ROCHLIN'S THEOREM

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ABSTRACT. These notes are prepared for a seminar talk in the seminar "Topological Manifolds 2" in Bonn, Summer term 2021. The goal is to give some context on Rochlin's theorem and to explain the proof by Matsumoto from [7].

1. Rochlin's theorem: History, proofs and applications

Let M be a smooth oriented closed 4-manifold[1] and consider its intersection form

$$H_2(M)/\text{torsion} \times H_2(M)/\text{torsion} \to \mathbb{Z}, \quad (x, y) \mapsto x \cdot y$$

which is acquired by applying Poincare duality to the cup product

$$\smile : H^2(M)/\text{torsion} \times H^2(M)/\text{torsion} \to H^4(M).$$

We denote the signature of M – by definition the signature of this pairing - by $\sigma(M)$.

Recall that a manifold admits an orientation if and only if its first Stiefel-Whitney class $w_1(M)$ vanishes. An oriented manifold admits a spin structure if and only if its second Stiefel-Whitney class $w_2(M)$ vanishes. Analogously, an oriented manifold which admits a spin structure will be called a *spin manifold*. We will not need much more about spin structures except for its relation to the second Stiefel-Whitney class. Now we can formulate Rochlin's theorem.

Theorem 1.1 (Rochlin). *Let M be a closed, oriented smooth spin 4-manifold. Then the signature of M is divisible by* 16.

The theorem is also sharp: an example of a spin 4-manifold with signature -16 is given by the zero set of the Fermat quartic which is defined as

$$K = \{[w : x : y : z] \in \mathbb{C}P^3 \mid w^4 + x^4 + y^4 + z^4 = 0\}.$$

The manifold K is a complex surface, commonly known as a K3 surface. The Topological properties of K are collected in [11]. In [13, p. 8], Kirby gives a different description of this manifold in terms of a handlebody decomposition.

[1]Most of our discussion (in particular, Rochlin's theorem) carries over to the PL-category.

Rochlin's original proof appeared in [1] in Russian. A French translation can be found in [7, p. 17 ff.]. We will explain the geometric proof by Matsumoto, also from [7, p. 119 ff.].

1.1. **Some historical remarks.** In his original paper [1], Rochlin proved his theorem with homotopy theoretic methods. In modern day language, he computed the that the image of the third J-homomorphism is isomorphic to $\mathbb{Z}/24$. A nice exposition of this proof can be found in [14].

In [6], Freedman and Kirby gave a geometric proof of Rochlin's theorem, which has a lot of similarities to the proof of Matsumoto which shall be given in detail in these notes. It is notable about the proof of Freedman and Kirby that they do something stronger: they construct and compute a bordism group of 4-manifolds with embedded characteristic surfaces. Rochlin's theorem can then be checked on generators of this bordism group.

1.2. **Non-realisability of surgery obstructions.** Let X be an w-oriented Poincaré-complex of dimension n. Can every surgery obstruction in $L_n(\mathbb{Z}\pi_1(X), w)$ be realized by a surgery problem with target X? A positive answer is provided by Wall's realization theorem [9, Theorems 5.8 and 6.5]. But can we do better and assume that the source of our surgery problem is a closed manifold? Here, the answer is *no* in general, and Rochlins theorem can be thought of as a first instance of this phenomenon.

If we take $k \geq 2$, $X = S^{4k}$, covered with the trivial bundle, then the index of the subgroup of realizable elements in $L_{4k}(\mathbb{Z}) \cong \mathbb{Z}$ (the isomorphism given dividing the signature ot a form by 8, thus mapping the E_8-form to $1 \in \mathbb{Z}$) can be computed explicitly. It is

$$\frac{2^{2k} \cdot (2^{2k-1} - 1) \cdot (3 - (-1)^k) \cdot (2k-1)! \cdot B_k}{2 \cdot 8 \cdot (2k)!} \cdot |\operatorname{Im} J_{4k-1}|,$$

where J_{4k-1} denotes the J-homomorphism and B_k the k-th Bernoulli number. The order of the image of the J-homomorphism is known, so this can be simplified to

$$\frac{(3 - (-1)^k)}{2} \cdot 2^{2k-2} \cdot (2^{2k-1} - 1) \cdot \operatorname{numerator}(B_k/(4k)).$$

These results are explained in [12, Chapter 11]. The original reference is [3].

2. Matsumoto's geometric proof

In the following we present a proof of Rochlin's theorem due to Matsumoto, which appeared in [7]. Actually, Matsumoto proves a congruence which is slightly stronger than Rochlin's theorem, for which we need the concept of characteristic homology classes.

Definition 2.1. Let M be a 4-manifold. A homology class $\xi \in H_2(M, \partial M; \mathbb{Z})$ is *characteristic* if for every element $a \in H_2(M; \mathbb{Z})$

$$\xi \cdot a \equiv a \cdot a \mod 2.$$

A *characteristic surface for M* is a properly embedded oriented surface $F \subset M$ whose fundamental class is a characteristic homology class.

In Lemma 2.2 can show the existence of a characteristic surface for an oriented 4-manifold M easily under the assumption that $H_1(M, \partial M; \mathbb{Z}) = 0$. Notice that if M is spin, we can do framed surgery on M to achieve this condition. Since the outcome of surgery on M is related to M via an oriented bordism, and since the signature is an oriented bordism invariant, it is reasonable to work with the assumption $H_1(M, \partial M; \mathbb{Z}) = 0$.

Lemma 2.2. *Every closed oriented 4-manifold M with $H_1(M; \mathbb{Z}) = 0$ admits a characteristic homology class. Such a class can be represented by a characteristic surface. If F_0, F_1 are characteristic surfaces which represent the same class in $H_2(M; \mathbb{Z})$, then there exists a orientable embedded 3-manifold $G \subset M \times I$ with $G \cap M \times \{i\} = F_i$.*

Proof. Notice that the second Stiefel-Whitney class $w_2(M)$ has the property that for all $\alpha \in H^2(M; \mathbb{Z}/2)$ we have

$$w_2(M) \smile \alpha = \alpha \smile \alpha.$$

Since $H_1(M; \mathbb{Z}) = 0$, the reduction homomorphism

$$r \colon H_2(M; \mathbb{Z}) \to H_2(M; \mathbb{Z}/2)$$

is surjective, as can be seen from the Bockstein sequence associated to the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

and the fact that $H_1(M; \mathbb{Z}) = 0$. Pick a preimage a of the Poincaré dual of $w_2(M)$.

In a 4-manifold, every class in $H_2(M; \mathbb{Z})$ can be represented by an embedded oriented surface, and two such surfaces can always be joined by an oriented 3-manifold in $M \times I$. This can be found in [2, Corollaire II.13, p. 43], but we also give a proof in the appendix, Theorem A.1. □

Here is another fundamental lemma which gives a good intuition for characteristic surfaces and will be needed later.

Lemma 2.3. *If $F \subset M$ is a characteristic surface in an oriented 4-manifold, then $M \setminus F$ admits a spin structure, i.e. has trivial second Stiefel-Whitney class.*

Proof. We denote a tubular neighborhood of F by $\nu(F)$. Note that the inclusions $F \to \nu(F)$ and $M \setminus \nu(F) \to M \setminus F$ are homotopy equivalences, so we may work with $M \setminus \nu(F)$. Consider the following commutative diagram.

$$\begin{tikzcd}
 & H^2(M;\mathbb{Z}/2) \arrow[r, "i^*"] \arrow[d, "PD"] & H^2(M \setminus \nu(F);\mathbb{Z}/2) \arrow[d, "PD"] \\
H_2(\nu(F);\mathbb{Z}/2) \arrow[r, "i_*"] & H_2(M;\mathbb{Z}/2) \arrow[r, "u"] & H_2(M, \nu(F);\mathbb{Z}/2)
\end{tikzcd}$$

We want to show that $w_2(M \setminus \nu(F)) = i^* w_2(M) = 0$. Since the vertical maps are isomorphisms by Poincare-Lefschetz duality (e.g. see [8, Corollary VI.8.4]), it suffices to see that $u(PD(w_2(M))) = 0$. But $PD(w_2(M)) = i_*[F]$ as shown in the proof of 2.2, and $u \circ i_* = 0$. $\square$

2.1. **The Arf invariant.** Let V be a finite dimensional $\mathbb{Z}/2$-vector space equipped with a nonsingular symmetric bilinear form $(x, y) \mapsto s(x, y)$. A *quadratic enhancement* of s is a function $q\colon V \to \mathbb{Z}/2$ satisfying

$$q(x + y) - q(x) - q(y) = s(x, y).$$

The data of a symmetric bilinear form over $\mathbb{Z}/2$ with a quadratic enhancement is called a *quadratic form.* We define the Arf invariant or, following Browder, "democratic invariant" of a quadratic form (V, s, q) to be

$$\operatorname{Arf}(V, s, q) = \begin{cases} 0 \text{ if } q \text{ sends strictly more elements of } V \text{ to } 0 \text{ than to } 1; \\ 1 \text{ else.} \end{cases}$$

Although this definition is nice and does not depend on any choices, there is another way to compute the Arf invariant, which will be more useful to us.

Lemma 2.4. *If $\{b_1, c_1, \ldots, b_n, c_n\}$ is a symplectic basis of (V, s, q), i.e. $s(b_i, b_j) = 0 = s(c_i, c_j)$ and $s(b_i, c_j) = \delta_{ij}$ for all $1 \le i, j \le n$, then*

$$\operatorname{Arf}(V, s, q) = \sum_i q(b_i)q(c_i) \in \mathbb{Z}/2.$$

This formula is quite famous, as it appears together with a picture of C. Arf on the Turkish 10 lira banknote.

2.2. **The Arf invariant of a characteristic homology class.** Let M be a compact, smooth, orientable 4-manifold and F a properly embedded connected surface in M. We assume that ∂F is empty or S^1 and that F is characteristic. The reason we are also interested in the case $\partial F = S^1$ is that if $M = B^4$, and F can be pushed into S^3 relative boundary, it can be thought of as a Seifert surface for ∂F and we can define the Arf invariant of a knot.

Furthermore, suppose $H_1(M) = 0$. We will define a quadratic enhancement of the $\mathbb{Z}/2$ intersection pairing on F. Let C be an immersed S^1 in F which is in general position. It bounds an immersed connected orientable surface D in M which is nowhere tangent to F. Its normal bundle ν_D is orientable since M is orientable, and it is even trivial as D is homotopy equivalent to a CW-complex of dimension 1, and a bundle over a one dimensional CW-complex is trivial if and only if it is orientable.

Pick a trivialization of ν_D. Of course, it induces a trivialization of $\nu_D|_{\partial D}$. Two such trivializations of ν_D differ up to homotopy by a map $g\colon D \to SO(2) = S^1$. Now $g|_{\partial D}$ is null-homotopic as it extends over D. Indeed, consider the diagram:

$$\begin{tikzcd} {[C, S^1]} \arrow[r, "\cong"] & H^1(C) \arrow[r, "\cong"] & \mathrm{Hom}(H_1(C), \mathbb{Z}) \\ {[D, S^1]} \arrow[r, "\cong"] \arrow[u, "i^*"] & H^1(D) \arrow[r, "\cong"] \arrow[u] & \mathrm{Hom}(H_1(D), \mathbb{Z}) \arrow[u, "\alpha^*"'] \end{tikzcd}$$

Suppose the class of a map $g\colon C \to S^1$ extends over D. Then it lies in the image of i^*. But α^* is induced by the inclusion of the boundary

$$H_1(C) \to H_1(D)$$

which is null-homologous since D is orientable. Hence, $\alpha^* = 0$, so $i^* = 0$. But then g is null-homotopic. So the trivialization on $\nu_D|_C$ is unique up to strong fiber homotopy.

The normal bundle ν_C of C in F determines a sub-line bundle in $\nu_D|_{\partial D}$. Using the trivialization of $\nu_D|_{\partial D}$, we can count the number of twists of ν_C modulo 2 and call this number $O(D)$. Denote by $D \cdot F$ the number of intersection points of the interior of D with F and $\mathrm{Self}(C)$ the number of self-intersections of C in F. We define

$$q(C) = O(D) + D \cdot F + \mathrm{Self}(C) \in \mathbb{Z}/2.$$

Lemma 2.5. *Let M be a compact, oriented smooth 4-manifold with $H_1(M;\mathbb{Z}) = 0$ and let $F \subset M$ be a characteristic surface. The function*

$$q\colon H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2$$

is well-defined and a quadratic enhancement of the intersection paring on $H_1(F;\mathbb{Z}/2)$.

Proof. We shall prove this in multiple steps.

- *The element $q(C)$ does not depend on the choice of the bounding surface D.*

  Let D and D' be surfaces bounding C. By spinning D around C as in [6, p. 87], which changes both $O(D)$ and $D \cdot F$ by one, we may assume that $\Sigma := D \cup D'$ is a smoothly immersed oriented closed surface.

  Both D and D' induce trivializations of $\nu_\Sigma|_C$, say τ and τ'. Changing the orientation of τ fiberwise (or exchanging the vector fields giving the trivialization) we obtain a new trivialization $-\tau$ and denote the difference of $-\tau$ and τ' by $d(-\tau, \tau') \in \pi_1(SO(2)) = \mathbb{Z}$.

  Recall that D and D' have trivial normal bundles so we can push each off itself inside M. The same is not true for Σ, but we can measure the failure of our ability to push Σ off itself with $d(-\tau,\tau')$.

$$\Sigma\cdot\Sigma = d(-\tau,\tau') - 2D\cdot D' \equiv d(-\tau,\tau') \mod 2.$$

Now since F is characteristic, we have $\Sigma\cdot\Sigma \equiv \Sigma\cdot F \mod 2$. But observe that

$$\Sigma\cdot F = D\cdot F + D'\cdot F.$$

Here we used that we can arrange that Σ does not intersect F in C, which can be achieved by making Σ transverse to C, so that Σ and C do not intersect.

By definition,

$$O(D') = O(D) + d(-\tau,\tau').$$

Thus,

$$\begin{aligned} O(D) + D\cdot F &\equiv O(D') - d(-\tau,\tau') + \Sigma\cdot F - D'\cdot F \\ &\equiv O(D') - d(-\tau,\tau') + d(-\tau,\tau') - D\cdot F \\ &= O(D') + D'\cdot F \mod 2. \end{aligned}$$

Hence, $q(C) = O(D) + D\cdot F + \mathrm{Self}(C)$ only depends on the immersed circle C.

- *The element $q(C)$ only depends on the homotopy class of the immersed circle C in general position.*
  First we consider the case where C and C' are regularly homotopic. We can choose a smooth regular homotopy $S^1\times I\to F$ between them. Then $S^1\times I\to F\times I$ is an immersion. We may, without changing it at the endpoints, assume that it is in general position, since being an immersion is an open and dense condition on the smooth maps $S^1\times I\to F\times I$. Hence, its self intersections are some arcs and circles, where the arcs either connect two self intersection points of C or C' respectively, or connect a self intersection point of C to a self intersection of C'. Thus, $\mathrm{Self}(C) = \mathrm{Self}(C')$.

  Let D be a surface bounding C. We will use the track of the regular homotopy $f\colon S^1\times I\to F$ to create a new surface bounding $f(S^1\times\{1\})$. Consider the bundle $f^*\nu(F)$ on $S^1\times I$. The surface D intersects F transversally, providing us with non-vanishing section s of $f^*\nu(F)|_{S^1\times\{0\}}$. As $f|_{S^1\times 0}$ is an immersion we also obtain a non-vanishing section t of $f^*TF|_{S^1\times 0}$. Pick an isomorphism

$$\Phi\colon f^*TF\to f^*\nu(F)$$

  carrying t to s. Since f is a regular homotopy, the derivative of $f|_{S^1\times\{r\}}$ applied to a standard trivializing vector field V on S^1 induces a continuous map

$$S^1 \times I \to f^*TF, \quad (z, r) \mapsto (z, r, df|_{S^1\times\{r\}}(V_x))$$

Consider the composition

$$g' \colon S^1 \times I \to f^*TF \xrightarrow{\Phi} f^*\nu(F).$$

Using a metric on $\nu(F)$, rescale g' to a map g such that the norm of $g(x, r)$ is $1-r$. We can glue $g(S^1 \times I)$ to D with a small collar removed to get a surface D' bounding $f(S^1 \times \{0\})$. The following picture illustrates our situation. Note that, although $f \colon S^1 \times I \to M$ is not an embedding, g embeds $S^1 \times I$. Now

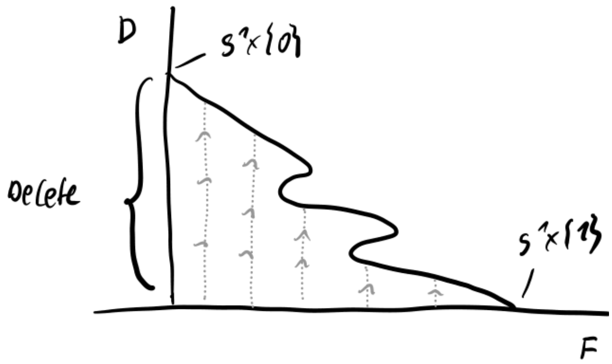

FIGURE 1. Constructing the new surface D'

$D \cdot F = D' \cdot F$ and $O(D) = O(D')$ are clear from the construction.

Second, notice that when C and C' are homotopic, we can apply the connected sum with a small figure 8 to C' until C' is also regularly homotopic to C.

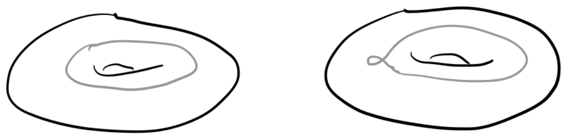

FIGURE 2. Adding a figure 8

The connected sum with a small figure 8 can be accompanied by a boundary conneced sum of D with a small disc bounding the figure 8. There will be no change in $D \cdot F$, $\mathrm{Self}(C)$ will increase by 1 for every figure 8 and $O(D)$ will also change by 1. Thus, $q(C) = q(C')$.

- *Let $x_0 \in F$ be a point. The function q induces a function $\pi_1(F, x_0) \to \mathbb{Z}/2$ which fulfills*
$$q(\omega * \omega') = q(\omega) + q(\omega') + [\omega] \cdot [\omega']$$
where $[\omega], [\omega'] \in H_1(F; \mathbb{Z}/2)$ are the images of $\omega, \omega' \in \pi_1(F, x_0)$ under $\pi_1(F, x_0) \to H_1(F; \mathbb{Z}/2)$.

  We can represent the elements $\omega, \omega' \in \pi_1(F)$ by circles $C, C*$ as above, together with paths γ, γ' from the base point to C or C' respectively. We can multiply (C, γ) and (C', γ') by taking $(C\sharp_{\gamma*(\gamma')^{-1}}C', \gamma)$. This corresponds to the multiplication in the fundamental group. As we saw above, q does not depend on the representative (C, γ) of an element in $\pi_1(F)$. Now we show that q is multiplicative. Consider elements $(C, \gamma), (C', \gamma')$ and surfaces D, D' bounding C, C'. Using the boundary connected sum, we can define a surface $D\sharp_\partial D'$ bounding $C\sharp_{\gamma*(\gamma')^{-1}}C'$ which fulfills $D\sharp_\partial D' \cdot F = D \cdot F + D' \cdot F$ and $O(D\sharp_\partial D') = O(D) + O(D')$. The self-intersections of $C\sharp_{\gamma*(\gamma')^{-1}}C'$ are
$$\mathrm{Self}(C) + \mathrm{Self}(C') + C \cdot C' \mod 2.$$
Note that although γ and γ' may intersect, the induced self-intersections of $C\sharp_{\gamma*(\gamma')^{-1}}C'$ have an even number, so they do not appear in the above sum.

  To conclude, we have
$$\begin{aligned} q(\omega * \omega*) &\equiv D\sharp_\partial D' \cdot F + O(D\sharp_\partial D') + \mathrm{Self}(C\sharp_{\gamma*(\gamma')^{-1}}C') \mod 2 \\ &\equiv D \cdot F + D' \cdot F + O(D) + O(D') \\ &\quad + \mathrm{Self}(C) + \mathrm{Self}(C') + C \cdot C' \mod 2 \\ &\equiv q(\omega) + q(\omega') + [\omega] \cdot [\omega'] \mod 2 \end{aligned}$$

- *The function q factors through $H_1(F; \mathbb{Z}/2)$.*
By the above, for $\omega, \omega' \in \pi_1(F, x_0)$ we have $q(\omega * \omega') = q(\omega' * \omega)$ since $[\omega] \cdot [\omega'] = [\omega'] \cdot [\omega] \mod 2$. Thus, q factors through $H_1(F; \mathbb{Z})$. Note for $C \in H_1(F; \mathbb{Z})$ we have $q(C + C) = 2q(C) + C \cdot C \equiv 0 \mod 2$. Since $H_1(F; \mathbb{Z}/2) = H_1(F; \mathbb{Z}) \otimes \mathbb{Z}/2$, we see that q also factors through $H_1(F; \mathbb{Z}/2)$.

This finishes the proof of the lemma and the construction of q. □

Now we can assign to the quadratic form $q\colon H_1(F; \mathbb{Z}/2) \to \mathbb{Z}/2$ its Arf invariant $\mathrm{Arf}(F)$. We will need some freedom regarding the characteristic surface F, so we prove the following lemma.

Lemma 2.6. *The element $\mathrm{Arf}(F) \in \mathbb{Z}/2$ only depends on the homology class $[F, \partial F] \in H_2(M, \partial M)$ and the isotopy class of the embedding $\partial F \to \partial M$.*

Before we prove this lemma, we consider the two extreme cases of applications, namely when one of the two dependencies of the Arf invariant becomes trivial.

Example 2.7. If M and F are closed, $\mathrm{Arf}(F)$ is determined by the class $[F] \in H_2(M;\mathbb{Z})$, so we can consider the Arf invariant of a characteristic homology class.

Example 2.8. If M is D^4 and F a Seifert surface of a knot $f\colon S^1 \to S^3 = \partial D^4$ which we push into D^4 to make it proper, this is known as the Arf invariant of the knot. In this case $[F, \partial F] = 0$, so it will really only depend on the isotopy class of f. We denote it by $\mathrm{Arf}(f)$.

Now we prove Lemma 2.6. The proof is quite technical and can safely be skipped when first reading these notes.

Proof of Lemma 2.6. Let F_0 and F_1 be characteristic surfaces satisfying $[F_0, \partial F_0] = [F_1, \partial F_1]$ and such that $\partial F_i \to \partial M$, $i = 0, 1$ are isotopic.

Then there is an orientable 3-manifold V embedded in $M \times I$ such that $V \cap M \times 0 = F_0$ and $V \cap M \times 1 = F_1$, and $\partial V = V \cap M \times 0 \cup M \times 1 \cup \partial F_0 \times I$. This result is proven in [13, Theorem II.1.1], but the proof is lacking some details, so we give a proof in the appendix.

As usual, F_1 is constructed from F_0 by successively attaching handles to $\mathrm{Int}(F_0) \times [0, \varepsilon]$. By the standard theory of handlebody decompositions, we may assume that only 1- and 2-handles are attached. Since a 2-handle corresponds to a 1-handle in the dual handlebody decomposition (starting at F_1) it suffices to show that attaching a 1-handle h leaves the Arf invariant unchanged.

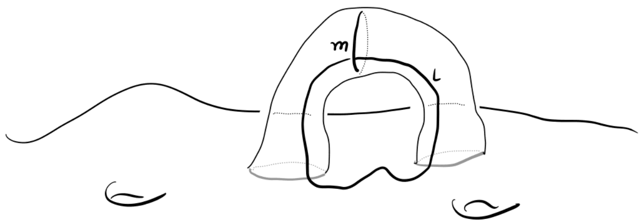

FIGURE 3. Adding a one-handle to a surface

The quadratic form q changes after attaching a 1-handle by addition of a copy of the quadratic form

$$(\mathbb{Z}/2[m] \oplus \mathbb{Z}/2[l], \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, q)$$

where m corresponds to the transverse circle and l runs through the attached handle, so m, l give a symplectic basis. We also have $q(m) = 0$ by choosing the obvious disc bounding m inside the handle, so that

$$\mathrm{Arf}(s) = q(m)q(l) = 0 \cdot q(l) = 0.$$

The Arf invariant is additive with respect to direct sums of quadratic forms, so

$$\mathrm{Arf}(F_0, \partial F_0) = \mathrm{Arf}(F_0, \partial F_0) + \mathrm{Arf}(s) = \mathrm{Arf}(F_1, \partial F_1). \qquad \square$$

2.3. **The Proof.** Now we are ready to formulate the main theorem of this discussion.

Theorem 2.9. *Let M be a closed oriented 4-manifold with $H_1(M;\mathbb{Z}) = 0$ and let ξ be an integral characteristic homology class. Then*

$$\mathrm{Arf}(\xi) = (\sigma(M) - \xi \cdot \xi)/8 \mod 2.$$

We will prove this theorem soon, first giving some applications. We start with Rochlin's theorem.

Corollary 2.10 (Rochlin's theorem)**.** *The signature of an oriented spin 4-manifold is divisible by 16.*

Proof. Kill the first homology group of M via framed surgery on embedded circles, preserving the spin structure and signature. Now the empty set is a characteristic surface and evidently has zero self-intersection and Arf invariant. Thus,

$$0 = \sigma(M)/8 \mod 2$$

which of course just means that 16 divides $\sigma(M)$. $\square$

Remark 2.11. Another cute application of Theorem 2.9 is the following. Consider the class $\xi = 3 \cdot [\mathbb{C}P^1] \in H_2(\mathbb{C}P^2;\mathbb{Z})$. Applying the congruence, we see that $\mathrm{Arf}(\xi) = 1$. Observe that the Arf invariant of a class that can be represented by an embedded sphere, is always trivial, as the sphere is simply connected. Hence, ξ cannot be represented by an embedded sphere. However, it is known that a smooth algebraic curve of degree d in $\mathbb{C}P^2$ represents the class $d \cdot [\mathbb{C}P^1]$. The genus of such an algebraic curve is known to be $\frac{(d-1)(d-2)}{2}$. In particular, $3 \cdot [\mathbb{C}P^1]$ can be represented by a smooth cubic, i.e. an elliptic curve which is diffeomorphic to a torus.

As a preliminary step to prove Theorem 2.9, we will need a lemma to gain some flexibility.

Let M be an oriented 4-manifold and $F \subset M$ an embedded oriented surface. A *meridian* of F in M is a circle C in $M \setminus F$ which bounds an embedded disc D such that F intersects the interior of D precisely once.

Lemma 2.12. *Let M be a closed, oriented 4-manifold with $H_1(M;\mathbb{Z}) = 0$ and let F be a characteristic surface. Every map $f\colon S^1 \to M$ is homotopic to an embedding not intersecting F. Additionally allowing f to change by a meridian of F we can arrange that doing surgery on f with the framing induced by the spin structure on $M - F$ preserves the property that F is characteristic and* $\mathrm{Arf}(F)$.

Proof. By Whitney approximation we may assume that f is a smooth embedding and by transversality, we may assume that $f(S^1)$ and F do not intersect.

Since $H_1(M) = 0$, we know that $f(S^1)$ bounds a surface D in M. Making D transverse to F, we can assume that D and F have transverse intersections, in k points, say. Using a meridian of F, we can change f to arrange for this count to be algebraically zero. Let M' be the result of surgery along f with the framing dictated by the spin structure. The second homology of M' is generated by the homology of $M \setminus f(S^1)$, by a small $S^2 \cong K \subset M$ around $f(S^1)$ and the newly added 2-handle together with the null homology D of $f(S^1)$ in $M \setminus \nu(f(S^1))$. For later purposes, we choose K to locally look like $S^2 \times \{0\} \subset \mathbb{R}^4$ in a submanifold chart of $f(S^1)$, where S^1 is embedded as $\{0\}^3 \times \mathbb{R}$, such that it does not intersect F. If $S \subset M \setminus f(S^1)$ is a surface, then its intersections with F satisfy

$$S \cdot S \equiv S \cdot F \mod 2$$

since this equation is satisfied in M. If S is the surface given by the null homology D of f together with the core of the newly added handle, then $S \cdot F = 0$. Now since we chose the framing of $f(S^1)$ for the surgery in a way such that $M' \setminus F$ admits a spin structure, the surface $S \subset M' \setminus F$ satisfies $S \cdot S \equiv 0 \mod 2$. Clearly, K can be pushed off itself, so that we have $K \cdot K = K \cdot F = 0$.

We have shown that the equation

$$S \cdot S \equiv S \cdot F \mod 2$$

holds on surfaces whose homology classes generate $H_2(M')$. Thus, F is still characteristic.

The Arf invariant of F does not change as well: for every circle $C \subset F$, we may choose a surface D bounding C which does not intersect $f(S^1)$ by transversality. Clearly, after surgery on $f(S^1)$, all of the numbers $D \cdot F$, $\mathrm{Self}(C)$ and $O(D)$ are preserved. □

Proof of Theorem 2.9. We want to prove that a closed, oriented manifold M with $H_1(M) = 0$ and a characteristic surface F fulfills

$$\mathrm{Arf}(F) = (\sigma(M) - F \cdot F)/8 \mod 2.$$

By Lemma 2.12 we can do finitely many surgery steps away from F to arrange that M is simply connected without changing the fact that F is characteristic or $\mathrm{Arf}(F)$.

Notice that both sides of the equation are additive with respect to the pairwise connected sum. Luckily, Wall calculated the Grothendieck group of simply-connected 4-manifolds with respect to $\mathbb{C}P^2$.

Theorem 2.13 (Wall, [4], 4.4 on page 148)**.** *Let M be a simply-connected 4-manifold. Then there are integers k, l, m, n such that*

$$M\sharp^k\mathbb{C}P^2\sharp^l(-\mathbb{C}P^2) \cong \sharp^m\mathbb{C}P^2\sharp^n(-\mathbb{C}P^2).$$

A characteristic surface for $\mathbb{C}P^2$ is $\mathbb{C}P^1$ with its standard embedding, and if has 0 Arf invariant, self-intersection 1 and signature 1, so that the formula is satisfied. A similar statement holds for $(-\mathbb{C}P^2, -\mathbb{C}P^1)$, hence we can apply Wall's theorem and only have to prove the statement for a characteristic surface in $\sharp^m \mathbb{C}P^2 \sharp^n(-\mathbb{C}P^2)$. Up to cobordism (which neither changes the Arf invariant nor self-intersections), the characteristic surface is a connected sum of surfaces in the $\mathbb{C}P^2$s and $-\mathbb{C}P^2$s, so we can apply additivity of the formula with respect to pairwise connected sums to reduce to the case of a characteristic surface in $\mathbb{C}P^2$ or $-\mathbb{C}P^2$ respectively. Both cases are equivalent, as can be seen by changing orientations, so we only have to consider the case of $\mathbb{C}P^2$.

A characteristic surface in $\mathbb{C}P^2$ corresponds to an odd class in $H_2(\mathbb{C}P^2)$, say $s \cdot [\mathbb{C}P^1]$. It is represented by the zero set of a generic homogenous polynomial of degree s. It is convenient to choose the following algebraic curve C:

$$\{[x : y : z] | x^s - y^{s-1} z = 0\}.$$

This curve is not smoothly embedded, but we will cut out the only singularity at $[0 : 0 : 1]$ and replace it by something smooth[2].

On the set where $z \neq 0$, we have the curve $x^s - y^{s-1} = 0$. Let B be a small ball around the origin with boundary $S \cong S^3$. Then $S \cap C$ is a torus knot of type $(s, s-1)$: take B to be the ball with radius 2, consider the map $S^1 \times S^1 \to B$ sending (z, w) to itself and observe that the $K(s-1, s) := \{(z^{s-1}, z^s) | z \in S^1\}$ is sent bijectively to the set $\{x^s - y^{s-1} = 0\} \cap \partial B$. Pick a Seifert surface $G \subset S$ for this knot. Then

$$\mathrm{Arf}(F) = \mathrm{Arf}(G) = \mathrm{Arf}(K(s, s-1)).$$

It is not to hard to compute the Arf invariant of a torus knot. It will be explained in Section 2.4. The result is

$$\mathrm{Arf}(K(s, s-1)) = (1 - s^2)/8 \mod 2.$$

But

$$(1 - s^2)/8 = (\sigma(\mathbb{C}P^2) - s[\mathbb{C}P^1] \cdot s[\mathbb{C}P^1])/8 \mod 2,$$

finishing the proof. □

Remark 2.14. Theorem 2.13 is also used as an input in the proof by Freedman-Kirby [6]. In [7, p. 128], Matsumoto gives an easy proof of this fact, using that the signature induces an isomorphism

$$\sigma \colon \Omega_4^{\mathrm{SO}} \to \mathbb{Z}$$

from the oriented cobordism group in dimension four to the integers.

[2]This still defines the same homology class as an algebraic curve close to C, e.g. the zero set of $x^s - y^{s-1}z + \epsilon y^s$.

2.4. **Explanation of the knot-theoretic facts.** We briefly explain the calculation of the Arf invariant of a torus knot.

Let (p, q) be coprime integers. The (p, q)-torus knot is the knot represented by the embedding

$$f_{p,q}\colon S^1 \hookrightarrow S^1 \times S^1 = T^2 \subset S^3$$

sending t to (t^p, t^q), where $T^2 \subset S^3$ is a standard unknotted embedding, e.g. $(z, w) \mapsto (z, w)/\sqrt{2}$. In the following, we will call this knot $K_{p,q}$.

We can compute the Alexander polynomial of the (p, q)-torus knot to be

$$\Delta_{K_{p,q}} = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}.$$

This is shown in [10, p. 178].

Hence, we can calculate the Arf invariant by plugging in -1 as shown in [5, p. 544], namely for a knot K with Alexander polynomial Δ_K we have

$$\mathrm{Arf}(K) = \begin{cases} 0 \text{ if } \Delta_K(-1) \equiv \pm 1 \mod 8; \\ 1 \text{ if } \Delta_K(-1) \equiv \pm 3 \mod 8. \end{cases}$$

First assume p and q to be odd:

$$\Delta_{K_{p,q}}(-1) = \frac{4}{4} = 1.$$

Assuming p to be even and q odd, and using de L'Hospitals rule:

$$\Delta_{K_{p,q}}(-1) = \frac{-(1-(-1)^{pq}) - (1-(-1))pq(-1)^{pq-1}}{-p(-1)^{p-1}(1-(-1)^q) - q(-1)^{q-1}(1-(-1)^p)} = \frac{2pq}{2p} = q.$$

Thus,

$$\mathrm{Arf}(K_{p,q}) = \begin{cases} 0 \text{ if } p, q \text{ both odd or one even and the other } \pm 1 \mod 8; \\ 1 \text{ if one of } p, q \text{ is even and the other } \pm 3 \mod 8. \end{cases}$$

In particular, the Arf invariant of the torus knot of type $(s, s-1)$ with s odd is

$$\frac{1-s^2}{8} \mod 2.$$

Indeed for s odd $s \equiv \pm 1 \mod 8$ if and only if $\frac{1-s^2}{8} \equiv 0 \mod 2$ and $s \equiv \pm 3 \mod 8$ if and only if $\frac{1-s^2}{8} \equiv 1 \mod 2$.

Appendix A. Representing homology classes by embedded surfaces

In the following, we want to show how homology classes in 4-manifolds can be represented by surfaces.

Denote by $\Omega_2^{SO}(M)$ the bordism group of oriented surfaces embedded in M, where two elements F_0, F_1 are bordant if there exists an oriented 3-manifold V embedded in $M \times I$ such that ∂V is the disjoint union of $F_1 \times 1$ and $F_0 \times 0$ with opposite orientation. Here the sum is defined as follows: we can make two embedded surfaces transverse, and at each intersection point take the connected sum.

Let $MSO(2)$ be the Thom space of the universal bundle over $BSO(2) = BU(1) = \mathbb{C}P^\infty$. Thom observed in [2, p. 50], that $MSO(2)$ is again $\mathbb{C}P^\infty$. The Pontryagin-Thom construction yields, for a smooth oriented 4-manifold, a bijection

$$\tau\colon \Omega_2^{SO}(M) \to [M, MSO(2)].$$

We describe τ. Let ν be the normal bundle of an embedded surface in M, identified with a tubular neighborhood. Now ν is classified by a bundle map into the universal bundle, and sending $M \setminus \nu$ to the point at infinity yields a map $M \to MSO(2)$. Here we used that the Thom space of a bundle over a compact space can be constructed by taking the one point compactification of the total space of the bundle.

Since there is a canonical homeomorphism $MSO(2) \cong \mathbb{C}P^\infty$, we can identify homotopy classes of maps $f \in [X, MSO(2)]$ with complex line bundles over X for any CW-complex X. Thus, taking the first Chern class of this bundle induces a map $[X, MSO(2)] \to H^2(X)$. Explicitly, this map is given by pulling back a generator of $H^2(MSO(2)) \cong \mathbb{Z}$.

Our goal is to prove the following theorem, which is a reformulation of [13, Theorem II.1.1].

Theorem A.1. *Let M be a closed, oriented, smooth 4-manifold. Then the following is a commutative diagram of isomorphisms:*

$$\begin{array}{ccc} \Omega_2^{SO}(M) & \xrightarrow{\tau} & [M, MSO(2)] \\ \big\downarrow{\scriptstyle e} & & \big\downarrow{\scriptstyle c_1} \\ H_2(M) & \xrightarrow{PD^{-1}} & H^2(M) \end{array}$$

Here e sends an embedded surface to the image of its fundamental class, PD^{-1} is the inverse of capping with the fundamental class of M and c_1 is the first Chern class.

Proof. Note that τ, c_1 and PD^{-1} are isomorphisms respectively by the Pontryagin-Thom theorem, $MSO(2)$ being a $K(\mathbb{Z}, 2)$ and Poincaré duality. Hence, it suffices to show that the diagram commutes to finish the proof. We will show $c_1 = PD^{-1} \circ e \circ \tau^{-1}$. The inverse of τ can be described as follows: since M is 4-dimensional, $[M, MSO(2)] = [M, \mathbb{C}P^2]$ and we can assume that a map $f\colon M \to \mathbb{C}P^2$ is smooth and transverse to $\mathbb{C}P^1$. Now $\tau^{-1}([f]) = [f^{-1}(\mathbb{C}P^1) \hookrightarrow M]$. The map f classifies a complex line bundle over M and the first Chern class of a complex line bundle is also the Euler class of this bundle. Now recall that the Euler class of an oriented bundle over an oriented manifold is always Poincaré dual to the fundamental class of the intersection of a generic section with the zero section, by [8, Chapter VI, Proposition 12.8]. □

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