Links Not Concordant to the Hopf Link

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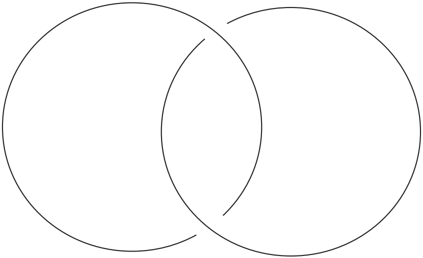
Joint work with Stefan Friedl.

The Hopf link:



Definition

Two n-component links $L, L' \colon S^1 \sqcup S^1 \cdots \sqcup S^1 \subseteq S^3$ are concordant if they cobound n embedded annuli

$$\bigsqcup_n S^1 \times I \subset S^3 \times I$$

with boundary $L \subset S^3 \times \{0\}$ and $L' \subset S^3 \times \{1\}$.

CAT $=$ TOP, all embeddings are to be locally flat.

Aim: given a 2-component link, find algebraic obstructions to it being concordant to the Hopf link H.

Find examples with unknotted components and so that previously known obstructions vanish.

We produce an obstruction theory in the spirit of the Casson–Gordon knot slicing obstructions.

Lemma

Any link which is concordant to a given 2 component link L has the same linking number as L.

Link exterior:

$$X_L := \mathrm{cl}(S^3 \setminus \nu L).$$

Using

$$\pi_1(X_L) \to H_1(X_L) \xrightarrow{\simeq} \mathbb{Z}^2,$$

define the *multi–variable Alexander polynomial* corresponding to the universal abelian cover:

$$\Delta_L(s,t) := \mathrm{ord}(H_1(X_L; \mathbb{Z}[t^{\pm 1}, s^{\pm 1}]))$$

Note that $\Delta_H(s,t) = 1$.

Theorem (Kawauchi)

If a 2 component link with linking number 1 is concordant to H then

$$\Delta_L(s,t) \equiv f(t,s)f(t^{-1},s^{-1}),$$

up to $\pm t^k s^l$.

Partial converse generalising Freedman:

Theorem (J. Davis 2004)

A 2 component link with multivariable Alexander polynomial 1 is topologically concordant to the Hopf link.

- Cha, Kim, Ruberman and Strle showed that this result does not hold in the smooth category.
- One imagines concordance to H admits a filtration analogous to the Cochran-Harvey-Horn positive and negative filtrations of knot concordance.

We form a closed 3 manifold

$$M_L := X_L \cup X_H$$

gluing along $\partial X_L \xrightarrow{\simeq} \partial X_H \cong S^1 \times S^1 \sqcup S^1 \times S^1$.

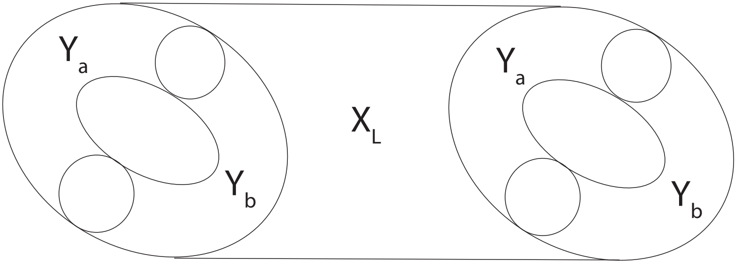
Note that $M_H \cong S^1 \times S^1 \times S^1$.

Pick a prime p, $a, b \in \mathbb{N}$ and a homomorphism to a p-group:

$$\varphi\colon H_1(M_L) \to A = \mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^b}$$

The maps $\varphi, \varphi|$ induce finite p^{a+b}-fold coverings M_L^{φ} and X_L^{φ}.

We define $Y_a, Y_b \subset \partial X_L$ as in the diagram.



Recall that X_L^φ is the p-primary cover induced by φ. Y_a^φ is the induced (disconnected) cover.

Lemma (Friedl,P)

There is a non-singular linking form

$$TH_1(X_L^\varphi, Y_a^\varphi) \times TH_1(X_L^\varphi, Y_a^\varphi) \to \mathbb{Q}/\mathbb{Z}.$$

If L is concordant to H with concordance $C \cong \bigsqcup_2 S^1 \times I$, and we define the exterior of the concordance

$$E_C := \mathrm{cl}(S^3 \times I \setminus \nu C)$$

then there is a metaboliser $P = P^\perp$ for the linking form given by

$$P := \ker\left(TH_1(X_L^\varphi, Y_a^\varphi) \to TH_1(E_C^\varphi, Y_a^\varphi)\right).$$

Definition

A Seifert surface $F \subset S^3$ is a compact, connected, orientable surface embedded in S^3 with $\partial F = K$. For $x, y \in H_1(F)$, define the Seifert form

$$V \colon H_1(F) \times H_1(F) \to \mathbb{Z}; \quad V(x,y) := \operatorname{lk}(x, y^+),$$

with y^+ a positive push off of y along the normal to F.

For $\omega \in S^1 \subset \mathbb{C}$, the *Levine-Tristram* signature $\sigma_K(\omega)$ of K at ω is the average of the one sided limits of the signature function:

$$\sigma_V(\omega) = \sigma\big((1-\omega)V + (1-\overline{\omega})V^T\big).$$

where V is a Seifert form for K.

These signatures are concordance invariants, and appear in the calculation of our obstructions.

A knot is *Algebraically Null Concordant* if there is a Seifert surface F and a basis of $H_1(F;\mathbb{Z})$ with respect to which the Seifert matrix takes the form:

$$\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

Casson and Gordon, in 1975, defined obstructions which detected the first knots which were found to be algebraically null concordant, but not concordant to the unknot.

We extend their methods to 2-component links with linking number 1.

Definition

The Witt group $L^0(\mathbb{F})$ of a field, characteristic $\neq 2$, with involution $\mathbb{F}$ is given by stable congruence classes of non-singular Hermitian forms on $\mathbb{F}$ vector spaces: $[(N, \lambda)]$, with addition:

$$[(N, \lambda)] + [(M, \eta)] = [(N \oplus M, \lambda \oplus \eta)],$$

inverses

$$-[(N, \lambda)] = [(N, -\lambda)],$$

and zero

$$\left(\mathbb{F} \oplus \mathbb{F}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \right) .$$

In our case $\mathbb{F} = \mathbb{C}(F)$, where F is free abelian of rank 3.

Choose homomorphisms:

$$\begin{array}{rcl} \phi\colon H_1(M_L^{\varphi}) & \to & H_1(M_L^{\varphi})/TH_1(M_L^{\varphi}) \to F \cong \mathbb{Z}^3 \\ \chi\colon H_1(M_L^{\varphi}) & \to & \mathbb{Z}_q \end{array}$$

with $\gcd(p,q) = 1$. We will define a Casson-Gordon style Witt group obstruction

$$\tau(M_L^{\varphi}, \phi, \chi) \in L^0(\mathbb{C}(F)) \otimes \mathbb{Q}.$$

Main Theorem (Friedl, P)

Suppose that L is concordant to H. For all

$$\varphi\colon H_1(M_L) \to A,$$

there exists a metaboliser

$$P = P^\perp \subseteq TH_1(X_L^\varphi, Y_a^\varphi)$$

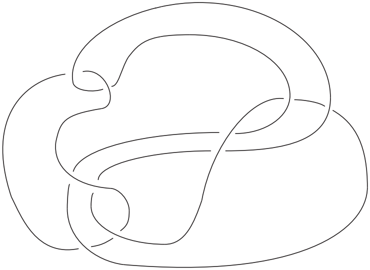
for the torsion linking form, such that for all

$$\chi\colon H_1(M_L^\varphi) \to \mathbb{Z}_q$$

which satisfy that $\chi|_{H_1(X_L^\varphi)}$ *factors as* $H_1(X_L^\varphi) \to H_1(X_L^\varphi, Y_a^\varphi) \xrightarrow{\delta} \mathbb{Z}_q$*, and that* $\delta(P) = \{0\}$*, we have*

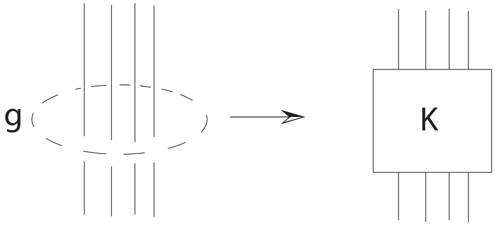
$$\tau(M_L^\varphi, \phi, \chi) = 0.$$

Example time. Temporarily, let L be:



- $\Delta_L(s,t) = (ts + 1 - s)(ts + 1 - t)$ $(\neq 1)$.
- L is concordant to H.
- Linking number is 1.
- Components are unknotted.

Idea: Take a satellite of L so that $\tau(M_L^{\varphi}, \phi, \chi)$ becomes non-zero. Choose a curve $g \subset X_L$ which is unknotted in S^3, and a knot K.



We obtain the satellite link $S := S(L, K, g)$.

Taking the signature: $\sigma \colon L^0(\mathbb{C}) \xrightarrow{\simeq} \mathbb{Z}$.

For $\zeta_i \in S^1 \subset \mathbb{C}$:

$$\tau(M_L^{\varphi}, \phi, \chi)(\zeta_1, \zeta_2, \zeta_3) \in L^0(\mathbb{C})$$

almost everywhere.

Define a homomorphism:

$$\sigma_{\mathbb{C}(F)} \colon L^0(\mathbb{C}(F)) \to \mathbb{R};$$

$$\sigma_{\mathbb{C}(F)}(\tau(s,t,u)) := \int_{S^1 \times S^1 \times S^1} \sigma\big(\tau(\zeta_1, \zeta_2, \zeta_3)\big).$$

Results of Litherland (1984) on Casson-Gordon invariants of satellite knots extend easily to our setting.

Theorem

$$\sigma_{\mathbb{C}(F)}(\tau(M_S^{\varphi}, \phi, \chi)) = \sigma_{\mathbb{C}(F)}(\tau(M_L^{\varphi}, \phi, \chi)) + \sum_{i,j} \sigma_K(\omega_{ij}) \in \mathbb{R},$$

where $\omega_{ij} := \chi(x^i y^j \cdot \widetilde{g})$, x, y *are generators of* $A = \mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^b}$ *and* $\sigma_K(\omega)$ *is the Levine-Tristram signature of* K *at* ω*.*

- Pick companion knot K with all positive/negative signatures (e.g. trefoil)
- Pick $\varphi\colon H_1(X_L;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 = A$.
- Pick axis curve g which maps non–zero under χ.

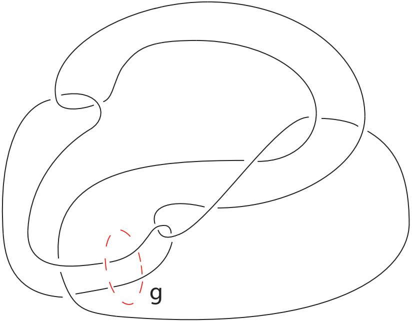
Then $\sigma_{\mathbb{C}(F)}(\tau(S^{\varphi}, \phi, \chi)) \neq 0$, so S is not concordant to H.

Note components of S are still unknotted.

Maple:

$$TH_1(X_L^{\varphi}, Y_a^{\varphi}) \cong \mathbb{Z}_9$$

Also used Maple to identify the curve g as shown below as a curve which lifts to a generator.



This completes the description of an example.

Jae Choon Cha has generalised the idea, to produce COT/CHL style obstructions.

Theorem (J.C.Cha)

There are links with two unknotted components which are height n Whitney tower/grope concordant to the Hopf link but not height $n.5$ Whitney tower/grope concordant to H.

We made a conjecture, which Jae Choon informs us has been confirmed by a graduate student of his:

Theorem (Min Hoon Kim)

Let L be a two component link which is height 3.5 Whitney tower/grope concordant to H. Then the conclusion of our main obstruction theorem is satisfied.

It may well be possible to also show that our link is not concordant to H by blowing down along one component and using Casson-Gordon knot slice obstructions; limitations of satellite construction.

However this trick however does not tell us anything about height 3.5 Whitney tower/grope concordance.

We now move on to explaining the obstruction theory i.e. defining $\tau(M_L^{\varphi}, \phi, \chi)$.

Characterisation of concordance to H:

Proposition

A two component link L is concordant to H if and only if M_L is the boundary of a topological 4-manifold W with

$$H_*(X_L;\mathbb{Z}) \xrightarrow{\simeq} H_*(W;\mathbb{Z}) \xleftarrow{\simeq} H_*(X_H;\mathbb{Z})$$

and

$$\pi_1(W) \cong \langle\langle \mu_1, \mu_2 \rangle\rangle$$

with μ_i a meridian of the ith component of L.

Proof.

$\Rightarrow$ Mayer-Vietoris and Seifert Van Kampen.

$\Leftarrow$ 4-dimensional h-cobordism theorem (Smale 1960, Freedman 1983). □

Usual strategy: find 4-manifolds W with

$$H_1(X_L;\mathbb{Z}) \xrightarrow{\simeq} H_1(W;\mathbb{Z})$$

but with larger H_2. Define obstructions to changing W to be a $\mathbb{Z}$-homology cobordism.

If independent of choice of W, we have an obstruction.

Element of bordism group

$$(M_L^{\varphi}, \phi \times \chi) \otimes 1 \in \Omega_3(F \times \mathbb{Z}_q) \otimes \mathbb{Q} \cong H_3(F; \Omega_0) \otimes \mathbb{Q}.$$

Recall $F \cong \mathbb{Z}^3$.

$$[M_L^{\varphi} \cup -M_H^{\varphi}] = 0 \in H_3(F;\Omega_0),$$

so there exists $s \in \mathbb{N}$ and W, with $\partial W = sM_L^{\varphi} \sqcup sM_H^{\varphi}$, such that ϕ and χ extend over W.

Use the ring homomorphism

$$\mathbb{Z}[F \times \mathbb{Z}_q] = \mathbb{Z}[\mathbb{Z}_q][F] \to \mathbb{Z}[\xi_q][F] \to \mathbb{C}[F] \to \mathbb{C}(F).$$

where $\xi_q \in S^1 \subset \mathbb{C}$ is a primitive root of unity.

We consider the intersection forms:

$$\lambda_{\mathbb{C}(F)} \colon H_2(W;\mathbb{C}(F)) \times H_2(W;\mathbb{C}(F)) \to \mathbb{C}(F)$$

and

$$\lambda_{\mathbb{Q}} \colon H_2(W;\mathbb{Q}) \times H_2(W;\mathbb{Q}) \to \mathbb{Q}.$$

We define

$$\tau(M_L^{\varphi}, \phi, \chi) := (\lambda_{\mathbb{C}(F)} - \mathbb{C}(F) \otimes \lambda_{\mathbb{Q}}) \otimes 1/s \in L^0(\mathbb{C}(F)) \otimes \mathbb{Q}.$$

Well-defined: independent of choice of W since $\lambda_{\mathbb{C}(F)}(V) = \lambda_{\mathbb{Q}}(V)$ for *closed* 4-manifolds V.

Recall:

Main Theorem (Friedl, P)

Suppose that L is concordant to H. For all $\varphi\colon H_1(M_L) \to A$, *there exists a metaboliser*

$$P = P^{\perp} \subseteq TH_1(X_L^{\varphi}, Y_a^{\varphi})$$

for the torsion linking form, such that for all $\chi\colon H_1(M_L^{\varphi}) \to \mathbb{Z}_q$ *which satisfy that* $\chi|_{H_1(X_L^{\varphi})}$ *factors as*

$$H_1(X_L^{\varphi}) \to H_1(X_L^{\varphi}, Y_a^{\varphi}) \xrightarrow{\delta} \mathbb{Z}_q,$$

and that $\delta(P) = \{0\}$, *we have* $\tau(M_L^{\varphi}, \phi, \chi) = 0$.

Key Step: $\mathbb{Z}$-homology controls the *size* of the torsion free part of the homology of certain covering spaces. The following theorem generalises Casson-Gordon's Lemma 4.

Theorem (Friedl,P)

Let $f\colon S \to Y$ be a map of finite CW complexes such that

$$f_*\colon H_*(S;\mathbb{Z}_q) \xrightarrow{\simeq} H_*(Y;\mathbb{Z}_q).$$

Let $\varphi\colon \pi_1(Y) \to A$ be a homomorphism onto an abelian p-group, $\gcd(p,q)=1$. φ induces covers $f^\varphi\colon S^\varphi \to Y^\varphi$, with S^φ the pull-back cover of Y^φ along f.

Let $\chi\colon \pi_1(Y^\varphi) \to \mathbb{Q}(\xi_q) \subset \mathbb{C}$ be a representation which factors through a q-group, and let $\phi\colon \pi_1(Y^\varphi) \to F$, a free abelian group, be another representation.
Then

$$f_*^\varphi\colon H_*(S^\varphi;\mathbb{C}(F)) \xrightarrow{\simeq} H_*(Y^\varphi;\mathbb{C}(F))$$

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The inclusion

$$(\partial X_L)^1 \to X_L$$

induces $\mathbb{Z}$-homology isomorphisms (linking number ± 1). Therefore

$$H_*(X_L^{\varphi}; \mathbb{C}(F)) \cong H_*((\partial X_L^{\varphi})^1; \mathbb{C}(F)) \cong 0.$$

By Mayer-Vietoris, $H_*(M_L^{\varphi}; \mathbb{C}(F)) \cong 0$.

Also needed this to see $\lambda_{\mathbb{C}(F)}(W)$ non–singular.

Let C be a concordance between L and H. Define:

$$W_C := E_C \cup X_H \times I.$$

Then

$$\partial W_C = M_L \sqcup M_H$$

and $M_L \to W_C$ induces a $\mathbb{Z}$ homology equivalence

$$H_*(M_L; \mathbb{Z}) \xrightarrow{\simeq} H_*(W_C; \mathbb{Z}).$$

Therefore

$$H_*(W_C; \mathbb{C}(F)) \cong H_*(M_L^{\varphi}; \mathbb{C}(F)) \cong 0.$$

No homology means no intersection form, and so τ vanishes.