) signed count of points of intersection. If the boundary of M is nonempty and $A' \subset M$ is properly embedded, then we can make the same definition and also write $A \cdot_{\infty,M} A' \in \mathbb{Z}[t^{\pm 1}]$.

Remark 2.2. We collect a couple of observations about equivariant intersections.

(1) Equivariant intersections are well defined on homology and in fact $A \cdot_{\infty,M} A' = \lambda([A'],[A])$, where λ denotes the equivariant intersection form

$$\lambda\colon H_k(M;\mathbb{Z}[t^{\pm 1}]) \times H_{n-k}(M;\mathbb{Z}[t^{\pm 1}]) \to \mathbb{Z}[t^{\pm 1}].$$

The reason for which $A \cdot_{\infty,M} A'$ equals $\lambda([A'],[A]) = \overline{\lambda([A],[A'])}$ instead of $\lambda([A],[A'])$ is due to the fact that we are following the conventions from [CP20, Section 2] in which the adjoint of a Hermitian form $\lambda\colon H \times H \to \mathbb{Z}[t^{\pm 1}]$ is defined by the equation $\widehat{\lambda}(y)(x) = \lambda(x,y)$. With these conventions λ is linear in the first variable and anti-linear in the second, whereas $\cdot_{\infty,M}$ is linear in the second variable and anti-linear in the first.

(2) When $\partial M \neq \emptyset$ and $A \subset M$ is a properly embedded submanifold with boundary, then again $A \cdot_{\infty,M} A' = \lambda^\partial([A'],[A])$ where this time λ^∂ denotes the pairing

$$\lambda^\partial\colon H_k(M;\mathbb{Z}[t^{\pm 1}]) \times H_{n-k}(M,\partial M;\mathbb{Z}[t^{\pm 1}]) \to \mathbb{Z}[t^{\pm 1}].$$

As previously λ^∂ is linear in the first variable and anti-linear in the second.

(3) The definition of the pairings λ and λ^∂ can be made with arbitrary twisted coefficients. In order to avoid extraneous generality, we simply mention that there are $\mathbb{Q}(t)$-valued pairings $\lambda_{\mathbb{Q}(t)}$ and $\lambda^\partial_{\mathbb{Q}(t)}$ defined on homology with $\mathbb{Q}(t)$-coefficients and that if $A, B \subset M^\infty$ are closed submanifolds of complementary dimension, then $\lambda_{\mathbb{Q}(t)}([A],[B]) = \lambda([A],[B])$ and similarly for properly embedded submanifolds with boundary.

2.2. **Equivariant linking.** We recall definitions and properties of equivariant linking numbers. Other papers that feature discussions of the topic include [PY04, BF14, KR20].

We assume for the rest of the section that Y is a 3-manifold and that $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ is an epimorphism such that the corresponding Alexander module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is torsion, i.e. $H_*(Y;\mathbb{Q}(t)) = 0$. We also write $p\colon Y^\infty \to Y$ for the infinite cyclic cover corresponding to $\ker(\varphi)$ so that $H_1(Y;\mathbb{Z}[t^{\pm 1}]) = H_1(Y^\infty)$. Given a simple closed curve $\widetilde{a} \subset Y^\infty$, we write $a^\infty := \bigcup_{k\in\mathbb{Z}} t^k \widetilde{a}$ for the union of all the translates of $\widetilde{a}$ and $a := p(\widetilde{a}) \subset Y$ for the projection of $\widetilde{a}$ down to Y. This way, the covering map $p\colon Y^\infty \to Y$ restricts to a covering map

$$Y^\infty \setminus \nu(a^\infty) \to Y \setminus \nu(a) =: Y_a.$$

Since the Alexander module of Y is torsion, a short Mayer-Vietoris argument shows that the vector space $H_*(Y_a;\mathbb{Q}(t)) = \mathbb{Q}(t)$ is generated by $[\widetilde{\mu}_a]$, the class of a meridian of $\widetilde{a} \subset Y^\infty$.

Definition 8. The *equivariant linking number* of two disjoint simple closed curves $\widetilde{a}, \widetilde{b} \subset Y^\infty$ is the unique rational function $\ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b}) \in \mathbb{Q}(t)$ such that

$$[\widetilde{b}] = \ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b})[\widetilde{\mu}_a] \in H_1(Y \setminus \nu(a);\mathbb{Q}(t)).$$

Observe that this linking number is only defined for *disjoint* pairs of simple closed curves. We give a second, more geometric, description of the equivariant linking number.

Remark 2.3. Since $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is torsion, for any simple closed curve $\widetilde{a}$ in Y^∞, there is some polynomial $p(t) = \sum_i c_i t^i$ such that $p(t)[\widetilde{a}] = 0$. Thus there is a surface $F \subset Y^\infty \smallsetminus \nu(a^\infty)$ with boundary consisting of the disjoint union of c_i parallel copies of $t^i \cdot \widetilde{a}'$ and d_j meridians of $t^j \cdot \widetilde{a}'$ where $\widetilde{a}'$ is some pushoff of $\widetilde{a}$ in $\partial\overline{\nu}(\widetilde{a})$ and $j \neq i$; we abusively write $\partial F = p(t)\widetilde{a}$.

Proposition 2.4. *Let Y be a 3-manifold, let $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ be an epimorphism such that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm1}])$ is torsion, and let $\widetilde{a},\widetilde{b} \subset Y^\infty$ be disjoint simple closed curves.*

Let F and $p(t)$ be respectively a surface and a polynomial associated to $\widetilde{a}$ as in Remark 2.3. The equivariant linking of $\widetilde{a}$ and $\widetilde{b}$ can be written as

$$\ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b}) = \frac{1}{p(t^{-1})}\sum_{k\in\mathbb{Z}}(F\cdot t^k\widetilde{b})t^{-k} = \frac{1}{p(t^{-1})}(F\cdot_{\infty,Y_a}\widetilde{b}). \tag{5}$$

In particular, this expression is independent of the choices of F and $p(t)$.

Proof. As in Subsection 2.1, write λ^∂ for the (homological) intersection pairing $H_1(Y_a;\mathbb{Z}[t^{\pm1}]) \times H_2(Y_a,\partial Y_a;\mathbb{Z}[t^{\pm1}]) \to \mathbb{Z}[t^{\pm1}]$ and $\lambda^\partial_{\mathbb{Q}(t)}$ for the pairing involving $\mathbb{Q}(t)$-homology.

Write $\ell := \ell k(\widetilde{a},\widetilde{b})$ so that $[\widetilde{b}] = \ell[\widetilde{\mu}_a] \in H_1(Y_a;\mathbb{Q}(t))$. From this and Remark 2.2, for a surface F as in the statement, we obtain

$$F\cdot_{\infty,Y_a}\widetilde{b} = \lambda^\partial([\widetilde{b}],[F]) = \lambda^\partial_{\mathbb{Q}(t)}([\ell\widetilde{\mu}_a],[F]) = \ell\lambda^\partial_{\mathbb{Q}(t)}([\widetilde{\mu}_a],[F]) = \ell(F\cdot_{\infty,Y_a}\widetilde{\mu}_a) = \ell p(t^{-1}).$$

The last equality here follows from inspection; since $F \hookrightarrow Y^\infty \smallsetminus \nu(a^\infty)$ has boundary along c_i copies of $t^i\cdot\widetilde{a}'$ and d_j copies of $t^j\widetilde{\mu}_a$, each meridian $t^i\cdot\mu_{\widetilde{a}}$ intersects F in c_i points. The result now follows after dividing out by $p(t^{-1})$. □

Just as for linking numbers in rational homology spheres, the equivariant linking number is not well defined on homology, unless the target is replaced by $\mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}]$. To describe the resulting statement, we briefly recall the definition of the Blanchfield form.

Remark 2.5. Using the same notation and assumptions as in Proposition 2.4, the Blanchfield form is a nonsingular sesquilinear, Hermitian pairing that can be defined as

$$\begin{aligned}\mathrm{Bl}_Y\colon H_1(Y;\mathbb{Z}[t^{\pm1}]) \times H_1(Y;\mathbb{Z}[t^{\pm1}]) &\to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}]\\ ([\widetilde{b}],[\widetilde{a}]) &\mapsto \left[\frac{1}{p(t)}(F\cdot_{\infty,Y_a}\widetilde{b})\right].\end{aligned} \tag{6}$$

We refer to [Pow16, FP17] for further background and homological definitions of this pairing.

We summarise this discussion and collect another property of equivariant linking in the next proposition.

Proposition 2.6. *Let Y be a 3-manifold and let $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ be an epimorphism such that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm1}])$ is torsion. For disjoint simple closed curves $\widetilde{a},\widetilde{b} \subset Y^\infty$, the equivariant linking number satisfies the following properties:*

1. *sesquilinearity: $\ell k_{\mathbb{Q}(t)}(p\widetilde{a},q\widetilde{b}) = \overline{p}q\ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b})$ for all $p,q \in \mathbb{Z}[t^{\pm1}]$;*
2. *symmetry: $\ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b}) = \overline{\ell k_{\mathbb{Q}(t)}(\widetilde{b},\widetilde{a})}$;*
3. *relation to the Blanchfield form: $[\ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b})] = \mathrm{Bl}_Y([\widetilde{b}],[\widetilde{a}]) \in \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}]$.*

Proof. The first property follows from (5). Before proving the second and third properties,we note that in (5) and (6), we can assume that $p(t) = p(t^{-1})$. Indeed, both formulae are independent of the choice of $p(t)$ and if $q(t)$ satisfies $q(t)[\widetilde{a}] = 0$, then so does $p(t) := q(t)q(t^{-1})$. The proof of the second assertion now follows as in [BF14, Lemma 3.3], whereas the third follows by inspecting (5) and (6). □

The reader will have observed that the formulas in Proposition 2.4 and 2.6 depend heavily on conventions chosen for adjoints, module structures, equivariant intersections and twisted homology. It is for this reason that the formulas presented here might differ (typically up to switching variables) from others in the literature.

2.3. **Parallels, framings, and longitudes.** Continuing with the notation and assumptions from the previous section, we fix some terminology regarding parallels and framings in infinite cyclic covers. The goal is to be able to describe a notion of integer surgery for appropriately nullhomologous knots in the setting of infinite cyclic covers. Our approach is inspired by [BL92, Boy93].

Definition 9. Let $\widetilde{K} \subset Y^\infty$ be a knot, let $p\colon Y^\infty \to Y$ be the covering map, and denote $K := p(\widetilde{K}) \subset Y$ the projection of $\widetilde{K}$.

(1) A *parallel* to $\widetilde{K}$ is a simple closed curve $\pi \subset \partial\overline{\nu}(\widetilde{K})$ that is isotopic to $\widetilde{K}$ in $\overline{\nu}(\widetilde{K})$.
(2) Given any parallel π of $\widetilde{K}$, we use $\overline{\nu}_\pi(\widetilde{K})$ to denote the parametrisation $S^1 \times D^2 \xrightarrow{\cong} \overline{\nu}(\widetilde{K})$ which sends $S^1 \times \{x\}$ to π for some $x \in \partial D^2$.
(3) A *framed link* is a link $\widetilde{L} \subset Y^\infty$ together with a choice of a parallel for each of its components.
(4) We say that the knot $\widetilde{K}$ *admits framing coefficient* $r(t) \in \mathbb{Q}(t)$ if there is a parallel π with $\ell k_{\mathbb{Q}(t)}(\widetilde{K}, \pi) = r(t)$. We remark that, unlike in the setting of homology with integer coefficients where every knot K admits any integer r as a framing coefficient, when we work with $\mathbb{Z}[t^{\pm1}]$-homology, a fixed knot $\widetilde{K}$ will have many $r(t) \in \mathbb{Q}(t)$ (in fact even in $\mathbb{Z}[t^{\pm1}]$) which it does not admit as a framing coefficient. We will refer to π as a *framing curve* of $\widetilde{K}$ with framing $r(t)$.
(5) A framed n-component link $\widetilde{L}$ which admits framing coefficients $\mathbf{r}(t) := (r_i(t))_{i=1}^n$, together with a choice of parallels realising those framing coefficients, is called an $\mathbf{r}(t)$-framed link.
(6) The *equivariant linking matrix* of an $\mathbf{r}(t)$-framed link $\widetilde{L}$ is the matrix $A_{\widetilde{L}}$ with diagonal term $(A_{\widetilde{L}})_{ii} = r_i(t)$ and off-diagonal terms $(A_{\widetilde{L}})_{ij} = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j)$ for $i \neq j$.
(7) For a link $\widetilde{L}$ in Y^∞, we define L^∞ to be the set of all the translates of $\widetilde{L}$. We also set
$$L := p(\widetilde{L}).$$
We say that $\widetilde{L}$ is in *covering general position* if the map $p\colon L^\infty \to L$ is a trivial $\mathbb{Z}$-covering isomorphic to the pullback cover
$$\begin{array}{ccc} L^\infty & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ L & \xrightarrow{c} & S^1 \end{array}$$
where c is a constant map. In particular each component of L^∞ is mapped by p, via a homeomorphism, to some component of L. From now on we will always assume that our links $\widetilde{L}$ are in covering general position. This assumption is to avoid pathologies, and holds generically.
(8) For an n-component link $\widetilde{L}$ which admits framing coefficients $\mathbf{r}(t) := (r_i(t))_{i=1}^n$, the $\mathbf{r}(t)$-*surgery* along $\widetilde{L}$ is the covering space $Y^\infty_{\mathbf{r}(t)}(\widetilde{L}) \to Y_{\mathbf{r}}(L)$ defined by Dehn filling $Y^\infty \setminus \nu(L^\infty)$ along all the translates of all the parallels $\pi_1^\infty, \ldots, \pi_n^\infty$ as follows:
$$Y^\infty_{\mathbf{r}(t)}(\widetilde{L}) = Y^\infty \setminus \left(\bigcup_{k\in\mathbb{Z}} \bigcup_{i=1}^n \left(t^k \overline{\nu}_{\pi_i}(\widetilde{K}_i) \right) \right) \cup \left(\bigcup_{k\in\mathbb{Z}} \bigcup_{i=1}^n (D^2 \times S^1) \right).$$
Since $\widetilde{L}$ is in covering general position, for all $\widetilde{K}_i$ the covering map $p|_{\widetilde{K}_i}\colon \widetilde{K}_i \to K_i$ is a homeomorphism, so $p|_{\overline{\nu}(\widetilde{K}_i)}\colon \overline{\nu}(\widetilde{K}_i) \to \nu(K_i)$ is a homeomorphism. Thus any parallel π_i of $\widetilde{K}_i$ projects to a parallel of K, so we may also define $\mathbf{r}$-surgery along L downstairs:
$$Y_{\mathbf{r}}(L) = Y \setminus \left(\bigcup_{i=1}^n \overline{\nu}_{p(\pi_i)}(p(\widetilde{K}_i)) \right) \cup \left(\bigcup_{i=1}^n (D^2 \times S^1) \right).$$
Observe that there is a naturally induced cover $Y^\infty_{\mathbf{r}(t)}(\widetilde{L}) \to Y_{\mathbf{r}}(L)$ obtained by restricting $p\colon Y^\infty \to Y$ to the link exterior and then extending it to the trivial disconnected $\mathbb{Z}$-cover over each of the surgery solid tori.

(9) The *dual framed link* $\widetilde{L}' \subset Y^\infty_{\mathbf{r}(t)}(\widetilde{L})$ associated to a framed link $\widetilde{L} \subset Y^\infty$ is defined as follows:
   - the i-th component $\widetilde{K}_i'$ of the underlying link $\widetilde{L}' \subset Y^\infty_{\mathbf{r}(t)}(\widetilde{L})$ is obtained by considering the core of the i-th surgery solid torus $D^2 \times S^1$.
   - The framing of $\widetilde{K}_i'$ is given by the S^1-factor $S^1 \times \{\text{pt}\}$ of the parametrised solid torus used to define $\widetilde{K}_i'$.

(10) We also define analogues of these notions (except (6) and (7)) for a link L in the 3-manifold Y, without reference to the cover.

The next lemma provides a sort of analogue for the Seifert longitude of a knot in S^3; it is inspired by [BL92, Lemma 1.2]. The key difference with the Seifert longitude is that in our setting this class, which we denote by $\lambda_{\widetilde{K}}$, is just a homology class in $H_1(\partial\overline{\nu}(\widetilde{K}); \mathbb{Q}(t))$; it will frequently not be represented by a simple closed curve.

Lemma 2.7. *For every knot $\widetilde{K} \subset Y^\infty$, there is a unique homology class $\lambda_{\widetilde{K}} \in H_1(\partial\overline{\nu}(\widetilde{K}); \mathbb{Q}(t))$ called the* longitude *of $\widetilde{K}$ such that the following two conditions hold.*

(1) The algebraic equivariant intersection number of $[\mu_{\widetilde{K}}]$ and $\lambda_{\widetilde{K}}$ is one:

$$\lambda_{\partial\overline{\nu}(K),\mathbb{Q}(t)}([\mu_{\widetilde{K}}], \lambda_{\widetilde{K}}) = 1.$$

(2) The class $\lambda_{\widetilde{K}}$ maps to zero in $H_1(Y_K; \mathbb{Q}(t))$.

For any parallel π of $\widetilde{K}$, this class satisfies

$$\lambda_{\widetilde{K}} := [\pi] - \ell k_{\mathbb{Q}(t)}(\widetilde{K}, \pi)[\mu_{\widetilde{K}}].$$

Proof. We first prove existence and then uniqueness. For existence, pick any parallel π to $\widetilde{K}$, i.e. any curve in $\partial\overline{\nu}(\widetilde{K})$ that is isotopic to $\widetilde{K}$ in $\overline{\nu}(\widetilde{K})$ and define

$$\lambda_{\widetilde{K}} := [\pi] - \ell k_{\mathbb{Q}(t)}(\widetilde{K}, \pi)[\mu_{\widetilde{K}}].$$

Here recall that the equivariant linking $r := \ell k_{\mathbb{Q}(t)}(\widetilde{K}, \pi)$ is the unique element of $\mathbb{Q}(t)$ such that $[\pi] = r[\mu_{\widetilde{K}}]$ in $H_1(Y_K; \mathbb{Q}(t))$. The two axioms now follow readily.

For uniqueness, we suppose that $\lambda_{\widetilde{K}}$ and $\lambda'_{\widetilde{K}}$ are two homology classes as in the statement of the lemma. Choose a parallel π of $\widetilde{K}$ and base $H_1(\partial\overline{\nu}(K); \mathbb{Q}(t))$ by the pair $(\mu_{\widetilde{K}}, \pi)$. This way, we can write $\lambda_{\widetilde{K}} = r_1[\mu_{\widetilde{K}}] + r_2[\pi]$ and $\lambda'_{\widetilde{K}} = r'_1[\mu_{\widetilde{K}}] + r'_2[\pi]$. The first condition on $\lambda_{\widetilde{K}}$ now promptly implies that $r_2 = r'_2 = 1$; formally

$$1 = \lambda_{\partial\overline{\nu}(K),\mathbb{Q}(t)}([\mu_{\widetilde{K}}], \lambda_{\widetilde{K}}) = r_2\lambda_{\partial\overline{\nu}(K),\mathbb{Q}(t)}([\mu_{\widetilde{K}}], [\pi]) = r_2$$

and similarly for r'_2. To see that $r_1 = r'_1$, observe that since $r_2 = r'_2$, we have that $\lambda_{\widetilde{K}} = \lambda'_{\widetilde{K}} + (r'_1 - r_1)[\mu_{\widetilde{K}}]$. Recall that $[\mu_{\widetilde{K}}]$ is a generator of the vector space $H_1(Y_K; \mathbb{Q}(t)) = \mathbb{Q}(t)$ and that $\lambda'_{\widetilde{K}}, \lambda'_{\widetilde{K}}$ are zero in $H_1(Y_K; \mathbb{Q}(t))$. We conclude that $(r'_1 - r_1) = 0$, as required. □

As motivation, observe that for a link $L = K_1 \cup \cdots \cup K_n \subset S^3$, the group $H_1(E_L; \mathbb{Z})$ is freely generated by the meridians μ_{K_i} and, if L is framed with integral linking matrix A, then the framing curves π_i can be written in this basis as $[\pi_i] = \sum_{j=1}^n A_{ij}[\mu_{K_j}] \in H_1(E_L; \mathbb{Z})$. The situation is similar in our setting.

Proposition 2.8. *Let $\widetilde{L} \subset Y^\infty$ be an n-component framed link in covering general position whose components have framing curves $\pi_1, \ldots, \pi_n$. Recall that $H_1(Y_L; \mathbb{Q}(t)) = \mathbb{Q}(t)^n$ is generated by the homology classes of the meridians $\mu_{\widetilde{K}_1}, \ldots, \mu_{\widetilde{K}_n}$. The homology classes of the π_i in $H_1(Y_L; \mathbb{Q}(t)) \cong \mathbb{Q}(t)^n$ are related to the meridians by the formula*

$$[\pi_i] = \sum_{j=1}^n (A_{\widetilde{L}})_{ij}[\mu_{\widetilde{K}_j}] \in H_1(Y_L; \mathbb{Q}(t)).$$

Proof. By definition of the equivariant linking matrix $A_{\widetilde{L}}$, we must prove that

$$[\pi_i] = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i)[\mu_{\widetilde{K}_i}] + \sum_{j \neq i} \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j)[\mu_{\widetilde{K}_j}] \in H_1(Y_L; \mathbb{Q}(t)) \tag{7}$$

for each i. Since the sum of the inclusion induced maps give rise to an isomorphism

$$H_1(Y_L; \mathbb{Q}(t)) \cong \bigoplus_{j=1}^{n} H_1(Y_{K_j}; \mathbb{Q}(t))$$

it suffices to prove the equality after applying the inclusion map $H_1(Y_L; \mathbb{Q}(t)) \to H_1(Y_{K_j}; \mathbb{Q}(t))$, for each j. Since π_i is a parallel of $\widetilde{K}_i$, applying Lemma 2.7, we have

$$[\pi_i] = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i)[\mu_{\widetilde{K}_i}] + \lambda_{\widetilde{K}_i} \in H_1(\partial Y_{K_i}; \mathbb{Q}(t)).$$

We consider the image of this homology class in $H_1(Y_{K_j}; \mathbb{Q}(t))$ for $j = 1, \ldots, n$. In the vector space $H_1(Y_{K_i}; \mathbb{Q}(t)) = \mathbb{Q}(t)[\mu_{\widetilde{K}_i}]$, the longitude class $\lambda_{\widetilde{K}_i}$ vanishes (again by Lemma 2.7). For $j \neq i$, the class $[\mu_{\widetilde{K}_i}]$ vanishes in $H_1(Y_{K_j}; \mathbb{Q}(t))$; thus the image of $[\pi_i]$ in $H_1(Y_{K_j}; \mathbb{Q}(t))$ is $\ell k_{\mathbb{Q}(t)}(\pi_i, \widetilde{K}_j)[\mu_{\widetilde{K}_j}] = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j)[\mu_{\widetilde{K}_j}]$. This concludes the proof of (7). □

From now on, we will be working with $\mathbb{Z}[t^{\pm 1}]$-coefficient homology both for Y and for the result $Y' := Y_{\mathbf{r}(t)}(L)$ of surgery on a framed link $L \subset Y$. Let W denote the trace of the surgery from Y to Y'. We therefore record a fact about the underlying coefficient systems for later reference.

Lemma 2.9. *The epimorphism $\varphi \colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ extends to an epimorphism $\pi_1(W) \twoheadrightarrow \mathbb{Z}$, which by precomposition with the inclusion map induces an epimorphism $\varphi' \colon \pi_1(Y') \twoheadrightarrow \mathbb{Z}$.*

Proof. Note that $\pi_1(W)$ is obtained from $\pi_1(Y)$ by adding relators that kill each of the $[K_i] \in \pi_1(Y)$ (indeed W is obtained by adding 2-handles to $Y \times [0,1]$ along the K_i). Since φ is trivial on the $K_i \subset Y$ (because they lift to Y^∞), we deduce that φ descends to an epimorphism on $\pi_1(W)$.

The composition $\pi_1(Y') \to \pi_1(W) \twoheadrightarrow \mathbb{Z}$ is also surjective because $\pi_1(W)$ is obtained from $\pi_1(Y')$ by adding relators that kill each of the $[K_i'] \in \pi_1(Y')$; indeed W is obtained by adding 2-handles to $Y' \times [0,1]$ along the dual knots K_i'. □

Remark 2.10. In particular note from the proof of Lemma 2.9 that the homomorphism $\varphi' \colon \pi_1(Y') \twoheadrightarrow \mathbb{Z}$ vanishes on the knots $K_i' \subset Y$ dual to the original $K_i \subset Y$.

The next lemma proves an infinite cyclic cover analogue of the following familiar statement: performing surgery on a framed link $L \subset S^3$ whose linking matrix is invertible over $\mathbb{Q}$ results in a rational homology sphere.

Lemma 2.11. *Let Y be a 3-manifold and let $\varphi \colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ be an epimorphism such that the Alexander module $H_1(Y; \mathbb{Z}[t^{\pm 1}])$ is torsion. If $\widetilde{L} \subset Y^\infty$ is an n-component framed link in covering general position, whose equivariant linking matrix $A_{\widetilde{L}}$ is invertible over $\mathbb{Q}(t)$, then the result Y' of surgery on L satisfies $H_1(Y'; \mathbb{Q}(t)) = 0$.*

Proof. The result will follow by studying the portion

$$\cdots \to H_2(Y, Y_L; \mathbb{Q}(t)) \xrightarrow{\partial} H_1(Y_L; \mathbb{Q}(t)) \to H_1(Y'; \mathbb{Q}(t)) \to H_1(Y', Y_L; \mathbb{Q}(t))$$

of the long exact sequence sequence of the pair (Y, Y_L) with $\mathbb{Q}(t)$-coefficients, and arguing that $H_1(Y', Y_L; \mathbb{Q}(t)) = 0$ and that ∂ is an isomorphism.

The fact that $H_1(Y', Y_L; \mathbb{Q}(t)) = 0$ can be deduced from excision, replacing (Y', Y_L) with the pair $(\sqcup^n S^1 \times D^2, \sqcup^n S^1 \times S^1)$. For the same reason, the vector space $H_2(Y, Y_L; \mathbb{Q}(t)) = \mathbb{Q}(t)^n$ is based by the classes of the discs $(D^2 \times \{\mathrm{pt}\})_i \subset (D^2 \times S^1)_i$ whose boundaries are the framing curves π_i. To conclude that ∂ is indeed an isomorphism, note that $H_1(Y_L; \mathbb{Q}(t)) = \mathbb{Q}(t)^n$ is generated by the $[\mu_{\widetilde{K}_i}]$ (because the Alexander module of Y is torsion) and use Proposition 2.8 to deduce that with respect to these bases, ∂ is represented by the equivariant linking matrix $A_{\widetilde{L}}$. Since this matrix is by assumption invertible over $\mathbb{Q}(t)$, we deduce that ∂ is an isomorphism. It follows that $H_1(Y'; \mathbb{Q}(t)) = 0$, as desired. □

The next lemma describes the framing on the dual of a framed link. The statement ressembles [BL92, Lemma 1.5] and [PY04, Theorem 1.1].

Lemma 2.12. *Let Y be a 3-manifold and let $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ be an epimorphism such that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm1}])$ is torsion. If $\widetilde{L} \subset Y^\infty$ is a framed link in covering general position whose equivariant linking matrix $A_{\widetilde{L}}$ is invertible over $\mathbb{Q}(t)$, then the equivariant linking matrix of the dual framed link $\widetilde{L}'$ is*

$$A_{\widetilde{L}'} = -A_{\widetilde{L}}^{-1}.$$

Proof. Consider the exterior $Y_L = Y'_{L'}$ and recall that $H_1(Y_L;\mathbb{Q}(t)) = \mathbb{Q}(t)^n$ is generated by the meridians $\mu_{\widetilde{K}_1},\ldots,\mu_{\widetilde{K}_n}$ of the link $\widetilde{L}$ because we assumed that $H_1(Y;\mathbb{Q}(t)) = 0$. Since we assumed that $H_1(Y;\mathbb{Q}(t)) = 0$ and $\det(A_{\widetilde{L}}) \neq 0$ we can apply Lemma 2.11 to deduce that $H_1(Y';\mathbb{Q}(t)) = 0$ and hence $H_1(Y_L;\mathbb{Q}(t)) = H_1(Y'_{L'};\mathbb{Q}(t))$ is also generated by the meridians $\mu_{\widetilde{K}'_1},\ldots,\mu_{\widetilde{K}'_n}$ of the link $\widetilde{L}'$.

Thus the vector space $H_1(Y_L;\mathbb{Q}(t)) = \mathbb{Q}(t)^n$ has bases both $\boldsymbol{\mu} = ([\mu_{\widetilde{K}_1}],\ldots,[\mu_{\widetilde{K}_n}])$ and $\boldsymbol{\mu}' = ([\mu_{\widetilde{K}'_1}],\ldots,[\mu_{\widetilde{K}'_n}])$, and we let B be the change of basis matrix between these two bases so that $B\boldsymbol{\mu} = \boldsymbol{\mu}'$. Here and in the remainder of this proof, we adopt the following convention: if C is a matrix over $\mathbb{Q}(t)^n$ and if $\boldsymbol{x} = (x_1,\ldots,x_n)$ is a collection of n vectors in $\mathbb{Q}(t)^n$, then we write $C\boldsymbol{x}$ for the collection of n vectors $Cx_1,\ldots,Cx_n$.

Recall that for $i = 1,\ldots,n$, the framing curves of the $\widetilde{K}_i$ and $\widetilde{K}'_i$ are respectively denoted by $\pi_i \subset Y^\infty$ and $\pi'_i \subset Y'^\infty$. Slightly abusing notation, we also write $[\pi_i]$ for the class of π_i in $H_1(Y_{K_i};\mathbb{Q}(t))$. We set $\boldsymbol{\pi} = ([\pi_1],\ldots,[\pi_n])$ and $\boldsymbol{\pi}' = ([\pi'_1],\ldots,[\pi'_n])$ and and use Proposition 2.8 to deduce that

$$\boldsymbol{\pi} = A_{\widetilde{L}}\boldsymbol{\mu}, \qquad \boldsymbol{\pi}' = A_{\widetilde{L}'}\boldsymbol{\mu}'$$

Inspecting the surgery instructions, we also have the relations

$$\boldsymbol{\mu}' = -\boldsymbol{\pi} \qquad \boldsymbol{\mu} = \boldsymbol{\pi}'.$$

We address the sign in Remark 2.13 below. Combining these equalities, we obtain

$$\begin{aligned}\boldsymbol{\mu} &= \boldsymbol{\pi}' = A_{\widetilde{L}'}\boldsymbol{\mu}' = A_{\widetilde{L}'}B\boldsymbol{\mu},\\ \boldsymbol{\mu}' &= -\boldsymbol{\pi} = -A_{\widetilde{L}}\boldsymbol{\mu} = -A_{\widetilde{L}}B^{-1}\boldsymbol{\mu}'.\end{aligned}$$

Unpacking the equality $A_{\widetilde{L}'}B\boldsymbol{\mu} = \boldsymbol{\mu}$, we deduce that $A_{\widetilde{L}'}B[\mu_{\widetilde{K}_i}] = [\mu_{\widetilde{K}_i}]$ for $i = 1,\ldots,n$. But since the $[\mu_{\widetilde{K}_1}],\ldots,[\mu_{\widetilde{K}_n}]$ form a basis for $\mathbb{Q}(t)^n$, this implies that $A_{\widetilde{L}'}B = I_n$. The same argument shows that $-A_{\widetilde{L}}B^{-1} = I_n$ and therefore both matrices $A_{\widetilde{L}}$ and $A_{\widetilde{L}'}$ are invertible, with $-A_{\widetilde{L}} = B = A_{\widetilde{L}'}^{-1}$. □

Remark 2.13. In the above proposition, we were concerned with the relationship between the curves $(\boldsymbol{\mu},\boldsymbol{\pi})$ and $(\boldsymbol{\mu}',\boldsymbol{\pi}')$, all of which represent classes in $H_1(\partial Y_L,\mathbb{Q}(t))$. We know from the surgery instructions that $g(\boldsymbol{\mu}) = \boldsymbol{\pi}'$. We are free to choose the collection of curves $g(\boldsymbol{\pi})$ so long as we choose each $g(\pi_i)$ to intersect π'_i geometrically once (as unoriented curves). We choose the unoriented curves $\pm\boldsymbol{\mu}'$. Since we know that the surgery was done to produce an oriented manifold, it must be the case that the gluing transformation $g\colon \partial Y_L \to \partial Y_L$ is orientation-preserving. The fact that g is orientation-preserving implies that it preserves intersections numbers, we deduce that $\delta_{ij} = \mu_i \cdot \pi_j = g(\mu_i)\cdot g(\pi_j) = \pi'_j \cdot (\pm\mu'_i)$. This forces $g(\boldsymbol{\pi}) = -\boldsymbol{\mu}'$.

3. Reidemeister torsion

We recall the definition of the Reidemeister torsion of a based chain complex as well as the corresponding definition for CW complexes. This will be primarly used in Subsection 4.3. References on Reidemeister torsion include [Tur01, Tur86, CF13].

Let $\mathbb{F}$ be a field. Given two bases u, v of a r-dimensional $\mathbb{F}$-vector space, we write $\det(u/v)$ for the determinant of the matrix taking v to u, i.e. the determinant of the matrix $A = (A_{ij})$ that satisfies $v^i = \sum_{j=1}^{r} A_{ij} u^j$. A *based chain complex* is a finite chain complex

$$C = \left(0 \to C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_0} C_0 \to 0\right)$$

of $\mathbb{F}$-vector spaces together with a basis c_i for each C_{i+1}. Given a based chain complex, fix a basis b_i for $B_i = \operatorname{im}(\partial_{i+1})$ and pick a lift $\widetilde{b}_i$ of b_i to C_i. Additionally, fix a basis h_i for each homology group $H_i(C)$ and let $\widetilde{h}_i$ be a lift of h_i to C_i. One checks that that $(b_i, \widetilde{h}_i, \widetilde{b}_{i-1})$ forms a basis of C_i.

Definition 10. Let C be a based chain complex over $\mathbb{F}$ and let $\mathscr{B} = \{h_i\}$ be a basis for $H_*(C)$. The *Reidemeister torsion* of $(C, \mathscr{B})$ is defined as

$$\tau(C, \mathscr{B}) = \frac{\prod_i \det((b_{2i+1}, \widetilde{h}_{2i+1}, \widetilde{b}_{2i})|c_{2i+1})}{\prod_i \det((b_{2i}, \widetilde{h}_{2i}, \widetilde{b}_{2i-1})|c_{2i})} \in \mathbb{F} \setminus \{0\}.$$

Implicit in this definition is the fact that $\tau(C, \mathscr{B})$ depends neither on the choice of the basis b_i, nor on the choice of the lifts $\widetilde{b}_i$, nor on the choice of the lifts $\widetilde{h}_i$ of the h_i. It does depend on $\mathscr{B} = \{h_i\}$.

When C is acyclic, we drop $\mathscr{B}$ from the notation and simply write $\tau(C)$.

Note that we are following Turaev's sign convention [Tur01, Tur86]; Milnor's convention [Mil62] yields the multiplicative inverse of $\tau(C, \mathscr{B})$ [Tur01, Remark 1.4 item 5]. The next result collects two properties of the torsion that will be used later on.

Proposition 3.1.

(1) *Suppose that* $0 \to C' \to C \to C'' \to 0$ *is a short exact sequence of based chain complexes and that* $\mathscr{B}', \mathscr{B}$, *and* $\mathscr{B}''$ *are bases for* $H_*(C'), H_*(C)$ *and* $H_*(C'')$ *respectively. If we view the associated homology long exact sequence as an acyclic complex* $\mathscr{H}$, *based by* $\mathscr{B}, \mathscr{B}'$, *and* $\mathscr{B}''$ *respectively, then*
$$\tau(C, \mathscr{B}) = \tau(C', \mathscr{B}')\tau(C'', \mathscr{B}'')\tau(\mathscr{H}).$$

(2) *If* $C = (0 \to C_1 \xrightarrow{\partial_0} C_0 \to 0)$ *is an isomorphism between* n*-dimensional vector spaces, so that* C *is an acyclic based chain complex, then*
$$\tau(C) = \det(A)^{-1}$$
where A *denotes the* $n \times n$*-matrix which represents* ∂_0 *with respect to the given bases.*

Proof. The multiplicativity statement is proved in [Mil62], The second statement follows from Definition 10; details are in [Tur01, Remark 1.4, item 3]. □

We now recall the definition of the torsion of a pair of CW complexes. We focus on the case where the spaces come with a map of their fundamental group to $\mathbb{Z}$. This is a special case of an analogous general theory for the case of an arbitrary group [Tur01], and for more general twisted coefficients [FV11].

Let (X, A) be a finite CW pair, let $\varphi \colon \pi_1(X) \to \mathbb{Z}$ be a homomorphism, and let $\mathscr{B}$ be a basis for the $\mathbb{Q}(t)$-vector space $H_*(X, A; \mathbb{Q}(t))$. Write $p \colon X^\infty \to X$ for the cover corresponding to $\ker(\varphi)$ and set $A^\infty := p^{-1}(A)$. The chain complex $C_*(X^\infty, A^\infty)$ can be based over $\mathbb{Z}[t^{\pm 1}]$ by choosing a lift of each cell of (X, A) and orienting it; this also gives a basis of $C_*(X, A; \mathbb{Q}(t)) = C_*(X^\infty, A^\infty) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t)$. We then define the torsion of (X, A, φ) as

$$\tau(X, A, \mathscr{B}) := \tau(C_*(X, A; \mathbb{Q}(t)), \mathscr{B}) \in \mathbb{Q}(t) \setminus \{0\}.$$

Again, we drop the $\mathscr{B}$ from the notation if $H_*(X, A; \mathbb{Q}(t)) = 0$. It is known that $\tau(X, A, \mathscr{B})$ is well defined up to multiplication by $\pm t^k$ with $k \in \mathbb{Z}$ and is invariant under simple homotopy equivalence preserving $\mathscr{B}$. In what follows, given $p(t), q(t) \in \mathbb{Q}(t)$, we write $p(t) \doteq q(t)$ to indicate that $p(t)$ and $q(t)$ agree up to multiplication by $\pm t^k$. Additionally, the invariant $\tau(X, A, \mathscr{B})$ only depends on the homeomorphism type of (X, A) [Cha74]. In particular, when (M, N) is a manifold pair, we can define $\tau(M, N, \mathscr{B})$ for any finite CW-structure on (M, N), We will only consider the Reidemeister torsion of 3-manifolds, and so every pair (M, N) we consider will admit a CW structure. It will not be relevant in this paper, but we note that it is possible to define Reidemeister torsion for topological 4-manifolds not known to admit a CW structure; see [FNOP19, Section 14] for a discussion.

Remark 3.2. The reason we consider Reidemeister torsion is its relation with Alexander polynomials; see Subsection 4.3 below. To this effect, we recall some relevant algebra. Let P be a $\mathbb{Z}[t^{\pm 1}]$-module with presentation

$$\mathbb{Z}[t^{\pm 1}]^m \xrightarrow{f} \mathbb{Z}[t^{\pm 1}]^n \to P \to 0.$$

Consider elements of the free modules $\mathbb{Z}[t^{\pm 1}]^m$ and $\mathbb{Z}[t^{\pm 1}]^n$ as row vectors and represent f by an $m \times n$ matrix A, acting on the right of the row vectors. By adding rows of zeros, corresponding to trivial relations, we may assume that $m \geq n$. The 0*-th elementary ideal* $E_0(P)$ of a finitely presented $\mathbb{Z}[t^{\pm 1}]$-module P is the ideal of $\mathbb{Z}[t^{\pm 1}]$ generated by all $n \times n$ minors of A. This definition is independent of the choice of the presentation matrix A. The *order* of P, denoted Δ_P, is then by definition a generator of the smallest principal ideal containing $E_0(P)$, i.e. the greatest common divisor of the minors. The order of P is well defined up to multiplication by units of $\mathbb{Z}[t^{\pm 1}]$ and if P admits a square presentation matrix, then $\Delta_P \doteq \det(A)$, where A is some square presentation matrix for P. It follows that for a $\mathbb{Z}[t^{\pm 1}]$-module P which admits a square presentation matrix, one has $P = 0$ if and only if $\Delta_P \doteq 1$. For more background on these topics, we refer the reader to [Tur01, Section 1.4].

4. Proof of Theorem 1.15.

Now we prove Theorem 1.15 from the introduction. For the reader's convenience, we recall the statement of this result.

Theorem 4.1. *Let Y be a 3-manifold with an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module is torsion, and let (H, λ) be a nondegenerate Hermitian form over $\mathbb{Z}[t^{\pm 1}]$ presenting Y. If $b \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$ is an isometry, then there is a 4-manifold M with $\pi_1(M) \cong \mathbb{Z}$, equivariant intersection form $\lambda_M \cong \lambda$, ribbon boundary Y and with $b_M = b$. If the form is odd, then M can be chosen to have either* $\operatorname{ks}(M) = 0$ *or* $\operatorname{ks}(M) = 1$.

For the remainder of the section, we let Y be a 3-manifold, let $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ be an epimorphism, and let $p\colon Y^\infty \to Y$ be the infinite cyclic cover associated to (Y, φ). We assume that $H_1(Y; \mathbb{Z}[t^{\pm 1}]) := H_1(Y^\infty)$ is $\mathbb{Z}[t^{\pm 1}]$-torsion. We first describe the strategy of the proof and then carry out each of the steps successively.

4.1. Plan.

Let $b\colon (\operatorname{coker}(\widehat{\lambda}), \partial\lambda) \to (H_1(Y; \mathbb{Z}[t^{\pm 1}]), -\operatorname{Bl}_Y)$ be an isometry. Precompose b with the projection $H^* \twoheadrightarrow \operatorname{coker}(\widehat{\lambda})$ to get an epimorphism $\pi\colon H^* \twoheadrightarrow H_1(Y; \mathbb{Z}[t^{\pm 1}])$. In particular, $0 \to H \xrightarrow{\widehat{\lambda}} H^* \xrightarrow{\pi} H_1(Y; \mathbb{Z}[t^{\pm 1}]) \to 0$ is a presentation of Y. Pick generators $x_1, \ldots, x_n$ for H and endow H^* with the dual basis $x_1^*, \ldots, x_n^*$. Write Q for the matrix of λ in this basis. Note that $Q = \overline{Q}^T$ since λ is Hermitian. The strategy to prove Theorem 4.1 is as follows.

- Step 1: Prove that one can represent the classes $\pi(x_1^*), \cdots, \pi(x_n^*)$ by an n-component framed link $\widetilde{L} = \widetilde{K}_1 \cup \cdots \cup \widetilde{K}_n$ with equivariant linking matrix $A_{\widetilde{L}} = -Q^{-T}$.
- Step 2: Argue that the result Y' of surgery on $L = p(\widetilde{L})$ satisfies $H_1(Y'; \mathbb{Z}[t^{\pm 1}]) = 0$.
- Step 3: There is a topological 4-manifold $B \simeq S^1$ with boundary Y' following [FQ90, Section 11.6].
- Step 4: Argue that the equivariant intersection form of the 4-manifold M defined below with boundary Y is represented by Q and prove that $b_M = b$. Here, the 4-manifold M

and its infinite cyclic cover M^∞ are defined via

$$-M^\infty := \left((Y^\infty \times [0,1]) \cup \bigcup_{i=1}^n \bigcup_{j_i \in \mathbb{Z}} t^{j_i} h_i^{(2)} \right) \cup_{Y'^\infty} -B^\infty$$
$$-M := \left((Y \times [0,1]) \cup \bigcup_{i=1}^n h_i^{(2)} \right) \cup_{Y'} -B,$$

where upstairs the 2-handles $h_i^{(2)}$ are attached along the link L^∞; downstairs, one attaches the 2-handles along the projection $L = p(L^\infty)$ of this link.

- Step 5: If λ is odd, then we use the star construction [FQ90, Sto94] to show that both values of the Kirby-Siebenmann invariant can occur.

4.2. **Step 1: constructing a link with the appropriate equivariant linking matrix.** We continue with the notation from the previous section. In particular, we have a presentation $0 \to H \xrightarrow{\widehat{\lambda}} H^* \xrightarrow{\pi} H_1(Y;\mathbb{Z}[t^{\pm 1}]) \to 0$ and a basis $x_1, \ldots, x_n$ for H with dual basis $x_1^*, \ldots, x_n^*$ for H^*. The aim of this section is to prove that it is possible to represent the generators $\pi(x_1^*), \ldots, \pi(x_n^*)$ of $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ by a framed link $\widetilde{L} = \widetilde{K}_1 \cup \cdots \cup \widetilde{K}_n \subset Y^\infty$ whose transposed equivariant linking matrix agrees with $-Q^{-1}$; see Proposition 4.4. In other words, we must have

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_j, \widetilde{K}_i) = -(Q^{-1})_{ij} \quad \text{and} \quad \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i) = -(Q^{-1})_{ii},$$

where π_i is the framing curve of $\widetilde{K}_i$. Since the Blanchfield form Bl_Y is represented by the $\mathbb{Q}(t)$-coefficient matrix $-Q^{-1}$ [CP20, Section 3], we know from Proposition 2.6 that any link representing the $\pi(x_i^*)$ must satisfy these relations up to adding a polynomial in $\mathbb{Z}[t^{\pm 1}]$. Most of this section therefore concentrates on showing that the equivariant linking (resp. framing) of an arbitrary framed link in Y^∞ can be changed by any polynomial (resp. symmetric polynomial) in $\mathbb{Z}[t^{\pm 1}]$, without changing the homology classes defined by the components of this link.

We start by showing how to modify the equivariant linking between distinct components of a link, without changing the homology class of the link.

Lemma 4.2. *Let $\widetilde{L} = \widetilde{K}_1 \cup \cdots \cup \widetilde{K}_n \subset Y^\infty$ be an n-component framed link in covering general position, with parallels $\pi_1, \ldots, \pi_n$. For every distinct i, j and every polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$, there is a framed link $\widetilde{L}' := \widetilde{K}_1 \cup \cdots \cup K_{i-1} \cup \widetilde{K}_i' \cup K_{i+1} \cup \cdots \cup \widetilde{K}_n$, also in covering general position, such that:*

(1) the knot $\widetilde{K}_i'$ is isotopic to $\widetilde{K}_i$ in Y^∞. In particular, $[\widetilde{K}_i'] = [\widetilde{K}_i]$ in $H_1(Y;\mathbb{Z}[t^{\pm 1}])$;

(2) the equivariant linking between $\widetilde{K}_i$ and $\widetilde{K}_j$ is changed by $p(t)$, i.e.

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \widetilde{K}_j) = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j) + p(t);$$

(3) the equivariant linking between $\widetilde{K}_i$ and $\widetilde{K}_\ell$ is unchanged for $\ell \neq i, j$;

(4) the framing coefficients are unchanged; that is, there is a parallel γ_i for $\widetilde{K}_i'$ such that

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \gamma_i) = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i).$$

Proof. Without loss of generality we can assume that $p(t) = mt^k$ for $m, k \in \mathbb{Z}$. The new knot $\widetilde{K}_i'$ is then obtained by band summing $\widetilde{K}_i$ with m meridians of $t^{-k}\widetilde{K}_j$, framed using the bounding framing induced by meridional discs. The first, third, and fourth properties of $\widetilde{K}_i'$ are immediate: clearly the linking of $\widetilde{K}_i$ with $\widetilde{K}_\ell$ is unchanged for $\ell \neq i, j$ and since the aforementioned meridians bound discs in Y^∞ over which the framing extends, we see that $\widetilde{K}_i'$ is framed isotopic (and in particular homologous) to $\widetilde{K}_i$ in Y^∞. It follows that the framing coefficient is unchanged.

The second property is obtained from a direct calculation using the sesquilinearity of equivariant linking numbers:

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \widetilde{K}_j) = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j) + m\ \ell k_{\mathbb{Q}(t)}(t^{-k}\mu_{\widetilde{K}_j}, \widetilde{K}_j) = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j) + mt^k. \qquad \square$$

Next, we show how to modify the framing of a framed link component by a symmetric polynomial $p = \overline{p}$, without changing the homology class of the link.

Lemma 4.3. *Let $\widetilde{L} = \widetilde{K}_1 \cup \cdots \cup \widetilde{K}_n \subset Y^\infty$ be an n-component framed link in covering general position. Fix a parallel π_i for $\widetilde{K}_i$. For each $i = 1, \ldots, n$ and every symmetric polynomial $p(t) = p(t^{-1})$, there exists a knot $\widetilde{K}'_i \subset Y^\infty$ and a parallel γ_i of $\widetilde{K}'_i$ such that*

(1) the knot $\widetilde{K}'_i$ is isotopic to $\widetilde{K}_i$ in $Y^\infty \setminus \cup_{j \neq i} \widetilde{K}_j$, and in particular, $[\widetilde{K}'_i] = [\widetilde{K}_i]$ in $H_1(Y; \mathbb{Z}[t^{\pm 1}])$;

(2) the framing coefficient of $\widetilde{K}_i$ is changed by $p(t)$, i.e.

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, \gamma_i) = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i) + p(t);$$

(3) the other linking numbers are unchanged: $\ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, \widetilde{K}_j) = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j)$ for all $j \neq i$.

Proof. We first prove the lemma when $p(t)$ has no constant term. In this case, it suffices to show how to change the self-linking number by $m(t^k + t^{-k})$ for $k \neq 0$. To achieve this, band sum $\widetilde{K}_i$ with m meridians of $t^k \widetilde{K}_i$. As in the proof of Lemma 4.2, the first and third properties of $\widetilde{K}_i$ are clear. To define γ_i and prove the second property, define $\mu'_{\widetilde{K}_i}$ to be a parallel of $\mu_{\widetilde{K}_i}$ with $\ell k_{\mathbb{Q}(t)}(\mu_{\widetilde{K}_i}, \mu'_{\widetilde{K}_i}) = 0$ in Y^∞. Define γ_i to be the parallel of $\widetilde{K}'_i$ obtained by banding π_i to m copies of $t^k \mu'_{\widetilde{K}_i}$, using bands which are push-offs of the bands used to define $\widetilde{K}'_i$, and parallel copies of the meridian chosen with the zero-framing with respect to the framing induced by the associated meridional disc. Using the sesquilinearity of equivariant linking numbers, we obtain

$$\begin{aligned}\ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, \gamma_i) &= \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i) + m\ \ell k_{\mathbb{Q}(t)}(t^k \mu_{\widetilde{K}_i}, \pi_i) + m\ \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, t^k \mu'_{\widetilde{K}_i}) + \ell k_{\mathbb{Q}(t)}(\mu_{\widetilde{K}_i}, \mu'_{\widetilde{K}_i}) \\ &= \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi) + m(t^k + t^{-k}).\end{aligned}$$

We have therefore shown how to modify the self-linking within a fixed homology class by a symmetric polynomial with no constant term.

The general case follows: thanks to the previous paragraph, it suffices to describe how to change the self-linking by a constant, and this can be arranged by varying the choice of the parallel γ_i i.e. by additionally winding an initial choice of γ_i around the appropriate number of meridians of $\widetilde{K}'_i$. □

By combining the previous two lemmas, we can now prove the main result of this section.

Proposition 4.4. *Let $0 \to H \xrightarrow{\widehat{\lambda}} H^* \xrightarrow{\pi} H_1(Y; \mathbb{Z}[t^{\pm 1}]) \to 0$ be a presentation of Y. Pick generators $x_1, \ldots, x_n$ for H and endow H^* with the dual basis $x_1^*, \ldots, x_n^*$. Let Q be the matrix of λ with respect to these bases. The classes $\pi(x_1^*), \ldots, \pi(x_n^*)$ can be represented by simple closed curves $\widetilde{K}_1, \ldots, \widetilde{K}_n \subset Y^\infty$ such that $\widetilde{L} = \widetilde{K}_1 \cup \cdots \cup \widetilde{K}_n$ is in covering general position and satisfies the following properties:*

(1) the equivariant linking of the $\widetilde{K}_i$ satisfy $\ell k_{\mathbb{Q}(t)}(\widetilde{K}_j, \widetilde{K}_i) = -(Q^{-1})_{ij}$ for $i \neq j$;

(2) there exist parallels $\gamma_1, \ldots, \gamma_n$ of $\widetilde{K}_1, \ldots, \widetilde{K}_n$ such that $\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \gamma_i) = -(Q^{-1})_{ii}$.

In particular the parallel γ_i represents the homology class $-(Q^{-1})_{ii}[\mu_{\widetilde{K}_i}] + \lambda_{\widetilde{K}_i} \in H_1(\partial \overline{\nu}(K_i); \mathbb{Q}(t))$ and the transpose of the equivariant linking matrix of $\widetilde{L}$ equals $-Q^{-1}$.

Proof. Represent the classes $\pi(x_1^*), \ldots, \pi(x_n^*)$ by an n-component link in Y^∞ that can be assumed to be in covering general position. Use $\widetilde{J}_1, \ldots, \widetilde{J}_n$ to denote the components of this link. Thanks to Lemma 4.2, we can assume that the equivariant linking numbers of these knots coincide with the off-diagonal terms of Q^{-1}; we can apply this lemma because for $i \neq j$ the rational functions $\ell k_{\mathbb{Q}(t)}(\widetilde{J}_j, \widetilde{J}_i)$ and the corresponding $-(Q^{-1})_{ij}$ both reduce mod $\mathbb{Z}[t^{\pm 1}]$ to $\mathrm{Bl}_Y(\pi(x_i^*), \pi(x_j^*))$ and thus differ by a Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$.

We arrange the framings and last assertion simultaneously. For brevity, from now on we write

$$r_i := -(Q^{-1})_{ii}.$$

By Lemma 2.7, for each i, the class $r_i[\mu_{\widetilde{J}_i}]+\lambda_{\widetilde{J}_i}$ can be rewritten as $(r_i-\ell k_{\mathbb{Q}(t)}(\widetilde{J}_i,\pi_i))[\mu_{\widetilde{J}_i}]+[\pi_i]$ for any choice of parallel π_i for $\widetilde{J}_i$. Note that $r_i-\ell k_{\mathbb{Q}(t)}(\widetilde{J}_i,\pi_i)$ is a Laurent polynomial: indeed both r_i and $\ell k_{\mathbb{Q}(t)}(\widetilde{J}_i,\pi_i)$ reduce mod $\mathbb{Z}[t^{\pm 1}]$ to $\mathrm{Bl}_Y(\pi([x_i^*]),\pi([x_i^*]))$.

Claim. *The polynomial $r_i-\ell k_{\mathbb{Q}(t)}(\widetilde{J}_i,\pi_i)$ is symmetric.*

Proof. We first assert that if σ is a parallel of $\widetilde{J}_i$, then $\ell k_{\mathbb{Q}(t)}(\sigma,\widetilde{J}_i)$ is symmetric. The rational function $\ell k_{\mathbb{Q}(t)}(\sigma,\widetilde{J}_i)$ is symmetric if and only if $\ell k_{\mathbb{Q}(t)}(\sigma,\widetilde{J}_i)=\overline{\ell k_{\mathbb{Q}(t)}(\sigma,\widetilde{J}_i)}$. By the symmetry property of the equivariant linking form mentioned in Proposition 2.6, this is equivalent to the equality $\ell k_{\mathbb{Q}(t)}(\sigma,\widetilde{J}_i)=\ell k_{\mathbb{Q}(t)}(\widetilde{J}_i,\sigma)$ and in turn this equality holds because the ordered link $(\sigma,\widetilde{J}_i)$ is isotopic to the ordered link $(\widetilde{J}_i,\sigma)$ in Y^∞. This concludes the proof of the assertion that $\ell k_{\mathbb{Q}(t)}(\sigma,\widetilde{J}_i)$ is symmetric.

We conclude the proof of the claim. Thanks to the assertion, it now suffices to prove that r_i is symmetric. To see this, note that since the matrix Q^{-1} is Hermitian (because Q is) we have $r_i(t^{-1})=-(\overline{Q^{-1}})_{ii}=-(\overline{Q^{-T}})_{ii}=-(Q^{-1})_{ii}=r_i(t)$, as required. □

We can now apply Lemma 4.3 to $p(t):=r_i-\ell k_{\mathbb{Q}(t)}(\widetilde{J}_i,\pi_i)$ (which is symmetric by the claim) to isotope the $\widetilde{J}_i$ to knots $\widetilde{K}_i$ (without changing the equivariant linking) and to find parallels $\gamma_1,\ldots,\gamma_n$ of $\widetilde{K}_1,\ldots,\widetilde{K}_n$ that satisfy the equalities $-(Q^{-1})_{ii}=r_i=\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i,\gamma_i)$. This proves the second item of the proposition and the assertions in the last sentence follow because $r_i[\mu_{\widetilde{K}_i}]+\lambda_{\widetilde{K}_i}=[\gamma_i]$ (by Lemma 2.7) and from the definition of the equivariant linking matrix. □

4.3. **Step 2: the result of surgery is a $\mathbb{Z}[t^{\pm 1}]$-homology $S^1\times S^2$.** Let $\widetilde{L}\subset Y^\infty$ be a framed link in covering general position. Let Y' be the effect of surgery on the framed link $L=p(\widetilde{L})$ with equivariant linking matrix $A_{\widetilde{L}}$ over $\mathbb{Q}(t)$. We assume throughout this subsection that $\det(A_{\widetilde{L}})\neq 0$. Our goal is to calculate the Alexander polynomial $\Delta_{Y'}$ in terms of Δ_Y and of the equivariant linking matrix of $\widetilde{L}\subset Y^\infty$. In Theorem 4.7 we will show that

$$\Delta_{Y'}\doteq\Delta_Y\det(A_{\widetilde{L}}). \tag{8}$$

We then apply this to the framed link $\widetilde{L}\subset Y^\infty$ that we built in Proposition 4.4; this framed link satisfies $\det(A_{\widetilde{L}})=\det(Q^{-T})\neq 0$. Continuing with the notation from that proposition, we have $\det(A_{\widetilde{L}})=\det(-Q^{-T})\doteq\frac{1}{\Delta_Y}$ (because Q presents $H_1(Y;\mathbb{Z}[t^{\pm 1}])$) so in this case (8) implies that $\Delta_{Y'}\doteq 1$, which in turn implies that Y' is a $\mathbb{Z}[t^{\pm 1}]$-homology $S^1\times S^2$; see Remark 3.2 and Proposition 4.8.

We start by outlining the proof of (8), which will be later recorded as Theorem 4.7.

Outline of proof of Theorem 4.7. Our plan is to compute the Reidemeister torsion $\tau(Y')$ in terms of the Reidemeister torsion $\tau(Y)$, and then, for $Z=Y,Y'$ to use the relation

$$\Delta_Z=\tau(Z)(t-1)^2 \tag{9}$$

from [Tur86, Theorem 1.1.2] to derive (8). We note that in our setting we are allowed to write $\tau(Y)$ and $\tau(Y')$ for the Reidemeister torsions without having to choose bases $\mathscr{B}$; this is because both $H_*(Y;\mathbb{Q}(t))=0$ and $H_*(Y';\mathbb{Q}(t))=0$, recall Lemma 2.11 and Section 3; here note that we can apply Lemma 2.11 because we are assuming that $\det(A_{\widetilde{L}})\neq 0$.

We will calculate $\tau(Y')$ from $\tau(Y)$ by studying the long exact sequence of the pairs (Y,Y_L) and (Y',Y_L) with $\mathbb{Q}(t)$ coefficients. More concretely, in Construction 2, we endow the $\mathbb{Q}(t)$-vector spaces $H_*(Y,Y_L;\mathbb{Q}(t))$, $H_*(Y',Y_L;\mathbb{Q}(t))$, and $H_*(Y_L;\mathbb{Q}(t))$ with bases that we denote by $\mathscr{B}_{Y,Y_L}$, $\mathscr{B}_{Y',Y_L}$, and $\mathscr{B}_{Y_L}$ respectively. In Lemma 4.5, we then show that

$$\tau(Y)\tau(\mathscr{H}_L)^{-1}\doteq\tau(Y_L,\mathscr{B}_{Y_L})\doteq\tau(Y')\tau(\mathscr{H}_{L'})^{-1},$$

where $\mathscr{H}_L$ and $\mathscr{H}_{L'}$ respectively denote the long exact sequences in $\mathbb{Q}(t)$-homology of the pairs (Y, Y_L) and (Y', Y_L). Finally, we prove that $\tau(\mathscr{H}_L) \doteq 1$ and $\tau(\mathscr{H}_{L'}) \doteq \det(A_{\widetilde{L}})$. From (9) and the previous equation we then deduce

$$\frac{\Delta_Y}{(t-1)^2 \cdot 1} \doteq \tau(Y)\tau(\mathscr{H}_L)^{-1} \doteq \tau(Y')\tau(\mathscr{H}_{L'})^{-1} \doteq \frac{\Delta_{Y'}}{(t-1)^2 \cdot \det(A_{\widetilde{L}})}.$$

The equality $\Delta_{Y'} \doteq \Delta_Y \det(A_{\widetilde{L}})$ follows promptly. □

We start filling in the details with our choice of bases for the previously mentioned $\mathbb{Q}(t)$-homology vector spaces.

Construction 2. We fix bases for $H_*(Y, Y_L; \mathbb{Q}(t))$, $H_*(Y', Y_L; \mathbb{Q}(t))$, and $H_*(Y_L; \mathbb{Q}(t))$, that we will respectively denote by $\mathscr{B}_{Y,Y_L}, \mathscr{B}_{Y',Y_L}$ and $\mathscr{B}_{Y_L}$.

- We base the $\mathbb{Q}(t)$-vector spaces $H_*(Y, Y_L; \mathbb{Q}(t))$ and $H_*(Y', Y_L; \mathbb{Q}(t))$. Excising $\mathring{Y}_L$, we obtain $H_i(Y, Y_L; \mathbb{Q}(t)) = \bigoplus_{i=1}^n H_i(D^2 \times S^1, S^1 \times S^1; \mathbb{Q}(t))$ where n is the number of components of L. Similarly, by excising $\mathring{Y}_L \cong \mathring{Y}_{L'}$, we have $H_i(Y', Y_L; \mathbb{Q}(t)) = \bigoplus_{i=1}^n H_i(S^1 \times D^2, S^1 \times S^1; \mathbb{Q}(t))$. Since the map $\pi_1(S^1) \to \mathbb{Z}$ determining the coefficients is trivial,
$$\bigoplus_{i=1}^n H_i(S^1 \times D^2, S^1 \times S^1; \mathbb{Q}(t)) \cong \bigoplus_{i=1}^n H^{3-i}(S^1; \mathbb{Q}(t)) \cong \bigoplus_{i=1}^n H^{3-i}(S^1; \mathbb{Z}) \otimes \mathbb{Q}(t).$$
These homology vector spaces are only non-zero when $i = 2, 3$. in which case they are isomorphic to $\mathbb{Q}(t)^n$.

  We now pick explicit generators for these vector spaces. Endow $S^1 \times S^1$ with its usual cell structure, with one 0-cell, two 1-cells and one 2-cell $e^2_{S^1 \times S^1}$. Note that $D^2 \times S^1$ is obtained from $S^1 \times S^1 \times I$ by additionally attaching a 3-dimensional 2-cell $e^2_{D^2 \times S^1}$ and 3-cell, $e^3_{D^2 \times S^1}$, where on the chain level $\partial e^3_{D^2 \times S^1} = e^2_{D^2 \times S^1} + e^2_{S^1 \times S^1} - e^2_{D^2 \times S^1} = e^2_{S^1 \times S^1}$. We now fix once and for all lifts of these cells to the covers. It follows that for $k = 2, 3$:
$$H_k(Y, Y_L; \mathbb{Q}(t)) = C_k(Y, Y_L; \mathbb{Q}(t)) = C_k(D^2 \times S^1, S^1 \times S^1; \mathbb{Q}(t)) = \bigoplus_{i=1}^n \mathbb{Q}(t)(\widetilde{e}^k_{D^2 \times S^1})_i$$
$$H_k(Y', Y_L; \mathbb{Q}(t)) = C_k(Y', Y_L; \mathbb{Q}(t)) = C_k(S^1 \times D^2, S^1 \times S^1; \mathbb{Q}(t)) = \bigoplus_{i=1}^n \mathbb{Q}(t)(\widetilde{e}^k_{S^1 \times D^2})_i.$$
- We now base $H_*(Y_L; \mathbb{Q}(t))$. Since $H_*(Y; \mathbb{Q}(t)) = 0$, a Mayer-Vietoris argument shows that $H_1(Y_L; \mathbb{Q}(t)) \cong \mathbb{Q}(t)^n$, generated by the meridians $\mu_{\widetilde{K}_i}$ of $\widetilde{L}$. Mayer-Vietoris also shows that the inclusion of the boundary induces an isomorphism $\mathbb{Q}(t)^n = H_2(\partial Y_L; \mathbb{Q}(t)) \cong H_2(Y_L; \mathbb{Q}(t))$. We can then base $H_2(Y_L; \mathbb{Q}(t))$ using fixed lifts of the aforementioned 2-cells $(e^2_{S^1 \times S^1})_i$ generating each of the torus factors of ∂Y_L. Summarising, we have
$$H_1(Y_L; \mathbb{Q}(t)) = \bigoplus_{i=1}^n \mathbb{Q}(t)\mu_{\widetilde{K}_i},$$
$$H_2(Y_L; \mathbb{Q}(t)) = \bigoplus_{i=1}^n \mathbb{Q}(t)(\widetilde{e}^2_{S^1 \times S^1})_i.$$

The next lemma reduces the calculation of $\Delta_{Y'}$ to the calculation of $\tau(\mathscr{H}_L)$ and $\tau(\mathscr{H}_{L'})$. Here, recall that $\tau(\mathscr{H}_L)$ and $\tau(\mathscr{H}_{L'})$ denote the torsion of the long exact sequences $\mathscr{H}_L$ and $\mathscr{H}_{L'}$ of the pairs (Y, Y_L) and (Y', Y_L), viewed as based acyclic complexes with bases $\mathscr{B}_{Y_L}, \mathscr{B}_{Y,Y_L}$, and $\mathscr{B}_{Y',Y_L}$.

Lemma 4.5. *If $H_1(Y; \mathbb{Q}(t)) = 0$ and $\det(A_{\widetilde{L}}) \neq 0$, then we have*
$$\tau(Y) \doteq \tau(Y_L, \mathscr{B}_{Y_L}) \cdot \tau(\mathscr{H}_L),$$
$$\tau(Y') \doteq \tau(Y_L, \mathscr{B}_{Y_L}) \cdot \tau(\mathscr{H}_{L'}).$$
In particular, we have
$$\Delta_{Y'} \cdot \tau(\mathscr{H}_L) \doteq \Delta_Y \cdot \tau(\mathscr{H}_{L'}).$$

Proof. We start by proving that the last statement follows from the first. First note that since the vector spaces $H_1(Y;\mathbb{Q}(t))$ and $H_1(Y';\mathbb{Q}(t))$ vanish (for the latter we use Lemma 2.11 which applies since $\det(A_{\widetilde{L}}) \neq 0$), the Alexander polynomials of Y and Y' are nonzero. Next, [Tur86, Theorem 1.1.2] implies that $\tau(Y)(t-1)^2 = \Delta_Y$ and similarly for Y'. Therefore $\Delta_{Y'}/\Delta_Y = \tau(Y')/\tau(Y)$. The first part of the lemma implies that $\tau(Y')/\tau(Y) = \tau(\mathscr{H}_{L'})/\tau(\mathscr{H}_L)$. Combining these equalities,

$$\frac{\Delta_{Y'}}{\Delta_Y} = \frac{\tau(Y')}{\tau(Y)} = \frac{\tau(\mathscr{H}_{L'})}{\tau(\mathscr{H}_L)},$$

from which the required statement follows immediately.

To prove the first statement of the lemma, it suffices to prove that $\tau(Y, Y_L, \mathscr{B}_{Y,Y_L}) = 1$ as well as $\tau(Y', Y_L, \mathscr{B}_{Y',Y_L}) = 1$: indeed, the required equalities then follow by applying the multiplicativity of Reidemeister torsion (the first item of Theorem 3.1) to the short exact sequences

$$0 \to C_*(Y_L;\mathbb{Q}(t)) \to C_*(Y;\mathbb{Q}(t)) \to C_*(Y, Y_L;\mathbb{Q}(t)) \to 0,$$

leading to $\tau(Y) = \tau(Y_L)\cdot\tau(Y,Y_L,\mathscr{B}_{Y,Y_L})\cdot\tau(\mathscr{H}_L) = \tau(Y_L)\cdot 1\cdot\tau(\mathscr{H}_L)$ as desired. And similarly for the pair (Y', Y_L).

We use Definition 10 to prove that $\tau(Y, Y_L, \mathscr{B}_{Y,Y_L}) = 1$; again the proof for L' is analogous. We endow Y and Y_L with cell structures for which Y_L and ∂Y_L are subcomplexes of Y, and Y is obtained from Y_L by attaching n solid tori to ∂Y_L. By definition of the relative chain complex, we have $C_*(Y, Y_L;\mathbb{Q}(t)) = C_*(Y;\mathbb{Q}(t))/C_*(Y_L;\mathbb{Q}(t))$. Since we are working with cellular chain complexes we deduce that

$$C_*(Y, Y_L;\mathbb{Q}(t)) = C_*(Y;\mathbb{Q}(t))/C_*(Y_L;\mathbb{Q}(t)) = \bigoplus_{i=1}^{n} C_*(D^2\times S^1;\mathbb{Q}(t))/C_*(S^1\times S^1;\mathbb{Q}(t)).$$

Using the cell structures described in Construction 2, $D^2\times S^1$ is obtained from $S^1\times S^1$ by attaching a 2-cell and a 3-cell. By the above sequence of isomorphisms, this shows that $C_i(Y, Y_L;\mathbb{Q}(t)) = 0$ for $i \neq 2, 3$ and gives a basis for $C_2(Y, Y_L;\mathbb{Q}(t))$ and $C_3(Y, Y_L;\mathbb{Q}(t))$. In fact, this also implies that $C_i(Y, Y_L;\mathbb{Q}(t)) = H_i(Y, Y_L;\mathbb{Q}(t))$ and that the differentials in the chain complex are zero, as was mentioned in Construction 2. Thus, the basis of $C_*(Y, Y_L;\mathbb{Q}(t))$ corresponds exactly to the way we based $H_*(Y, Y_L;\mathbb{Q}(t))$ in Construction 2. Therefore the change of basis matrix is the identity and so the torsion is equal to 1. This concludes the proof of the lemma. □

Our goal is now to show that $\tau(\mathscr{H}_L) \doteq 1$ and $\tau(\mathscr{H}_{L'}) \doteq \det(A_{\widetilde{L}})$. We start by describing the long exact sequences $\mathscr{H}_L$ and $\mathscr{H}_{L'}$.

Lemma 4.6. *Assume that $H_1(Y_L;\mathbb{Q}(t)) = 0$ and $\det(A_{\widetilde{L}}) \neq 0$. The only nontrivial portions of the long exact sequence of the pairs (Y, Y_L) and $(Y, Y_{L'})$ with $\mathbb{Q}(t)$-coefficients are of the following form:*

$$\mathscr{H}_L = \Big(0 \to H_3(Y, Y_L;\mathbb{Q}(t)) \xrightarrow{\partial_3^L} H_2(Y_L;\mathbb{Q}(t)) \to 0 \to H_2(Y, Y_L;\mathbb{Q}(t)) \xrightarrow{\partial_2^L} H_1(Y_L;\mathbb{Q}(t)) \to 0\Big),$$

$$\mathscr{H}_{L'} = \Big(0 \to H_3(Y', Y_L;\mathbb{Q}(t)) \xrightarrow{\partial_3^{L'}} H_2(Y_L;\mathbb{Q}(t)) \to 0 \to H_2(Y', Y_L;\mathbb{Q}(t)) \xrightarrow{\partial_2^{L'}} H_1(Y_L;\mathbb{Q}(t)) \to 0\Big).$$

Additionally, with respect to the bases of Construction 2,

- *the homomorphism $\partial_2^{L'}$ is represented by $-A_{\widetilde{L}}^{-1}$, i.e. minus the inverse of the equivariant linking matrix for $\widetilde{L}$;*
- *the homomorphisms ∂_2^L, ∂_3^L, and $\partial_3^{L'}$ are represented by identity matrices.*

Proof. Since Y^∞ and Y'^∞ are connected, we have $H_0(Y;\mathbb{Z}[t^{\pm1}]) = \mathbb{Z}$ and $H_0(Y';\mathbb{Z}[t^{\pm1}]) = \mathbb{Z}$, so $H_0(Y;\mathbb{Q}(t)) = 0$ and $H_0(Y';\mathbb{Q}(t)) = 0$. Since we are working with field coefficients, Poincaré duality and the universal coefficient theorem imply that $H_3(Y;\mathbb{Q}(t)) = 0$ and $H_3(Y';\mathbb{Q}(t)) = 0$. As observed in Construction 2 above, by excision, the only non-zero relative homology groups of (Y, Y_L) and (Y', Y_L) are

$$H_i(Y, Y_L;\mathbb{Q}(t)) = \mathbb{Q}(t)^n \qquad\text{and}\qquad H_i(Y', Y_L;\mathbb{Q}(t)) = \mathbb{Q}(t)^n$$

for $i = 2, 3$. Next, since by assumption $H_1(Y;\mathbb{Q}(t)) = 0$, duality and the universal coefficient theorem imply that $H_2(Y;\mathbb{Q}(t)) = 0$. Since we proved in Lemma 2.11 that $H_1(Y';\mathbb{Q}(t)) = 0$, (here we used $\det(A_{\widetilde{L}}) \neq 0$) the same argument shows that $H_2(Y';\mathbb{Q}(t)) = 0$. This establishes the first part of the lemma.

We now prove the statement concerning ∂_2^L and $\partial_2^{L'}$. Recall from Construction 2 that we based the vector spaces $H_2(Y, Y_L;\mathbb{Q}(t))$ and $H_2(Y', Y_L;\mathbb{Q}(t))$ by meridional discs to the $\widetilde{K}_i$ and $\widetilde{K}_i'$ respectively. The map ∂_2^L takes each disc to its boundary, the meridian $\mu_{\widetilde{K}_i}$; since these meridians form our chosen basis for $H_1(Y_L;\mathbb{Q}(t))$, we deduce that ∂_2^L is represented by the identity matrix. The map $\partial_2^{L'}$ also takes each meridional disc to its boundary, the meridian $\widetilde{\mu}_{K_i'}$ to the dual knot. It follows that $\partial_2^{L'}$ is represented by the change of basis matrix B such that $\boldsymbol{\mu}' = B\boldsymbol{\mu}$. But during the proof of Lemma 2.12 we saw that $B = -A_{\widetilde{L}}^{-1}$.

Finally, we prove that ∂_3^L and $\partial_3^{L'}$ are represented by identity matrices. In Construction 2, we based $H_3(Y, Y_L;\mathbb{Q}(t))$ and $H_3(Y', Y_L;\mathbb{Q}(t))$ using respectively (lifts of) the 3-cells of the $(D^2 \times S^1)_i$ and $(S^1 \times D^2)_i$. Now both ∂_3^L and $\partial_3^{L'}$ take these 3-cells to their boundaries. But as we noted in Construction 2, these boundaries are (algebraically) the 2-cells $(e^2_{S^1\times S^1})_i$. In other words both ∂_3^L and $\partial_3^{L'}$ map our choice of ordered bases to our other choice of ordered bases, and are therefore represented in these bases by identity matrices, as required. This concludes the proof of Lemma 4.5. □

As we now understand the exact sequences $\mathscr{H}_L$ and $\mathscr{H}_{L'}$ we can calculate their torsions, leading to the proof of the main result of this subsection.

Theorem 4.7. *If $H_1(Y_L;\mathbb{Q}(t)) = 0$ and $\det(A_{\widetilde{L}}) \neq 0$, then we have*

$$\Delta_{Y'} \doteq \det(A_{\widetilde{L}})\Delta_Y.$$

Proof. Use the bases from Construction 2. Combine the second item of Theorem 3.1 with Lemma 4.6 to obtain:

$$\tau(\mathscr{H}_L) \doteq \frac{\det(\partial_3^L)}{\det(\partial_2^L)} \doteq 1 \text{ and } \tau(\mathscr{H}_{L'}) \doteq \frac{\det(\partial_3^{L'})}{\det(\partial_2^{L'})} \doteq \det(A_{\widetilde{L}}).$$

We deduce that $\tau(\mathscr{H}_{L'})/\tau(\mathscr{H}_L) \doteq \det(A_{\widetilde{L}})$. Apply Lemma 4.5 to obtain

$$\frac{\Delta_{Y'}}{\Delta_Y} \doteq \frac{\tau(\mathscr{H}_{L'})}{\tau(\mathscr{H}_L)} \doteq \det(A_{\widetilde{L}}).$$

Rearranging yields the desired equality. □

As a consequence, we complete the second step of the plan from Subsection 4.1.

Proposition 4.8. *Let $0 \to H \xrightarrow{\widehat{\lambda}} H^* \xrightarrow{\pi} H_1(Y;\mathbb{Z}[t^{\pm 1}]) \to 0$ be a presentation of Y. Pick generators $x_1, \ldots, x_n$ for H and endow H^* with the dual basis $x_1^*, \ldots, x_n^*$. Let Q be the matrix of λ with respect to these bases. The classes $\pi(x_1^*), \ldots, \pi(x_n^*)$ can be represented by a framed link $\widetilde{L}$ in covering general position with equivariant linking matrix $A_{\widetilde{L}} = -Q^{-T}$. In addition, the 3-manifold Y' obtained by surgery on Y along L satisfies $H_1(Y';\mathbb{Z}[t^{\pm 1}]) = 0$.*

Proof. The existence of $\widetilde{L}$ representing the given generators and with equivariant linking matrix $A_{\widetilde{L}} = -Q^{-T}$ is proved in Proposition 4.4. Since Q^T presents $H_1(Y;\mathbb{Z}[t^{\pm 1}])$, we have $\det(Q) \doteq \Delta_Y$ and therefore $\det(A_{\widetilde{L}}) \doteq \frac{1}{\Delta_Y}$. Theorem 4.7 now implies that $\Delta_{Y'} \doteq 1$.

A short argument is now needed to use Remark 3.2 in order to conclude $H_1(Y';\mathbb{Z}[t^{\pm 1}]) = 0$: we require that this torsion module admits a square presentation matrix, i.e. has projective dimension at most 1, denoted $\mathrm{pd}(H_1(Y';\mathbb{Z}[t^{\pm 1}])) \leq 1$. Here recall that that a $\mathbb{Z}[t^{\pm 1}]$-module P *has projective dimension at most k* if $\mathrm{Ext}^i_{\mathbb{Z}[t^{\pm 1}]}(P;V) = 0$ for every $\mathbb{Z}[t^{\pm 1}]$-module V and every $i \geq k+1$, and that for a short exact sequence $0 \to A \to B \to C \to 0$ of $\mathbb{Z}[t^{\pm 1}]$-modules, the associated long exact sequence in $\mathrm{Ext}(-;V)$ groups implies that:

(a) if $\operatorname{pd}(C) \leq 1$ and A is free, then $\operatorname{pd}(B) \leq 1$;
(b) if $\operatorname{pd}(B) \leq 1$ and A is free, then $\operatorname{pd}(C) \leq 1$.

The following paragraph proves that $\operatorname{pd}(H_1(Y';\mathbb{Z}[t^{\pm 1}])) \leq 1$. As $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ and $H_1(Y';\mathbb{Z}[t^{\pm 1}])$ are torsion (for the latter recall Lemma 2.11), a duality argument implies that $H_2(Y;\mathbb{Z}[t^{\pm 1}]) = \mathbb{Z}$ and $H_2(Y';\mathbb{Z}[t^{\pm 1}]) = \mathbb{Z}$ (see e.g. the first item of [CP20, Lemma 3.2]). Since these modules are torsion and since excision implies that

$$H_2(Y, Y_L;\mathbb{Z}[t^{\pm 1}]) = \mathbb{Z}[t^{\pm 1}]^n \quad \text{and} \quad H_2(Y', Y_L;\mathbb{Z}[t^{\pm 1}]) = \mathbb{Z}[t^{\pm 1}]^n$$
$$H_1(Y, Y_L;\mathbb{Z}[t^{\pm 1}]) = 0 \quad \text{and} \quad H_1(Y', Y_L;\mathbb{Z}[t^{\pm 1}]) = 0,$$

we deduce that the maps $H_2(Y;\mathbb{Z}[t^{\pm 1}]) \to H_2(Y, Y_L;\mathbb{Z}[t^{\pm 1}])$ and $H_2(Y';\mathbb{Z}[t^{\pm 1}]) \to H_2(Y', Y_L;\mathbb{Z}[t^{\pm 1}])$ are both trivial leading to the short exact sequences

$$0 \to H_2(Y, Y_L;\mathbb{Z}[t^{\pm 1}]) \to H_1(Y_L;\mathbb{Z}[t^{\pm 1}]) \to H_1(Y;\mathbb{Z}[t^{\pm 1}]) \to 0,$$
$$0 \to H_2(Y', Y_L;\mathbb{Z}[t^{\pm 1}]) \to H_1(Y_L;\mathbb{Z}[t^{\pm 1}]) \to H_1(Y';\mathbb{Z}[t^{\pm 1}]) \to 0.$$

Next we apply the facts (a) and (b) on projective dimension given above. Since the torsion module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is presented by (H, λ), it has projective dimension at most 1 and since $H_2(Y, Y_L;\mathbb{Z}[t^{\pm 1}])$ is free, the first short exact sequence implies that $H_1(Y_L;\mathbb{Z}[t^{\pm 1}])$ has projective dimension at most 1. Since $H_2(Y', Y_L;\mathbb{Z}[t^{\pm 1}])$ is free, the second short exact sequence now implies that $\operatorname{pd}(H_1(Y';\mathbb{Z}[t^{\pm 1}])) \leq 1$ as required.

As explained above, since $\operatorname{pd}(H_1(Y';\mathbb{Z}[t^{\pm 1}])) \leq 1$ and $\Delta_{Y'} \doteq 1$, Remark 3.2 now allow us to conclude that $H_1(Y';\mathbb{Z}[t^{\pm 1}]) = 0$, as required. □

4.4. **Step 3: every $\mathbb{Z}[t^{\pm 1}]$-homology $S^1 \times S^2$ bounds a homotopy circle.** The goal of this subsection is to prove the following theorem, which is a generalisation of a key step in the proof that Alexander polynomial one knots are topologically slice.

Theorem 4.9. *Let Y be a 3-manifold with an epimorphism $\pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module vanishes, i.e. $H_1(Y;\mathbb{Z}[t^{\pm 1}]) = 0$. Then there exists a 4-manifold B with a homotopy equivalence $g\colon B \xrightarrow{\simeq} S^1$ so that $\partial B \cong Y$ and $\pi_1(Y) \twoheadrightarrow \pi_1(B) \xrightarrow{g_*} \pi_1(S^1) \cong \mathbb{Z}$ agrees with φ.*

Proof. This proof can be deduced by combining various arguments from [FQ90, Section 11.6], so we only outline the main steps. We start by describing the general strategy. First we use framed bordism theory to find some 4-manifold W whose boundary is Y, with a map to S^1 realising φ. This map might not be a homotopy equivalence, but we then we use surgery theory to show that W is bordant rel. boundary to a homotopy circle.

We use framed bordism to find some 4-manifold whose boundary is Y, with a map to S^1 realising φ, as in [FQ90, Lemma 11.6B]. Every oriented 3-manifold admits a framing of its tangent bundle. Using the axioms of a generalised homology theory, we have

$$\Omega_3^{\mathrm{fr}}(B\mathbb{Z}) \cong \Omega_3^{\mathrm{fr}} \oplus \Omega_2^{\mathrm{fr}} \cong \mathbb{Z}/24 \oplus \mathbb{Z}/2.$$

We consider the image of (Y, φ) in $\Omega_3^{\mathrm{fr}}(B\mathbb{Z})$. The first summand can be killed by changing the choice of framing of the tangent bundle of Y; see [FQ90, proof of Lemma 11.6B] for details. The second summand is detected by an Arf invariant, which vanishes thanks to the assumption that $H_1(Y;\mathbb{Z}[t^{\pm 1}]) = 0$; details are again in [FQ90, proof of Lemma 11.6B]. Therefore there exists a framed 4-manifold W with framed boundary Y, such that the map $Y \to S^1$ associated with φ extends over W.

We now use surgery theory to show that W is bordant rel. boundary to a homotopy circle. Consider the mapping cylinder

$$X := \mathscr{M}(Y \xrightarrow{\varphi} S^1). \tag{10}$$

We claim that (X, Y) is a Poincaré pair. The argument is similar to [FQ90, Proposition 11.C]. As $X \simeq S^1$, the connecting homomorphism from the exact sequence of the pair (X, Y) gives an isomorphism $\partial\colon H_4(X, Y) \cong H_3(Y) \cong \mathbb{Z}$. We then define the required fundamental class as $[X,Y] := \partial^{-1}([Y]) \in H_4(X,Y)$. Using $H_1(Y;\mathbb{Z}[t^{\pm 1}]) = 0$, one can now use the same argument as in [FT05, Lemma 3.2] to show that the following cap product is an isomorphism:

$$- \cap [X,Y] \colon H^i(X,Y;\mathbb{Z}[t^{\pm 1}]) \to H_{4-i}(X;\mathbb{Z}[t^{\pm 1}]).$$

This concludes the proof of the fact that (X,Y) is a Poincaré pair.

The end of the argument follows from the exactness of the surgery sequence for (X,Y) as in [FQ90, Proposition 11.6A] but we outline some details for the reader unfamiliar with surgery theory. Since (X,Y) is a Poincaré pair, we can consider its set $\mathscr{N}(X,Y)$ of normal invariants. The set $\mathscr{N}(X,Y)$ consists of normal bordism classes of degree one normal maps to X that restrict to a homeomorphism on the boundary, where a bordism restricts to a product cobordism homeomorphic to $Y \times I$ between the boundaries. The next paragraph uses the map $W \to S^1$ to define an element of $\mathscr{N}(X,Y)$.

Via the homotopy equivalence $X \simeq S^1$, the map $Y \to S^1 \simeq X$ extends to $F \colon W \to S^1 \simeq X$. It then follows from the naturality of the long exact sequence of the pairs (W,Y) and (X,Y) that F has degree one. We therefore obtain a degree one map $(F, \mathrm{Id}_Y) \colon (W,Y) \to (X,Y)$. To upgrade (F,Id_Y) to a degree one normal map, take a trivial (stable) bundle $\xi \to X$ over the codomain. Normal data is determined by a (stable) trivialisation of $TW \oplus F^*\xi$. The framing of W provides a trivialisation for the first summand, while any choice of trivialisation for $F^*\xi$ can be used for the second summand. We therefore have a degree one normal map

$$\big((F,\mathrm{Id}_Y) \colon (W,Y) \to (X,Y)\big) \in \mathscr{N}(X,Y).$$

Our goal is to do surgery on the interior of the domain (W,Y) to convert F into a homotopy equivalence $(F',\mathrm{Id}_Y) \colon (B,Y) \to (X,Y)$. Since the fundamental group $\mathbb{Z}$ is a good group, surgery theory says that this is possible if and only if $\ker(\sigma)$ is nonempty [FQ90, Section 11.3]. Here

$$\sigma \colon \mathscr{N}(X,Y) \to L_4(\mathbb{Z}[t^{\pm 1}])$$

is the surgery obstruction map. Essentially, it takes the intersection pairing on $H_2(W;\mathbb{Z}[t^{\pm 1}])$ and considers it in the Witt group of nonsingular, Hermitian, even forms over $\mathbb{Z}[t^{\pm 1}]$ up to stable isometry, where stabilisation is by hyperbolic forms

$$\left(\mathbb{Z}[t^{\pm 1}] \oplus \mathbb{Z}[t^{\pm 1}], \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

Shaneson splitting [Sha69] implies that $L_4(\mathbb{Z}[t^{\pm 1}]) \cong L_4(\mathbb{Z}) \oplus L_3(\mathbb{Z}) \cong L_4(\mathbb{Z}) \cong 8\mathbb{Z}$. The last isomorphism is given by taking the signature. We take the connected sum of $W \to X$ with copies of $(E_8 \to S^4)$ or $(-E_8 \to S^4)$, to arrange that the signature becomes zero. Then the resulting normal map $W \#^{\ell} Z \to X$ has trivial surgery obstruction in $L_4(\mathbb{Z}[t^{\pm 1}])$ (i.e. lies in $\ker(\sigma)$) and therefore is normally bordant to a homotopy equivalence $(F',\mathrm{Id}_Y) \colon (B,Y) \to (X,Y)$, as desired. Since the mapping cylinder X from (10) is a homotopy circle, so is B. This concludes the proof of the theorem. □

4.5. **Step 4: constructing a 4-manifold that induces the given boundary isomorphism.** We begin by recalling the notation and outcome of Proposition 4.8. Let $b \in \mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)$ be an isometry of linking forms. Pulling this back to H, we obtain a presentation

$$0 \to H \xrightarrow{\widehat{\lambda}} H^* \xrightarrow{\pi} H_1(Y;\mathbb{Z}[t^{\pm 1}]) \to 0$$

of Y. Pick generators $x_1, \ldots, x_n$ for H and endow H^* with the dual basis $x_1^*, \ldots, x_n^*$. Let Q be the matrix of λ with respect to these bases. By Propositions 4.4 and 4.8, the classes $\pi(x_1^*), \ldots, \pi(x_n^*)$ can be represented by a framed link $\widetilde{L} \subset Y^\infty$ in covering general position with transposed equivariant linking matrix $-Q^{-1}$ and the 3-manifold Y' obtained by surgery on $L = p(\widetilde{Y})$ satisfies $H_1(Y';\mathbb{Z}[t^{\pm 1}]) = 0$. Applying Theorem 4.9, there is a topological 4-manifold B with boundary Y' and such that $B \simeq S^1$.

We now define a 4-manifold M with boundary Y as follows: begin with $Y \times I$ and attach 2-handles to $Y \times \{1\}$ along the framed link $L := p(\widetilde{L})$ (here recall that $p \colon Y^\infty \to Y$ denotes the covering map), so that the resulting boundary is Y'. Call this 2-handle cobordism W, and observe that $\partial^- W = -Y$. We can now cap $\partial^+ W \cong Y'$ with $-B$. Since $W \cup -B$ has boundary $-Y$, we define M to be $-W \cup B$. We can then consider the corresponding $\mathbb{Z}$-cover:

$$-M^\infty := \Big((Y^\infty \times [0,1]) \cup \bigcup_{i=1}^{n} \bigcup_{j_i \in \mathbb{Z}} t^{j_i} h_i^{(2)}\Big) \cup_{Y'^\infty} -B^\infty = W^\infty \cup_{Y'^\infty} -B^\infty$$

$$-M := \Big((Y \times [0,1]) \cup \bigcup_{i=1}^{n} h_i^{(2)}\Big) \cup_{Y'} -B =: W \cup_{Y'} -B,$$

in which the 2-handles are attached along the framed link $\widetilde{L}$ upstairs and its framed projection L downstairs.

We begin by verifying some properties of M.

Lemma 4.10. *The 4-manifold M has fundamental group $\mathbb{Z}$ and ribbon boundary Y.*

Proof. We first prove that $\pi_1(M) \cong \mathbb{Z}$. A van Kampen argument shows that $\pi_1(M)$ is obtained from $\pi_1(B)$ by modding out the $[\iota(\widetilde{K}_i')]$ where $\widetilde{K}_1', \ldots, \widetilde{K}_n'$ denote the components of the framed link dual to $\widetilde{L}$ and where $\iota\colon \pi_1(Y') \to \pi_1(B)$ is the inclusion induced map. Recall from Lemma 2.9 and Remark 2.10 that the epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ induces an epimorphism $\varphi'\colon \pi_1(Y') \twoheadrightarrow \mathbb{Z}$ and that $\varphi'([K_i']) = 0$ for $i = 1, \ldots, n$. Since Theorem 4.9 ensures that ι agrees with φ', we deduce that the classes $[\iota(\widetilde{K}_i')]$ are trivial and therefore $\pi_1(M) \cong \pi_1(B) \cong \mathbb{Z}$.

Next we argue that M has ribbon boundary. Since the inclusion induced map $\pi_1(Y) \to \pi_1(W)$ is surjective, it suffices to prove that the inclusion induced map $\pi_1(W) \to \pi_1(M)$ is surjective. This follows from van Kampen's theorem: as $\pi_1(Y') \to \pi_1(B)$ is surjective, so is $\pi_1(W) \to \pi_1(M)$. □

It is not too hard to compute, as we will do in Proposition 4.11 below, that $H_2(M; \mathbb{Z}[t^{\pm 1}])$ is f.g. free of rank n. To complete step 4, we must prove the following two claims.

(1) The equivariant intersection form λ_M of M is represented by Q; i.e. λ_M is isometric to λ.
(2) The 4-manifold M satisfies $b_M = b \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$.

The proof of the first claim follows a standard outline; for the hasty reader we will give the outline here, and for the record we provide a detailed proof at the end of the subsection.

Proof outline of claim (1). Since by setup the transposed equivariant linking matrix of the framed link $\widetilde{L}$ is $-Q^{-1}$, Proposition 2.12 shows that the transposed equivariant linking matrix of the dual link $\widetilde{L}'$ is Q. Thus, it suffices to show that λ_M is presented by the transposed equivariant linking matrix of $\widetilde{L}'$.

While it was natural initially to build W^∞ by attaching 2-handles to $Y^\infty \times I$, in what follows it will be more helpful to view $-W^\infty$ as being obtained from $Y' \times I$ by attaching 2-handles to the framed link $\widetilde{L}'$ dual to $\widetilde{L}$. In particular, the components of $\widetilde{L}'$ bound the cores of the 2-handles.

Recall that $H_1(Y'; \mathbb{Z}[t^{\pm 1}]) = 0$ by Proposition 4.8 and that $H_2(B; \mathbb{Z}[t^{\pm 1}]) = 0$ by Proposition 4.9. Let Σ_i denote a surface in Y'^∞ with boundary $\widetilde{K}_i'$, and let F_i be the surface in M formed by Σ_i capped with the core of the 2-handle attached along $\widetilde{K}_i'$. The proof that $H_2(M; \mathbb{Z}[t^{\pm 1}])$ is freely generated by the $[F_i]$ and that the equivariant intersection form λ_M is represented by the transposed equivariant linking matrix of $\widetilde{L}'$ (which we showed above is Q), is now routine; the details are expanded in Propositions 4.11 and 4.13 below. □

As promised, the section now concludes with a detailed proof of the claims. Firstly in Construction 3, we give the detailed construction of the surfaces F_i that were mentioned in the proof outline. Secondly, in Proposition 4.11 we show that these surfaces lead to a basis of $H_2(M; \mathbb{Z}[t^{\pm 1}])$. Thirdly, in Proposition 4.13 we conclude the proof of the first claim by showing that with respect to this basis, λ_M is represented by the transposed equivariant linking matrix of $\widetilde{L}'$. Finally, in Proposition 4.14, we prove the second claim.

Construction 3. For $i = 1, \dots, n$, we define the closed surfaces $F_i \subset -W^\infty \subset M^\infty$ that were mentioned in the outline. As $H_1(Y'; \mathbb{Z}[t^{\pm 1}]) = 0$ (by Step 2), each component $\widetilde{K}'_i$ of $\widetilde{L}'$ bounds a surface $\Sigma_i \subset Y'^\infty$. Additionally, each $\widetilde{K}'_i$ (considered in $Y' \times \{1\}$) bounds the core of one of the (lifted) 2-handles in the dual handle decomposition of $-W$. Define the surface $F_i \subset -W^\infty \subset M^\infty$ by taking the union of Σ_i with this core.

The next proposition shows that the surfaces F'_i give a basis for $H_2(M; \mathbb{Z}[t^{\pm 1}])$. It is with respect to this basis that we will calculate λ_M in Proposition 4.13 below.

Proposition 4.11. *The following isomorphisms hold:*

$$H_2(-W; \mathbb{Z}[t^{\pm 1}]) = \mathbb{Z} \oplus \bigoplus_{i=1}^{n} \mathbb{Z}[t^{\pm 1}][F_i], \qquad \text{and} \qquad H_2(M; \mathbb{Z}[t^{\pm 1}]) = \bigoplus_{i=1}^{n} \mathbb{Z}[t^{\pm 1}][F_i].$$

Proof. These follow by standard arguments using Mayer-Vietoris, which we outline now.

The first equality follows from the observation that $-W^\infty$ is obtained from $Y'^\infty \times [0,1]$ by attaching the dual 2-handles to the $h_i^{(2)}$. Morally, since $H_1(Y'; \mathbb{Z}[t^{\pm 1}]) = 0$ (Step 2), each dual 2-handle contributes a free generator. The additional $\mathbb{Z}$ summand comes from $H_2(Y' \times [0,1]; \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z}$. More formally, one applies Mayer-Vietoris with $\mathbb{Z}[t^{\pm 1}]$-coefficients to the decomposition of W as the union of $Y' \times [0,1]$ with the dual 2-handles, which since the dual 2-handles are contractible and $H_1(Y'; \mathbb{Z}[t^{\pm 1}]) = 0$ yields the short exact sequence:

$$0 \to H_2(Y' \times [0,1]; \mathbb{Z}[t^{\pm 1}]) \to H_2(-W; \mathbb{Z}[t^{\pm 1}]) \xrightarrow{\partial} H_1(\overline{\nu}(L'); \mathbb{Z}[t^{\pm 1}]) \to 0.$$

Since $\varphi'([L']) = 0$, $H_1(\overline{\nu}(L'); \mathbb{Z}[t^{\pm 1}]) \cong \bigoplus_{i=1}^n \mathbb{Z}[t^{\pm 1}]$, generated by the $[K'_i]$. Mapping each $[K'_i]$ to $[F_i]$ determines a splitting.

For the second equality, note that since B is a homotopy circle and $g_* \colon \pi_1(B) \to \mathbb{Z}$ is an isomorphism, B has no (reduced) $\mathbb{Z}[t^{\pm 1}]$-homology. The Mayer-Vietoris exact sequence associated to the decomposition $M = -W \cup_{Y' \times \{1\}} B$ therefore yields the short exact sequence

$$0 \to H_2(Y'; \mathbb{Z}[t^{\pm 1}]) \to H_2(-W; \mathbb{Z}[t^{\pm 1}]) \to H_2(M; \mathbb{Z}[t^{\pm 1}]) \to 0.$$

Appealing to our computation of $H_2(-W; \mathbb{Z}[t^{\pm 1}])$, we deduce that $H_2(M; \mathbb{Z}[t^{\pm 1}])$ is freely generated by the $[F_i]$. □

We will use the following lemma during the proof of Proposition 4.13 below: since $H_2(B; \mathbb{Z}[t^{\pm 1}]) = 0$, the equivariant intersection form of two surfaces can be calculated using any other surfaces with the same boundary.

Lemma 4.12. *For surfaces Σ and Σ' embedded in B^∞ with boundary a common knot in Y'^∞, and a properly embedded surface G in B^∞ with boundary disjoint from $\partial\Sigma = \partial\Sigma'$, we have that*

$$\Sigma \cdot_{\infty,B} G = \Sigma' \cdot_{\infty,B} G.$$

Proof. Observe that $\Sigma \cdot_{\infty,B} G = \Sigma' \cdot_{\infty,B} G$ if and only if $(\Sigma \cup -\Sigma') \cdot_{\infty,B} G = 0$. Using both Remark 2.2 and the fact that $(\Sigma \cup -\Sigma')$ determines a class $[(\Sigma \cup -\Sigma')]$ in $H_2(B; \mathbb{Z}[t^{\pm 1}]) = 0$, we get $(\Sigma \cup -\Sigma') \cdot_{\infty,B} G = \lambda_B^\partial([G], [\Sigma \cup -\Sigma']) = 0$, as required. □

Now we prove the first claim of the previously mentioned outline.

Proposition 4.13. *With respect to the basis of $H_2(M; \mathbb{Z}[t^{\pm 1}])$ given by the $[F_1], \dots, [F_n]$, the equivariant intersection form λ_M of M is given by the transposed equivariant linking matrix of the framed link $\widetilde{L}'$ dual to $\widetilde{L}$.*

Proof. Recall from Construction 3 that for $i = 1, \dots, n$, the surface $F_i \subset -W^\infty \subset M^\infty$ was obtained as the union of a surface $\Sigma_i \subset Y'^\infty$ whose boundary is $\widetilde{K}'_i$ with the core of a (lifted) 2-handle in the dual handle decomposition of W. For $i = 1, \dots, n$, define F'_i to be a surface isotopic to F_i obtained by pushing the interior of Σ_i into B^∞. Let Σ'_i be such a push-in. Since F_i

and F'_i are isotopic for every $i = 1, \dots, n$, we can use the F'_i to calculate λ_M. Fix real numbers $0 < s_1 < \cdots < s_n < 1$. We model Σ'_i in the coordinates of a collar neighborhood $\partial B \times [0,1]$ as

$$\Sigma'_i := (\partial\Sigma_i \times [0, s_i]) \cup (\Sigma_i \times \{s_i\}).$$

We start by calculating the equivariant intersection form $\lambda_M([F'_i],[F'_j])$ for $i \neq j$. Since the aforementioned cores of the dual 2-handles are pairwise disjoint, we obtain

$$\overline{\lambda_M([F'_i],[F'_j])} = F'_i \cdot_{\infty,M} F'_j = \Sigma'_i \cdot_{\infty,B} \Sigma'_j.$$

Recall that we use $A_{\widetilde{L}'}$ to be the linking matrix of the framed link L'. It therefore remains to show that $\Sigma'_i \cdot_{\infty,B} \Sigma'_j = (A_{\widetilde{L}'})_{ij}$.

To show this, we begin by picking surfaces $G_i, G_j \subset B$ with $\partial G_i = r(K'_i)$ and $\partial G_j = r(K'_j)$, where we use r to denote reversing the orientation on the knot. We then consider $G_i \cup \Sigma'_i$ and $G_j \cup \Sigma'_j$ as closed surfaces in B. Since $H_2(B;\mathbb{Z}[t^{\pm1}]) = 0$, the equivariant intersection form λ_B is identically zero and, in particular, we obtain

$$\lambda_B([G_i \cup \Sigma'_i],[G_j \cup \Sigma'_j]) = 0. \tag{11}$$

We now calculate $\lambda_B([G_i \cup \Sigma'_i],[G_j \cup \Sigma'_j])$ in a second, more geometric, manner

$$\begin{aligned}
0 &= \overline{\lambda_B([G_i \cup \Sigma'_i],[G_j \cup \Sigma'_j])} \\
&= (G_i \cup \Sigma'_i) \cdot_{\infty,B} (G_j \cup \Sigma'_j) \\
&= (G_i \cdot_{\infty,B} G_j) + (\Sigma'_i \cdot_{\infty,B} G_j) + (G_i \cdot_{\infty,B} \Sigma'_j) + (\Sigma'_i \cdot_{\infty,B} \Sigma'_j) \\
&= (G_i \cdot_{\infty,B} G_j) + (\Sigma_i \cdot_{\infty,\partial B} \partial G_j) + (\partial G_i \cdot_{\infty,\partial B} \Sigma_j) + (\partial\Sigma_i \cdot_{\infty,\partial B} \Sigma_j) \\
&= (G_i \cdot_{\infty,B} G_j) + \ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, r(\widetilde{K}'_j)) + \overline{\ell k_{\mathbb{Q}(t)}(r(\widetilde{K}'_i), \widetilde{K}'_j)} + \overline{\ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, \widetilde{K}'_j)} \\
&= (G_i \cdot_{\infty,B} G_j) - \ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, \widetilde{K}'_j) \\
&= (\Sigma'_i \cdot_{\infty,B} \Sigma'_j) - \ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, \widetilde{K}'_j).
\end{aligned}$$

In the above computation, the first equality is (11), the second is the definition of λ_B, the third is given by inspection of location of intersections. For the fourth equality, we may and shall assume that G_i and G_j intersect a collar neighborhood $\partial B \times [0,1]$ of $\partial B = Y'$ as products $\partial G_i \times [0,1]$ and $\partial G_j \times [0,1]$, in the coordinates of the collar. Then for the third term, since $\partial G_i \cap \partial\Sigma_j = \emptyset$, by inspecting the location of intersections we have:

$$G_i \cdot_{\infty,B} \Sigma'_j = (\partial G_i \times [0,1]) \cdot_{\infty,B} (\Sigma_j \times \{s_j\}) = \partial G_i \cdot_{\infty,\partial B} \Sigma_j.$$

The second term is similar. For the fourth term, assume without loss of generality that $i > j$, and so $s_i > s_j$. Also note that $\partial\Sigma_i \cap \partial\Sigma_j = \emptyset$. By further inspection of locations of intersections, it follows that:

$$\Sigma'_i \cdot_{\infty,B} \Sigma'_j = (\partial\Sigma_i \times [0, s_i]) \cdot_{\infty,B} (\Sigma_j \times \{s_j\}) = \partial\Sigma_i \cdot_{\infty,\partial B} \Sigma_j.$$

The fifth equality uses the definition of the equivariant linking number in $\partial B = Y'$, the sixth is the fact that $\ell k_{\mathbb{Q}(t)}$ changes sign if one reverses the orientation in one of its entries, and for the last we apply Lemma 4.12, which stated that, since $H_2(B;\mathbb{Z}[t^{\pm1}]) = 0$, the intersection $G_i \cdot_{\infty,B} G_j = -G_i \cdot_{\infty,B} -G_j$ can be calculated using any two surfaces in B that have the same boundaries as G_i and G_j. For $i \neq j$, we have therefore proved that

$$\lambda_M([F'_j],[F'_i]) = \Sigma'_i \cdot_{\infty,B} \Sigma'_j = \ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, \widetilde{K}'_j).$$

It remains to prove that $\lambda_M([F'_i],[F'_i]) = (A_{\widetilde{L}'})_{ii}$. Recall that by definition, the dual knot $\widetilde{K}'_i$ is framed by $\pi'_i = -\mu_{\widetilde{K}_i}$, which means that $(A_{\widetilde{L}'})_{ii} = \ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, \pi'_i)$. Perform a small push-off of the surface $\Sigma'_i \subset B^\infty$ to obtain a surface $\Sigma''_i \subset B^\infty$ isotopic to $\Sigma'_i \subset B^\infty$ with boundary $\partial\Sigma''_i = \pi'_i$. As in the $i \neq j$ case, we have $\lambda_M([F'_i],[F'_i]) = \Sigma'_i \cdot_\infty \Sigma''_i$, and therefore it suffices to prove that

$$\Sigma'_i \cdot_{\infty,B} \Sigma''_i = \ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, \pi'_i).$$

To do this, as before, pick properly embedded surfaces G_i and G_i'' in B^∞ whose respective boundaries are $r(\widetilde{K}_i')$ and $r(\pi_i')$. Form the closed surfaces $G_i \cup \Sigma_i'$ and $G_i'' \cup \Sigma_i''$ and exactly as above we obtain

$$0 = (G_i \cup \Sigma_i') \cdot_{\infty,B} (G_i'' \cup \Sigma_i'') = \Sigma_i' \cdot_{\infty,B} \Sigma_i'' - \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \pi_i').$$

Rearranging, we conclude that $\Sigma_i' \cdot_{\infty,B} \Sigma_i'' = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \pi_i')$, as required. We have therefore shown that the equivariant intersection form of M is represented by the transposed linking matrix $A^T_{\widetilde{L}'}$, and this concludes the proof of the proposition. □

Finally, we prove the second claim of our outline, thus completing step 4.

Proposition 4.14. *Let Y be a 3-manifold with an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module is torsion, and let (H, λ) be a nondegenerate Hermitian form presenting Y. If $b \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$ is an isometry, then there is a 4-manifold M with $\pi_1(M) \cong \mathbb{Z}$, equivariant intersection form $\lambda_M \cong \lambda$, ribbon boundary Y and with $b_M = b$.*

Proof. Let M be the 4-manifold with boundary Y constructed as described above. The manifold $M = -W \cup_{Y'} B$ comes with a homeomorphism $g\colon \partial M \cong Y$, because $-W$ is obtained from $Y \times [0,1]$ by adding 2-handles. We already explained why M has intersection form isometric to λ but we now make the isometry more explicit. Define an isomorphism $F\colon H \to H_2(M; \mathbb{Z}[t^{\pm 1}])$ by mapping x_i to $[F_i]$, where the $F_i \subset M^\infty$ are the surfaces built in Construction 3. This is an isometry because, by combining Proposition 4.13 with Lemma 2.12, we get

$$\lambda_M([F_i], [F_j]) = (A_{\widetilde{L}'})_{ji} = -(A^{-1}_{\widetilde{L}})_{ji} = Q_{ij} = \lambda(x_i, x_j).$$

We now check that $b_M = b$ by proving that $b = g_* \circ D_M \circ \partial F$. This amounts to proving that the bottom square of the following diagram commutes (we refer to Construction 1 if a refresher on the notation is needed):

$$\begin{array}{ccccc}
H^* & \xrightarrow{F^{-*},\cong} & H_2(M;\mathbb{Z}[t^{\pm1}])^* & \xrightarrow{\mathrm{PD}\circ\mathrm{ev}^{-1},\cong} & H_2(M,\partial M;\mathbb{Z}[t^{\pm1}]) \\
\downarrow{\scriptstyle \mathrm{proj}} & & \downarrow{\scriptstyle \mathrm{proj}} & & \downarrow{\scriptstyle \delta_M} \\
\operatorname{coker}(\widehat{\lambda}) & \xrightarrow{\partial F,\cong} & \operatorname{coker}(\widehat{\lambda}_M) & \xrightarrow{D_M,\cong} & H_1(\partial M;\mathbb{Z}[t^{\pm1}]) \\
\downarrow{\scriptstyle =} & & & & \downarrow{\scriptstyle g_*,\cong} \\
\operatorname{coker}(\widehat{\lambda}) & & \xrightarrow{\quad b,\cong \quad} & & H_1(Y;\mathbb{Z}[t^{\pm1}]).
\end{array}$$

The top squares of this diagram commute by definition of ∂F and D_M. Since the top vertical maps are surjective, the commutativity of the bottom square is now equivalent to the commutativity of the outer square. It therefore remains to prove that $g_* \circ \delta_M \circ (\mathrm{PD} \circ \mathrm{ev}^{-1}) \circ F^{-*} = \pi$; (recall that by definition $\pi = b \circ \mathrm{proj}$). In fact, it suffices to prove this on the x_i^* as they form a basis of H^*. Writing c_i for the core of the two handles attached to $Y \times [0,1]$, we have

$$g_* \circ \delta_M \circ (\mathrm{PD} \circ \mathrm{ev}^{-1}) \circ F^{-*}(x_i^*) = g_* \circ \delta_M \circ (\mathrm{PD} \circ \mathrm{ev}^{-1})([F_i]^*) = g_* \circ \delta_M([\widetilde{c}_i]) = [\widetilde{K}_i] = \pi(x_i^*).$$

Here we use successively the definition of F, the geometric interpretation of $\mathrm{PD} \circ \mathrm{ev}^{-1}$, the fact that $\widetilde{g}(\partial \widetilde{c}_i) = \widetilde{K}_i$ and the definition of the $\widetilde{K}_i$. Therefore the outer square commutes as asserted. This concludes the proof that $b = g_* \circ D_M \circ \partial F$ and therefore $b_M = b$, as required. □

4.6. **Step 5: fixing the Kirby-Siebenmann invariant and concluding.** The conclusion of Theorem 4.1 will follow promptly from Proposition 4.14 once we recall how, in the odd case, it is possible to modify the Kirby-Siebenmann invariant of a given 4-manifold with fundamental group $\mathbb{Z}$. This is achieved using the star construction, a construction which we now recall following [FQ90] and [Sto93]. In what follows, $*\mathbb{C}P^2$ denotes the Chern manifold, i.e. the unique simply-connected topological 4-manifold homotopy equivalent to $\mathbb{C}P^2$ but with $\operatorname{ks}(*\mathbb{C}P^2) = 1$.

Let M be a topological 4-manifold with (potentially empty) boundary, good fundamental group π and such that the second Stiefel-Whitney class of the universal cover $w_2(\widetilde{M})$ is nontrivial. There is a 4-manifold $*M$, called the *star partner of* M that is rel. boundary homotopy equivalent to M but has the opposite Kirby-Siebenmann invariant from that of M [FQ90, Theorem 10.3 (1)]. See [Tei97] or [KPR22, Propostion 5.8] for a more general condition under which a star partner exists.

Remark 4.15. For fundamental group $\mathbb{Z}$, every non-spin 4-manifold has $w_2(\widetilde{M}) \neq 0$. To see this, we use the exact sequence

$$0 \to H^2(B\pi;\mathbb{Z}/2) \to H^2(M;\mathbb{Z}/2) \xrightarrow{p^*} H^2(\widetilde{M};\mathbb{Z}/2)^{\pi},$$

where $\pi := \pi_1(M)$. This can be deduced from the Leray-Serre spectral sequence for the fibration $\widetilde{M} \to M \to B\pi$; see e.g. [KLPT17, Lemma 3.17]. For $\pi = \mathbb{Z}$ the first term vanishes, so p^* is injective. By naturality, $p^*(w_2(M)) = w_2(\widetilde{M})$, so $w_2(M) \neq 0$ implies $w_2(\widetilde{M}) \neq 0$ as desired. It follows that for a non-spin 4-manifold M with fundamental group $\mathbb{Z}$, [FQ90, Theorem 10.3] applies and there is a star partner.

To describe $*M$, consider the 4-manifold $W := M\#(*\mathbb{C}P^2)$ and note that the inclusions $M \hookrightarrow W$ and $*\mathbb{C}P^2 \hookrightarrow W$ induce a splitting

$$\pi_2(M) \oplus (\pi_2(*\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi]) \xrightarrow{\cong} \pi_2(W). \tag{12}$$

By [FQ90, Theorem 10.3 (1)] (cf. [KPR22, Proposition 5.8]) there exists a 4-manifold $*M$ and an orientation-preserving homeomorphism

$$h\colon W \xrightarrow{\cong} *M\#\mathbb{C}P^2$$

that respects the splitting on π_2 displayed in (12). The star partner $*M$ is also unique up to homeomorphism, by [Sto94, Corollary 1.2].

To be more precise about the splitting of π_2, fix a basis $[\alpha]$ of $\pi_2(\mathbb{C}P^2) = \mathbb{Z}$ and $[\alpha']$ of $\pi_2(*\mathbb{C}P^2) = \mathbb{Z}$ so that if $\iota\colon \pi_2(*\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi] \to \pi_2(M\#(*\mathbb{C}P^2)) = \pi_2(W)$ denotes the split isometric injection induced by the inclusion, then the following diagram commutes

$$\begin{array}{ccc} \pi_2(*\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi] & \xhookrightarrow{\iota} & \pi_2(W) \\ \downarrow \cong & & \cong \downarrow h_* \\ \pi_2(\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi] & \xhookrightarrow{\text{incl}_*} & \pi_2(*M\#\mathbb{C}P^2), \end{array}$$

where the bottom horizontal map is also a split isometric injection induced by the inclusion, and the left vertical map is the unique isomorphism that takes $[\alpha']$ to $[\alpha]$, induced by a homotopy equivalence $*\mathbb{C}P^2 \simeq \mathbb{C}P^2$. Since both horizontal maps in this diagram are split, h_* induces an isomorphism $\pi_2(M) \cong \pi_2(*M)$ and it is in this sense that h respects the splitting displayed in (12).

We briefly recall why M and $*M$ are orientation-preserving homotopy equivalent rel. boundary. This will ensure that their automorphism invariants agree. The argument is due to Stong [Sto94, Section 2].

Proposition 4.16. *If M is a topological 4-manifold with boundary, good fundamental group π and whose universal cover has nontrivial second Stiefel-Whitney class, then M is orientation-preserving homotopy equivalent rel. boundary to its star partner $*M$.*

Proof. Set once again $W := M\# * \mathbb{C}P^2$ and use α and α' to denote spheres whose homotopy classes generate $\pi_2(\mathbb{C}P^2)$ and $\pi_2(*\mathbb{C}P^2)$. As we explained above, there exists a homeomorphism

$$h\colon W \xrightarrow{\cong} *M\#\mathbb{C}P^2$$

such that $h([\alpha']) = \alpha$ (here, we suppressed the inclusion maps from the notation). Observe that the 4-complexes $\mathbb{C}P^2 \cup_\alpha e^3$ and $\mathbb{C}P^2 \cup_{\alpha'} e^3$ obtained by attaching 3-cells along α and α' have the homotopy type of S^4. To see this take the standard degree one map $\mathbb{C}P^2 \to S^4$. The image of the

2-skeleton is null-homotopic and so the map extends to a map $\mathbb{C}P^2 \cup_\alpha e^3 \to S^4$. This is homotopy equivalence by the Hurewicz and Whitehead theorems.

Thinking respectively of M and $*M$ as $M\#S^4$ and $*M\#S^4$, the required homotopy equivalence is then obtained as the composition

$$M \simeq M\#(*\mathbb{C}P^2 \cup_{\alpha'} e^3) \cong W \cup_{\alpha'} e^3 \xrightarrow{h\cup \mathrm{Id},\cong} (*M\#\mathbb{C}P^2) \cup_\alpha e^3 \cong *M\#(\mathbb{C}P^2 \cup_\alpha e^3) \simeq *M.$$

This homotopy equivalence is rel. boundary because these are interior connected sums. □

We are ready to prove Theorem 4.1, whose statement we now recall for the reader's convenience. Let Y be a 3-manifold with an epimorphism $\pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module is torsion, and let (H, λ) be a form presenting Y. If $b \in \mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/\mathrm{Aut}(\lambda)$ is an isometry, then there is a 4-manifold M with $\pi_1(M) \cong \mathbb{Z}$, equivariant intersection form λ_M, ribbon boundary Y and with $b_M = b$. If the form is odd, then M can be chosen to have either $\mathrm{ks}(M) = 0$ or $\mathrm{ks}(M) = 1$. We now conclude the proof of this theorem.

Proof of Theorem 4.1. In Proposition 4.14, we proved the existence of a manifold M with equivariant intersection form λ_M, ribbon boundary Y and with $b_M = b$. It remains to show that if λ is odd, then M can be chosen to have either $\mathrm{ks}(M) = 0$ or $\mathrm{ks}(M) = 1$. This is possible by using the star partner $*M$ of M. Indeed Proposition 4.16 implies that M and $*M$ are homotopy equivalent rel. boundary and therefore Remark 1.16 ensures that $b_{*M} = b_M$ is unchanged. □

5. Application to $\mathbb{Z}$-surfaces in 4-manifolds

Recall that a *$\mathbb{Z}$-surface* refers to a locally flat, embedded surface in a 4-manifold whose complement has infinite cyclic fundamental group. In this section we apply our classification of 4-manifolds with fundamental group $\mathbb{Z}$ to the study of $\mathbb{Z}$-surfaces in simply connected 4-manifolds and prove Theorems 1.6, 1.9, and 1.10 from the introduction. In Subsection 5.1, we focus on $\mathbb{Z}$-surfaces with boundary up to equivalence rel. boundary. In the shorter Subsections 5.2 and 5.3, we respectively study surfaces with boundary up to equivalence (not necessarily rel. boundary) and closed surfaces. Subsection 5.4 lists some open problems.

5.1. Surfaces with boundary up to equivalence rel. boundary.

Let N be a simply-connected 4-manifold with boundary homeomorphic to S^3. We fix once and for all a particular homeomorphism $h\colon \partial N \cong S^3$. Let $K \subset S^3$ be a knot. Thus K and h determine a knot in ∂N, which we also denote by K. The goal of this subsection is to give an algebraic description of the set of $\mathbb{Z}$-surfaces in N with boundary K up to equivalence rel. boundary.

We begin with some conventions. Given a properly embedded $\mathbb{Z}$-surface $\Sigma \subset N$ in a simply-connected 4-manifold, denote its exterior by $N_\Sigma := N \smallsetminus \nu(\Sigma)$. Throughout this section, we will refer to embedded surfaces simply as Σ, and abstract surfaces as $\Sigma_{g,b}$, where g is the genus and b is the number of boundary components; we may sometimes write Σ_g when $b = 0$. Recall that throughout, $\Sigma_{g,b}$ and N will be oriented. This data determines orientations on S^3, K, and every meridian of an embedding of $\Sigma_{g,b}$. Observe that the $\pi_1(N_\Sigma) \cong \mathbb{Z}$ hypothesis implies that $[\Sigma, \partial\Sigma] = 0 \in H_2(N, \partial N)$ by [CP20, Lemma 5.1], so the relative Euler number of the normal bundle of Σ, with respect to the zero-framing of $\nu(\partial N)$, vanishes [CP20, Lemma 5.2]. From now on, we choose a framing $\nu(\Sigma) \cong \Sigma \times \mathring{D}^2 \cong \Sigma \times \mathbb{R}^2$ compatible with the orientation and with the property that for each simple closed curve $\gamma_k \subset \Sigma$, we have $\gamma_k \times \{e_1\} \subset N \setminus \Sigma$ is nullhomologous in $N \setminus \Sigma$. We will refer to such a framing as a *good framing.* As such, when $\partial\Sigma = K \subset \partial N$ we can identify the boundary of N_Σ as

$$\partial N_\Sigma \cong E_K \cup_\partial (\Sigma_{g,1} \times S^1) =: M_{K,g},$$

where the gluing ∂ takes λ_K to $\partial\Sigma \times \{\mathrm{pt}\}$.

We call two locally flat surfaces $\Sigma, \Sigma' \subset N$ with boundary $K \subset \partial N \cong S^3$ *equivalent rel. boundary* if there is an orientation-preserving homeomorphism of pairs $(N, \Sigma) \cong (N, \Sigma')$ that is pointwise the identity on $\partial N \cong S^3$. Note that if $\Sigma \subset N$ is a $\mathbb{Z}$-surface with boundary K, then N_Σ has ribbon boundary $\partial N_\Sigma \cong M_{K,g}$ [CP20, Lemma 5.4] and $H_1(M_{K,g};\mathbb{Z}[t^{\pm1}]) \cong H_1(M_K;\mathbb{Z}[t^{\pm1}]) \oplus \mathbb{Z}^{2g}$ is torsion because the Alexander module $H_1(M_K;\mathbb{Z}[t^{\pm1}])$ of K is torsion [CP20, Lemma 5.5]. Additionally, note that the equivariant intersection form λ_{N_Σ} of a surface exterior N_Σ must present $M_{K,g}$.

Consequently, as we did for manifolds, it is natural to fix a form (H,λ) that presents $M_{K,g}$ and to consider the set $\mathrm{Surf(g)}^0_\lambda(N,K)$ of genus g $\mathbb{Z}$-surfaces in N with boundary K and $\lambda_{N_\Sigma} \cong \lambda$.

Definition 11. For a nondegenerate Hermitian form (H,λ) over $\mathbb{Z}[t^{\pm1}]$ that presents $M_{K,g}$, set

$$\mathrm{Surf(g)}^0_\lambda(N,K) := \{\mathbb{Z}\text{-surfaces } \Sigma \subset N \text{ for } K \text{ with } \lambda_{N_\Sigma} \cong \lambda\}/ \text{ equivalence rel. } \partial.$$

There is an additional necessary condition for this set to be nonempty. For conciseness, we write $\lambda(1) := \lambda \otimes_{\mathbb{Z}[t^{\pm1}]} \mathbb{Z}_\varepsilon$, where $\mathbb{Z}_\varepsilon$ denotes $\mathbb{Z}$ with the trivial $\mathbb{Z}[t^{\pm1}]$-module structure. This way, if $A(t)$ is a matrix that represents λ, then $A(1)$ represents $\lambda(1)$. Additionally, recall that if W is a $\mathbb{Z}$-manifold, then $\lambda_W(1) \cong Q_W$, where Q_W denotes the standard intersection form of W; see e.g. [CP20, Lemma 5.10]. Thus, if we take $W = N_\Sigma$ and assume that $\lambda \cong \lambda_{N_\Sigma}$, then

$$\lambda(1) \cong \lambda_{N_\Sigma}(1) \cong Q_{N_\Sigma} \cong Q_N \oplus (0)^{\oplus 2g},$$

where the last isometry follows from a Mayer-Vietoris argument. Thus, for the set $\mathrm{Surf(g)}^0_\lambda(N,K)$ to be nonempty, it is also necessary that $\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}$.

For the final piece of setup for the statement of the main result of the section, we describe an action of $\mathrm{Homeo}^+(\Sigma_{g,1},\partial)$ on the set $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_{M_{K,g}})$ as follows. First, a rel. boundary homeomorphism $x\colon \Sigma_{g,1} \to \Sigma_{g,1}$ induces an isometry $x''_*\colon \mathrm{Bl}_{M_{K,g}} \cong \mathrm{Bl}_{M_{K,g}}$ as follows. Extend x to a self homeomorphism x' of $\Sigma_{g,1} \times S^1$ by defining $x'(s,\theta) = (x(s),\theta)$. Then extend x' by the identity over E_K; in total one obtains a self homeomorphism x'' of $M_{K,g}$. Now lift this homeomorphism to the covers and take the induced map on H_1 to get $x''_*\colon \mathrm{Bl}_{M_{K,g}} \cong \mathrm{Bl}_{M_{K,g}}$. The required action is now by postcomposition; for $f \in \mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_{M_{K,g}})$, define $x \cdot f := x''_* \circ f$. The main result of this section proves Theorem 1.6 from the introduction.

Theorem 5.1. *Let N be a simply-connected 4-manifold with boundary $\partial N \cong S^3$ and let $K \subset S^3$ be a knot. Given a nondegenerate Hermitian form (H,λ) over $\mathbb{Z}[t^{\pm1}]$, the following assertions are equivalent:*

(1) the Hermitian form (H,λ) presents $M_{K,g}$ and satisfies $\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}$;
(2) the set $\mathrm{Surf(g)}^0_\lambda(N,K)$ is nonempty and there is a bijection

$$\mathrm{Surf(g)}^0_\lambda(N,K) \approx \mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_{M_{K,g}})/(\mathrm{Aut}(\lambda) \times \mathrm{Homeo}^+(\Sigma_{g,1},\partial)).$$

Remark 5.2. We collect some remarks concerning Theorem 5.1.

- If (H,λ) presents $M_{K,g}$, then there is a non-canonical bijection

$$\frac{\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_{M_{K,g}})}{(\mathrm{Aut}(\lambda) \times \mathrm{Homeo}^+(\Sigma_{g,1},\partial))} \approx \frac{\mathrm{Aut}(\partial\lambda)}{(\mathrm{Aut}(\lambda) \times \mathrm{Homeo}^+(\Sigma_{g,1},\partial))}.$$

  In addition, we have the isomorphism $\mathrm{Aut}(\partial\lambda) \cong \mathrm{Aut}(\mathrm{Bl}_{M_{K,g}}) \cong \mathrm{Aut}(\mathrm{Bl}_K) \oplus \mathrm{Sp}_{2g}(\mathbb{Z})$ where the latter is the group of automorphisms of the symplectic intersection pairing of $\Sigma_{g,1}$ [CP20, Propositions 5.6 and 5.7]. The group $\mathrm{Homeo}^+(\Sigma_{g,1},\partial)$ acts trivially on the first summand and transitively on the second. Therefore one can express the quotients above as

$$\mathrm{Aut}(\mathrm{Bl}_K)/\mathrm{Aut}(\lambda),$$

  where the action of $\mathrm{Aut}(\lambda)$ on $\mathrm{Aut}(\mathrm{Bl}_K)$ arises by restricting the action of $\mathrm{Aut}(\lambda)$ on $\mathrm{Aut}(\partial\lambda) \cong \mathrm{Aut}(\mathrm{Bl}_{M_{K,g}}) \cong \mathrm{Aut}(\mathrm{Bl}_K) \oplus \mathrm{Sp}_{2g}(\mathbb{Z})$ to the first summand. We stress again that the isomorphism $\mathrm{Aut}(\partial\lambda) \cong \mathrm{Aut}(\mathrm{Bl}_{M_{K,g}})$ is not canonical. The set $\mathrm{Aut}(\mathrm{Bl}_K)/\mathrm{Aut}(\lambda)$ was mentioned in Theorem 1.6 from the introduction.
- The action of $\mathrm{Homeo}^+(\Sigma_{g,1},\partial)$ on $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_{M_{K,g}})$ factors through the corresponding mapping class group $\mathrm{Mod}^+(\Sigma_{g,1},\partial) := \pi_0(\mathrm{Homeo}^+(\Sigma_{g,1},\partial))$. In particular, Theorem 5.1 could have equally well been stated using $\mathrm{Mod}^+(\Sigma_{g,1},\partial)$ instead of $\mathrm{Homeo}^+(\Sigma_{g,1},\partial)$.

- Our surface set $\mathrm{Surf(g)}^0_\lambda(N,K)$ is defined up to equivalence, hence Theorem 5.1 only gives a classification of surfaces up to equivalence (instead of ambient isotopy). This is because we prove Theorem 5.1 as a consequence of Theorem 1.1 and the equivalence on $\mathscr{V}^0_\lambda(M_{K,g})$ is up to *any* homeomorphism rel. boundary, not just homeomorphisms in a prescribed isotopy class. As a consequence, when N admits homeomorphisms which are not isotopic to the identity rel. boundary, there can be $\mathbb{Z}$-surfaces that are equivalent rel. boundary but not ambient isotopic. Here is an example.

  Let $K \subset S^3$ be a knot with nontrivial Alexander polynomial Δ_K, that bounds a $\mathbb{Z}$-disc in a punctured $\mathbb{C}P^2$ with intersection form represented by the 1×1 matrix (Δ_K). Let N be given by the boundary connected sum with another punctured $\mathbb{C}P^2$ (so that N is a punctured $\mathbb{C}P^2\#\mathbb{C}P^2$), and denote the same $\mathbb{Z}$-disc considered in N by D. There is a self-homeomorphism $\tau\colon N \to N$ that induces $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ on $H_2(N)\cong\mathbb{Z}^2$. Isotope τ to be the identity on $\partial N\cong S^3$. The discs D and $\tau(D)$ are equivalent rel. boundary. But a short computation shows that the equivariant intersection forms of the exteriors are $\left(\begin{smallmatrix}\Delta_K&0\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\0&\Delta_K\end{smallmatrix}\right)$ respectively. A straightforward computation shows that every $\mathbb{Z}[t^{\pm1}]$-isometry between these two forms augments over $\mathbb{Z}$ to $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. It follows that there is no ambient isotopy between D and $\tau(D)$.

Theorem 5.1 will be proved in three steps.

(1) We define a map Θ from a set of equivalence classes of embeddings $\Sigma_{g,1}\hookrightarrow N$, which we denote $\mathrm{Emb}^0_\lambda(\Sigma_{g,1},N;K)$ and which we will define momentarily, to the set of manifolds $\mathscr{V}^0_\lambda(M_{K,g})$ from Definition 5. By Theorem 1.1, $\mathscr{V}^0_\lambda(M_{K,g})$ corresponds bijectively to the set of isometries $\mathrm{Iso}(\partial\lambda,-\mathrm{Bl}_{M_{K,g}})/\mathrm{Aut}(\lambda)$.
(2) We prove that the map Θ is a bijection, by defining a map Ψ in the other direction, from the set of manifolds to the set of embeddings, and showing that both $\Theta\circ\Psi$ and $\Psi\circ\Theta$ are the identity maps.
(3) We describe the set of surfaces $\mathrm{Surf(g)}^0_\lambda(N,K)$ as a quotient of $\mathrm{Emb}^0_\lambda(\Sigma_{g,1},N;K)$ by $\mathrm{Homeo}^+(\Sigma_{g,1},\partial)$. We then use the bijection from the first step to define an action of $\mathrm{Homeo}^+(\Sigma_{g,1},\partial)$ on $\mathscr{V}^0_\lambda(M_{K,g})$ and $\mathrm{Iso}(\partial\lambda,-\mathrm{Bl}_{M_{K,g}})/\mathrm{Aut}(\lambda)$ leading to the bijection in Theorem 5.1. This step is largely formal.

Step (1)*: From embeddings to manifolds.* For the first step, we give some definitions and construct the map which will be the bijection in Theorem 5.1.

Consider the following set:

$$\mathrm{Emb}^0_\lambda(\Sigma_{g,1},N;K)=\frac{\{e\colon\Sigma_{g,1}\hookrightarrow N \mid e(\Sigma_{g,1}) \text{ is a } \mathbb{Z}\text{-surface for } K \text{ with } \lambda_{N_{e(\Sigma_{g,1})}}\cong\lambda\}}{\text{equivalence rel. } \partial}.$$

Two embeddings e_1,e_2 are *equivalent rel. boundary* if there exists a homeomorphism $\Phi\colon N\to N$ that is the identity on $\partial N\cong S^3$ and satisfies $\Phi\circ e_1=e_2$.

In what follows, we let $\varphi\colon\pi_1(M_{K,g})\twoheadrightarrow\mathbb{Z}$ be the epimorphism such that the induced map $\varphi'\colon H_1(M_{K,g})\twoheadrightarrow\mathbb{Z}$ is the unique epimorphism that maps the meridian of K to 1 and the other generators to zero. When we write $\mathscr{V}^0_\lambda(M_{K,g})$, it is with respect to this epimorphism φ. Recall also that we have a fixed homeomorphism $h\colon\partial N\to S^3$; whenever we say $\partial N\cong S^3$, it is with this fixed h.

In addition to our homeomorphism $h\colon\partial N\to S^3$, we fix once and for all the following data.

- A closed tubular neighborhood $\overline{\nu}(K)\subset\partial N$. Since we have already fixed h, and since we are abusively using K for both the knot K in ∂N and for the image $h(K)$ in S^3, this choice of $\overline{\nu}(K)\subset\partial N$ also determines a particular neighborhood $\overline{\nu}(K)\subset S^3$. We will use E_K exclusively to denote the complement of $\nu(K)$ in S^3.
- A homeomorphism $D\colon\partial\Sigma_{g,1}\times S^1\to\partial\overline{\nu}(K)$ that takes $\partial\Sigma_{g,1}\times\{1\}$ to the 0-framed longitude of K and $\{\mathrm{pt}\}\times S^1$ to the meridian of K such that

$$M_{K,g}=E_K\cup_D\Sigma_{g,1}\times S^1.$$

These choices can change the bijection, however we are interested only in the existence of a bijection, so this is not an issue.

Next we define the map which will be the bijection in Theorem 5.1.

Construction 4. We construct a map $\Theta\colon \operatorname{Emb}^0_\lambda(\Sigma_{g,1}, N; K) \to \mathscr{V}^0_\lambda(M_{K,g})$.

Let $e\colon \Sigma_{g,1} \hookrightarrow N$ be an embedding that belongs to $\operatorname{Emb}^0_\lambda(\Sigma_{g,1}, N; K)$. We will assign to e a pair $(N_{e(\Sigma_{g,1})}, f)$, where $f\colon \partial N_{e(\Sigma_{g,1})} \to M_{K,g}$ is a homeomorphism. The pair we construct will depend on several choices, but we will show that the outcome is independent of these choices up to equivalence in $\mathscr{V}^0_\lambda(M_{K,g})$.

To cut down on notation we set $\Sigma := e(\Sigma_{g,1})$ and describe the choices on which our pair (N_Σ, f) will a priori depend.

(1) An embedding $\iota\colon \overline{\nu}(\Sigma) \hookrightarrow N$ of the normal bundle of Σ such that $\iota(\overline{\nu}(\Sigma)) \cap \partial N$ agrees with our fixed tubular neighbourhood of K.
(2) A good framing $\gamma\colon \overline{\nu}(\Sigma) \cong \Sigma_{g,1} \times D^2$ such that $h| \circ \iota \circ \gamma^{-1} = D$:

$$\begin{array}{ccc} \partial\Sigma_{g,1} \times S^1 & \xrightarrow{\ D\ } & \partial\overline{\nu}(K) \subset E_K \\ \Big\downarrow{\scriptstyle \gamma^{-1}} & & \Big\uparrow{\scriptstyle h|} \\ \gamma^{-1}(\partial\Sigma_{g,1} \times S^1) & \xrightarrow{\ \iota|\ } & \iota(\gamma^{-1}(\partial\Sigma_{g,1} \times S^1)) \subset \partial N \setminus \nu(K). \end{array} \tag{13}$$

In this diagram, $h|$ denotes the restriction of our fixed identification $h\colon \partial N \cong S^3$ and $D\colon \partial\Sigma_{g,1} \times S^1 \to \partial\overline{\nu}(K)$ is the homeomorphism that we fixed above.

We also record some of the notation that stems from these choices.

- The boundary of the surface exterior N_Σ decomposes as

$$\partial N_\Sigma \cong \big(\partial N \setminus \nu(K)\big) \cup \Big(\partial\iota(\overline{\nu}(\Sigma)) \smallsetminus (\iota(\nu(\Sigma)) \cap \partial N)\Big). \tag{14}$$

  Here the first part of this union is homeomorphic to a knot exterior, while the second is homeomorphic to $\Sigma_{g,1} \times S^1$.
- Restricting our fixed homeomorphism $h\colon \partial N \cong S^3$ to the knot exterior part in (14), we obtain the homeomorphism

$$h|\colon \partial N \smallsetminus \nu(K) \to E_K \subset M_{K,g}.$$

- On the circle bundle part of (14), we consider the homeomorphism

$$\gamma| \circ \iota^{-1}\colon \Big(\partial\iota(\overline{\nu}(\Sigma)) \smallsetminus (\iota(\nu(\Sigma)) \cap \partial N)\Big) \to \Sigma_{g,1} \times S^1 \subset M_{K,g}.$$

  Here by the slightly abusive notation ι^{-1}, we mean that since $\iota\colon \overline{\nu}(\Sigma) \hookrightarrow N$ is an embedding, it is a homeomorphism onto its image, whence the inverse.

The diagram in (13) ensures that $h|$ and $\gamma| \circ \iota^{-1}$ can be glued together to give rise to the homeomorphism we have been building towards:

$$f_\gamma\colon \partial N_\Sigma \to M_{K,g}, \qquad f_\gamma := (h|) \cup (\gamma| \circ \iota^{-1}). \tag{15}$$

Set $\Theta(e) := (N_\Sigma, f_\gamma)$. We need to verify that Θ gives rise to a map $\operatorname{Emb}^0_\lambda(\Sigma_{g,1}, N; K) \to \mathscr{V}^0_\lambda(M_{K,g})$. In other words, we need to check that modulo homeomorphisms rel. boundary, $\Theta(e)$ does not depend on the embedding $\iota\colon \overline{\nu}(\Sigma) \hookrightarrow N$ nor on the particular choice of the good framing γ subject to the condition in (13). We also have to verify that equivalent embeddings produce equivalent manifolds.

- First we show that the construction is independent of γ and ι. Pick another embedding $\iota'\colon \overline{\nu}(e(\Sigma_{g,1})) \hookrightarrow N$ of the normal bundle and another good framing $\gamma'\colon \overline{\nu}(e(\Sigma_{g,1})) \cong \Sigma_{g,1} \times D^2$ with the same hypothesis about compatibility with D. This leads to boundary homeomorphisms $f_\gamma := (h|) \cup (\gamma| \circ \iota^{-1})$ and $f_{\gamma'} := (h|) \cup (\gamma'| \circ \iota'^{-1})$ and we must show that the following pairs are equivalent rel. boundary:

$$(N_{e_\iota(\Sigma_{g,1})}, f_\gamma) \text{ and } (N_{e_{\iota'}(\Sigma_{g,1})}, f_{\gamma'}). \tag{16}$$

For a moment we are keeping track of the embeddings ι and ι' in our notation for exteriors. More explicitly, we set $N_{e_\iota(\Sigma_{g,1})} := N \setminus \iota(\nu(e(\Sigma_{g,1})))$ and similarly for ι'.

By uniqueness of tubular neighbourhoods [FQ90, Theorem 9.3D], there is an isotopy of embeddings $\Gamma_t\colon \Sigma_{g,1}\times D^2 \hookrightarrow N$ such that $\Gamma_0 = \iota\circ\gamma^{-1}$ and $\Gamma_1 = \iota'\circ\gamma'^{-1}$ that fixes a neighborhood of $\partial\Sigma_{g,1}\times D^2$. Then by the Edwards-Kirby isotopy extension theorem [EK71], there is an isotopy of homeomorphisms $F_t\colon N \to N$ with $F_1\circ\iota\circ\gamma^{-1} = \iota'\circ\gamma'^{-1}$ and $F_0 = \mathrm{Id}_N$ and such that F_t is the identity on a neighborhood of the boundary ∂N for every $t\in[0,1]$. We will argue that this F_1 restricted to the exteriors $N_{e_\iota(\Sigma_{g,1})}$ and $N_{e_{\iota'}(\Sigma_{g,1})}$ gives a rel. boundary homeomorphism between the pairs in (16).

We wish to argue that the restriction of F_1 to the surface exteriors identifies $(N_{e_\iota(\Sigma_{g,1})}, f_\gamma)$ with $(N_{e_{\iota'}(\Sigma_{g,1})}, f_{\gamma'})$ as elements of $\mathscr{V}^0_\lambda(M_{K,g})$. Consider the following diagram:

$$\begin{array}{ccccccc}
M_{K,g} & \xleftarrow{f_\gamma=(h|)\cup(\gamma|\circ\iota^{-1})} & \partial N_{e_\iota(\Sigma_{g,1})} & \xrightarrow{\subset} & N_{e_\iota(\Sigma_{g,1})} & \xrightarrow{\subset} & N \\
\downarrow{=} & & \downarrow{F_1} & & \downarrow{F_1} & & \downarrow{F_1} \\
M_{K,g} & \xleftarrow{f_{\gamma'}=(h|)\cup(\gamma'|\circ\iota'^{-1})} & \partial N_{e_{\iota'}(\Sigma_{g,1})} & \xrightarrow{\subset} & N_{e_{\iota'}(\Sigma_{g,1})} & \xrightarrow{\subset} & N.
\end{array}$$

The right two squares certainly commute, while the left square commutes because the homeomorphism $F_1\colon N\to N$ is rel. boundary and because, by construction, $\gamma|\circ\iota^{-1} = F_1\circ\gamma'|\circ\iota'^{-1}$. In total, we have:

$$F_1\circ f_{\gamma'} = F_1\circ\Big((h|)\cup(\gamma'|\circ\iota'^{-1})\Big) = (F_1\circ h|)\cup(F_1\circ\gamma'|\circ\iota'^{-1}) = h|\cup(\gamma\circ\iota^{-1}) = f_\gamma. \tag{17}$$

- We now show that the map Θ from Construction 4 is well defined up to rel. boundary homeomorphisms of N; recall that this is the equivalence relation on the domain $\mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K)$. Assume that $e, e'\colon \Sigma_{g,1}\hookrightarrow N$ are embeddings that are homeomorphic rel. boundary via a homeomorphism $F\colon N\to N$. Pick good framings γ,γ' for $\overline{\nu}(e(\Sigma_{g,1}))$ and $\overline{\nu}(e'(\Sigma_{g,1}))$ as well as an embedding $\iota'\colon \overline{\nu}(e'(\Sigma_{g,1}))\hookrightarrow N$. We now consider the embedding $\iota := F^{-1}\circ\iota'\circ(\gamma')^{-1}\circ\gamma$. The following diagram commutes:

$$\begin{array}{ccccccc}
\Sigma_{g,1}\times D^2 & \xrightarrow{\gamma^{-1},\cong} & \overline{\nu}(e(\Sigma_{g,1})) & \xrightarrow{\iota,\cong} & \iota(\overline{\nu}(e'(\Sigma_{g,1}))) & \xrightarrow{\subset} & N \\
\downarrow{=} & & & & \downarrow{F|} & & \downarrow{F} \\
\Sigma_{g,1}\times D^2 & \xrightarrow{\gamma'^{-1},\cong} & \overline{\nu}(e'(\Sigma_{g,1})) & \xrightarrow{\iota',\cong} & \iota'(\overline{\nu}(e(\Sigma_{g,1}))) & \xrightarrow{\subset} & N.
\end{array} \tag{18}$$

As in Construction 4, the choice of framings leads to boundary homeomorphisms

$$f = (h|)\cup(\gamma|\circ\iota^{-1})\colon \partial N_{e_\iota(\Sigma_{g,1})} \xrightarrow{\cong} M_{K,g},$$
$$f' = (h|)\cup(\gamma'|\circ\iota'^{-1})\colon \partial N_{e'_{\iota'}(\Sigma_{g,1})} \xrightarrow{\cong} M_{K,g}.$$

As in (17), using the diagram from (18) and the fact that F is a rel. boundary homeomorphism, we deduce that $F| = f'^{-1}\circ f$ and that F restricts to a rel. boundary homeomorphism

$$F|\colon N_{e_\iota(\Sigma_{g,1})}\to N_{e'_{\iota'}(\Sigma_{g,1})}.$$

We conclude that $(N_{e(\Sigma_{g,1})}, f)$ is equivalent to $(N_{e'(\Sigma_{g,1})}, f')$ in $\mathscr{V}^0_\lambda(M_{K,g})$.

This concludes the verification that the map Θ from Construction 4 is well defined.

Remark 5.3. From now on, we continue to use the notation $\Sigma := e(\Sigma_{g,1})$ and we omit the choice of an embedding $\iota\colon\overline{\nu}(\Sigma_{g,1})\hookrightarrow N$ from the notation since we have shown that $\Theta(e)$ is independent of the choice of embedding ι up to equivalence in $\mathscr{V}^0_\lambda(M_{K,g})$. In practice this means that we will simply write $\overline{\nu}(\Sigma)\subset N$. Since we omit ι from the notation, we also allow ourselves to think of (the inverse of) a good framing γ as giving an embedding

$$\gamma^{-1}\colon \Sigma_{g,1}\times D^2\hookrightarrow\overline{\nu}(\Sigma)\subset N.$$

Similarly, given a choice of such a good framing, we now write the homeomorphism from (15) as

$$f_\gamma\colon \partial N_\Sigma \to M_{K,g}, \qquad f_\gamma := (h|) \cup (\gamma|), \tag{19}$$

once again omitting ι from the notation. We sometimes also omit the choice of the framing γ from the notation, writing instead $\Theta(e) = (N_\Sigma, f)$.

Step (2)*: From manifolds to embeddings.* We set up some notation aimed towards proving that Θ is a bijection when the form λ is even, and that Θ is a bijection when λ is odd and the Kirby-Siebenmann is fixed. Set $\varepsilon := \operatorname{ks}(N)$ and write $\mathscr{V}^{0,\varepsilon}_\lambda(M_{K,g})$ for the subset of those manifolds in $\mathscr{V}^0_\lambda(M_{K,g})$ whose Kirby-Siebenmann invariant equals ε. Observe that by additivity of the Kirby-Siebenmann invariant (see e.g. [FNOP19, Theorem 8.2]), if λ is odd and $\Sigma \subset N$ is a $\mathbb{Z}$-surface, then $\operatorname{ks}(N_\Sigma) = \operatorname{ks}(N) = \varepsilon$, so the image of Θ lies in $\mathscr{V}^{0,\varepsilon}_\lambda(M_{K,g})$. The next proposition is the next step in the proof of Theorem 5.1.

Proposition 5.4. *Let N be a simply-connected 4-manifold with boundary $\partial N \cong S^3$, let $K \subset S^3$ be a knot and let (H, λ) be a nondegenerate Hermitian form.*

(1) If λ is even, then the map Θ from Construction 4 determines a bijection

$$\operatorname{Emb}^0_\lambda(\Sigma_{g,1}, N; K) \to \mathscr{V}^0_\lambda(M_{K,g}).$$

(2) If λ is odd, then the map Θ from Construction 4 determines a bijection

$$\operatorname{Emb}^0_\lambda(\Sigma_{g,1}, N; K) \to \mathscr{V}^{0,\varepsilon}_\lambda(M_{K,g}).$$

Proof. We construct an inverse Ψ to the assignment $\Theta\colon e \mapsto (N_{e(\Sigma_{g,1})}, f)$ from Construction 4; this will in fact take up most of the proof. Let (W, f) be a pair, where W is a 4-manifold with fundamental group $\pi_1(W) \cong \mathbb{Z}$, equivariant intersection form $\lambda_W \cong \lambda$ and, in the odd case, Kirby-Siebenmann invariant $\operatorname{ks}(W) = \varepsilon$, and $f\colon \partial W \cong M_{K,g}$ is a homeomorphism.

The inverse $\Psi(W, f)$ is an embedding $\Sigma_{g,1} \hookrightarrow N$ defined as follows. Glue $\Sigma_{g,1} \times D^2$ to W via the homeomorphism $f^{-1}|_{\Sigma_{g,1}\times S^1}$. This produces a 4-manifold $\widehat{W}$ with boundary $\partial\widehat{W} = (\partial W \setminus f^{-1}(\Sigma_{g,1} \times S^1)) \cup (\partial\Sigma_{g,1} \times D^2)$, together with an embedding

$$\times\{0\}\colon \Sigma_{g,1} \hookrightarrow \widehat{W} \qquad x \mapsto (x, 0) \in \Sigma_{g,1} \times \{0\} \subset \Sigma_{g,1} \times D^2.$$

Note for now that $\partial\Sigma_{g,1} \times \{0\} \subset \partial\widehat{W}$ bounds a genus g $\mathbb{Z}$-surface in $\widehat{W}$ (with exterior W).

We will use the homeomorphism $f\colon \partial W \to M_{K,g}$ to define a homeomorphism $f'\colon \partial\widehat{W} \to \partial N$ and then use Freedman's classification of compact simply-connected 4-manifolds with S^3 boundary, to deduce that this homeomorphism extends to a homeomorphism $F\colon \widehat{W} \to N$. We will then take our embedding to be

$$\Psi(W, f) := F \circ (\times\{0\})\colon \Sigma_{g,1} \hookrightarrow N.$$

The next paragraphs flesh out the details of this construction. Namely, firstly we build $f'\colon \partial\widehat{W} \to \partial N$ and secondly we argue it extends to a homeomorphism $F\colon \widehat{W} \to N$.

- Towards building this f', first observe that we get a natural homeomorphism $\partial\widehat{W} \to S^3$ as follows. Restricting f gives a homeomorphism $f|\colon \partial W \setminus f^{-1}(\Sigma_{g,1} \times S^1) \cong S^3 \setminus \nu(K)$. We can also define a homeomorphism

$$\vartheta\colon \partial\Sigma_{g,1} \times D^2 \to \overline{\nu}(K) \tag{20}$$

  which sends $\partial\Sigma_{g,1} \times \{\mathrm{pt}\}$ to λ_K and $\{\cdot\} \times \partial D^2$ to μ_K, where λ_K and μ_K denote the Seifert longitude and meridian of $K \subset S^3$. Specifying the image of these curves determines a map $\partial\Sigma_{g,1} \times \partial D^2 \to \overline{\nu}(K)$ (up to isotopy) which then extends to $\vartheta\colon \partial\Sigma_{g,1} \times D^2 \to \overline{\nu}(K)$ because μ_K bounds a disc in $\overline{\nu}(K)$. Note that ϑ is well defined up to isotopy. The union $f| \cup \vartheta\colon \partial\widehat{W} \to S^3$ is continuous on $\partial\Sigma_{g,1} \times \partial D^2$ because the gluing used to define $\widehat{W}$ was f^{-1}.

  Then $h^{-1} \circ (f| \cup \vartheta)$ gives the required homeomorphism

$$f' := h|^{-1} \circ (f| \cup \vartheta)\colon \partial\widehat{W} \to \partial N.$$

Further, we observe that $f'(\partial\Sigma_{g,1}) = K$.

- To prove that this homeomorphism extends to a homeomorphism $\widehat{W} \cong N$, we will appeal to Freedman's theorem that for every pair of simply-connected topological 4-manifolds with boundary homeomorphic to S^3, the same intersection form, and the same Kirby-Siebenmann invariant, every homeomorphism between the boundaries extends to a homeomorphism between the 4-manifolds [Fre82]. We check now that the hypotheses are satisfied.

  First, we argue that $\widehat{W}$ is simply-connected. The hypothesis that W lies in $\mathscr{V}^0_\lambda(M_{K,g})$ implies that there is an isomorphism $\widehat{\varphi}\colon \pi_1(W) \xrightarrow{\cong} \mathbb{Z}$ such that $\varphi = \widehat{\varphi}\circ\kappa$, where κ is the inclusion induced map $\pi_1(M_{K,g}) \to \pi_1(W)$ (see Definition 5). Since we required that $\varphi(\mu_K)$ generates $\mathbb{Z}$, we must have that $\kappa(\mu_K)$ generates $\pi_1(W) \cong \mathbb{Z}$. Since gluing $\Sigma_{g,1} \times D^2$ along $\Sigma_{g,1} \times S^1$ has the effect of killing $\kappa(\mu_K)$, we conclude that $\widehat{W}$ is simply-connected as claimed.

  Next we must show that $Q_{\widehat{W}}$ is isometric to Q_N. A Mayer-Vietoris argument establishes the isometry $Q_{\widehat{W}} \oplus (0)^{\oplus 2g} \cong Q_W$. It then follows from our assumption on the Hermitian form (H, λ) that we have the isometries
$$Q_{\widehat{W}} \oplus (0)^{\oplus 2g} \cong Q_W \cong \lambda_W(1) \cong \lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}.$$
  This implies that $Q_{\widehat{W}} \cong Q_N$ because both forms are nonsingular (indeed $\partial\widehat{W} \cong \partial N \cong S^3$).

  In the even case, we deduce that both $\widehat{W}$ and N are spin. In the odd case, using the additivity of the Kirby-Siebenmann invariant (see e.g. [FNOP19, Theorem 8.2]), we have $\operatorname{ks}(\widehat{W}) = \operatorname{ks}(W) = \varepsilon = \operatorname{ks}(N)$.

  Therefore $\widehat{W}$ and N are simply-connected topological 4-manifolds with boundary S^3, with the same intersection form and the same Kirby-Siebenmann invariant. Freedman's classification of simply-connected 4-manifolds with boundary S^3 now ensures that the homeomorphism $f'\colon \partial\widehat{W} \to \partial N$ extends to a homeomorphism $F\colon \widehat{W} \to N$ that induces the isometry $Q_{\widehat{W}} \cong Q_N$ and fits into the following commutative diagram

$$\begin{array}{ccccc} (\partial W \setminus f^{-1}(\Sigma_{g,1} \times S^1)) \cup (\partial\Sigma_{g,1} \times D^2) & \xrightarrow{=} & \partial\widehat{W} & \xrightarrow{\subset} & \widehat{W} \\ \downarrow{\scriptstyle h|^{-1}\circ(f|\cup\vartheta|)} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle F} \\ (\partial N \setminus \nu(K)) \cup \overline{\nu}(K) & \xrightarrow{=} & \partial N & \xrightarrow{\subset} & N. \end{array} \tag{21}$$

As mentioned above, we obtain an embedding as
$$\Psi(W, f) := \Big(e\colon \Sigma_{g,1} \xrightarrow{\times\{0\}} \widehat{W} \xrightarrow{F,\cong} N\Big). \tag{22}$$
This concludes the construction of our embedding $\Psi(W, f)$.

We must check that this construction gives rise to a map $\Psi\colon \mathscr{V}^0_\lambda(M_{K,g}) \to \operatorname{Emb}^0_\lambda(\Sigma_{g,1}, N; K)$. In other words, we verify that, up to homeomorphisms of N rel. boundary, the embedding e from (22) depends neither on the choice of isometry $Q_{\widehat{W}} \cong Q_N$ nor the choice of ϑ from (20) nor the homeomorphism $\widehat{W} \cong N$ extending our boundary homeomorphism nor on the homeomorphism rel. boundary type of (W, f).

- The precise embedding e depends on the homeomorphism $\widehat{W} \cong N$. This homeomorphism in turn depends on the choice of isometry $Q_{\widehat{W}} \cong Q_N$. However for any two choices of homeomorphisms $\widehat{W} \cong N$, the resulting embeddings are equivalent, as can be seen by composing one choice of homeomorphism with the inverse of the other. So the equivalence class of the surface $\Psi(W, f)$ does not depend on the choice of isometry $Q_{\widehat{W}} \cong Q_N$ nor on the choice of homeomorphism $\widehat{W} \cong N$ realizing this isometry.
- Next, we show that the definition is independent of the choice of $\vartheta\colon \partial\Sigma_{g,1} \times D^2 \to \overline{\nu}(K)$ within its isotopy class. If $\vartheta_0, \vartheta_1\colon \partial\Sigma_{g,1} \times D^2 \to \overline{\nu}(K)$ are isotopic, then so are the resulting homeomorphisms $f'_0 := (f| \cup \vartheta_0|), f'_1 := (f| \cup \vartheta_1|)\colon \partial\widehat{W} \to \partial N$ via an isotopy f'_s.

Claim. *There is an isotopy* $F_s\colon \widehat{W} \to N$ *extending* f'_s.

Proof. Pick a homeomorphism $F_0\colon \widehat{W} \to N$ extending f'_0; when we constructed $\Psi(W, f)$, we argued that such an F_0 exists. There are collars $\partial\widehat{W} \times [0,1]$ and $\partial N \times [0,1]$ such that $F_0|_{\partial\widehat{W}\times[0,1]} = f'_0 \times [0,1]$. Here it is understood that the boundaries of $\widehat{W}$ and N are respectively given by $\partial\widehat{W} \times \{0\}$ and $\partial N \times \{0\}$.

The idea is to implant the isotopy f'_s between f'_0, f'_1 in these collars in order to obtain an isotopy between F_0 and a homeomorphism F_1 that retricts to f'_1 on the boundary. To carry out this idea, consider the restriction

$$F_0|\colon \widehat{W} \setminus (\partial\widehat{W} \times [0,1]) \to N \setminus (\partial N \times [0,1]).$$

Define an isotopy of homeomorphisms between the collars via the formula

$$\begin{aligned} G_s\colon \partial\widehat{W} \times [0,1] &\to \partial N \times [0,1] \\ (x,t) &\mapsto (f'_{(1-t)s}(x), t). \end{aligned}$$

Since $G_s(x,1) = (f'_0(x), 1)$ for every s, we obtain the required isotopy as $F_s := G_s \cup F_0$. By construction F_i restricts to f'_i on the boundary for $i = 0,1$, thus concluding the proof of the claim. □

Thanks to the claim, we can use F_0 and F_1 to define the embeddings $e_0 := F_0 \circ (\times\{0\})$ and $e_1 := F_1 \circ (\times\{0\})$. This way, $F_1 \circ F_0^{-1}\colon N \to N$ is an equivalence rel. boundary between e_0 and e_1 so that the definition of Ψ is independent of the choice of ϑ within its isotopy class.

- Next we check the independence of the rel. boundary homeomorphism type of (W, f). If we have (W_1, f_1) and (W_2, f_2) that are equivalent rel. boundary, then there is a homeomorphism $\Phi\colon W_1 \to W_2$ that satisfies $f_2 \circ \Phi| = f_1$. This homeomorphism extends to $\widehat{\Phi} := \Phi \cup \mathrm{Id}_{\Sigma_{g,1}\times D^2}\colon \widehat{W_1} \to \widehat{W_2}$ and therefore to a homeomorphism $N \to N$ that is, by construction rel. boundary. A formal verification using this latter homeomorphism then shows that the embeddings $\Psi(W_1, f_1)$ and $\Psi(W_2, f_2)$ are equivalent rel. boundary.

Now we prove that the maps Θ and Ψ are mutually inverse.

- First we prove that $\Psi \circ \Theta = \mathrm{Id}$. Start with an embedding $e\colon \Sigma_{g,1} \hookrightarrow N$ and write $\Theta(e) = (N_{e(\Sigma_{g,1})}, f)$ with $f = (h|) \cup (\gamma|)\colon \partial N_{e(\Sigma_{g,1})} \to M_{K,g}$ the homeomorphism described in Construction 4. Then $\Psi(\Theta(e))$ is an embedding

$$\Sigma_{g,1} \xrightarrow{\times\{0\}} N_{e(\Sigma_{g,1})} \cup_f (\Sigma_{g,1} \times D^2) \xrightarrow{F,\cong} N.$$

We showed that the equivalence class of this embedding is independent of the homeomorphism F that extends f. It suffices to show that we can make choices so that $\Psi(\Theta(e))$ recovers e. This can be done explicitly as follows. Choose $\vartheta := h\circ\gamma^{-1}\colon \partial\Sigma_{g,1}\times D^2 \to \overline{\nu}(K)$. Then we have $f' = \mathrm{Id}_{\partial N\setminus\nu(K)} \cup (h^{-1}\circ(h\circ\gamma^{-1})) = \mathrm{Id}_{\partial N\setminus\nu(K)} \cup \gamma|^{-1}$ where the notation is as in (21) (with $W = N_{e(\Sigma_{g,1})}$). We already know an extension of f', namely $\mathrm{Id}_{N_{e(\Sigma_{g,1})}} \cup \gamma^{-1}$, which we take to be F. Thus $\Psi(\Theta(e)) = \gamma^{-1}|_{\Sigma_{g,1}\times\{0\}}\colon \Sigma_{g,1} \hookrightarrow N$ which, by definition of a normal bundle, agrees with the initial embedding e.
- Next we prove that $\Theta \circ \Psi = \mathrm{Id}$. This time we start with a pair (W, f) consisting of a 4-manifold W and a homeomorphism $f\colon \partial W \to M_{K,g}$. Then $\Psi(W, f)$ is represented by an embedding $e\colon \Sigma_{g,1} \xrightarrow{\times\{0\}} \widehat{W} \xrightarrow{F,\cong} N$. Recall that we write $h\colon \partial N \to S^3$ for our preferred homeomorphism and that by construction, on the boundaries, F restricts to

$$h|^{-1} \circ (f| \cup \vartheta)\colon \partial\widehat{W} \to \partial N$$

where (the isotopy class of) $\vartheta\colon \partial\Sigma_{g,1} \times D^2 \to \overline{\nu}(K)$ satisfies the properties listed below equation (20).

We frame $\Sigma_{g,1}\times\{0\}\subset\widehat{W}$ via the unique homeomorphism $\mathrm{fr}\colon\overline{\nu}(\Sigma_{g,1}\times\{0\})\to\Sigma_{g,1}\times D^2$ that makes the following diagram commute:

$$\begin{tikzcd} \overline{\nu}(\Sigma_{g,1}\times\{0\}) \arrow[rr, "\mathrm{fr}"] \arrow[dr, "\mathrm{incl}"'] & & \Sigma_{g,1}\times D^2 \arrow[dl, "\mathrm{incl}"] \\ & \widehat{W}=W\cup(\Sigma_{g,1}\times D^2). & \end{tikzcd}$$

We then frame $e(\Sigma_{g,1})\subset N$ via

$$\gamma:=\mathrm{fr}\circ F^{-1}|\colon\overline{\nu}(e(\Sigma_{g,1}))\cong\Sigma_{g,1}\times D^2.$$

This framing is good thanks to the definition of $\varphi\colon\pi_1(M_{K,g})\to\mathbb{Z}$ as the unique epimorphism that maps the meridian of K to 1 and the other generators to zero: indeed this implies that the curves on $\Sigma_{g,1}\times\{0\}$ are nullhomologous in W and therefore the same thing holds for $e(\Sigma_{g,1})\subset N$. It can be verified that this framing satisfies the condition from (13).

We then obtain $\Theta(\Psi(W,f))=(N_\Sigma:=N\setminus\nu(e(\Sigma_{g,1})),h|\cup\gamma|)$, where, as dictated by Construction 4, the boundary homeomorphism is $h|\cup\gamma|\colon\partial N_\Sigma\to M_{K,g}$. Here we are making use of the fact that up to equivalence, we can choose any framing in the definition of Θ.

We have to prove that $(N_\Sigma,h|\cup\gamma|)$ is homeomorphic rel. boundary to (W,f). We claim that the restriction of $F\colon\widehat{W}\to N$ gives the required homeomorphism. To see this, consider the following diagram

$$\begin{tikzcd} M_{K,g} \arrow[d, "="] & (\partial W\setminus f^{-1}(\Sigma_{g,1}\times S^1))\cup(f^{-1}(\Sigma_{g,1}\times S^1)) \arrow[l, "f,\cong"'] \arrow[r, "="] \arrow[d, "f':=(h|^{-1}\circ f|)\cup F|"] & \partial W \arrow[r, "\subset"] \arrow[d, "F|"] & W \arrow[r, "\subset"] \arrow[d, "F|"] & \widehat{W} \arrow[d, "F"] \\ M_{K,g} & (\partial N\setminus\nu(K))\cup(\partial\overline{\nu}(\Sigma)\setminus(\nu(\Sigma)\cap\partial N)) \arrow[l, "h|\cup\gamma|"'] \arrow[r, "="] & \partial N_\Sigma \arrow[r, "\subset"] & N_\Sigma \arrow[r, "\subset"] & N. \end{tikzcd}$$

The right two squares certainly commute. In the second-from-left square, we have just expanded out ∂W and ∂N_Σ, as well as written $F|$ explicitly on the regions where we have an explicit description from the construction of Ψ. So this square commutes.

It remains to argue that the left square commutes. By construction $F|_{\partial\widehat{W}}=f'=(h^{-1}\circ f|)\cup\vartheta)$. Thus on the knot exteriors, we have that $F|=h^{-1}\circ f|$ and so the left portion of the square commutes on the knot exteriors.

Now it remains to prove that $\gamma|\circ F|=f$. By definition of $\gamma=\mathrm{fr}\circ F^{-1}$, we must show that $\mathrm{fr}|=f|$ on $f^{-1}(\Sigma_{g,1}\times S^1)$. First note that fr has domain $\overline{\nu}(\Sigma_{g,1}\times\{0\})\subset\widehat{W}=W\cup(\Sigma_{g,1}\times D^2)$, so it appears we are attempting to compare maps which have different domains. However, the definition of $\widehat{W}$ identifies the portion of the boundary of $\overline{\nu}(\Sigma_{g,1})$ that we are interested in with $f^{-1}(\Sigma_{g,1}\times S^1)\subset\partial W$ via $f^{-1}|\circ\mathrm{fr}|$, so it makes sense to compare f on $f^{-1}(\Sigma_{g,1}\times S^1)$ with $\mathrm{fr}|$ on $\mathrm{fr}|^{-1}\circ f|_{f^{-1}(\Sigma_{g,1}\times S^1)}$. These maps are tautologically equal. Therefore the left hand side of the diagram commutes and this concludes the proof that $\Theta\circ\Psi=\mathrm{Id}$.

We have shown that Θ and Ψ are mutually inverse, and so both are bijections. This completes the proof of Proposition 5.4. □

Step (3)*: From embeddings to submanifolds.* Now we deduce a description of $\mathrm{Surf(g)}^0_\lambda(N,K)$ from Proposition 5.4. Note that $\mathrm{Surf(g)}^0_\lambda(N,K)$ arises as the orbit set

$$\mathrm{Surf(g)}^0_\lambda(N,K)=\mathrm{Emb}^0_\lambda(\Sigma_{g,1},N;K)/\operatorname{Homeo}^+(\Sigma_{g,1},\partial),$$

where the left action of $x\in\operatorname{Homeo}^+(\Sigma_{g,1},\partial)$ on $e\in\mathrm{Emb}^0_\lambda(\Sigma_{g,1},N;K)$ is defined by $x\cdot e=e\circ x^{-1}$. There is a surjective map $\mathrm{Emb}^0_\lambda(\Sigma_{g,1},N;K)\to\mathrm{Surf(g)}^0_\lambda(N,K)$ that maps an embedding $e\colon\Sigma_{g,1}\hookrightarrow N$ onto its image. One then verifies that this map descends to a bijection on the orbit set.

Next, we note that $\operatorname{Homeo}^+(\Sigma_g, \partial)$ acts on the sets $\mathscr{V}^0_\lambda(M_{K,g})$ and $\mathscr{V}^{0,\varepsilon}_\lambda(M_{K,g})$ as follows. A rel. boundary homeomorphism $x\colon \Sigma_{g,1} \to \Sigma_{g,1}$ extends to a self homeomorphism x' of $\Sigma_{g,1} \times S^1$ by defining $x'(s,\theta) = (x(s),\theta)$. Then extend x' by the identity over E_K; in total one obtains a self homeomorphism x'' of $M_{K,g}$. The required action is now by postcomposition: for (W,f) representing an element of $\mathscr{V}^0_\lambda(M_{K,g})$ or $\mathscr{V}^{0,\varepsilon}_\lambda(M_{K,g})$, define $x \cdot (W,f) := (W, x'' \circ f)$.

The following proposition is now a relatively straightforward consequence of Proposition 5.4.

Proposition 5.5. *Let N be a simply-connected 4-manifold with boundary $\partial N \cong S^3$, let $K \subset S^3$ be a knot and let (H,λ) be a nondegenerate Hermitian form.*

(1) If λ is even, then the map Θ from Construction 4 descends to a bijection

$$\operatorname{Surf(g)}^0_\lambda(N,K) \to \mathscr{V}^0_\lambda(M_{K,g})/\operatorname{Homeo}^+(\Sigma_{g,1},\partial).$$

(2) If λ is odd, then the map Θ from Construction 4 descends to a bijection

$$\operatorname{Surf(g)}^0_\lambda(N,K) \to \mathscr{V}^{0,\varepsilon}_\lambda(M_{K,g})/\operatorname{Homeo}^+(\Sigma_{g,1},\partial).$$

Proof. Thanks to Proposition 5.4, it is enough to check that $\Theta(x\cdot e) = x\cdot\Theta(e)$ for $x \in \operatorname{Homeo}(\Sigma_{g,1},\partial)$ and $e\colon \Sigma_{g,1} \hookrightarrow N$ an embedding representing an element of $\operatorname{Emb}^0_\lambda(\Sigma_{g,1}, N; K)$. By definition of Θ, we know that $\Theta(x\cdot e)$ is $(N_{e(x^{-1}(\Sigma_{g,1}))}, f_{e\circ x^{-1}})$ and $x\cdot\Theta(e) = (N_{e(\Sigma_{g,1})}, x''\circ f_e)$ where the $f_e, f_{e\circ x^{-1}}$ are homeomorphisms from the boundaries of these surface exteriors to $M_{K,g}$ that can be constructed, up to equivalence rel. boundary, using any choice of good framing; recall Construction 4. In what follows, we will make choices of framings so that the pairs $\Theta(x\cdot e) = (N_{e(x^{-1}(\Sigma_{g,1}))}, f_{e\circ x^{-1}})$ and $x\cdot\Theta(e) = (N_{e(\Sigma_{g,1})}, x''\circ f_e)$ are equivalent rel. boundary.

Pick a good framing $\gamma\colon \overline{\nu}(e(\Sigma_{g,1})) \cong \Sigma_{g,1}\times D^2$ so that $\Theta(e) = (N_{e(\Sigma_{g,1})}, f_e) = (N_{e(\Sigma_{g,1})}, h| \cup \gamma|)$. Since $\gamma^{-1}\colon \Sigma_{g,1}\times D^2 \hookrightarrow N$ satisfies $\gamma^{-1}|_{\Sigma_{g,1}\times\{0\}} = e$, we deduce that $\gamma^{-1}\circ(x^{-1}\times \operatorname{Id}_{D^2})$ gives an embedding of the normal bundle of $e\circ x^{-1}$. We can therefore choose the inverse $\gamma_{e\circ x} := (x\times\operatorname{Id}_{D^2})\circ\gamma$ as a good framing for the embedding $e\circ x^{-1}$. Using this choice of good framing to construct $f_{e\circ x^{-1}}$, we have $\Theta(e\circ x^{-1}) = (N_{e\circ x^{-1}(\Sigma_{g,1})}, h| \cup ((x\times\operatorname{Id}_{D^2})\circ\gamma|))$. Using these observations and the fact that x is rel. boundary, we obtain

$$\begin{aligned}\Theta(x\cdot e) = \Theta(e\circ x^{-1}) &= (N_{e\circ x^{-1}(\Sigma_{g,1})}, h| \cup ((x\times\operatorname{Id}_{D^2})\circ\gamma|)) \\ &= (N_{e\circ x^{-1}(\Sigma_{g,1})}, x''\circ(h|\cup\gamma|)) = x\cdot(N_{e(\Sigma_{g,1})}, f_e) = x\cdot\Theta(e).\end{aligned}$$

This proves that the pairs $\Theta(x\cdot e) = (N_{e(x^{-1}(\Sigma_{g,1}))}, f_{e\circ x^{-1}})$ and $x\cdot\Theta(e) = (N_{e(\Sigma_{g,1})}, f_e)$ are equivalent rel. boundary and thus concludes the proof of the proposition. □

We now deduce our description of the surface set, thus proving the main result of this section.

Proof of Theorem 5.1. Proposition 5.5 shows that if λ is even then the map Θ from Construction 4 induces a bijection

$$\operatorname{Surf(g)}^0_\lambda(N,K) \to \mathscr{V}^0_\lambda(M_{K,g})/\operatorname{Homeo}^+(\Sigma_{g,1},\partial)$$

while if λ is odd, for $\varepsilon := \operatorname{ks}(N)$, the map Θ induces a bijection

$$\operatorname{Surf(g)}^0_\lambda(N,K) \to \mathscr{V}^{0,\varepsilon}_\lambda(M_{K,g})/\operatorname{Homeo}^+(\Sigma_{g,1},\partial).$$

Thus the theorem will follows once we show that the map $b\colon V^0_\lambda(M_{K,g}) \to \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_{M_{K,g}})/\operatorname{Aut}(\lambda)$ from Construction 1 intertwines the $\operatorname{Homeo}^+(\Sigma_{g,1},\partial)$-actions, i.e. satisfies $b_{x\cdot(W,f)} = x\cdot b_{(W,f)}$ for every $x\in\operatorname{Homeo}^+(\Sigma_{g,1},\partial)$ and for every pair (W,f) representing an element of $V^0_\lambda(M_{K,g})$.

This follows formally from the definitions of the actions: on the one hand, for some isometry $F\colon \lambda\cong\lambda_W$, we have $b_{x\cdot(W,f)} = b_{(W,x''\circ f)} = x''_*\circ f_*\circ D_W\circ\partial F$; on the other hand, we have $x\cdot b_{(W,f)}$ is $x\cdot(f_*\circ D_W\circ\partial F)$ and this gives the same result. This concludes the proof of Theorem 5.1. □

5.2. **Surfaces with boundary up to equivalence.** The study of surfaces up to equivalence (instead of equivalence rel. boundary) presents additional challenges: while there is still a map $\Theta\colon \mathrm{Emb}_\lambda(\Sigma_{g,1}, N; K) \to \mathscr{V}_\lambda(M_{K,g})$, the proof of Proposition 5.4 (in which we constructed an inverse Ψ of Θ) breaks down because if W and W' are homeomorphic $\mathbb{Z}$-fillings of $M_{K,g}$, it is unclear whether we can always find a homeomorphism $W \cup (\Sigma_{g,1} \times D^2) \cong W' \cup (\Sigma_{g,1} \times D^2)$. We nevertheless obtain the following result.

Theorem 5.6. *Let N be a simply-connected 4-manifold with boundary $\partial N \cong S^3$, let K be a knot such that every isometry of Bl_K is realised by an orientation-preserving homeomorphism $E_K \to E_K$ and let (H, λ) be a nondegenerate Hermitian form over $\mathbb{Z}[t^{\pm 1}]$. The following assertions are equivalent:*

(1) the Hermitian form λ presents $M_{K,g}$ and $\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}$;
(2) up to equivalence, there exists a unique genus g surface $\Sigma \subset N$ with boundary K and whose exterior has equivariant intersection form λ, i.e. $|\operatorname{Surf(g)}_\lambda(N, K)| = 1$.

Proof. We already proved the fact that the second statement implies the first, so we focus on the converse. We can apply Theorem 5.1 to deduce that $\operatorname{Surf(g)}^0_\lambda(N, K)$ is nonempty, this implies in particular that $\operatorname{Surf(g)}_\lambda(N, K)$ is nonempty. Since this set is nonempty, we assert that the hypothesis on K ensures we can apply [CP20, Theorem 1.3] to deduce that $|\operatorname{Surf(g)}_\lambda(N, K)| = 1$.

In contrast to Theorem 5.6, the statement of [CP20, Theorem 1.3] contains the additional condition that the orientation-preserving homeomorphism $f\colon E_K \to E_K$ be the identity on ∂E_K. We show that this assumption is superfluous, so that we can apply [CP20, Theorem 1.3] without assuming that $f|_{\partial E_K} = \mathrm{Id}_{\partial E_K}$.

First, note that since f realises an isometry of Bl_K, it is understood that f preserves a basepoint x_0 and satisfies $f([\mu_K]) = [\mu_K]$, where $[\mu_K] \in \pi_1(E_K, x_0)$ is the based homotopy class of a meridian of K. An application of the Gordon-Luecke theorem [GL89] now implies that $f|_{\partial E_K}$ is isotopic to $\mathrm{Id}_{\partial E_K}$; this isotopy can be assumed to be basepoint preserving by [FM12, page 57]. Implanting this basepoint preserving isotopy in a collar neighborhood of ∂E_K implies that f itself is basepoint preserving isotopic to a homeomorphism $E_K \to E_K$ that restricts to the identity on ∂E_K. This completes the proof that the extra assumption in the statement of [CP20, Theorem 1.3] can be assumed to hold without loss of generality. □

5.3. **Closed surfaces.** We now turn our attention to closed $\mathbb{Z}$-surfaces. Let X be a closed simply-connected 4-manifold and let $\Sigma \subset X$ be a closed $\mathbb{Z}$-surface with genus g, whose normal bundle we frame as in the case with boundary. With this framing, we can now identify the boundary of $X_\Sigma := X \setminus \nu(\Sigma)$ as

$$\partial X_\Sigma \cong \Sigma_g \times S^1.$$

Two such surfaces Σ and Σ' are *equivalent* if there exists an orientation-preserving homeomorphism $(X, \Sigma) \cong (X, \Sigma')$. Again as in the case of surfaces with boundary, X_Σ has ribbon boundary and $H_1(\Sigma_g \times S^1; \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z}^{2g}$ is torsion. Additionally, note that the equivariant intersection form λ_{X_Σ} of a surface exterior X_Σ must present $\Sigma_g \times S^1$.

Definition 12. For a nondegenerate Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ presenting $\Sigma_g \times S^1$, set

$$\operatorname{Surf(g)}_\lambda(X) := \{\mathbb{Z}\text{-surface } \Sigma \subset X \text{ with } \lambda_{X_\Sigma} \cong \lambda\}/ \text{ equivalence}.$$

As for $\mathbb{Z}$-surfaces with nonempty boundary, in order for $\operatorname{Surf(g)}_\lambda(X)$ to be nonempty it is additionally necessary that $\lambda(1) \cong Q_X \oplus (0)^{\oplus 2g}$. It was proved in [CP20, Theorem 1.4] that whenever $\operatorname{Surf(g)}_\lambda(X)$ is nonempty, it contains a single element. We improve this statement to include an existence clause.

Theorem 5.7. *Let X be a closed simply-connected 4-manifold. Given a nondegenerate Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$, the following assertions are equivalent:*

(1) the Hermitian form λ presents $\Sigma_g \times S^1$ and $\lambda(1) \cong Q_X \oplus (0)^{\oplus 2g}$;
(2) there exists a unique (up to equivalence) genus g $\mathbb{Z}$-surface $\Sigma \subset X$ whose exterior has equivariant intersection form λ; i.e. $|\operatorname{Surf(g)}_\lambda(X)| = 1$.

Proof. Use $U \subset S^3$ to denote the unknot and use N to denote the simply-connected 4-manifold with boundary S^3 obtained from X by removing a small open 4-ball. Note that $M_{U,g} = \Sigma_g \times S^1$ and that $Q_N = Q_X$. Since the Blanchfield form of U is trivial, Theorem 5.6 applies; this shows us that item (1) in Theorem 5.7 is equivalent to the existence of a unique (up to equivalence) genus g surface $\Sigma \subset N$ with boundary U and equivariant intersection form λ, in other words:

$$|\operatorname{Surf(g)}_\lambda(N,U)| = 1.$$

It remains to prove that this is equivalent to $|\operatorname{Surf(g)}_\lambda(X)| = 1$. We will prove this by demonstrating that the sets $\operatorname{Surf(g)}_\lambda(X)$ and $\operatorname{Surf(g)}_\lambda(N,U)$ are in bijective correspondence.

Given a closed genus g $\mathbb{Z}$-surface $\Sigma \subset X$, a $\mathbb{Z}$-surface $\mathring{\Sigma} \subset N$ with boundary U can be obtained by removing a $(\mathring{D}^4, \mathring{D}^2)$-pair from (X, Σ). Because $\lambda_{X_\Sigma} \cong \lambda_{N_{\mathring{\Sigma}}}$ and because an equivalence from Σ to Σ' in X, restricts to an equivalence from $\mathring{\Sigma}$ to $\mathring{\Sigma}'$ in N, this puncturing operation gives rise to a map

$$\operatorname{Surf(g)}_\lambda(X) \to \operatorname{Surf(g)}_\lambda(N,U). \tag{23}$$

The surjectivity of this map is straightforward: a pair (N, Σ) where Σ has boundary U can be capped off by a pair (D^4, D^2) to get a closed surface in X, and so it remains to prove that the aforementioned assignment is injective.

We give two arguments for this fact. The first argument is elementary in the sense that it does not rely on any heavy machinery. Suppose that two surfaces $\mathring{\Sigma}$ and $\mathring{\Sigma}'$ are identified in $\operatorname{Surf(g)}_\lambda(N,U)$, i.e. there is a homeomorphism of pairs $F\colon (N, \mathring{\Sigma}) \to (N, \mathring{\Sigma}')$. Observe that $F|_\partial$ does not have to be the identity homeomorphism, but it does have to take U to U. In a moment we will construct a homeomorphism $G\colon D^4 \to D^4$ such that $G|_\partial = F|_\partial$ and such that $G(D^2) = D^2$ where D^2 is the standard disc for U in D^4. Then, capping off $(N, \mathring{\Sigma})$ with (D^4, D^2) and capping off $(N, \mathring{\Sigma}')$ with (D^4, D^2), we obtain closed surfaces (X, Σ) and (X, Σ') mapping to our $(N, \mathring{\Sigma})$ and $(N, \mathring{\Sigma}')$. Since $F \cup -G$ gives a homeomorphism (X, Σ) to (X, Σ'), we see that the map is indeed injective.

We now construct the homeomorphism $G\colon D^4 \to D^4$ such that $G|_\partial = F|_\partial$ and $G(D^2) = D^2$ where D^2 is the standard disc for U in D^4. Assume that in the standard radial height function on D^4, the standard disc D^2 is given by $D^2_U \cup (U \times [\frac{1}{2}, 1])$, where D^2_U is the standard disc for U in $S^3 \times \{\frac{1}{2}\}$. Let $G'\colon D^4 \to D^4$ be the homeomorphism obtained by applying Alexander's trick to $F|_\partial$. Then $G'(D^2)$ is given by $F|_\partial(D^2_U) \cup (U \times [\frac{1}{2}, 1])$. Since every genus 0 Seifert surface for U is isotopic rel. boundary to D^2_U in $S^3 \times \{\frac{1}{2}\}$ [BZ67, page 241], there is in particular a homeomorphism $H\colon D^4 \to D^4$ which is the identity on the boundary and which sends $G'(D^2)$ to D^2. Take $G := H \circ G'$.

Alternatively, one can prove injectivity using an argument mentioned in [CP20, Subsection 1.3] and again in [CP20, proof of Theorem 5.11] but we outline it here for completeness. If $\mathring{\Sigma}$ and $\mathring{\Sigma}'$ are equivalent $\mathbb{Z}$-surfaces in N, that have boundary U and whose exteriors have isometric equivariant intersection pairings, then (N, Σ) and $(N, \mathring{\Sigma})$ are equivalent rel. boundary thanks to [CP20, Theorem 1.3]. That Σ and Σ' are equivalent in X now follows by capping off with a (D^4, D^2)-pair and extending the homeomorphism by the identity.

Thus, the map from (23) is a bijection and, as we explained above, this concludes the proof. □

5.4. **Problems and open questions.** We conclude with some problems in the theory of $\mathbb{Z}$-surfaces, both in the closed case and in the case with boundary. In what follows, we set

$$\mathscr{H}_2 := \begin{pmatrix} 0 & t-1 \\ t^{-1}-1 & 0 \end{pmatrix}.$$

We start with closed surfaces in closed manifolds where the statements are a little cleaner.

Problem 1. *Fix a closed, simply-connected 4-manifold X. Characterise the nondegenerate Hermitian forms (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ that arise as λ_{X_Σ} where $\Sigma \subset X$ is a closed $\mathbb{Z}$-surface of genus g.*

It is known that if λ is as in Problem 1, then it must present $\Sigma_g \times S^1$, that $\lambda(1) \cong Q_X \oplus (0)^{\oplus 2g}$ and that $\lambda \oplus \mathscr{H}_2^{\oplus n} \cong Q_X \oplus \mathscr{H}_2^{\oplus(g+n)}$ for some $n \geq 0$. The necessity of the first two conditions was mentioned in Subsection 5.3 while the necessity of third was proved in [CP20, Proposition 1.6].

Here is what is known about Problem 1:

- if $X = S^4$ and $g \neq 1, 2$, then $\lambda \cong \mathscr{H}_2^{\oplus g}$ [CP20, Section 7];
- if $b_2(X) \geq |\sigma(X)| + 6$, then [Sun15, Theorem 7.2] implies that $\lambda \cong Q_X \oplus \mathscr{H}_2^{\oplus g}$.

This leads to the following question, a positive answer to which would solve Problem 1.

Question 1. *Let X be a closed simply-connected 4-manifold and let (H, λ) be a nondegenerate Hermitian form over $\mathbb{Z}[t^{\pm 1}]$. Is it the case that if λ presents $\Sigma_g \times S^1$, $\lambda(1) \cong Q_X \oplus (0)^{\oplus 2g}$ and $\lambda \oplus \mathscr{H}_2^{\oplus n} \cong Q_X \oplus \mathscr{H}_2^{\oplus(g+n)}$ for some $n \geq 0$, then $\lambda \cong Q_X \oplus \mathscr{H}_2^{\oplus g}$?*

If the answer to Question 1 were positive, then using Theorem 5.7 one could completely classify closed $\mathbb{Z}$-surfaces in closed simply-connected 4-manifolds: for every $g \geq 0$, in a closed simply-connected 4-manifold X, there would exist a unique $\mathbb{Z}$-surface of genus g in X up to equivalence.

Next, we discuss the analogous (but more challenging) problem for surfaces with boundary.

Problem 2. *Fix a simply-connected 4-manifold N with boundary S^3. Characterise the nondegenerate Hermitian forms (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ that arise as λ_{N_Σ} where $\Sigma \subset N$ is a $\mathbb{Z}$-surface of genus g with boundary a fixed knot K. For brevity, we call such forms (N, K, g)*-realisable.

It is known that if λ is (N, K, g)-realisable, then it must present $M_{K,g}$, satisfy $\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}$ as well as $\lambda \oplus \mathscr{H}_2^{\oplus n} \cong Q_N \oplus \mathscr{H}_2^{\oplus(g+n)}$ for some $n \geq 0$. The necessity of the first two conditions was mentioned in Subsection 5.1 while the necessity of third was proved in [CP20, Proposition 1.6].

Less appears to be known about Problem 2 than about Problem 1: to the best of our knowledge, the only result on the topic is that when $N = D^4$, $g \neq 1, 2$ and K has Alexander polynomial one, then $\lambda \cong \mathscr{H}_2^{\oplus g}$ [CP20, Section 7].

We conclude by listing consequences of a solution to Problem 2.

(1) Using Theorem 5.1, a solution to Problem 2 would make it possible to fully determine the classification of properly embedded $\mathbb{Z}$-surfaces in a simply-connected 4-manifold N with boundary S^3 up to equivalence rel. boundary: for every $g \geq 0$, there would be precisely one $\mathbb{Z}$-surface of genus g in N with boundary K for every element of $\operatorname{Aut}(\mathrm{Bl}_K)/\operatorname{Aut}(\lambda)$, where λ ranges across all (N, K, g)-realisable forms.
(2) If one dropped the rel. boundary condition, then one might conjecture that for every $g \geq 0$, in a simply-connected 4-manifold N with boundary S^3, there is precisely one $\mathbb{Z}$-surface of genus g with boundary K for every element of $\operatorname{Aut}(\partial\lambda)/\big(\operatorname{Aut}(\lambda) \times \operatorname{Homeo}^+(E_K, \partial)\big)$, where λ ranges across (N, K, g)-realisable forms. If the conjecture were true, then a solution to Problem 2 would provide a complete description of the set of properly embedded $\mathbb{Z}$-surfaces in a simply-connected 4-manifold N with boundary S^3, up to equivalence.

6. Ubiquitous exotica

In this section we demonstrate the failure of our topological classification to hold in the smooth setting. In Subsection 6.1 we set up some preliminaries we will require about Stein 4-manifolds and corks. In Subsection 6.2 we give the proofs of Theorems 1.12 and 1.13 from the introduction. In this section, all manifolds and embeddings are understood to be smooth.

6.1. **Background on Stein structures and corks.** We will be concerned with arranging that certain compact 4-manifolds with boundary admit a Stein structure. The unfamiliar reader can think of this as a particularly nice symplectic structure. Abusively, we will say that any smooth 4-manifold which admits a Stein structure is Stein. The reason for this sudden foray into geometry is to take advantage of restrictions on the genera of smoothly embedded surfaces representing certain

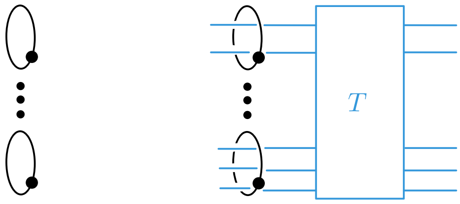

FIGURE 1. The left hand side shows a handle diagram for a boundary connected sum of $S^1 \times D^3$. On the right hand side, the tangle diagram T satisfies the conventions of a front diagram.

homology classes in Stein manifolds. These restrictions will aid us in demonstrating that two 4-manifolds are not diffeomorphic. In this section, we will recall both a combinatorial condition for ensuring that a 4-manifold is Stein and the restrictions on smooth representatives of certain homology classes in Stein manifolds. We use the conventions and setup of [Gom98] throughout.

We begin by recalling a criterion to ensure that a handle diagram with an unique 0-handle and no 3 or 4-handles describes a Stein 4-manifold. Recall that we can describe $\natural_{i=1}^r S^1 \times B^3$ using the dotted circle notation for 1-handles as in the left frame of Figure 1. It is not hard to show that any link in $\#_{i=1}^r S^1 \times S^2$ can be isotoped into the position shown in the right frame of Figure 1, where inside the tangle marked T we require that the diagram meet the conventions of a front diagram for the standard contact structure on S^3. For details on front diagrams, see [Etn03]; stated briefly this amounts to isotoping the diagram so that all vertical tangencies are replaced by cusps and so that at each crossing the more negatively sloped strand goes over. We note that front diagrams require oriented links; we can choose orientations on our 2-handle attaching spheres arbitrarily, since orienting the link does not affect the 4-manifold. Thus any handle diagram with a unique 0-handle and no 3 or 4-handles can be isotoped into the form of the right frame of Figure 1; we say that such a diagram is in *Gompf standard form*.

For a diagram in Gompf standard form, let L_i^T denote the tangle diagram obtained by restricting the i^{th} component L_i of the diagram of L to T. For a diagram in Gompf standard form, the *Thurston-Bennequin number* $TB(L_i)$ of L_i is defined as

$$TB(L_i^D) = w(L_i^T) - c(L_i^T)$$

where $w(L_i^T)$ denotes the writhe of the tangle and $c(L_i^T)$ denotes the number of left cusps.

In this setup, the following criterion is helpful to prove that handlebodies are Stein.

Theorem 6.1 ([Eli90, Gom98], see also Theorem 11.2.2 of [GS99])**.** *A smooth 4-manifold X with boundary is Stein if and only if it admits a handle diagram in Gompf standard form such that the framing f_i on each 2-handle attaching curve L_i has $f_i = TB(L_i) - 1$.*

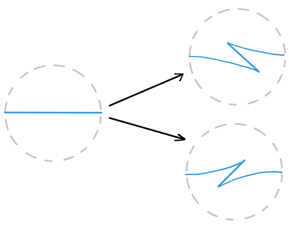

FIGURE 2. Stabilising a front diagram.

Remark 6.2. The 'if' direction of the Theorem 6.1 holds under the weaker hypothesis that each 2-handle attaching curve L_i has $f_i \leq TB(L_i) - 1$. To see this, observe that any 2-handle L_i can be locally isotoped via the *stabilisations* demonstrated in Figure 2 and observe that stabilisation preserves the condition on T and lowers the Thurston-Bennequin number of L_i by one. The claim now follows since we can stabilise any 2-handle in a diagram Gompf standard form to lower its Thurston-Bennequin number without changing the smooth 4-manifold described.

We will also make use of the following special case of the adjunction inequality for Stein manifolds.

Theorem 6.3 ([LM97])**.** *In a Stein manifold X, any homology class $\alpha \in H_2(X)$ with $\alpha \cdot \alpha = -1$ cannot be represented by a smoothly embedded sphere.*

Proof. The proof can be deduced by combining [LM97, Theorem 3.2] with [Bru96, FM97]; further exposition can be found in [AM97, Theorems 1.2 and 1.3]. □

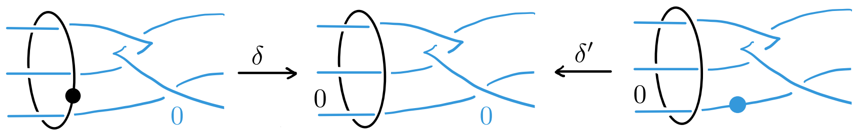

FIGURE 3. Two fillings of the boundary of the Akbulut cork, with boundary homeomorphism $\delta'^{-1} \circ \delta$. Here and throughout the rest of the paper, all handle diagrams drawn in this horizontal format should be braid closed.

In order to handily construct pairs of homeomorphic 4-manifolds, we will make use of *cork twisting*. Define C to be the contractible 4-manifold in the left frame of Figure 3, which is commonly refered to as the *Akbulut cork*. Observe that ∂C admits another contractible filing C' given by the right frame of Figure 3, and that there is a natural homeomorphism $\tau := (\delta')^{-1} \circ \delta \colon \partial C \to \partial C'$ demonstrated in the figure. Using the work of Freedman [Fre82], the homeomorphism τ extends to a homeomorphism $T \colon C \to C'$. As a result, for any 4-manifold W with $\iota \colon C \hookrightarrow W$, one can construct a new 4-manifold $W' := W \smallsetminus \iota(C) \cup_{(\iota|_\partial) \circ \tau^{-1}} C'$ and, combining the identity homeomorphism $\mathrm{Id}_{W \smallsetminus \iota(C)}$ with T, one sees that W and W' are homeomorphic.

Historically, the literature has been concerned with two types of exotic phenomena. If smooth 4-manifolds X, X' with boundary admit a homeomorphism $F \colon X \to X'$ but no diffeomorphism $G \colon X \to X'$ such that $G|_\partial$ is isotopic to $F|_\partial$, we call X and X' *relatively exotic*. If smooth 4-manifolds X, X' admit a homeomorphism $F \colon X \to X'$ but no diffeomorphism $G \colon X \to X'$ we call X and X' *absolutely exotic*. It is easier to build relatively exotic pairs in practice. Fortunately, work of Akbulut and Ruberman shows that all relative exotica contains absolute exotica.

Theorem 6.4 (Theorem A of [AR16])**.** *Let M and M' be smooth 4-manifolds and let $F \colon M \to M'$ be a homeomorphism whose restriction to the boundary is a diffeomorphism that does not extend to a diffeomorphism $M \to M'$. Then M (resp. M') contains a smooth codimension 0 submanifold V (resp. V') which is orientation-preserving homotopy equivalent to M (resp. M') such that V is homeomorphic but not diffeomorphic to V'.*

Note that V and V' necessarily have nonempty boundaries since they are codimension zero submanifolds of manifolds with boundary. We remark that Akbulut-Ruberman's theorem is only stated when M is diffeomorphic to M' (hence by applying a reference identification, they can in fact just call both manifolds M). However their proof works verbatim when M and M' are just homeomorphic smooth manifolds, which is the hypothesis we take above. Additionally, Akbulut-Ruberman do not include the emphasis that the homotopy equivalence is orientation preserving, but this follows immediately from their proof.

6.2. **Proof of Theorems 1.12 and 1.13.** We prove Theorem 1.12 from the introduction.

Theorem 6.5. *For every Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ there exists a pair of smooth 4-manifolds M and M' with ribbon boundary and fundamental group $\mathbb{Z}$, such that:*

1. *there is a homeomorphism $F \colon M \to M'$;*
2. *F induces an isometry $\lambda_M \cong \lambda_{M'}$, and both forms are isometric to λ;*

(3) there is no diffeomorphism from M to M'.

In other words, every Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ is exotically realisable.

Proof. Let $A(t)$ be a matrix representing the given form λ, so that $A(1)$ is an integer valued matrix. Choose any framed link $L = \cup L_i \subset S^3$ with linking matrix $A(1)$ and let M_1 be the 4-manifold obtained from D^4 by attaching $A(1)_{ii}$-framed 2-handles to D^4 along L_i. Let M_2 be the 4-manifold obtained from M_1 by attaching a 1-handle (which we will think of as removing the tubular neighborhood of a trivial disc for an unknot split from L). Thus $\pi_1(M_2) \cong \mathbb{Z}$ and both the integer valued intersection form Q_{M_2} and the equivariant intersection form λ_{M_2} are represented by a matrix for $\lambda(1)$.

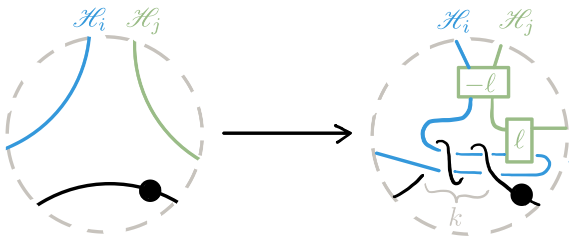

FIGURE 4. Arbitrary Hermitian forms can be realised as equivariant intersection forms by repeatedly performing the following local move, which we illustrate for $k = 2$.

Now we will modify the handle diagram of M_2 in a way which will preserve the fundamental group and integer valued intersection form, but will result in an M_3 with equivariant intersection form $\lambda_{M_3} \cong \lambda$. For pairs i, j with $i < j$, for each monomial ℓt^k in the polynomial $A(t)_{ij}$, perform the local modification exhibited in Figure 4. Observe (for later use) that this move does not change the framed link type of the link of attaching spheres of 2-handles. Furthermore, the modification does not change the fundamental group or the integer valued intersection form of M_2. We exhibit in Figure 5 what the cover looks like locally after the modification.

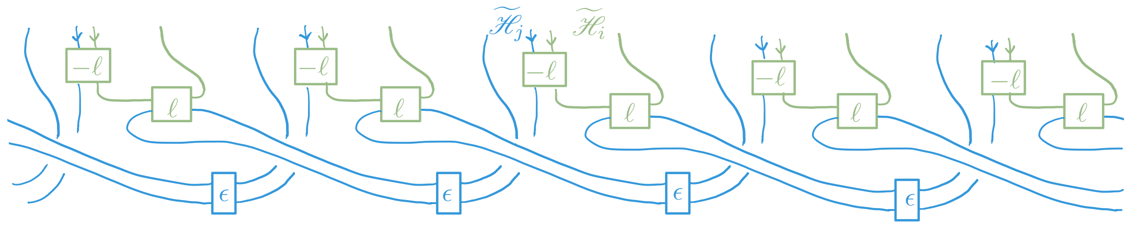

FIGURE 5. A local picture of the cover after our local modification with $k = 2$. When $k > 0$ the twist parameter ϵ is $1 - k$, when $k < 0$ it is $-k - 1$.

Recall from Remark 2.2 that for elements $[\widetilde{a}], [\widetilde{b}] \in H_2(M_2, \mathbb{Z}[t^{\pm 1}])$ the equivariant intersection form satisfies

$$\lambda_{M_2}([\widetilde{b}], [\widetilde{a}]) = \sum_k (\widetilde{a} \cdot_{M_3^\infty} t^k \widetilde{b}) t^{-k}.$$

Thus we see that after each iteration of the local move we have that $\lambda_{M_2'}(t)_{ij} = \lambda_{M_2}(t)_{ij} - \ell + \ell t^k$ and $\lambda_{M_2'}(t)_{ji} = \lambda_{M_2}(t)_{ji} - \ell + \ell t^{-k}$.

For pairs $i = j$, for each monomial ℓt^k with $k > 0$ in the polynomial $A(t)_{ii}$, again perform the local modification in Figure 4. In this case, one finds that

$$\lambda_{M_2'}(t)_{ii} = \lambda_{M_2}(t)_{ii} - 2\ell + \ell t^k + \ell t^{-k}. \tag{24}$$

The non-constant terms of (24) are straightforward to deduce. The constant term is computed by considering a parallel of $\mathscr{H}_i$ downstairs which is 0-framed in the modification region, lifting the framing curve into the cover, and then computing the linking of the lift of the framing with $\widetilde{\mathscr{H}_i}$.

Once these modifications are complete, we obtain a 4-manifold M_3 with λ_{M_3} agreeing with λ everywhere except *a priori* on the constant terms of each $A(t)_{ij}$. Observe however that since these local modifications do not change the integer valued intersection form $\lambda(1)$, we have that λ_{M_3} must also agree with λ on the constant terms of each $A(t)_{ij}$. Thus, when we are finished, we have a smooth 4-manifold M_3 with no 3-handles, $\pi_1(M_3) \cong \mathbb{Z}$ and $\lambda_{M_3} \cong \lambda$.

FIGURE 6. The knot K in $S^1 \times S^2$. A handle diagram for the 4-manifold X is obtained from this diagram by dotting the black unknot and attaching a 0-framed 2-handle to K.

Next we will modify the 2-handles of our handle diagram $\mathscr{H}$ of M_3 to get a Stein 4-manifold M_4 with the same fundamental group and equivariant intersection form as M_3. We will do this by getting the handle diagram into a form where we can apply Eliashberg's Theorem 6.1, which requires arranging that each 2-handle has a suitably large Thurston-Bennequin number. To begin, isotope $\mathscr{H}$ into Gompf standard form, so that we think of the 2-handles of $\mathscr{H}$ as a Legendrian link in the standard tight contact structure on $S^1 \times S^2$. If any of the 2-handle attaching curves do not have any cusps, stabilise once so that they do. Let $A_3(t)$ be the equivariant linking matrix of $\mathscr{H}$; note that $A_3(t) = A(t)$ is a matrix representing the equivariant intersection form λ. Let K be the knot in $S^1 \times S^2$ exhibited in Figure 6. Observe that if we use K to describe a 4-manifold X via attaching a 0-framed 2-handle to $S^1 \times B^3$ along K, then $\pi_1(X) \cong \mathbb{Z}$ and the equivariant intersection form λ_X is represented by the size one matrix (0). Observe further that K has a Legendrian representative $\mathscr{K}$ (illustrated in Figure 6) in the standard tight contact structure on $S^1 \times S^2$ with $\mathrm{TB}(\mathscr{K}) = 1$. In our handle diagram $\mathscr{H}$ of M_3, let $\mathring{K}$ be a copy of K in $S^1 \times S^2$ which is split from all of the 2-handles of $\mathscr{H}$, as depicted in the left frame of Figure 7.

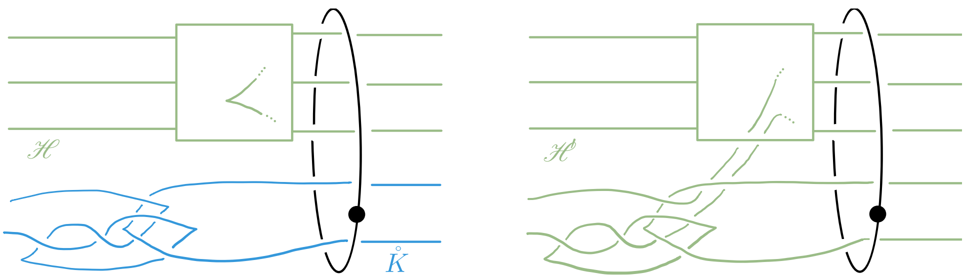

FIGURE 7. The connect sum band can be taken with a sufficiently positive slope that choosing it to pass under any strands in the tangle T causes the diagram to remain in Gompf standard form.

Now for any handle $\mathscr{H}_i$ of $\mathscr{H}$ with $A_3(1)_{ii} > \mathrm{TB}(\mathscr{H}_i) - 2$ form $\mathscr{H}_i'$ by taking the connected sum of $\mathscr{H}_i$ with a split copy of $\mathring{K}$ in the manner depicted in Figure 7. Frame $\mathscr{H}_i'$ using the same diagrammatic framing instruction that was used to frame $\mathscr{H}_i$. One computes readily from the right frame of Figure 7 that $\mathrm{TB}(\mathscr{H}_i') = \mathrm{TB}(\mathscr{H}_i) + 1$. Repeat this process until $A_3(1)_{ii} \leq \mathrm{TB}(\mathscr{H}_i) - 2$ for all 2-handles. Let M_4 be the resulting 4-manifold. Then M_4 is Stein by Theorem 6.1 and

Remark 6.2. Further, since X contributes neither to the equivariant intersection form nor to π_1, we have that M_4 has the same equivariant intersection form and π_1 as M_3. We record (for later use) the observation that the link in S^3 consisting of the attaching spheres of the 2-handles is unchanged by these modifications; one can see this by ignoring the 1-handle in Figure 7 and doing a bit of isotopy.

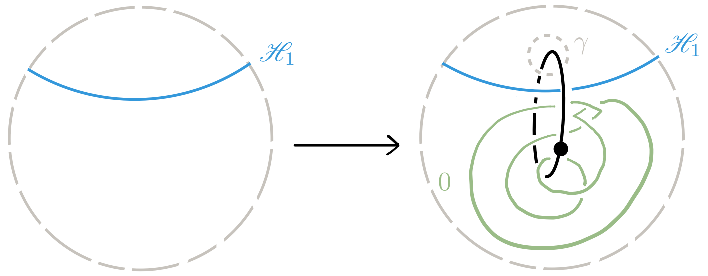

FIGURE 8. The local modification performed on the handle $\mathscr{H}_1$ of the manifold M_3.

Now we will make a final modification to M_4 to get a 4-manifold $M_5 =: M$ which we can cork twist to get M'. Choose any 2-handle, without loss of generality we choose $\mathscr{H}_1$, and perform the local modification described in Figure 8; the resulting 4-manifold is our M.

One can readily check that this local modification does not impact π_1 or the equivariant intersection form. Further, this local diagram can be readily converted to Gompf standard form, (see the blue and green handles of Figure 9) where we have $A_3(1)_{ii} \leq \mathrm{TB}(\mathscr{H}_i) - 1$ for all 2-handles, hence M is Stein. By construction, M contains a copy of the Akbulut cork C. Because M has no 3-handles, M has ribbon boundary.

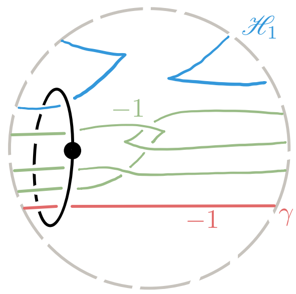

FIGURE 9. A handle diagram for the manifold W in Gompf standard form.

Now define M' to be the 4-manifold obtained from M by twisting C. Since there is a homeomorphism $T: C \to C$ extending the twist homeomorphism $\tau: \partial C \to \partial C$, there is a natural homeomorphism $F: M \to M'$; let f denote the restriction $f: \partial M \to \partial M'$.

It remains to show that M and M' are not diffeomorphic. We will begin by showing the relative statement, i.e. there is no diffeomorphism $G: M \to M'$ such that $G|_\partial = f$. It would be convenient if at this point we could distinguish M and M' directly by showing that one is Stein and one is not. Unfortunately, both are Stein. So instead we will consider auxiliary manifolds W and W' constructed as follows. Suppose for a contradiction that there were such a diffeomorphism G. Construct a 4-manifold W by attaching a (-1)-framed 2-handle to M along γ (where γ is the curve in ∂M marked in Figure 8) and a second 4-manifold W' from M' by attaching a 2-handle to M' with attaching sphere and framing given by $(f(\gamma), -1)$. [1] Notice that the image under f

[1]The (-1)-framing instruction for $f(\gamma)$ requires a diagram of $f(\gamma)$ in $\partial M'$. Because f is a dot-zero homeomorphism, we can use the exact same diagram as we used for γ in ∂M.

of a (-1)-framing curve for γ is in fact a (-1)-framing curve for $f(\gamma)$. The diffeomorphism G extends to give a diffeomorphism $\widehat{G}\colon W \to W'$. In Figure 9, we have exhibited the natural handle diagram for W in Gompf standard form, from which Theorem 6.1 implies that W admits a Stein structure.

We will finish showing that f does not extend by demonstrating that W' does not admit any Stein structure, thus W cannot be diffeomorphic to W'. Since W' is obtained from W by reversing the dot and the zero on the handles of C, $f(\gamma)$ is just a meridian of a 2-handle of M'. Thus the final 2-handle of W' is attached along a curve which bounds a disc in M', implying that there is a (-1)-framed sphere embedded in W'. But the adjunction inequality for Stein manifolds (recall Theorem 6.3) indicates that no 4-manifold which admits a Stein structure can contain an embedded sphere with self-intersection -1. Hence, W is not diffeomorphic to W', thus there cannot be a diffeomorphism $G\colon M \to M'$ extending f.

Now we would like to extend this to a statement about absolute exotica. To do so, we apply Theorem 6.4 to our M, M', and f to produce a pair of smooth 4-manifolds V and V' (both of which have nonempty boundary) which are homeomorphic but not diffeomorphic. Since V and V' are orientation-preserving homotopy equivalent to M and M' respectively, the equivariant intersection forms λ_V and $\lambda_{V'}$ are also isometric to λ, and both V and V' have fundamental group $\mathbb{Z}$. Since V and V' are homeomorphic, so are ∂V and $\partial V'$. □

Next, we prove Theorem 1.13 from the introduction. If one wants to show that any 2-handlebody N with boundary S^3 contains a pair of exotic $\mathbb{Z}$-discs one can run the same proof, where in the first line $\mathscr{H}'$ is chosen to be a handle diagram for N; this was mentioned in Remark 1.14.

Theorem 6.6. *For every Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ such that $\lambda(1)$ is realised as the intersection form of a smooth simply-connected 4-dimensional 2-handlebody N with $\partial N \cong S^3$, there exists a pair of smooth $\mathbb{Z}$-discs D and D' in N with the same boundary and the following properties:*

(1) the equivariant intersection forms λ_{N_D} and $\lambda_{N_{D'}}$ are isometric to λ;
(2) D is topologically isotopic to D' rel. boundary;
(3) D is not smoothly equivalent to D' rel. boundary.

Proof. Let $\mathscr{H}'$ be a handle diagram for a 2-handlebody with S^3 boundary and such that Q_N isometric to $\lambda(1)$. Let D be the standard disc for a local unknot in ∂N, and as usual let N_D be its exterior, which has handle diagram $\mathscr{H} := \mathscr{H}' \cup 1\text{−handle}$.

Akin to the proof of Theorem 1.12, we will now modify the linking of the handles of $\mathscr{H}$ to get a Stein manifold with equivariant intersection form λ. However, we also want to do so in such a way that the manifold presented by $\mathscr{H}$ is still $N_{D'}$ for some smooth disc D' properly embedded in N.

We claim that if we modify only the linking of the 2-handles with the 1-handle, and not the linking of the 2-handles with each other nor the knot type or framing of the 2-handles, we will have that $\mathscr{H}$ presents such an $N_{D'}$. To prove the claim, first observe that X is the exterior of a disc in N if and only if N can be obtained from X by adding on a single 2-handle. Observe that adding a 0-framed 2-handle to the meridian of a 1-handle in dotted circle notation allows us to erase both the new 2-handle and the 1-handle. Thus, if our modifications only change the way the 2-handles of N link the new one-handle, we will still have the property that after a single 2-handle addition we obtain N, thus our manifold is the exterior of a disc embedded in N. This concludes the proof of the claim.

Now observe that all of the modifications we performed in the proof of Theorem 1.12 to get from M_2 to M_4 modified only the linking of the 2-handles with the 1-handle, and not the linking of the 2-handles with each other nor the knot type or framing of the 2-handles. Thus we can again perform those same modifications to our $\mathscr{H}$ to obtain a smooth $\mathbb{Z}$-disc D' properly embedded in N such that the resulting $\mathscr{H}$ is a handle diagram for $N_{D'}$ in Gompf standard form satisfying

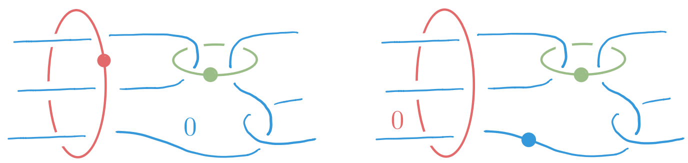

FIGURE 10. In both frames the red and blue handles give a nonstandard handle diagram for D^4, and in both frames the green knot $K \subset S^3$ bounds a disc disjoint from the 1-handle; these are our two discs Σ and Σ' for K in D^4. The handle diagrams here present D^4_Σ and $D^4_{\Sigma'}$.

Eliashberg's criteria and such that the equivariant intersection form of the exterior is $\lambda_{N_{D'}} \cong \lambda$. Notice in particular that $N_{D'}$ is Stein.

Now let Σ and Σ' be the pair of slice discs for K in D^4 exhibited in Figure 10. These discs were constructed following the techniques of [Hay20]. It is elementary to check from the exhibited handle diagrams that both discs have $\pi_1(D^4_\Sigma) = \pi_1(D^4_{\Sigma'}) = \mathbb{Z}$ and are ribbon. It is then a consequence of [CP21, Theorem 1.2] that Σ is topologically isotopic to Σ' rel. boundary.

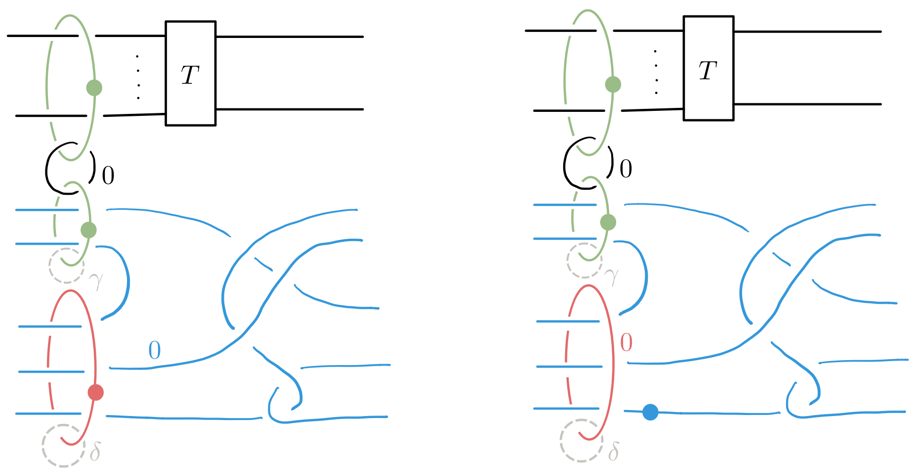

FIGURE 11. The left frame gives a handle diagram for N_R, and the right for $N_{R'}$. The top black 2-handles and tangle T represent the handle diagram of $N_{D'}$ in Gompf standard form which we already constructed.

We will construct discs R and R' in N by taking the boundary connect sum of pairs $(N, R) := (N, D')\natural(D^4, \Sigma)$ and $(N, R') := (N, D')\natural(D^4, \Sigma')$. We demonstrate natural handle decompositions for N_R and $N_{R'}$ in Figure 11. It is straightforward to confirm that $\pi_1(N_R) \cong \pi_1(N_{R'}) \cong \mathbb{Z}$. Further, since Σ is topologically isotopic to Σ' in D^4 rel. boundary, R is topologically isotopic in N to R' rel. boundary. Since Σ and Σ' are $\mathbb{Z}$-discs in D^4, their exteriors are aspherical [CP20, Lemma 2.1] and so both λ_{N_Σ} and $\lambda_{N_{\Sigma'}}$ are trivial. It is then not hard to show that band summing D' with Σ or Σ' does not change the equivariant intersection form, so $\lambda_{N_R} \cong \lambda_{N_{R'}} \cong \lambda_{N_{D'}}$.

It remains to show that R is not smoothly equivalent to R' rel. boundary. If R were equivalent to R' rel. boundary then there would be a diffeomorphism $F \colon N_R \to N_{R'}$ which is the identity on the boundary. Let γ and δ be the curves in $\partial N_R = \partial N_{R'}$ demonstrated in Figure 11, and let W (similarly W') be formed from N_R by attaching (-1)-framed 2-handles along γ and δ.

If a diffeomorphism $F \colon N_R \to N_{R'}$ extending the identity exists, then W is diffeomorphic to W'. Observe that W' does not admit a Stein structure, because the 2-handle along δ naturally introduces a (-1)-framed 2-sphere embedded in W', which violates the Stein adjunction inequality in Theorem 6.3. However, W admits the handle decomposition given in Figure 12, which is in

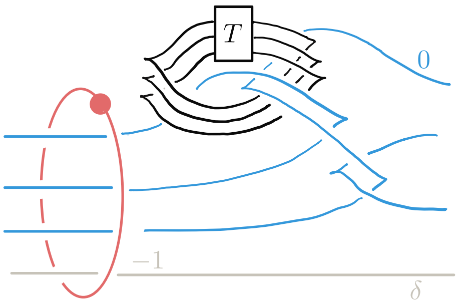

FIGURE 12. The black 2-handles here have both framing and TB one less than they had in Figure 11; since we had already arranged that the tangle T in Figure 11 satisfied the framing criteria of Theorem 6.1, this handle diagram also satisfies the criteria.

Gompf standard form, so Theorem 6.1 ensures that W admits a Stein structure. Therefore W is not diffeomorphic to W', so there can be no such F, so R is not smoothly equivalent to R' rel. boundary.

□

Remark 6.7. In the above proof, R is smoothly isotopic to R' *not* rel. boundary, because Σ is smoothly isotopic to Σ' not rel. boundary. If we wanted to produce R and R' which are not smoothly isotopic (without a boundary condition), we could have instead used a Σ and Σ' which are not isotopic rel. boundary and run a similar argument. Such Σ and Σ' are produced in [Hay20]; we have not pursued this here because the diagrams are somewhat more complicated.

7. Nontrivial boundary automorphism set

We prove that there are examples of pairs (Y, φ) for which the set of 4-manifolds with fixed boundary Y and equivariant intersection form, up to homeomorphism, can have arbitrarily large cardinality. This was alluded to in Example 1.5. The main step in this process is to find examples of Hermitian forms (H, λ) for which $\mathrm{Aut}(\partial\lambda)/\mathrm{Aut}(\lambda)$ can become arbitrarily large. The most direct way to achieve this is when H has rank 1. Indeed, in this case, $\mathrm{Aut}(\partial\lambda)/\mathrm{Aut}(\lambda)$ can be described in terms of certain units of $\mathbb{Z}[t^{\pm 1}]/\lambda$, as we now make precise.

Given a ring R with involution $x \mapsto \overline{x}$, the group of *unitary units* $U(R)$ refers to those $u \in R$ such that $u\overline{u} = 1$. For example, when $R = \mathbb{Z}[t^{\pm 1}]$, all units are unitary and are of the form $\pm t^k$ with $k \in \mathbb{Z}$.

In what follows, we make no distinction between rank one Hermitian forms and symmetric Laurent polynomials. The next lemma follows by unwinding the definition of $\mathrm{Aut}(\partial\lambda)$; see also [CP20, Remark 1.16].

Lemma 7.1. *If $\lambda \in \mathbb{Z}[t^{\pm 1}]$ is a symmetric Laurent polynomial, then*
$$\mathrm{Aut}(\partial\lambda)/\mathrm{Aut}(\lambda) = U(\mathbb{Z}[t^{\pm 1}]/\lambda)/U(\mathbb{Z}[t^{\pm 1}]).$$

Given a symmetric Laurent polynomial $P \in \mathbb{Z}[t^{\pm 1}]$, use n_P to denote the number of ways P can be written as an unordered product ab of symmetric polynomials $a, b \in \mathbb{Z}[t^{\pm 1}]$ such that there exists $x, y \in \mathbb{Z}[t^{\pm 1}]$ with $ax + by = 1$, where the factorisations ab and $(-a)(-b)$ are deemed equal.

Lemma 7.2. *If $P \in \mathbb{Z}[t^{\pm 1}]$ is a symmetric Laurent polynomial, then $U(\mathbb{Z}[t^{\pm 1}]/2P)/U(\mathbb{Z}[t^{\pm 1}])$ contains at least n_P elements.*

Proof. A first verification shows that if P factorises as $P = ab$ where $a, b \in \mathbb{Z}[t^{\pm 1}]$ are symmetric polynomials and satisfy $ax + by = 1$, then
$$\Phi(a, b) := -ax + by$$

is a unitary unit in $\mathbb{Z}[t^{\pm 1}]/2P$, i.e. belongs to $U(\mathbb{Z}[t^{\pm 1}]/2P)$:

$$\begin{aligned}(-ax+by)\overline{(-ax+by)} &= a\overline{a}x\overline{x} + b\overline{b}y\overline{y} - ax\overline{b}\overline{y} - \overline{a}\overline{x}by = a\overline{a}x\overline{x} + b\overline{b}y\overline{y} - ab(x\overline{y} + \overline{x}y) \\ &\equiv a\overline{a}x\overline{x} + b\overline{b}y\overline{y} + ab(x\overline{y} + \overline{x}y) = (ax+by)\overline{(ax+by)} = 1.\end{aligned}$$

It can also be verified that $\Phi(a,b)$ depends neither on the ordering of a,b nor on the choice of x,y. The former check is immediate from the definition of Φ because $-1 \in U(\mathbb{Z}[t^{\pm 1}])$. We verify that the assignment does not depend on the choice of x,y. Assume that $ax+by = 1 = ax'+by'$ for $x,x',y,y' \in \mathbb{Z}[t^{\pm 1}]$. We deduce that $ax' = 1 = ax \bmod b$ and $by' = 1 = by \bmod a$. But now $x' \equiv (ax)x' = x(ax') = x \bmod b$ and similarly $y' = y \bmod a$ so that $x' = x+kb$ and $y' = y+\ell a$ for $k,l \in \mathbb{Z}[t^{\pm 1}]$. Expanding $ax'+by' = 1$, it follows that $k = -l$. Therefore

$$-ax'+by' = -a(x+kb) + b(y-ka) \equiv -ax+by.$$

We will prove that if $\Phi(a,b) = v \cdot \Phi(a',b')$ for some unit $v \in U(\mathbb{Z}[t^{\pm 1}])$, then $(a,b) = \pm(a',b')$ or $(a,b) = \pm(b',a')$. It then follows that for any two ways (a,b) and (a',b') of factorising P, distinct up to sign and up to reordering, the resulting elements $\Phi(a,b)$ and $\Phi(a',b')$ are distinct in $U(\mathbb{Z}[t^{\pm 1}]/2P)/U(\mathbb{Z}[t^{\pm 1}])$, from which the proposition follows.

Assume that $x,x',y,y' \in \mathbb{Z}[t^{\pm 1}]$ are such that $ax+by = 1 = a'x'+b'y'$ and $-ax+by = -a'x'+b'y' \bmod 2P$. Add $2ax+2a'x'v$ to both sides of the congruence $-ax+by = v(-a'x'+b'y') \bmod 2P$. Using that $ax+by = 1$ and $a'x'+b'y' = 1$, we obtain the congruence

$$2ax + v = 2a'x'v + 1 \bmod 2P. \tag{25}$$

Similarly, add $-2by+2a'x'v$ to both sides of $-ax+by = v(-a'x'+b'y') \bmod 2P$. Using that $ax+by = 1$ and $a'x'+b'y' = 1$, we obtain the equation

$$-2by + v = 2a'x'v - 1 \bmod 2P. \tag{26}$$

We deduce from the previous two equations that $v+1$ and $v-1$ are divisible by 2. Since $v = \pm t^k$, we deduce that $\pm t^k \pm 1$ is divisible by 2 and so $v = \pm 1$.

First, we treat the case where the unit is $v = 1$.

Claim 1. *We have* (i) *a divides a', and* (ii) *a' divides a.*

Proof. As $v = 1$, (25) implies that $2ax = 2a'x' \bmod 2P$. Writing $2P = 2ab$, and simplifying the 2s, we deduce that a divides $a'x'$. Similarly, writing $2P = 2a'b'$, and simplifying the 2s, we deduce that a' divides ax. Next, multiply the equations $1 = ax+by$ (resp. $1 = a'x'+b'y'$) by a (resp. a') to obtain

$$a = a^2x + aby$$
$$a' = {a'}^2x' + a'b'y'.$$

Since a' divides ax and $ab = P = a'b'$, it follows that a' divides a. The same reasoning with the second equation shows that a divides a'. This concludes the proof of the claim. ☐

Using the claim we have $a = ua'$ for some unit u; this unit is necessarily symmetric since both a and a' are symmetric. It follows that $a'b' = ab = ua'b$ with $u = \pm 1$. We deduce $b' = ub$ and therefore $b = b'/u$. Thus $(a,b) = u \cdot (a',b')$ as required, in the case $v = 1$.

Next, we treat the case where the unit is $v = -1$.

Claim 2. *We have* (i) *b divides a', and* (ii) *a' divides b.*

Proof. As $v = -1$, (26) implies that $-2by = 2a'x' \bmod 2P$. Writing $2P = 2ab$, and simplifying the 2s, we deduce that b divides $a'x'$. Similarly, writing $2P = 2a'b'$, and simplifying the 2s, we deduce that a' divides by. Next, multiply the equations $1 = ax+by$ (resp. $1 = a'x'+b'y'$) by b (resp. a') to obtain

$$b = abx + b^2y$$
$$a' = {a'}^2x' + a'b'y'.$$

Since a' divides by and $ab = P = a'b'$, it follows that a' divides b. The same reasoning with the second equation shows that b divides a'. This concludes the proof of the claim. $\square$

Using the claim we have $b = ua'$ for some unit u; this unit is necessarily symmetric since both b and a' are symmetric. It follows that $a'b' = ab = uaa'$ with $u = \pm 1$. We deduce $b' = ua$ and therefore $a = b'/u$. Thus $(a, b) = u \cdot (b', a')$ as required, in the case that $v = -1$. This completes the proof that $\Phi(a, b) = v \cdot \Phi(a', b')$ implies $(a, b) = \pm(a', b')$ or $(a, b) = \pm(b', a')$, which completes the proof of the proposition. $\square$

Over $\mathbb{Z}$, it is not difficult to show that if N is an integer that can be factored as a product of n distinct primes, then $U(\mathbb{Z}/N)/U(\mathbb{Z})$ contains precisely 2^{n-1} elements. Using Lemma 7.2, the next example shows that a similar lower bound (which is not in general sharp) holds over $\mathbb{Z}[t^{\pm 1}]$.

Example 7.3. The reader can check that if P is an integer than can be factored as a product $p_1 \cdots p_n$ of n distinct primes, then $n_P = 2^{n-1}$. Lemma 7.2 implies that $U(\mathbb{Z}[t^{\pm 1}]/2P)/U(\mathbb{Z}[t^{\pm 1}])$ contains at least 2^{n-1} elements.

Remark 7.4. In order to produce examples, there is no need to restrict P an integer. Take $P = q_1 \cdots q_n$, where the q_i are symmetric Laurent polynomials such that for every i, j, there exists $x, y \in \mathbb{Z}[t^{\pm 1}]$ with $q_i x + q_j y = 1$. The latter condition implies, via a straightforward induction on n, that there exists such x, y for any pair of polynomials $q_{i_1} \cdots q_{i_k}$ and $q_{i_{k+1}} \cdots q_{i_n}$ with $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$ obtained from factoring P. Then by applying Φ we can obtain examples of P such that $U(\mathbb{Z}[t^{\pm 1}]/2P)/U(\mathbb{Z}[t^{\pm 1}])$ has cardinality at least 2^{n-1}. However, this level of generality is not strictly necessary, as Example 7.3, in which P is an integer, suffices to prove Proposition 7.5 below.

We now prove the main result of this section that was mentioned in Example 1.5 from the introduction: there are examples of pairs (Y, φ) for which the set of 4-manifolds with fixed boundary Y and equivariant intersection form, up to homeomorphism, can have arbitrarily large cardinality.

Proposition 7.5. *For every $m \geq 0$, there is a pair (Y, φ) and a Hermitian form (H, λ) so that $\mathscr{V}^0_\lambda(Y)$ and $\mathscr{V}_\lambda(Y)$ have at least m elements.*

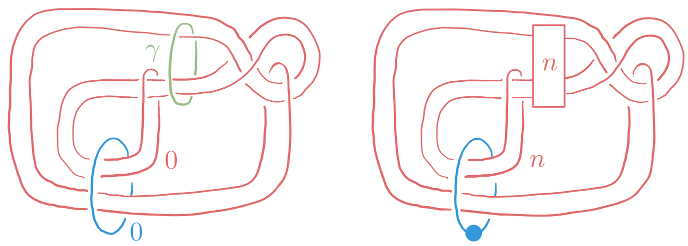

FIGURE 13. Left frame: the complement of γ is a hyperbolic 3-manifold Z with trivial mapping class group. Right frame: This 4-manifold W_n has ribbon boundary, $\pi_1(W_n) \cong \mathbb{Z}$, equivariant intersection form (n) and, for n sufficiently large, boundary ∂W_n with trivial mapping class group.

Proof. Since the cardinality of $\mathscr{V}^0_\lambda(Y)$ is greater than that of $\mathscr{V}_\lambda(Y)$, it suffices to prove that the latter set can be made arbitrarily large. However since proof involving $\mathscr{V}^0_\lambda(Y)$ is substantially less demanding, we include it as a quick warm up.

Set $\lambda := 2P$ where P is an integer than can be factored as a product $p_1 \cdots p_k$ of k distinct primes with $2^{k-1} \geq m$. Example 7.3 and Proposition 7.2 imply that $U(\mathbb{Z}[t^{\pm 1}]/\lambda)/U(\mathbb{Z}[t^{\pm 1}])$ has at least 2^{k-1} elements. By Proposition 7.1, this means that $\operatorname{Aut}(\partial\lambda)/\operatorname{Aut}(\lambda)$ has at least 2^{k-1} elements. As in the proof of Theorem 6.5, construct a smooth 4-manifold W with $\pi_1(W) \cong \mathbb{Z}$, ribbon boundary, and equivariant intersection form λ. In our setting, where $\lambda := 2P$, the manifold produced will be $X_\lambda(U)\natural(S^1 \times D^3)$, where $X_\lambda(U)$ is the manifold obtained by attaching a λ-framed 2-handle to D^4 along the unknot U. Let Y' be the boundary of this 4-manifold and let $\varphi\colon \pi_1(Y') \to \pi_1(W) \cong \mathbb{Z}$ be the inclusion induced map. Since λ presents Y', Theorem 1.1 implies that $\mathscr{V}^0_\lambda(Y')$ has at least $2^{k-1} \geq m$ elements, as required.

We now turn to the statement involving $\mathscr{V}_\lambda(Y)$.

Claim. *There is an integer $N > 0$ so that for any $n > N$, there exists a smooth 4-manifold W_n with $\pi_1(W_n) \cong \mathbb{Z}$, ribbon boundary, equivariant intersection form (n) and such that ∂W_n has trivial mapping class group.*

Proof. Let L be the 3-component link in the left frame of Figure 13 and let Z be the 3-manifold obtained from L by 0-surgering both the red and blue components, and removing a tubular neighborhood of the green component γ. Using verified computations in Snappy inside of Sage, we find that Z is hyperbolic and has trivial mapping class group.[2] By Thurston's hyperbolic Dehn surgery theorem [Thu02, Theorem 5.8.2], there exists $N > 0$ such that for $n > N$, the manifold Z_n obtained by $-1/n$ filling γ is hyperbolic and has trivial symmetry group; for the mapping class group part of this statement, see for example [DHL15, Lemma 2.2].

Let W_n be the 4-manifold described in the right frame of Figure 13 and observe that $\partial W_n \cong Z_n$. It is not difficult to verify that W_n has ribbon boundary, $\pi_1(W_n) \cong \mathbb{Z}$ and equivariant intersection form (n). This concludes the proof of the claim. □

We conclude the proof of the proposition. Fix $m \geq 0$ and choose an integer P such that

- P can be factored as a product $p_1 \cdots p_k$ of k distinct primes with $2^{k-1} \geq m$.
- $2P > N$ where N is as in the claim.

Since $2P > N$, the claim implies that $Y := \partial W_{2P}$ has trivial mapping class group. The proof is now concluded as in the warm up, but we spell out the details. As we already mentioned, W_{2P} has equivariant intersection form $\lambda := 2P$. Example 7.3 and Proposition 7.2 imply that $U(\mathbb{Z}[t^{\pm 1}]/\lambda)/U(\mathbb{Z}[t^{\pm 1}])$ has at least 2^{k-1} elements. By Proposition 7.1, this means that $\operatorname{Aut}(\partial\lambda)/\operatorname{Aut}(\lambda)$ has at least 2^{k-1} elements. Since Y has trivial mapping class group, either of Theorem 1.1 or Theorem 1.2 implies that $\mathscr{V}_\lambda(Y) = \mathscr{V}^0_\lambda(Y)$ has at least $2^{k-1} \geq m$ elements. □

References

[Akb91] Selman Akbulut. A fake compact contractible 4-manifold. *J. Differential Geom.*, 33(2):335–356, 1991.

[AM97] Selman Akbulut and Rostislav Matveyev. Exotic structures and adjunction inequality. *Turkish J. Math.*, 21(1):47–53, 1997.

[AP08] Anar Akhmedov and Doug Park. Exotic smooth structures on small 4-manifolds. *Invent. Math.*, 173(1):209–223, 2008.

[AP10] Anar Akhmedov and Doug Park. Exotic smooth structures on small 4-manifolds with odd signatures. *Invent. Math.*, 181(3):577–603, 2010.

[AR16] Selman Akbulut and Daniel Ruberman. Absolutely exotic compact 4-manifolds. *Comment. Math. Helv.*, 91(1):1–19, 2016.

[BF14] Maciej Borodzik and Stefan Friedl. On the algebraic unknotting number. *Trans. London Math. Soc.*, 1(1):57–84, 2014.

[BF15] Maciej Borodzik and Stefan Friedl. The unknotting number and classical invariants, I. *Algebr. Geom. Topol.*, 15(1):85–135, 2015.

[BKK+21] Stefan Behrens, Boldizár Kalmaár, Min Hoon Kim, Mark Powell, and Arunima Ray. *The disc embedding theorem.* Oxford University press. 2021.

[BL92] Steven Boyer and Daniel Lines. Conway potential functions for links in **Q**-homology 3-spheres. *Proc. Edinburgh Math. Soc. (2)*, 35(1):53–69, 1992.

[2]Transcripts of the computation are available at [Pic22].

[Boy86] Steven Boyer. Simply-connected 4-manifolds with a given boundary. *Trans. Amer. Math. Soc.*, 298(1):331–357, 1986.

[Boy93] Steven Boyer. Realization of simply-connected 4-manifolds with a given boundary. *Comment. Math. Helv.*, 68(1):20–47, 1993.

[Bru96] Rogier Brussee. The canonical class and the C^∞ properties of Kähler surfaces. *New York J. Math.*, 2:103–146, 1996.

[BZ67] Gerhard Burde and Heiner Zieschang. Neuwirthsche Knoten und Flächenabbildungen. *Abh. Math. Sem. Univ. Hamburg*, 31:239–246, 1967.

[CCPS21a] Anthony Conway, Diarmuid Crowley, Mark Powell, and Joerg Sixt. Simply connected manifolds with large homotopy stable classes. 2021. https://arxiv.org/abs/2109.00654.

[CCPS21b] Anthony Conway, Diarmuid Crowley, Mark Powell, and Joerg Sixt. Stably diffeomorphic manifolds and modified surgery obstructions. 2021. https://arxiv.org/abs/2109.05632.

[CF13] Jae Choon Cha and Stefan Friedl. Twisted torsion invariants and link concordance. *Forum Math.*, 25(3):471–504, 2013.

[Cha74] Thomas Chapman. Topological invariance of Whitehead torsion. *Amer. J. Math.*, 96:488–497, 1974.

[CN20] Anthony Conway and Matthias Nagel. Stably slice disks of links. *J. Topol.*, 13(3):1261–1301, 2020.

[Con22] Anthony Conway. Homotopy ribbon discs with a fixed group. 2022. https://arxiv.org/abs/2201.04465.

[CP20] Anthony Conway and Mark Powell. Embedded surfaces with infinite cyclic knot group. 2020. https://arxiv.org/abs/2009.13461.

[CP21] Anthony Conway and Mark Powell. Characterisation of homotopy ribbon discs. *Adv. Math.*, 391:Paper No. 107960, 29, 2021.

[CS11] Diarmuid Crowley and Joerg Sixt. Stably diffeomorphic manifolds and $l_{2q+1}(\mathbb{Z}[\pi])$. *Forum Math.*, 23(3):483–538, 2011.

[DHL15] Nathan M. Dunfield, Neil R. Hoffman, and Joan E. Licata. Asymmetric hyperbolic L-spaces, Heegaard genus, and Dehn filling. *Math. Res. Lett.*, 22(6):1679–1698, 2015.

[DMS22] Irving Dai, Abishek Mallck, and Matthew Stoffregen. Equivariant knots and knot floer homology. 2022. https://arxiv.org/abs/2201.01875.

[EK71] Robert D. Edwards and Robion C. Kirby. Deformations of spaces of imbeddings. *Ann. Math. (2)*, 93:63–88, 1971.

[Eli90] Yakov Eliashberg. Topological characterization of Stein manifolds of dimension > 2. *Internat. J. Math.*, 1(1):29–46, 1990.

[EMM19] John B. Etnyre, Hyunki Min, and Anubhav Mukherjee. On 3-manifolds that are boundaries of exotic 4-manifolds, 2019. https://arxiv.org/abs/1901.07964.

[Etn03] John B. Etnyre. Introductory lectures on contact geometry. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume 71 of *Proc. Sympos. Pure Math.*, pages 81–107. Amer. Math. Soc., Providence, RI, 2003.

[FKV88] Sergey Finashin, Matthias Kreck, and Oleg Viro. Nondiffeomorphic but homeomorphic knottings of surfaces in the 4-sphere. In *Topology and geometry—Rohlin Seminar*, volume 1346 of *Lecture Notes in Math.*, pages 157–198. Springer, Berlin, 1988.

[FL18] Peter Feller and Lukas Lewark. On classical upper bounds for slice genera. *Selecta Math. (N.S.)*, 24(5):4885–4916, 2018.

[FL19] Peter Feller and Lukas Lewark. Balanced algebraic unknotting, linking forms, and surfaces in three- and four-space. 2019. https://arxiv.org/abs/1905.08305.

[FM97] Robert Friedman and John W. Morgan. Algebraic surfaces and Seiberg-Witten invariants. *J. Algebraic Geom.*, 6(3):445–479, 1997.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[FNOP19] Stefan Friedl, Matthias Nagel, Patrick Orson, and Mark Powell. A survey of the foundations of four-manifold theory in the topological category. *ArXiv 1910.07372*, 2019.

[FP17] Stefan Friedl and Mark Powell. A calculation of Blanchfield pairings of 3-manifolds and knots. *Mosc. Math. J.*, 17(1):59–77, 2017.

[FQ90] Michael H. Freedman and Frank Quinn. *Topology of 4-manifolds*, volume 39 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1990.

[Fre82] Michael Hartley Freedman. The topology of four-dimensional manifolds. *J. Differential Geometry*, 17(3):357–453, 1982.

[Fre84] Michael H. Freedman. The disk theorem for four-dimensional manifolds. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 647–663, Warsaw, 1984. PWN.

[FS97] Ronald Fintushel and Ronald J. Stern. Surfaces in 4-manifolds. *Math. Res. Lett.*, 4(6):907–914, 1997.

[FT05] Stefan Friedl and Peter Teichner. New topologically slice knots. *Geom. Topol.*, 9:2129–2158, 2005.

[FV11] S. Friedl and S. Vidussi. A survey of twisted Alexander polynomials. In *The mathematics of knots*, volume 1 of *Contrib. Math. Comput. Sci.*, pages 45–94. Springer, Heidelberg, 2011.

[GL89] Cameron Gordon and John Luecke. Knots are determined by their complements. *J. Amer. Math. Soc.*, 2(2):371–415, 1989.

[Gom98] Robert E. Gompf. Handlebody construction of Stein surfaces. *Ann. of Math. (2)*, 148(2):619–693, 1998.

[GS99] Robert E. Gompf and András I. Stipsicz. *4-manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999.

[Hay20] Kyle Hayden. Exotic ribbon disks and symplectic surfaces. 2020. https://arxiv.org/abs/2003.13681.

[HK88] Ian Hambleton and Matthias Kreck. On the classification of topological 4-manifolds with finite fundamental group. *Math. Ann.*, 280(1):85–104, 1988.

[HKK+21] Kyle Hayden, Alexandra Kjuchukova, Siddhi Krishna, Maggie Miller, Mark Powell, and Nathan Sunukjian. Brunnian exotic surface links in the 4-ball,. 2021. https://arxiv.org/abs/2106.13776.

[HKT09] Ian Hambleton, Matthias Kreck, and Peter Teichner. Topological 4-manifolds with geometrically two-dimensional fundamental groups. *J. Topol. Anal.*, 1(2):123–151, 2009.

[HLSX19] Michael J. Hopkins, Jianfeng Lin, XiaoLin Danny Shi, and Zhouli Xu. Intersection forms of spin 4-manifolds and the pin(2)-equivariant mahowald invariant, 2019. https://arxiv.org/abs/1812.04052.

[HS20] Neil R. Hoffman and Nathan S. Sunukjian. Null-homologous exotic surfaces in 4-manifolds. *Algebr. Geom. Topol.*, 20(5):2677–2685, 2020.

[HS21] Kyle Hayden and Isaac Sundberg. Khovanov homology and exotic surfaces in the 4-ball. 2021. https://arxiv.org/abs/2108.04810.

[IMT21] Nobuo Iida, Anubhav Mukherjee, and Masaki Taniguchi. An adjunction inequality for the bauer-furuta type invariants, with applications to sliceness and 4-manifold topology. 2021. https://arxiv.org/abs/2102.02076.

[JMZ21] András Juhász, Maggie Miller, and Ian Zemke. Transverse invariants and exotic surfaces in the 4-ball. *Geom. Topol.*, 25(6):2963–3012, 2021.

[Kim06] Hee Jung Kim. Modifying surfaces in 4-manifolds by twist spinning. *Geom. Topol.*, 10:27–56, 2006.

[KLPT17] Daniel Kasprowski, Markus Land, Mark Powell, and Peter Teichner. Stable classification of 4-manifolds with 3-manifold fundamental groups. *J. Topol.*, 10(3):827–881, 2017.

[KMRS21] Alexandra Kjuchukova, Allison N Miller, Arunima Ray, and Sakalli Sumeyra. Slicing knots in definite 4-manifolds. 2021. https://arxiv.org/abs/2112.14596.

[KPR22] Daniel Kasprowski, Mark Powell, and Arunima Ray. Counterexamples in 4-manifold topology. 2022. https://arxiv.org/abs/2203.13332.

[KR08a] Hee Jung Kim and Daniel Ruberman. Smooth surfaces with non-simply-connected complements. *Algebr. Geom. Topol.*, 8(4):2263–2287, 2008.

[KR08b] Hee Jung Kim and Daniel Ruberman. Topological triviality of smoothly knotted surfaces in 4-manifolds. *Trans. Amer. Math. Soc.*, 360(11):5869–5881, 2008.

[KR20] Hee Jung Kim and Daniel Ruberman. Topological spines of 4-manifolds. *Algebr. Geom. Topol.*, 20(7):3589–3606, 2020.

[LM97] Paolo Lisca and Gordana Matić. Tight contact structures and Seiberg-Witten invariants. *Invent. Math.*, 129(3):509–525, 1997.

[Mar13] Thomas E. Mark. Knotted surfaces in 4-manifolds. *Forum Math.*, 25(3):597–637, 2013.

[Mil62] John Milnor. A duality theorem for Reidemeister torsion. *Ann. of Math. (2)*, 76:137–147, 1962.

[MMP20] Ciprian Manolescu, Marco Marengon, and Lisa Piccirillo. Relative genus bounds in indefinite 4-manifolds. 2020. https://arxiv.org/abs/2012.12270.

[MMSW19] Ciprian Manolescu, Marco Marengon, Sucharit Sarkar, and Michael Willis. A generalization of rasmussen's invariant, with applications to surfaces in some four-manifolds. 2019. https://arxiv.org/abs/1910.08195.

[MP21] Ciprian Manolescu and Lisa Piccirillo. From zero surgeries to candidates for exotic definite four-manifolds. 2021. https://arxiv.org/abs/2102.04391.

[Pic22] Lisa Piccirillo. Verified computations for "4-manifolds with boundary and fundamental group $\mathbb{Z}$". 2022. https://doi.org/10.7910/DVN/IJIIVC.

[Pow16] Mark Powell. Twisted Blanchfield pairings and symmetric chain complexes. *Q. J. Math.*, 67(4):715–742, 2016.

[PY04] Józef H. Przytycki and Akira Yasuhara. Linking numbers in rational homology 3-spheres, cyclic branched covers and infinite cyclic covers. *Trans. Amer. Math. Soc.*, 356(9):3669–3685, 2004.

[Ran81] Andrew Ranicki. *Exact sequences in the algebraic theory of surgery*, volume 26 of *Mathematical Notes*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1981.

[Sha69] Julius L. Shaneson. Wall's surgery obstruction groups for $G \times \mathbb{Z}$. *Ann. of Math. (2)*, 90:296–334, 1969.

[Sto93] Richard Stong. Simply-connected 4-manifolds with a given boundary. *Topology Appl.*, 52(2):161–167, 1993.

[Sto94] Richard Stong. Uniqueness of connected sum decompositions in dimension 4. *Topology Appl.*, 56(3):277–291, 1994.

[Sun15] Nathan S. Sunukjian. Surfaces in 4-manifolds: concordance, isotopy, and surgery. *Int. Math. Res. Not. IMRN*, (17):7950–7978, 2015.

[SW00] Richard Stong and Zhenghan Wang. Self-homeomorphisms of 4-manifolds with fundamental group $\mathbb{Z}$. *Topology Appl.*, 106(1):49–56, 2000.

[Tei97] Peter Teichner. On the star-construction for topological 4-manifolds. In *Geometric topology. 1993 Georgia international topology conference, August 2–13, 1993, Athens, GA, USA*, pages 300–312. Providence, RI: American Mathematical Society; Cambridge, MA: International Press, 1997.

[Thu02] William Thurston. The geometry and topology of three-manifolds, 2002. http://www.msri.org/publications/books/gt3m.

[Tur86] Vladimir Turaev. Reidemeister torsion in knot theory. *Uspekhi Mat. Nauk*, 41(1(247)):97–147, 240, 1986.

[Tur01] Vladimir Turaev. *Introduction to combinatorial torsions*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001. Notes taken by Felix Schlenk.

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