Block bundles: I

By C. P. ROURKE and B. J. SANDERSON*

Introduction

The purpose of this and two subsequent papers is to establish for the piecewise-linear category a tool analogous to the vector bundle in the differential category.

The fundamental problem is this. Given a smooth submanifold $M^n \subset Q^{n+q}$, one can define a normal bundle to M in Q. It is a q-vector bundle uniquely determined up to isomorphism. What is the correct PL analogue of this result?

The answer we present here is the normal block bundle, and the classic construction is the following. Suppose M, Q are triangulated by complexes K, L with K full in L. Form the dual complexes K^*, L^*. Each t-cell $\sigma \in K^*$ is contained in a unique $(t + q)$-cell of L^*, called the block over σ. These blocks fit together to form a normal block bundle to M in Q.

In § 1, we give the precise definition of a block bundle and prove several fundamental results. We also define cartesian product, Whitney sum, and induced block bundle.

In § 2, we construct a universal block bundle using an analogue of the grassmannian.

In § 3, we introduce the semi-simplicial group $\widetilde{PL}_q$, which plays the same role for block bundles as the orthogonal group for vector bundles, and we associate a principal $\widetilde{PL}_q$-bundle to each block bundle having a simplicial base complex.

In § 4, we prove existence of normal block bundles and a "tubular neighbourhood" theorem which implies uniqueness. This leads to the classification of regular neighbourhoods of manifolds.

In § 5, we are concerned with connections with other types of bundle, and we have obstruction theories for the existence of normal disc-, plane-, and micro-bundles.

In § 6, we give an obstruction theory for the problem of smoothing a PL submanifold of a differential manifold.

In future papers, we will prove the transversality theorems announced in [14], and investigate the groups of PL immersions and embeddings of spheres

* The second named author was partially supported during the summer of 1965 by the Centro de Investigacion del I.P.N., Mexico. D.F.

in spheres, with and without normal bundle, etc. (cf. Haefliger [3], Levine [12]).

Since this paper was written, we have proved that normal microbundles do not always exist in the PL category (see [14B]); thus block bundles appear to be the best possible bundle theory for this category. The normal block bundle forms a natural extension of the existing most powerful PL tool, namely the regular neighbourhood.

Notation

We work in the category of polyhedra and piecewise linear (PL) maps. If no statement is made to the contrary, all maps and spaces are in this category. In particular, we use "homeomorphism" to mean an isomorphism in this category.

A *polyhedron* is a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ a (maximal) family of PL related locally finite triangulations. For a more general definition, see Zeeman [19]. If $t\colon K \to X$ is a triangulation of X, then we write $|K| = X$. Also, for obvious reasons, we confuse the space X with the polyhedron $(X, \mathcal{F})$.

We have the following standard objects; *euclidean n-space* R^n, which may be identified with the n-fold cartesian product of R^1. There are inclusions $R^n \subset R^{n+1} \subset \cdots$, and R^∞ is the topological union. Of course R^∞ is not a polyhedron but it has a "polystructure", see [19]. The *n-cube* $I^n = [-1, +1]^n \subset R^n$, and the *unit interval* $I = [0, 1]$. The *$(n-1)$-sphere* $\Sigma^{n-1} = \dot{I}^n$ or ∂I^n, the boundary of I^n. The *n-simplex* $\Delta^n \subset R^n$ and $\Lambda_n = \mathrm{cl}(\dot{\Delta}^n - \Delta^{n-1})$. An *n-ball* or *cell* (*n-sphere*) is a homeomorph of $I^n(\Sigma^n)$.

Let X, Y be polyhedra. An *isotopy of X in Y* is a level preserving embedding $F\colon X \times I \to Y \times I$. An *isotopy of X* is a level preserving homeomorphism $F\colon X \times I \to X \times I$ such that $F \mid X \times \{0\} = 1$ (the identity map). (Note the distinction between an isotopy of X and an isotopy of X in X). An *ambient isotopy of X in Y* is an isotopy of X in Y which is the restriction of an isotopy of Y. An isotopy (ambient isotopy, etc.) of X mod a subpolyhedron X', is an isotopy F such that $F_t \mid X' = F_0 \mid X'$ all $t \in I$.

An *n-manifold* is a polyhedron each point of which has an n-ball neighbourhood. All embeddings of manifolds will be assumed to be *locally flat* (i.e., the image is a submanifold). Let M^n, Q^{n+q} be manifolds, then an embedding $f\colon M \to Q$ is *proper* if $f^{-1}(\partial Q) = \partial M$, and *admissible* if $f^{-1}(\partial Q)$ is an $(n-1)$-dimensional submanifold of ∂M. Let $M \subset Q$ be a proper submanifold. A *collar* of $(\partial Q, \partial M)$ in (Q, M) is an embedding $h\colon (\partial Q, \partial M) \times I \to (Q, M)$ such that $h \mid (\partial Q, \partial M) \times \{0\} = 1$ and $h^{-1}M = \partial M \times I$. There are similar definitions of collars in more general situations. Collars always exist [18].

By a *complex*, we mean a collection K of cells which cover a polyhedron X satisfying;

(i) $\sigma, \tau \in K$ implies $\partial\sigma$, $\sigma \cap \tau$ are unions of cells in K.

(ii) $\operatorname{int}\sigma \cap \operatorname{int}\tau = \emptyset$.

Again we write $|K| = X$. If K', K are complexes such that $|K'| = |K|$, then K' is a *subdivision* of K if, whenever $\sigma \in K'$, there exists $\tau \in K$ such that $|\sigma| \subset |\tau|$ (here σ and τ are identified with the subcomplexes they determine). It can be shown that any complex K has a subdivision given by a triangulation of $|K|$. If K and L are complexes, then $K \times L$ is a complex, and, if J is a subcomplex of K ($J \subset K$), and J' a subdivision, then $J' + K$ (defined by $\sigma \in J' + K$ if and only if $\sigma \in J'$ or, $\sigma \in K$ and $|J| \cap \operatorname{int}|\sigma| = \emptyset$) is also a complex.

A collapse from X to Y is written $X \searrow Y$, and an elementary collapse $X \overset{e}{\searrow} Y$ as usual (see Zeeman [19] for details).

We use the terminology c.s.s. *set* (or *complex*) in its usual sense (see, for example, Kan [9A]), in particular, a c.s.s. set always possesses degeneracy functions, we also use the terminologies *pointed* c.s.s. *set*, c.s.s. *group*, c.s.s. *map*, etc. In our treatment of semi-simplicial sets, we will rely on our note [14C] which showed how to work without assuming the existence of degeneracies. A semi-simplicial set with no degeneracy functions will be called a Δ-*set*, similarly there are *pointed* Δ-*sets*, Δ-*groups*, Δ-*maps*, etc. See [14C] for the precise definitions, and also for notions of homotopy equivalence, and an analogue of J.H.C. Whitehead's theorem.

1. Block bundle theory

A *q-block bundle* ξ^q/K consists of a total space $E(\xi)$ and a complex K such that $|K| \subset E(\xi)$, satisfying

(i) For each n-cell $\sigma_i \in K$, there exists an $(n+q)$-ball $\beta_i \subset E(\xi)$ such that

$$(\beta_i, \sigma_i) \cong (I^{n+q}, I^n).$$

β_i is called the *block* over σ_i.

(ii) $E(\xi)$ is the union of the blocks β_i.

(iii) The interiors of the blocks are disjoint.[1]

(iv) Let $L = \sigma_i \cap \sigma_j$, then $\beta_i \cap \beta_j$ is the union of the blocks over cells of L.

Block bundles $\xi^q, \eta^q/K$ are *isomorphic*, written $\xi \cong \eta$, if there exists a homeomorphism

$$h: E(\xi) \longrightarrow E(\eta)$$

[1] By an oversight, this condition was omitted from our announcement [14], we are indebted to R.Z. Goldstien for pointing out the omission.

such that $h \mid K = 1$ and $h(\beta_i(\xi)) = \beta_i(\eta)$ for each $\sigma_i \in K$.

ξ^q/K is *trivial* if it is isomorphic with *the trivial block bundle* ε^q/K, where $E(\varepsilon^q/K) = |K| \times I^q$ and $\beta_i(\varepsilon^q) = \sigma_i \times I^q$ for each $\sigma_i \in K$.

If $L \subset K$ and ξ/K, then the *restriction* $\xi \mid L$ is defined by $\beta_i(\xi \mid L) = \beta_i(\xi)$ for each $\sigma_i \in L$.

Let $\sigma \in K$ and ξ/K. We may also regard σ as a subcomplex of K. An embedding

$$h: \sigma \times I^q \longrightarrow E(\xi \mid \sigma) \subset E(\xi)$$

is a *chart* for ξ if it is an isomorphism of ε^q/σ with $\xi \mid \sigma$. An *atlas* for ξ^q/K is a collection $\{h_i\}_{\sigma_i \in K}$ of charts.

THEOREM 1.1. *Given ξ^q/K such that $|K| \cong I^n$, then $\xi^q \cong \varepsilon^q$.*

The proof of 1.1 is by induction on n. We shall assume the result in dimensions less than n and mark theorems depending on this assumption with a star.

LEMMA 1.2. *Suppose $(B_i^{n+q}, B_i^n) \cong (I^{n+q}, I^n)$ $i = 1, 2$, and $h: \dot{B}_1^{n+q} \cup B_1^n \longrightarrow \dot{B}_2^{n+q} \cup B_2^n$ is a homeomorphism ($hB_1^n = B_2^n$). Then there exists a homeomorphism $h': B_1^{n+q} \longrightarrow B_2^{n+q}$ extending h.*

For a proof see [19; Lem. 18]. This lemma is needed to prove the following key proposition.

PROPOSITION 1.3.* *Suppose $|K| \cong I^n$, and K has just one n-cell σ^n. Let σ^{n-1} be any $(n-1)$-cell in K, and let L be the subcomplex of K consisting of all cells except σ^n and σ^{n-1}. Suppose given ξ^q/K and an isomorphism*

$$t: E(\varepsilon^q/L) \longrightarrow E(\xi^q \mid L).$$

Then t extends to an isomorphism

$$t': E(\varepsilon^q/K) \longrightarrow E(\xi^q/K).$$

PROOF. Consider $(E(\xi \mid L), L)$. It is an unknotted ball pair by hypothesis. Similarly $(\dot{\beta}^{n+q} - \operatorname{int}(E(\xi \mid L)), \sigma^{n-1})$, where β^{n+q} is the block over σ^n, is an unknotted ball pair by [8, Cor. 8], since it is the complement of the first pair in the unknotted sphere pair $(\dot{\beta}^{n+q}, \dot{\sigma}^n)$. Hence, by Lemma 1.2, there is a homeomorphism $t_1: \partial(\sigma^n \times I^q) \longrightarrow \dot{\beta}^{n+q}$ extending t and the identity on σ^{n-1}.

Let β^{n+q-1} be the block over σ^{n-1}. By Theorem 1.1, there exists an isomorphism

$$h: \sigma^{n-1} \times I^q \longrightarrow E(\xi \mid \sigma^{n-1}) = \beta^{n+q-1},$$

and it follows that β^{n+q-1} is a regular neighbourhood of σ^{n-1} in $M = \dot{\beta}^{n+q} - \operatorname{int} E(\xi \mid L)$ which meets the boundary regularly. Similarly $t_1(\sigma^{n-1} \times I^q)$ is a regular neighbourhood of σ^{n-1} in M, and both meet the boundary regularly in the same set. (see Figure 1.)

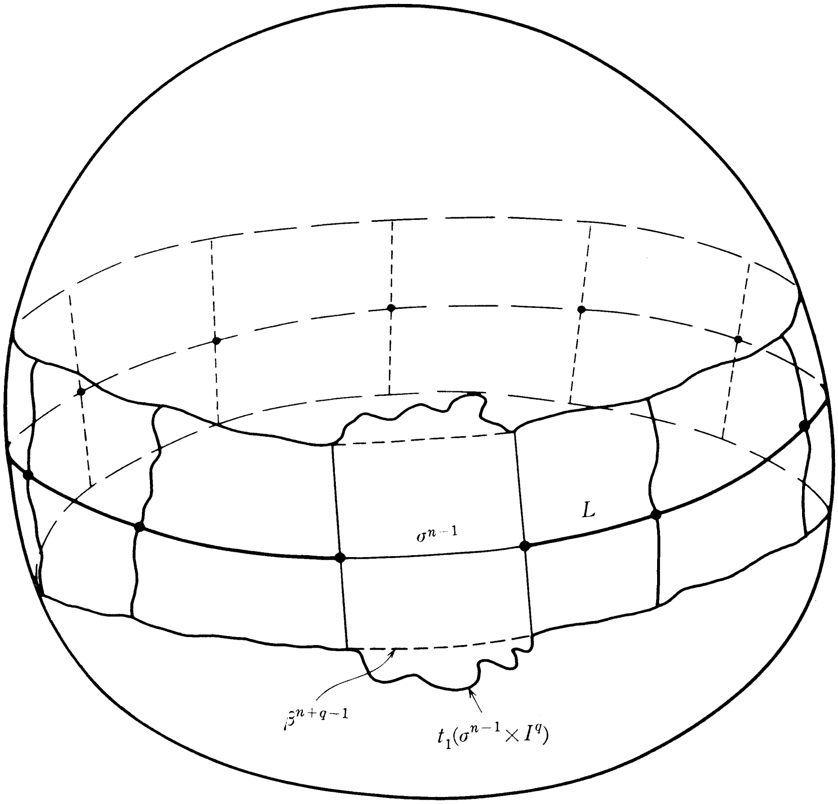

Figure 1

Hence it follows from [9] that there is an isotopy F_s of M mod $\dot{M} \cup \sigma^{n-1}$, such that $F_1 t_1(\sigma^{n-1} \times I^q) = \beta^{n+q-1}$. Now $F_1 t_1 \mid M$, t on $L \times I^q$, and the identity on K, extend by Lemma 1.2 to the required isomorphism

$$t'\colon E(\varepsilon^q/K) \longrightarrow E(\xi^q/K).$$

PROPOSITION 1.4.* *Any* ξ^q/K, $\dim K \leqq n$, *has an atlas.*

PROOF. We have to show that $\xi^q \mid \sigma$ is trivial for each $\sigma \in K$. If $\dim \sigma < n$, the result follows at once from Theorem 1.1. Let σ^n be an n-cell of K, and let σ^{n-1} be an $(n-1)$-cell in $\dot{\sigma}^n$. Define L as in Proposition 1.3; $\xi \mid L$ is trivial by Theorem 1.1, and any trivialization extends to a trivialization of $\xi \mid \sigma^n$ by Proposition 1.3, completing the proof.

Definition. Suppose K' is a subdivision of K and ξ/K. ξ'/K' is a *subdivision* of ξ if $\beta_i(\xi) = \bigcup \beta_j(\xi')$, where the union is taken over all j such that $\sigma_j \in K'$, and $|\sigma_j| \subset |\sigma_i|$.

Now suppose given ξ/K, $L \subset K$ and η/L' a subdivision of $\xi \mid L$. We define $\eta + \xi/L' + K$ by $E(\eta + \xi) = E(\xi)$ and

$$\beta_i(\eta + \xi) = \begin{cases} \beta_i(\eta) & \text{if } \sigma_i \in L' \\ \beta_i(\xi) & \text{if } \sigma_i \in K - L. \end{cases}$$

THEOREM 1.5.* *Suppose given ξ/K and K' a subdivision of K,* $\dim K \leqq n$. *Then there exists a subdivision ξ'/K' of ξ.*

PROOF. Suppose $\xi_r/(K^r)'$ is a subdivision of $\xi \mid K^r$, where K^r denotes the r-skeleton of K. We wish to extend this subdivision over K^{r+1}. Without loss of generality, we may suppose K^{r+1} contains a single $(r+1)$-cell, and K^r is its boundary. Now $\xi_r + \xi/(K^r)' + K^{r+1}$ is trivial by Proposition 1.4, and a choice of trivialization determines ξ_{r+1}.

Remarks. (a) Results 1.4, 1.5, and 1.6 (below) are true for $\dim K = \infty$ once the induction is complete.

(b) Uniqueness of subdivision up to isomorphism will be proved later (Theorem 1.9), and uniqueness up to isotopy in § 4.

THEOREM 1.6.* *Suppose given ξ_i^q/K, $i = 1, 2$, $\dim K \leqq n$ and $|K| \searrow |L|$ (L a subcomplex of K). Then any isomorphism*

$$h\colon E(\xi_1 \mid L) \longrightarrow E(\xi_2 \mid L)$$

extends to an isomorphism

$$h\colon E(\xi_1) \longrightarrow E(\xi_2).$$

PROOF. Let K', L' be subdivisions of K, L such that there is a simplicial collapse $K' \overset{s}{\searrow} L'$, by [19; Th. 4] applied to the (finite) complexes $\mathrm{cl}\,(K - L)$, $\mathrm{cl}\,(K - L) \cap L$.

Let ξ_1'/K' be a subdivision of ξ_1 (by Theorem 1.5), and let ζ/L' be the subdivision of $\xi_2 \mid L$ determined by ξ_1' and h. Then again by Theorem 1.5, there is a subdivision ξ_2'/K' of $\zeta + \xi_2/L' + K$. If we prove the theorem for ξ_i', it will follow for ξ_i on taking unions of blocks. We prove this by induction on the simplicial collapse.

Now if $K_r' \overset{s}{\searrow} K_{r+1}'$, then we can extend an isomorphism of $\xi_1' \mid K_{r+1}'$ with $\xi_2' \mid K_{r+1}'$ to an isomorphism of $\xi_1' \mid K_r'$ with $\xi_2' \mid K_r'$ by Proposition 1.3, since $\xi_i' \mid \mathrm{cl}\,(K_r' - K_{r+1}')$ is trivial by Proposition 1.4.

COROLLARY 1.7.* *If $|K| \searrow 0$, $\dim K \leqq n$, then any ξ/K is trivial.* In particular, we have Theorem 1.1 in dimension n, *completing the induction.*

COROLLARY 1.8. *Suppose given ξ^q, $\eta^q/K \times I$ and an isomorphism*

$$h\colon E(\xi \mid K \times \{0\}) \longrightarrow E(\eta \mid K \times \{0\}).$$

Then there is an isomorphism

$$h'\colon E(\xi) \longrightarrow E(\eta)$$

which extends h.

PROOF. Let $\sigma \in K$. Then $\sigma \times I \searrow \sigma \times \{0\} \cup \dot{\sigma} \times I$, and the result follows from Theorem 1.6 by induction up the skeleton of K.

THEOREM 1.9. *Suppose ξ', η' are subdivisions of ξ/K. Then ξ' and η' are isomorphic.*

PROOF. Define $\xi \times I/K \times I$ by taking $E(\xi \times I) = E(\xi) \times I$, and blocks $\beta_i \times A$ over $\sigma_i \times A$, for $A = \{0\}, \{1\}, I$.

Subdivide $\xi \times \{0\}/K' \times \{0\}$ to $\xi' \times \{0\}$ and $\xi \times \{1\}/K' \times \{1\}$ to $\eta' \times \{1\}$, and extend over $K' \times I$, using Theorem 1.5, to give $\zeta/K' \times I$.

There exists an isomorphism $h\colon E(\xi' \times I) \rightarrow E(\zeta)$ extending the identity on $E(\xi' \times \{0\})$ by Corollary 1.8. Restricting h to $E(\xi' \times \{1\})$ gives the required isomorphism.

We can now make the following definition.

Definition. Suppose given ξ/L and η/K with $|L| = |K|$. We write $\xi \sim \eta$, ξ is *equivalent* to η, if there exist subdivisions ξ', η' such that $\xi' \cong \eta'$.

The relation is symmetric and reflexive (trivially), and transitive, for suppose $\xi/K \sim \eta/L \sim \zeta/J$. Then, by definition, there exist $\xi' \cong \eta_1/L_1$ and $\zeta' \cong \eta_2/L_2$. Let L' be a common subdivision of L_1 and L_2, and let η_1', η_2' be subdivisions of η_1, η_2 over L'. Now $\eta_1' \cong \eta_2'$ by Theorem 1.9, since they are both subdivisions of η, and since they are also subdivisions of ξ and ζ, the result follows.

Definition. Given ξ/K', K' a subdivision of K, the *amalgamation bundle* η/K is formed by defining

$$\beta_i(\eta) = \bigcup \beta_j(\xi)$$

where the union is taken over all $\sigma'_j \in K'$ such that $|\sigma'_j| \subset |\sigma_i|$. It follows from Theorem 1.1 that $(\beta_i(\eta), \sigma_i)$ is an unknotted ball pair. Thus condition (i) of the definition of a block bundle is satisfied. Conditions (ii), (iii) and (iv) are easily checked. Note that whilst amalgamation and subdivision are inverse operations, subdivision is not well defined on representatives but only on isomorphism classes (Theorem 1.9).

Now let $I_q(K)$ denote the set of isomorphism classes of q-block bundles over K, and $I_q(X)$ the set of equivalence classes over X. Associating to each block bundle its equivalence class defines a function $\alpha_K\colon I_q(K) \rightarrow I_q(|K|)$.

THEOREM 1.10. *Let $X = |K|$, then*

$$\alpha_K\colon I_q(K) \longrightarrow I_q(X)$$

is a bijection .

PROOF. To prove α_K is onto, suppose given ξ/L with $|L| = X$. Let K' be a common subdivision of K and L, and let ξ'/K' be a subdivision of ξ. Amalgamating blocks, we have ζ/K with $\zeta \sim \xi$.

Now suppose given ξ/K, η/K such that $\xi \sim \eta$. An isomorphism of ξ' with η' is also an isomorphism of ξ with η, and therefore α_K is injective.

Remark. The bijection $\alpha_L^{-1}\alpha_K\colon I_q(K) \to I_q(L)$ for $|K| = |L|$, may be described as follows. Take any common subdivision J of K and L, subdivide over J and then amalgamate.

We now define further operations on block bundles exhibiting a strong analogy with vector bundle theory.

Restriction. Let u/X be an equivalence class, and let Y be a closed subspace of X. Then there exist triangulations of X with Y a subcomplex, and the restriction $u\,|\,Y$ is a well-defined equivalence class by Theorems 1.9 and 1.10.

Remark. Restriction over an arbitrary subspace will be defined below, as a special case of induced bundle.

Cartesian product. Suppose given ξ/K, η/L. The *cartesian product* $\xi \times \eta/K \times L$ is defined by

(i) $E(\xi \times \eta) = E(\xi) \times E(\eta)$

(ii) $\beta_{ij}(\xi \times \eta) = \beta_i(\xi) \times \beta_j(\eta)$, $\qquad \sigma_i \in K$, $\sigma_j \in L$.

We leave it to the reader to check that this is indeed a block bundle, and that the operation is well-defined on equivalence classes.

We denote the product of u/X with the class of ε^0/K ($|K| = Y$) by $u \times Y$.

Whitney sum. Given equivalence classes u/X, v/X, the *Whitney sum* $u \oplus v/X$ is defined to be $u \times v\,|\,\Delta$, where $\Delta = \{(x, x) \in X \times X\}$ is identified with X by the diagonal map.

Induced bundle. Let $f\colon X \to Y$, and suppose given u/Y. The induced class f^*u/X is defined by

$$f^*u = X \times u\,|\,\Gamma f\,,$$

where $\Gamma f = \{(x, fx) \in X \times Y\}$ is identified with X by the projection.

THEOREM 1.11. *Let $u/X \times I$ be an equivalence class, then*

$$u = \big(u\,|\,X \times \{0\}\big) \times I\,.$$

PROOF. Pick a representative $\xi/K \times I$ for u. It is unique by Theorem 1.10, and the result follows by Corollary 1.8.

From the definition of induced bundle and Theorem 1.11 we have

COROLLARY 1.12. *Let $f, g\colon X \to Y$ be homotopic maps, and let u/Y be an equivalence class. Then $f^*u = g^*u$.*

At this point it is not clear that $(fg)^* = g^*f^*$, and in order to prove this fact, we introduce another definition of induced bundle, which is also needed in § 2.

Let K be a simplicial complex, and suppose ξ^q/K is a *subbundle* of the infinite dimensional trivial bundle ε^∞/K. More precisely, $E(\xi) \subset K \times R^\infty$, and

$$E(\xi) \cap \sigma_i \times R^\infty = \beta_i(\xi) \qquad \text{each } \sigma_i \in K.$$

Let $f\colon L \to K$ be a simplicial map. We define $f^\#\xi/L$ as follows.

(i) $E(f^\#\xi) = (f \times 1)^{-1}E(\xi)$, where $f \times 1\colon L \times R^\infty \to K \times R^\infty$.

(ii) $\beta_i(f^\#\xi) = \sigma_i \times R^\infty \cap E(f^\#\xi)$, each $\sigma_i \in L$.

The following proposition links the two definitions, and together with Theorem 1.10, proves that $f^\#\xi$ does not depend on the embedding of $E(\xi)$ in $K \times R^\infty$.

PROPOSITION 1.13. *Suppose ξ/K is a subbundle of ε^∞/K, and $f\colon L \to K$ a simplicial map. Then $f^\#\xi$ is a member of the equivalence class of f^*u, where u is the equivalence class of ξ.*

PROOF. Let $\pi\colon L \times K \to K$ be the projection on K, and let $(L \times K)'$ be a subdivision of $L \times K$ on which π is simplicial, and $\Gamma f\colon L \to (L \times K)'$, $\Gamma f(x) = (x, fx)$, is simplicial [19; Lem. 1]. Consider the commutative diagram

$$\begin{tikzcd} L \times R^\infty \arrow[rr, "f \times 1"] \arrow[dr, "\Gamma f \times 1"'] & & K \times R^\infty \\ & (L \times K)' \times R^\infty \arrow[ur, "\pi \times 1"'] & \end{tikzcd} .$$

Then $\Gamma f \times 1\big(E(f^\#\xi)\big)$ is the restriction of $\pi^\#\xi$ to $\Gamma f(L)$, and indeed, $\Gamma f \times 1$ is an isomorphism between these two bundles. Thus it suffices to prove that $\pi^\#\xi \sim L \times \xi$. But $E(\pi^\#\xi) = E(L \times \xi)$, and the block of $L \times \xi$ over $\sigma \times \tau$, $\sigma \in L$, $\tau \in K$, is a union of blocks of $\pi^\#\xi$, which proves the result.

Remarks. (i) In § 2 we shall prove that any block bundle is isomorphic with a subbundle of ε^∞.

(ii) Suppose $K \subset L$ are simplicial complexes, and ξ/L, and let $i\colon K \to L$ be the inclusion. Then $i^\#\xi \cong \xi \mid K$. Hence restriction bundles are a special case of induced bundles, and we can define restriction over an arbitrary subset, coherently, as the induced bundle by the inclusion map.

(iii) Since it is clear that $(fg)^\# = g^\#f^\#$, and since any map is homotopic to a map which is simplicial with respect to some triangulation, we have, by Corollary 1.11 and Proposition 1.13,

COROLLARY 1.14. $(fg)^* = g^*f^*$ *for any maps f and g.*

2. The classifying space $B\widetilde{PL}_q$

In this section, we construct a locally finite simplicial complex $B\widetilde{PL}_q$ and a universal block bundle $\gamma^q/B\widetilde{PL}_q$.

Let $\mathcal{G}_q$ be the c.s.s. complex whose k-simplexes are subbundles of $\varepsilon^\infty/\Delta^k$ ($\xi^q/\Delta^k \in \mathcal{G}_q$ if for each face Δ^s of Δ^k (including Δ^k), $E(\xi) \cap \Delta^s \times R^\infty = E(\xi \mid \Delta^s)$).

Now let $\lambda: \Delta^l \to \Delta^k$ be a monotone simplicial map, and suppose $\sigma \in \mathcal{G}_q^{(k)}$. Then $\lambda^\# \sigma \in \mathcal{G}_q^{(l)}$, which defines the associated operation.

Let K be a locally finite simplicial complex, and suppose an ordering of its vertices has been chosen (so that each simplex is totally ordered). K may now be regarded as a Δ-set with typical k-simplex a monotone simplicial embedding $\sigma: \Delta^k \to K$. K also determines a c.s.s. set $\boldsymbol{K}$ with typical k-simplex a monotone simplicial *map* $\sigma: \Delta^k \to K$. A Δ-*map* $f: K \to \mathcal{G}^q$ is a dimension preserving function which commutes with the face maps (i.e., regarding Δ-sets as contravariant functors, a Δ-map is just a natural transformation). Note that f and the degeneracies in $\mathcal{G}_q$ determine a unique c.s.s. extension of f from $\boldsymbol{K}$ to $\mathcal{G}_q$.

PROPOSITION 2.1. *Let $f: \dot{\Delta}^k \times I^q \to \dot{\Delta}^k \times R^\infty$ be an isomorphism onto a subbundle. Then there is an extension $f': \Delta^k \times I^q \to \Delta^k \times R^\infty$, which is also an isomorphism onto a subbundle.*

PROOF. First extend f to a map which is the identity on $\Delta^k \times \{0\}$. Then shift f into general position keeping $\dot{\Delta}^k \times I^q \cup \Delta^k \times \{0\}$ fixed [19; Ch. 6]. This replaces f by an embedding with the required properties.

COROLLARY 2.2. *Suppose given ξ^q/K and $L \subset K$, simplicial complexes. Then any isomorphism $f: E(\xi \mid L) \to L \times R^\infty$ onto a subbundle extends to an isomorphism $f': E(\xi) \to K \times R^\infty$ onto a subbundle.*

PROOF. The isomorphism is constructed inductively working up the skeleton of $K - L$ using Propositions 1.4 and 2.1.

COROLLARY 2.3. *$\mathcal{G}_q$ is a Kan complex.*

PROOF. Any block bundle over Λ_k is trivial by Theorem 1.1, and so the result follows by 2.2 with $K = \Delta^k$ and $L = \Lambda_k$.

A *map* $f: X \to \mathcal{G}_q$, where X is a polyhedron, is an ordered triangulation K of X and a Δ-map $f: K \to \mathcal{G}_q$. A *homotopy* is a map of $X \times I$ to $\mathcal{G}_q$. Let $[X, \mathcal{G}_q]$ denote the set of homotopy classes; (cf. [14C] for the connection between this and the usual definition of homotopy in c.s.s. sets).

THEOREM 2.4. *There is a bijection*

$$\varphi: [X, \mathcal{G}_q] \to I_q(X) .$$

PROOF. Let $f: K \to \mathcal{G}_q$ be a Δ-map with $|K| = X$, and let $\sigma^k \in K$. $\sigma^k \times R^\infty$ can be identified with $\Delta^k \times R^\infty$ by means of the ordering of the vertices of K. Thus

$f\sigma$ can be regarded as a subbundle of $\sigma \times R^\infty$, and so f determines a subbundle of $K \times R^\infty$. This defines φ. Now suppose $f: L \to \mathcal{G}_q$ is a homotopy of X in $\mathcal{G}_q$. By the same construction, this gives a block bundle over L, and by Theorem 1.11, the restrictions over $X \times \{0\}$, $X \times \{1\}$ are equivalent. Thus φ is well defined.

Corollary 2.2 implies that φ is onto. To prove φ is 1–1, we need the following lemma, from which the result follows by 2.2.

LEMMA 2.5. *Suppose K_0, K_1 are triangulations of $X \times \{0\}$, $X \times \{1\}$ respectively, then there is a triangulation L of $X \times I$ extending K_0, K_1.*

PROOF. Let K be a common subdivision of K_0 and K_1, and triangulate $X \times \{1/2\}$ by K. We now have a cell complex consisting of simplexes of K_0, K_1, and K, and cells $\sigma \times [0, 1/2]$, $\sigma \in K_0$, and $\sigma' \times [1/2, 1]$, $\sigma' \in K_1$. This complex can be subdivided to the required simplicial complex L by inductively starring at the levels $X \times \{1/4\}$, $X \times \{3/4\}$.

PROPOSITION 2.6. *$I_q(\Sigma^n)$ is countable.*

PROOF. Each block bundle ξ^q/K, $|K| = \Sigma^n$, has a finite description, since $E(\xi)$ may be triangulated so that the blocks and simplexes of K are subcomplexes [19; Th. 1]. Thus there are only a countable number of block bundles up to a homeomorphism of total space, which preserves blocks and simplexes of K setwise (and therefore need not be the identity on Σ_n). Let $h: E(\xi) \to E(\eta)$ be such a homeomorphism, then, by definition of $(\ )^\#$, $\xi \cong (h \mid \Sigma^n)^\# \eta$, and since $h \simeq 1$, the result follows by 1.12.

It follows from Proposition 2.6 that the homotopy groups $\pi_i(\mathcal{G}_q)$ are all countable. We now proceed as in [13]. Using [17; Th. 13], one can show that there exists a locally finite simplicial complex $B\widetilde{PL}_q$ and a homotopy equivalence

$$f: B\widetilde{PL}_q \longrightarrow \mathcal{G}_q\ .$$

Let $\gamma^q/B\widetilde{PL}_q$ be the bundle given by f and Theorem 2.4.

Now let $g: X \to |B\widetilde{PL}_q|$ be a map. Then we have the induced bundle $g^*\gamma^q/X$ defined up to equivalence. By Corollary 1.12, $g^*\gamma^q$ depends only on the homotopy class of g. Write $[X, B\widetilde{PL}_q]$ for the set of homotopy classes of maps of X into $|B\widetilde{PL}_q|$. The above construction gives a function

$$T(X): [X, B\widetilde{PL}_q] \longrightarrow I_q(X)$$

which, by Theorem 2.4, is an isomorphism of sets.

Let $f: X \to Y$ be a map; define $I_q(f): I_q(Y) \to I_q(X)$ by $I_q(f)u = f^*u$. Then T is a natural transformation by Corollary 1.14, and we have proved

THEOREM 2.7. *The transformation*

$$T: [\quad , B\widetilde{PL}_q] \longrightarrow I_q(\quad)$$

given by $T(g) = g^*\gamma$, *is a natural equivalence of functors.*

3. The associated semi-simplicial principal bundle

In this section, we associate a semi-simplicial principal bundle with a block bundle, and construct a universal principal bundle.

Let $\widetilde{PL}_q$ be the Δ-group whose k-simplexes are self-isomorphism of ε^q/Δ^k. Face operators are defined by restriction. Degeneracy operators can be defined, but the natural choice is not piecewise-linear (by the standard mistake), and, if modified, fails to give homomorphisms. For more detail see [14A; § 2]. It follows from the fact that $|\Delta^k| \cong |\Lambda_k \times I|$, that $\widetilde{PL}_q$ satisfies the extension condition.

Now let K be a simplicial complex with ordered vertices (so that the vertices of any simplex are totally ordered), and suppose given a block bundle ξ^q/K.

The associated principal bundle ξ_P is defined as follows

(i) The total Δ-set is defined by

$$E^{(s)} = \{h_i;\ h_i\colon \Delta^s \times I^q \longrightarrow E(\xi \mid \sigma_i) \text{ is a chart for } \xi\}$$

$(h_i \mid \Delta^s \times \{0\}$ is the identification with σ_i determined by the vertex ordering$)$.

(ii) The projection $p\colon E^{(s)} \longrightarrow K^{(s)}$ is given by $p(h_i) = \sigma_i$.

(iii) The action $E^{(s)} \times \widetilde{PL}_q^{(s)} \longrightarrow E^{(s)}$ is composition.

The group $\widetilde{PL}_q^{(s)}$ premutes the elements of $E^{(s)}$ freely with orbit set $K^{(s)}$, and it follows that ξ_P is indeed a Δ-principal bundle with base K, and if $\dim K = k$, the group of ξ_P is the k-skeleton $\widetilde{PL}_q^k$.

Conversely, given ξ_P, a principal bundle with base K and group $\widetilde{PL}_q^k$, we construct a block bundle over K as follows. Let $s\colon K \longrightarrow E(\xi_P)$ be any dimension preserving function satisfying $ps = 1$. Now $\partial_i s(\sigma^t) = s(\partial_i \sigma^t)\cdot F(i, \sigma^t)$ for some uniquely determined $F \in \widetilde{PL}_q^{(t-1)}$. Take the topological sum of trivial bundles ε^q/σ for each $\sigma \in K$. Paste these together by identifying each $\varepsilon^q/\partial_i\sigma$ with $(\varepsilon^q/\sigma) \mid \partial_i\sigma$ by the isomorphism given by $F(i, \sigma)$ and the ordering of the vertices of σ. It may be checked that these identifications are compatible when one passes to a face of σ and this defines a block bundle ξ^q/K.

Let $IP_q^k(K)$ be the set of isomorphism classes of principal $\widetilde{PL}_q^k$-bundles over K ($\dim K = k$). The constructions above are inverse and (cf. Milnor [13]) the following is easily proved.

PROPOSITION 3.1. *Let K^k be a simplicial complex with ordered vertices. The process of assigning to a block bundle its associated principal bundle determines a bijection*

$$I_q(K) \longrightarrow IP_q^k(K) \; .$$

Now let $\boldsymbol{K}$ be the c.s.s. complex determined by K (an s-simplex of $\boldsymbol{K}$ is a monotone simplicial map $\lambda\colon \Delta^s \to K$). We now show how to associate a principal $\widetilde{PL}_q$-bundle (base $\boldsymbol{K}$) with a subbundle ξ/K of ε^∞/K (recall that any bundle is isomorphic with a subbundle of ε^∞ by Corollary 2.2).

DEFINITION. The principal $\widetilde{PL}_q$-bundle $\boldsymbol{\xi}$ with base $\boldsymbol{K}$ associated with a subbundle ξ^q/K of ε^∞/K is defined by

(i) $E^{(s)} = \{h_i;\, h_i\colon \Delta^s \times I^q \to \lambda_i^\sharp(\xi)$ is an isomorphism (where $\lambda_i \in \boldsymbol{K}^{(s)}$)$\}$.

(ii) $p\colon E^{(s)} \to \boldsymbol{K}^{(s)}$ is given by $ph_i = \lambda_i$.

(iii) $E^{(s)} \times \widetilde{PL}_q^{(s)} \to E^{(s)}$ is composition.

Remark. A principal $\widetilde{PL}_q$-bundle over $\boldsymbol{K}$ determines a principal $\widetilde{PL}_q^k$-bundle over K ($\dim K = k$) by restriction and hence a block bundle ξ^q/K. We shall show that this process induces a bijection on isomorphism classes.

Owing to the lack of degeneracies in $\widetilde{PL}_q$, this isomorphism is difficult to construct directly, and in order to do this, we introduce the universal principal bundle (cf. [14C] for an alternative proof).

Let $\tilde{V}_q$ be the Δ-set whose k-simplexes are block and zero preserving embeddings

$$f\colon \Delta^k \times I^q \to \Delta^k \times R^\infty$$

(i.e., $f(\Delta^k \times I^q) \cap \Delta^s \times R^\infty = f(\Delta^s \times I^q)$ for each face Δ^s of Δ^k, and $f \mid \Delta^k \times \{0\} = 1$). Face operators are defined by restriction and degeneracies can be defined inductively by modifying the natural choice.

The following is a direct consequence of Proposition 2.1.

PROPOSITION 3.2. (a) *Any Δ-map $f\colon \dot{\Delta}^k \to \tilde{V}_q$ has an extension $f'\colon \Delta^k \to \tilde{V}_q$.*

(b) *$\tilde{V}_q$ is a Kan complex.*

An action $\tilde{V}_q \times \widetilde{PL}_q \to \tilde{V}_q$ is defined by composition making $\tilde{V}_q$ the total space of a principal $\widetilde{PL}_q$-bundle $\boldsymbol{\gamma}^q$ with base $\mathcal{G}_q$, and an easy argument analogous to the proof of Theorem 2.4 shows that $\boldsymbol{\gamma}^q$ is a universal bundle, (using Proposition 3.2).

Now let $f\colon K \to \mathcal{G}_q$ be a map. f (and the degeneracies in $\mathcal{G}_q$) determine a unique c.s.s. extension $f'\colon \boldsymbol{K} \to \mathcal{G}_q$, and conversely f' determines f.

Let ξ^q/K, which is a subbundle of ε^∞, be the bundle given by f and Theorem 2.4, then it is easy to see that the principal $\widetilde{PL}_q^k$ and $\widetilde{PL}_q$ bundles induced by f and f' respectively are isomorphic to the principal bundles associated with ξ/K.

Let $IP_q(\boldsymbol{K})$ denote the set of isomorphism classes of principal $\widetilde{PL}_q$-bundles over $\boldsymbol{K}$. Then we have proved

THEOREM 3.3. *The process of assigning to a q-subbundle of ε^∞/K its associated $\widetilde{PL}_q$-bundle over $\boldsymbol{K}$ determines a bijection*

$$I_q(K) \to IP_q(\boldsymbol{K}) .$$

This completes the transition to principal bundles.

4. Classification of regular neighbourhoods

Suppose $M^n \subset Q^{n+q}$ is a proper submanifold of Q, and both M and Q are compact. We say Q is an *abstract regular neighbourhood* of M if $Q \searrow M$.

Now suppose given ξ/K, then by using an atlas for ξ, we see that $E(\xi)$ may be collapsed to K by induction on the blocks of ξ starting with the top dimension.

Now suppose $|K| = M^n$; it follows easily from Theorem 1.1 that $E(\xi)$ is a manifold, and $M \subset E(\xi)$ is a proper submanifold. Hence E(ξ) is an abstract regular neighbourhood of M.

In this section we prove the converse to this, namely that any abstract regular neighbourhood of $|K| = M^n$ is the total space of a block bundle over K (Theorem 4.3), and we prove uniqueness by proving a full analogue of the smooth tubular neighbourhood theorem (Theorem 4.4). These results, together with the classification Theorems of § 2 and 3, give a homotopy classification of regular neighbourhoods (Corollary 4.7).

We begin by proving a stronger version of the uniqueness of subdivision theorem, 1.9.

THEOREM 4.1. *Suppose ξ', η'/K' are subdivisions of ξ/K and that $\xi' \mid L' = \eta' \mid L'$ (L' being the induced subdivision of a subcomplex $L \subset K$).*

Then there is an isotopy of $E(\xi)$ mod $K \cup E(\xi \mid L)$ *realizing an isomorphism of ξ' with η', and keeping the blocks of ξ setwise fixed.*

We will need the following new machinery:

Cellular shelling. The complex K is said to *shell cellularly* to $L \subset K$ if we can collapse K to L by elementary collapses across n-cells of K. If $|L|$ is an n-cell of K, we simply say K shells cellularly, and in this case it follows that $|K| \cong I^n$.

LEMMA 4.2. *Let K be a cell complex such that $|K| \cong I^n$. Then there exists a subdivision K' of K which shells cellularly.*

For a proof, see Zeeman [20; Lemmas 1 and 2].

PROOF OF THEOREM 4.1. If σ_i is a cell of K, then any isotopy of $E(\xi \mid \dot{\sigma}_i)$ mod $\dot{\sigma}_i$ may be extended to an isotopy of $\beta_i(\xi)$ mod σ_i, by using a collar of $(E(\xi \mid \dot{\sigma}_i), \dot{\sigma}_i)$ in $(\beta_i(\xi), \sigma_i)$. This remark enables us to use an inductive argument up the skeleton of $K - L$, and the theorem reduces to the case when $|K| \cong I^n$, K has one n-cell and $|L|$ is its boundary. This case we prove by induction on n.

Now suppose K'' is a subdivision of K'. Then by Theorem 1.5, there exist subdivisions $\xi'', \eta''/K''$ of ξ', η' such that $\xi'' \mid L'' = \eta'' \mid L''$, and any isotopy of $E(\xi)$ mod $K \cup E(\xi \mid L)$, realizing an isomorphism of ξ'' with η'', also realizes an isomorphism of ξ' with η'.

From this and Lemma 4.2, we may assume that K' shells cellularly, and induct on the number of steps of the shelling. If there are no steps (K' has one n-cell) there is nothing to prove, so let the first step be $K' \searrow K_1$.

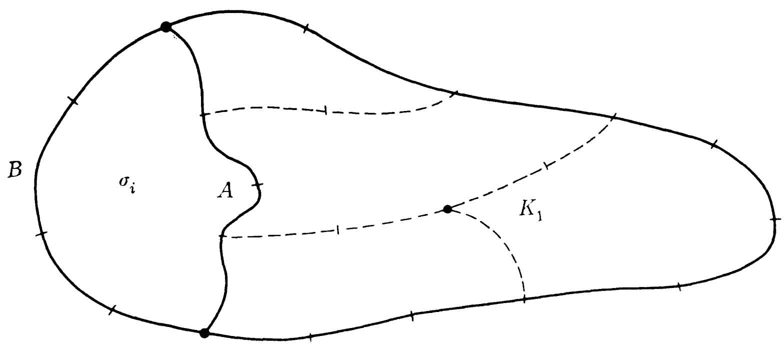

Figure 2

We use the following notation (cf. Figure 2). cl $(K' - K_1)$ = an n-cell σ_i, $\sigma_i \cap K_1$ = an $(n-1)$-ball A, $\sigma_i \cap \partial K'$ = an $(n-1)$-ball B, A and B both being subcomplexes of K'.

Now $\beta_i(\xi')$ and $\beta_i(\eta')$ are both regular neighbourhoods in $E(\xi)$ of $\sigma_i \cup E(\xi' \mid B)$ mod $K_1 \cup E(\xi \mid (L - \text{int } B))$, which meet the boundary regularly (recall that ξ', η' are both trivial by Theorem 1.1). Therefore there is an isotopy of $E(\xi)$ mod $K \cup E(\xi \mid L)$ carrying $\beta_i(\xi')$ onto $\beta_i(\eta')$, [8, Th. 3]. So we may assume that $\beta_i(\xi') = \beta_i(\eta')$.

Now $\xi' \mid A$, $\eta' \mid A$ agree over $\dot{A} \subset L'$ by hypothesis. Therefore by induction, there is an isotopy of $E(\xi' \mid A)$ mod $A \cup E(\xi' \mid \dot{A})$ realizing an isomorphism of $\xi' \mid A$ with $\eta' \mid A$. By a similar remark to that at the beginning of the proof, we can extend this isotopy over $E(\xi)$, keeping $K \cup E(\xi \mid L)$ fixed. So we may assume that $\xi' \mid A = \eta' \mid A$.

Now $\xi' \mid K_1$, $\eta' \mid K_1$ agree over ∂K_1 and K_1 shells cellularly in one less step than K'. Therefore by induction there is an isotopy of $E(\xi' \mid K_1)$ mod $E(\xi' \mid \dot{K}_1) \cup K_1$ carrying $\xi' \mid K_1$ to $\eta' \mid K_1$. Extending this isotopy by the identity over $E(\xi' \mid \sigma_i)$ completes the proof.

The following theorems are the main results of this section.

THEOREM 4.3. (a) *Let* Q^{n+q} *be an abstract regular neighbourhood of a compact manifold* M^n, *and suppose given complexes* $L \subset K$ *such that* $(M, \dot{M}) = (|K|, |L|)$.

Then there is a block bundle ξ^q/K *with* $E(\xi) = Q$.

(b) *Suppose further that* η^q/L *is given such that* $E(\eta) \subset \dot{Q}$.

Then ξ *may be chosen so that* $\xi \mid L = \eta$.

THEOREM 4.4. (a) *Let* $M^n \subset V^{n+q}$ *be a compact proper submanifold, and* $(M, \dot{M}) = (|K|, |L|)$ *for complexes* $L \subset K$. *Suppose* $E(\xi^q/K), E(\eta^q/K) \subset V$.

Then there is an isotopy of $E(\xi)$ mod M *in* V *realizing an isomorphism of* ξ *with* η.

(b) *Suppose further that* $E(\xi)$, $E(\eta)$ *meet* $\dot{V}$ *regularly*, i.e., *in* $E(\xi \mid L)$, $E(\eta \mid L)$ *respectively.*

Then the isotopy of (a) *may be taken to be ambient.*

(c) *Suppose further that* $\xi \mid L = \eta \mid L$.

Then the isotopy of (b) *may be taken to be* mod $\dot{V} \cup M$.

Remark. For the sake of clarity of exposition, the statement and proof of 4.4 are given for a proper submanifold M of V. The proof however readily adapts to prove the following extension for an admissible submanifold.

ADDENDUM 4.5. *Let* $M^n \subset V^{n+q}$ *be a compact admissible submanifold, and suppose* $(|K|, |L|, |J|) = (M, \dot{M}, \dot{M} \cap \dot{V})$, $J \subset L \subset K$. *Then Theorem* 4.4 *is true for this pair, on replacing* L *by* J *in the conditions of* (b) *and* (c).

Now let $\mathfrak{N}_q(M^n)$ denote the set of homeomorphism classes (homeomorphism mod M) of abstract regular neighbourhoods of M of dimension $n + q$.

Equivalent block bundles are homeomorphic, and so we have a function

$$R\colon I_q(M) \longrightarrow \mathfrak{N}_q(M) .$$

COROLLARY 4.6. *R is a bijection.*

PROOF. R is onto by Theorem 4.3, and $1-1$ by Theorem 4.4 (a) (using 1.10).

COROLLARY 4.7. *There is a bijection*

$$\mathfrak{N}_q(M) \longrightarrow [M, B\widetilde{PL}_q] .$$

PROOF: Apply Corollary 4.6 and Theorem 2.6.

We prove Theorems 4.3 and 4.4 in reverse order.

PROOF OF THEOREM 4.4. We will first prove the implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a), and then prove (c).

(b) $\Rightarrow$ (a). $E(\xi)$ is a regular neighbourhood of M mod $\dot{M}$ in V, so by the relative regular neighbourhood theorems [8], we may assume that $N =$

$\partial E(\xi) \cap \dot{V}$ is a regular neighbourhood of $\dot{M}$ in $\dot{V}$. Now $E(\xi \mid L)$ and N are regular neighbourhoods of $\dot{M}$ in $\partial E(\xi)$. Thus there exists an isotopy of $\partial E(\xi)$ mod $\dot{M}$ throwing $E(\xi|L)$ onto N. This isotopy may be extended to an isotopy of $E(\xi)$ mod M by means of a collar of $(\partial E(\xi), \dot{M})$ in $(E(\xi), M)$.

Similar remarks apply to η, and (a) now follows from (b).

(c) $\Rightarrow$ (b). Applying (c) to $\dot{M} \subset \dot{V}$, there is an isotopy of $\dot{V}$ mod $\dot{M}$ realizing an isomorphism of $\xi \mid L$ with $\eta \mid L$. This isotopy can be extended to an isotopy of V mod M by means of a collar of $(\dot{V}, \dot{M})$ in (V, M), hence (b) follows from (c).

PROOF OF (c). We will prove (c) by induction on n. It is sufficient to prove the result for any complex J with $|J| = M$. For suppose K_1 is a common subdivision of K, J. Use Theorem 1.5 to subdivide ξ, η to ξ', η' over K_1 so that $\xi' \mid L = \eta' \mid L$.

Let $\bar{\xi}, \bar{\eta}$ be the amalgamations over J. If $\bar{\xi}, \bar{\eta}$ are ambient isotopic mod $\dot{V} \cup M$, then, using a collar of $\partial E(\xi) \cap \mathrm{cl}\,(V - E(\xi))$ in $\mathrm{cl}\,(V - E(\xi))$ and Theorem 4.1, we see that ξ', η' are ambient isotopic mod $\dot{V} \cup M$, which proves that ξ, η are, as required.

Choose simplicial complexes $B \subset A$ such that $(|A|, |B|) = (M, \dot{M})$.

We will prove (c) for the dual complex A^* of A (constructed below), which has the advantage of also being a handle decomposition of M.

Let A'' be the second derived of A and, for each simplex $\alpha^t \in A'$, define the *dual cell* $\sigma^{n-t} \in A^*$ corresponding to α^t to be $\bigcap \mathrm{st}(P, A'')$, where the intersection is taken over vertices P of α^t. A^* is the cell complex consisting of these cells together with their intersections with $\dot{M}$.

Note that B^*, which consists of cells dual to simplexes of B', is a subcomplex of A^*.

Each n-cell of A^* corresponds to a simplex of A (since it is dual to a vertex of A'). Suppose $\alpha_0, \alpha_1, \cdots, \alpha_r$ are the simplexes of A in order of increasing dimension, and let $\sigma_0, \cdots, \sigma_r$ be the corresponding n-cells of A^*.

The σ_i give a handle decomposition of M with the index of σ_i being $\dim \alpha_i$ (for more details, the reader is referred to Zeeman [19; Ch. 9]).

Define

$$\begin{aligned} R_s &= \sigma_0 \cup \sigma_1 \cdots \cup \sigma_s \\ M_s &= \mathrm{cl}(M - R_s) \\ V_s &= \mathrm{cl}(V - E(\xi \mid R_s)) \end{aligned}$$

(see Figure 3).
(In the figure, σ_s corresponds to α_s of dimension 1.)

By elementary properties of the handle decomposition of M (and the

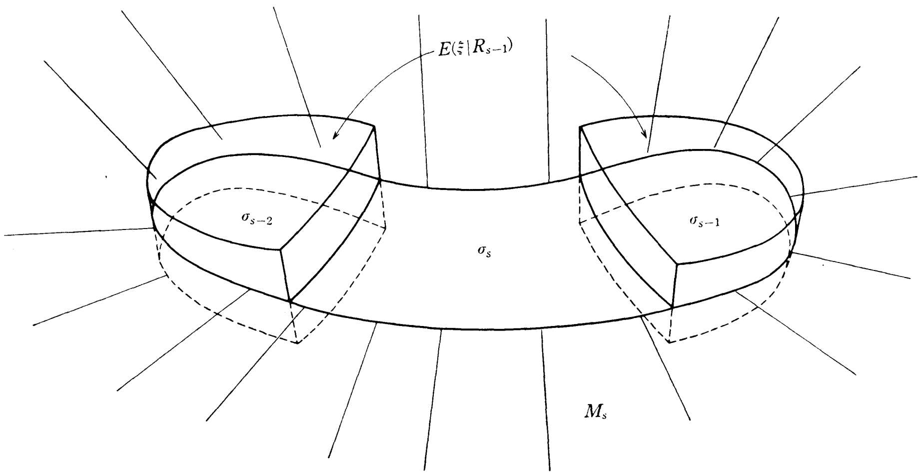

Figure 3

analogous one of $\dot{M}$), these are all manifolds admissibly embedded in each other.

Assuming $\xi \mid R_{s-1} = \eta \mid R_{s-1}$, we shall show that there exists an isotopy of $V \bmod \dot{V} \cup M \cup E(\xi \mid R_{s-1})$, realizing an isomorphism of $\xi \mid R_s$ with $\eta \mid R_s$. Thus the isotopy required by the theorem may be constructed inductively.

Consider $\beta_s(\xi)$, $\beta_s(\eta)$ (the blocks over σ_s). These are both regular neighbourhoods of σ_s mod M_s in V_{s-1}, which meet $\dot{V}_{s-1}$ regularly, and in the same set. Therefore by Hudson and Zeeman [9] there exists an isotopy of $V_{s-1} \bmod \dot{V}_{s-1} \cup M_{s-1}$ carrying $\beta_s(\xi)$ to $\beta_s(\eta)$. This isotopy extends trivially to an isotopy of $V \bmod \dot{V} \cup M$. Thus we may assume that $\beta_s(\xi) = \beta_s(\eta)$.

At this point, $\xi \mid R_s$, $\eta \mid R_s$ agree except over the part of $\dot{\sigma}_s$ not in R_{s-1}, and we appeal to induction on n to make them agree here:

By induction on n, $\xi \mid (\dot{\sigma}_s \cap \dot{M}_s)$, $\eta \mid (\dot{\sigma}_s \cap \dot{M}_s)$ are ambient isotopic in $\dot{\beta}_s \cap \dot{V}_s \bmod \dot{\sigma}_s \cup \partial(\dot{\beta}_s \cap \dot{V}_s)$. Extend this isotopy by means of collars of $(\dot{\beta}_s \cap \dot{V}_s, \dot{\sigma}_s \cap \dot{M}_s)$ in (β_s, σ_s) and (V_s, M_s), and then trivially to an isotopy of $V \bmod \dot{V} \cup M$, and this completes the proof.

Proof of 4.3(a). In view of Theorem 1.10 and the fact that equivalent block bundles are homeomorphic, it is sufficient to find ξ^q/J with $E(\xi) = Q$, for any complex J with $|J| = M$.

By the "classical" regular neighbourhood theorem, it is also sufficient to find ξ/J with $E(\xi) = N$, any regular neighbourhood of M in Q.

Choose simplicial complexes $A \subset B$ such that $(|A|, |B|) = (M, Q)$, and define the dual complexes A^*, B^* as in the proof of Theorem 4.4, above. We will prove the theorem for $J = A^*$.

Let $\alpha^t \in A'$ be a simplex, then we have corresponding dual cells $\sigma^{n-t} \in A^*$, $\beta^{n+q-t} \in B^*$, and if $\alpha^t \subset \dot{M}$, then we also have

$$\tilde{\sigma}^{n-t-1} = \sigma^{n-t} \cap \dot{M}$$
$$\tilde{\beta}^{n+q-t-1} = \beta^{n+q-t} \cap \dot{Q} \,.$$

Now the pair (β, σ) is isomorphic with the pair cone on $(\mathrm{lk}(\alpha^t, B'), \mathrm{lk}(\alpha^t, A'))$. Since M is a submanifold of Q, it follows that (β, σ), and similarly $(\tilde{\beta}, \tilde{\sigma})$, is an unknotted ball pair.

Thus the cells β over σ form the blocks of a block bundle over A^*, with total space $N =$ the simplicial neighbourhood $N(A, B'')$, the other conditions for a block bundle being apparent.

PROOF OF 4.3(b). It follows from Theorem 4.3 (a) that there exists $E(\xi/K) = Q$. Applying Theorem 4.4 (b) to $\xi \mid L$ and $\eta \mid L$ in $\dot{Q}$, and extending the resulting isotopy by means of a collar of $(\dot{Q}, \dot{M})$ in (Q, M), we have the result.

5. Connections with microbundles, etc.

In this section, we define Δ-groups similar to $\widetilde{PL}_q$, which correspond to sphere block bundles, open block bundles, and micro-block bundles. We show that all these theories coincide by exhibiting homotopy equivalences between their groups, and we deduce obstruction theories for existence of normal closed, open, and micro-bundles of M in Q. We also show that a block bundle gives rise to a fibre space, with fibre a sphere.

Definitions. $\widetilde{PL}_q(\Sigma)$ is the Δ-group of which a k-simplex is a block-preserving homeomorphism

$$\Delta^k \times \Sigma^{q-1} \longrightarrow \Delta^k \times \Sigma^{q-1} \,.$$

$\widetilde{PL}_q(R)$ is the Δ-group of which a k-simplex is a block and zero-preserving homeomorphism

$$\Delta^k \times R^q \longrightarrow \Delta^k \times R^q \,.$$

$\widetilde{PL}_q(\mu)$ is the Δ-group of which a k-simplex is a germ of block and zero-preserving homeomorphisms defined on a neighbourhood of $\Delta^k \times \{0\}$ in $\Delta^k \times I^q$.

Face operators are defined by restriction (degeneracies cannot be defined to be homeomorphisms, for the same reason as for $\widetilde{PL}_q$, but they are again unnecessary). That these are all Kan complexes is clear from the fact that $\Delta^k \cong \Lambda_k \times I$. Suppose given a block-preserving homeomorphism

$$h \colon \Delta^k \times \Sigma^{q-1} \longrightarrow \Delta^k \times \Sigma^{q-1}$$

and an extension $h' \colon \dot{\Delta}^k \times I^q \to \dot{\Delta}^k \times I^q$ of $h \mid \dot{\Delta}^k \times \Sigma^{q-1}$ which is zero-preserving. Then, $h \cup h'$ can be extended to $h_1 \colon \Delta^k \times I^q \to \Delta^k \times I^q$ by conical extension in the linear cone structure on $\Delta^k \times I^q$ with vertex the barycentre of $\Delta^k \times \{0\}$, and h_1 is thus also zero-preserving.

Using this construction, we get (by induction up the skeleton of $\widetilde{PL}_q(\Sigma)$) a canonical embedding

$$\gamma\colon \widetilde{PL}_q(\Sigma) \longrightarrow \widetilde{PL}_q\ .$$

THEOREM 5.1. *$\gamma(\widetilde{PL}_q(\Sigma))$ is a deformation retract of $\widetilde{PL}_q$.*

In proving Theorem 5.1, and later theorems like it, we shall use the following useful criterion, see [14C].

LEMMA 5.2. *Let $\boldsymbol{K} \subset \boldsymbol{L}$ be Δ-sets satisfying the Kan condition. Suppose that for any Δ-map $f\colon \Lambda_k \to \boldsymbol{L}$ such that $f(\dot{\Lambda}_k) \subset \boldsymbol{K}$, there exists an extension $f'\colon \Delta^k \to \boldsymbol{L}$ such that $f'(\Delta^{k-1}) \subset \boldsymbol{K}$.*

Then $\boldsymbol{K}$ is a deformation retract of $\boldsymbol{L}$.

PROOF OF 5.1. Let $f\colon \Lambda_k \times I^q \to \Lambda_k \times I^q$ be the homeomorphism determined by a Δ-map $f_1\colon \Lambda_k \to \widetilde{PL}_q$, and suppose $f_1(\dot{\Lambda}_k) \subset \gamma(\widetilde{PL}_q(\Sigma))$. Extend $f \mid \Lambda_k \times \Sigma^{q-1}$ to $f_2 : \Delta^k \times \Sigma^{q-1} \longrightarrow \Delta^k \times \Sigma^{q-1}$, using the extension condition for $\widetilde{PL}_q(\Sigma)$. Now extend $f \cup f_2$ over $\Delta^k \times I^q$ by conical extension (as in the definition of γ) first over $\Delta^{k-1} \times I^q$, then over $\Delta^k \times I^q$. Applying Lemma 5.2, we have the result.

Let $g\colon \widetilde{PL}_q \to \widetilde{PL}_q(\mu)$ assign to each homeomorphism its germ.

THEOREM 5.3. *$g\gamma\colon \widetilde{PL}_q(\Sigma) \to \widetilde{PL}_q(\mu)$ is a (homomorphic) embedding with $g\gamma(\widetilde{PL}_q(\Sigma))$ a deformation retract of $\widetilde{PL}_q(\mu)$.*

PROOF. $g\gamma$ is an embedding since two conical homeomorphisms agree near the vertex if and only if they agree.

A Δ-map $f\colon \Lambda_k \to \widetilde{PL}_q(\mu)$, such that $f(\dot{\Lambda}_k) \subset g\gamma(\widetilde{PL}_q(\Sigma))$ determines a block and zero-preserving embedding

$$f_1\colon \Lambda_k \times I^q(\varepsilon) \longrightarrow \Lambda_k \times R^q \qquad \text{some } \varepsilon\ ,$$

$(I^q(\varepsilon) = [-\varepsilon, +\varepsilon]^q)$, and we may assume that $f_1 \mid \dot{\Lambda}_k \times I^q(\varepsilon)$ extends to $f'\colon \dot{\Lambda}_k \times I^q \to \dot{\Lambda}_k \times I^q$ in $\gamma\widetilde{PL}_q(\Sigma)$. Now $\Lambda_k \times (I^q - \operatorname{int} I^q(\varepsilon))$ is a collar of

$$\dot{\Lambda}_k \times (I^q - \operatorname{int} I^q(\varepsilon)) \cup \Lambda_k \times \dot{I}^q(\varepsilon)\ ,$$

so we can extend $f_1 \cup f'$ to an embedding

$$f_2\colon \Lambda_k \times I^q \longrightarrow \Lambda_k \times R^q$$

by using a collar of the image of $f_1 \cup f'$ (f_2 is not necessarily block-preserving except near $\Lambda_k \times \{0\}$).

Now $f_2(\Lambda_k \times I^q)$ and $\Lambda_k \times I^q$ are both regular neighbourhoods of $\Lambda_k \times \{0\}$

in $\Lambda_k \times R^q$ which meet the boundary regularly and in the same set, and are therefore isotopic mod $\dot{\Lambda}_k \times R^q \cup \Lambda_k \times \{0\}$ [9].

Use this isotopy, and the fact that $\Delta^k \cong \Lambda_k \times I$ to extend f_2 to an emdedding $f_3\colon \Delta^k \times I^q \to \Delta^k \times R^q$ which is zero-preserving, block-preserving over Δ^{k-1} (and near $\Delta^k \times \{0\}$), and such that $f_3(\Delta^{k-1} \times I^q) = \Delta^{k-1} \times I^q$. Define f_4, with domain $\partial(\Delta^k \times I^q)$, by

$$f_4 \mid \Lambda_k \times I^q \cup \Delta^k \times \Sigma^{q-1} = f_3 \mid \Lambda_k \times I^q \cup \Delta^k \times \Sigma^{q-1}\ ,$$

and by $f_4 \mid \Delta^{k-1} \times I^q =$ the conical extension of $f_3 \mid \partial(\Delta^{k-1} \times I^q)$.

Now f_4 and the identity on $\Delta^k \times \{0\}$ extend over $\Delta^k \times I^q$, by Lemma 1.2. Applying Lemma 5.2, we have the result.

Any homeomorphism $f\colon \Delta^k \times I^q \to \Delta^k \times I^q$, such that $f(\Delta^k \times \dot{I}^q) = (\Delta^k \times \dot{I}^q)$, can be extended to a homeomorphism $f'\colon \Delta^k \times R^q \to \Delta^k \times R^q$ by the fact that $R^q - \operatorname{int} I^q$ is an (open) collar of $\dot{I}^q$. Thus we have a homomorphic embedding

$$c\colon \widetilde{PL}_q \longrightarrow \widetilde{PL}_q(R)\ .$$

THEOREM 5.4. *$c(\widetilde{PL}_q)$ is a deformation retract of $\widetilde{PL}_q(R)$.*

PROOF. Suppose $f\colon \Lambda_k \to \widetilde{PL}_q(R)$ determines a homeomorphism $f_1\colon \Lambda_k \times R^q \to \Lambda_k \times R^q$ such that $f(\dot{\Lambda}_k) \subset c(\widetilde{PL}_q)$, we have to extend f_1 over $\Delta^k \times R^q$ such that $f_1 \mid \Delta^{k-1} \times R^q$ is in $c(\widetilde{PL}_q)$. Since $\Delta^k \cong \Lambda_k \times I$, it is only necessary to show that f_1 is isotopic mod $\dot{\Lambda}_k \times R^q \cup \Lambda_k \times \{0\}$ to a homeomorphism $\Lambda_k \times R^q \to \Lambda_k \times R^q$ which is obtained from a homeomorphism of $\Lambda_k \times I^q$ by "collaring" (as in the definition of c).

The isotopy is constructed in two steps.

Step 1. By the regular neighbourhood argument used in Theorem 5.3, we can isotope f_1 mod $\dot{\Lambda}_k \times R^q \cup \Lambda_k \times \{0\}$ to a homeomorphism f_2, which preserves $\Lambda_k \times I^q$.

Step 2. Now isotope f_2 to $c(f_2 \mid \Lambda_k \times I^q)$ by "combing" the collar, cf. [7].

Remark 5.5. (i) We have shown that the following scheme of Δ-homomorphisms are all homotopy equivalences (g' is defined by taking germs)

$$\begin{tikzcd}
\widetilde{PL}_q(\Sigma) \arrow[rr, "\gamma"] & & \widetilde{PL}_q \arrow[rr, "c"] \arrow[dr, "g"'] & & \widetilde{PL}_q(R) \arrow[dl, "g'"] \\
& & & \widetilde{PL}_q(\mu) &
\end{tikzcd}\ .$$

(ii) Homomorphisms $i\colon \widetilde{PL}_q \to \widetilde{PL}_q(R)$ and $b\colon \widetilde{PL}_q \to \widetilde{PL}_q(\Sigma)$ are defined by "ignoring the boundary" and "restricting to the boundary" respectively. It is not hard to show that b and γ are homotopy inverse to each other, and that i and c are homotopic.

We now define an *open* (*micro, sphere*) block bundle to be the realization, constructed as in § 3, of a principal $\widetilde{PL}_q(R)$ $(\widetilde{PL}_q(\mu),\ \widetilde{PL}_q(\Sigma))$ bundle.

Given a block bundle ξ, define $c_*(\xi)$ to be the open block bundle obtained by adding an open collar to $E(\xi)$, and $i_*(\xi)$ to be the open block bundle obtained by removing the boundary of $E(\xi)$. The significance of Remark 5.5 (ii) is that $c_*(\xi)$ and $i_*(\xi)$ are isomorphic, and it follows from Theorem 5.4 that i_* induces a bijection on isomorphism classes (coinciding with that induced by c_*). We shall use the notion of open block bundle in § 6 for comparison with vector bundles.

Remark 5.5 (i) shows that $I_q(K)$ is isomorphic with the set of isomorphism classes of open, sphere, or micro-block bundles over K. This does not immediately show that the *theories* coincide, one also needs to know that the notions of induced bundle are the same. However, if we define induced open, sphere, and micro-block bundles in the same way as for ordinary block bundles (namely as the restriction of the cartesian product, cf. § 1), then this is immediate.

More definitions. $PL_q(I)$ is the subgroup of $\widetilde{PL}_q$ consisting of simplexes σ^k such that $p\sigma = \sigma p$, where $p\colon \Delta^k \times I^q \longrightarrow \Delta^k$ is the canonical projection. $PL_q(R) \subset \widetilde{PL}_q(R)$, $PL_q \subset \widetilde{PL}_q(\mu)$, and $PL_q(\Sigma) \subset \widetilde{PL}_q(\Sigma)$ are defined similarly.

Remark. It is well known that degeneracies can be defined in $PL_q(R)$, PL_q, and $PL_q(\Sigma)$ making them c.s.s. groups.

Let $M^n \subset Q^{n+q}$ be a compact submanifold. A *closed tube* on M in Q is a neighbourhood N^{n+q} of $\operatorname{int} M$ in Q, provided with a projection $p\colon N \longrightarrow M$, making N the total space of an I^q-bundle with fixed zero section M, cf. Hirsch [5]. Two tubes are *concordant* if they are the restrictions of a tube on $M \times I$ in $Q \times I$. There are similar definitions for open tubes and micro-bundles. We will examine the connection between closed tubes and block bundles; a similar exposition holds for open tubes, open block bundles, and micro-bundles, micro-block bundles.

Define $\widetilde{PL}_q/PL_q(I)$ to be the complex of right cosets of $PL_q(I)$ in $\widetilde{PL}_q$ (equivalence classes under the equivalence $\sigma \sim \sigma' \Leftrightarrow \sigma^{-1}\sigma' \in PL_q(I)$), which has a natural right $\widetilde{PL}_q$ action. Given a q-block bundle ξ/K^k, where K is a simplicial complex with ordered vertices, we can form the associated $\widetilde{PL}{}^k_q/PL^k_q(I)$-bundle as follows. A k-simplex is an equivalence class of charts $g\colon \Delta^k \times I^q \longrightarrow E(\xi)$ under the equivalence, $g \sim g' \Leftrightarrow g^{-1}g' \in PL_q(I)$. $\widetilde{PL}{}^k_q$ acts on the right by composition.

A cross-section of the associated bundle chooses for each simplex in K a compatible equivalence class, and gives us a $PL^k_q(I)$ bundle over K by taking as simplexes the members of these equivalence classes. This defines a closed tube over K exactly as in the construction part of Theorem 3.1. Conversely, given a closed tube N/K, we can form a block bundle by taking as blocks the restrictions over simplexes; the associated $\widetilde{PL}_q^k/PL_q^k(I)$-bundle has a natural cross-section, namely map each simplex to the equivalence class of fibre-preserving charts.

From these remarks, and the results of § 4, we deduce by an easy argument

THEOREM 5.6. *Suppose M and Q are as above, and K is any triangulation of M. Then concordance classes of closed tubes (respectively, open tubes, micro-bundles) on M in Q correspond bijectively with homotopy classes of cross-sections of the bundle over $\boldsymbol{K}$ with fibre $\widetilde{PL}_q/PL_q(I)$ (respectively, $\widetilde{PL}_q(R)/PL_q(R)$, $\widetilde{PL}_q(\mu)/PL_q$) associated with any normal block bundle ξ/K to M in Q.*

Remark 5.7. (i) By Theorem 5.6, we have obstruction theories for the existence of closed tubes (open tubes, micro-bundles) with coefficients in $\pi_k(\widetilde{PL}_q, PL_q(I))$ $(\pi_k(\widetilde{PL}_q(R), PL_q(R)), \pi_k(\widetilde{PL}_q(\mu), PL_q)))$.

(ii) The obstruction theories all have non-zero coefficients by [5, 14B]. Note that $PL_q \simeq PL_q(R)$ [10] and $PL_q(R) \not\simeq PL_q(I)$ [1].

We conclude this section by proving that a block bundle determines a fibre space with fibre a sphere, cf. Fadell [2].

Definition. $\widetilde{G}_q$ is the Δ-set of block-preserving degree ± 1 *maps*

$$\Delta^k \times \Sigma^{q-1} \longrightarrow \Delta^k \times \Sigma^{q-1} .$$

G_q is the subcomplex of fibre-preserving maps (i.e., commuting with projection on Δ^k).

THEOREM 5.8. *G_q is a deformation retract of $\widetilde{G}_q$.*

PROOF. Let $f\colon \Lambda_k \times \Sigma^{q-1} \to \Lambda_k \times \Sigma^{q-1}$ be block-preserving, and fibre-preserving over $\dot{\Lambda}_k$.

Let π_1, π_2 be the projections of $\Lambda_k \times \Sigma^{q-1}$ on Λ_k and Σ^{q-1} respectively. The homotopy

$$f_t(x) = \big(t\pi_1(x) + (1 - t)\pi_1(f_1(x)), \pi_2 f(x)\big)$$

enables us to extend f over $\Delta^k \cong \Lambda_k \times I$, such that the restriction over Δ^{k-1} is fibre preserving.

Thus the result follows from Lemma 5.2.

COROLLARY 5.9. *A block bundle determines a unique fibre space with* *fibre a $(q-1)$-sphere.*

PROOF. Consider the sequence of homeomorphisms

$$\widetilde{PL}_q \xrightarrow{b} \widetilde{PL}_q(\Sigma) \subset \widetilde{G}_q \supset G_q \ .$$

The result follows from Theorem 5.8 and the existence of principal classifying bundles (§ 3 and Stasheff [14D]) cf. [14C] (section on Δ-monoids) for more detail.

Remark. The last result can also be obtained by a construction based on Fadell [2]; one merely has to restrict his paths to lie entirely in the block corresponding to the base simplex in which the path starts. Denoting the resulting fibre space by $G(\xi)$, one then has a map $g: G(\xi) \rightarrow E_0(\xi) = E(\xi) - K$ by restricting to end points of the paths. One easily checks that g is a homotopy equivalence, and that the following diagram commutes up to homotopy

$$\begin{array}{ccc} E_0(\xi) & \xleftarrow[g]{} & G(\xi) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ |K| & = & |K| \ . \end{array}$$

Here π is the *projection* given by the collapse $E(\xi) \searrow K$ (cf. § 4), and p is the projection of the fibre space.

6. Application to smoothing

In this section we compare the structure of a block bundle with that of a vector bundle, and deduce an obstruction theory for smoothing a PL submanifold of a PL manifold with compatible differential structure. We will need a piecewise-differential (PD) analogue of $\widetilde{PL}_q(R)$.

Definition. $\widetilde{PD}_q$ is the Δ-set of which a k-simplex is a PD isomorphism

$$\sigma: E(\varepsilon^q(R)/\Delta^k) \longrightarrow E(\varepsilon^q(R)/\Delta^k) \subset R^{k+q} \ ,$$

where $\varepsilon^q(R)$ denotes the trivial open q-block bundle. Face operators are defined by restriction (degeneracies may be defined but will not be needed).

Note that $\widetilde{PL}_q(R)$ is a subcomplex of $\widetilde{PD}_q$ and acts on the right by composition.

PROPOSITION 6.1. *$\widetilde{PD}_q$ satisfies the extension condition, and $\widetilde{PL}_q(R)$ is a deformation retract of $\widetilde{PD}_q$.*

PROOF. Let $f: \Lambda_k \rightarrow \widetilde{PD}_q$. We want to show that there exists an extension $f': \Delta^k \rightarrow \widetilde{PD}_q$ such that, if $f(\dot{\Lambda}_k) \subset \widetilde{PL}_q(R)$, then $f'(\Delta^{k-1}) \subset \widetilde{PL}_q(R)$. This proves both parts of the proposition. Now f determines a PD homeomorphism

$$f_1\colon \Lambda_k \times R^q \longrightarrow \Lambda_k \times R^q\ ,$$

which is isotopic (Whitehead [16]) to a PL homeomorphism f_2, and if $f_1 \mid \dot{\Lambda}_k \times R^q$ is PL, the isotopy may be taken mod $\dot{\Lambda}_k \times R^q$. Using the fact that $\Delta^k \cong \Lambda_k \times I$, the result is proved.

Definition. O_q, the c.s.s. analogue of the orthogonal group $O(q)$, is defined as follows. A k-simplex is an orthogonal vector bundle isomorphism

$$\sigma\colon \Delta^k \times R^q \rightarrow \Delta^k \times R^q\ ,$$

which is also a diffeomorphism (i.e., σ extends to a diffeomorphism of a neighbourhood of $\Delta^k \times R^q$ in R^{k+q}). Face and degeneracy operators are defined in the usual way. Note that O_q is a subcomplex of $\widetilde{PD}_q$, and acts on the left by composition.

Remark. The definition of O_q given here differs from that given in Lashof and Rothenberg [11], which used PD homeomorphisms. This is to ensure that O_q acts on $\widetilde{PD}_q$.

PROPOSITION 6.2. *O_q satisfies the extension condition, and is a deformation retract of the singular complex of $O(q)$.*

PROOF. The proposition follows easily from the following lemma and the fact that O_q may be identified with the complex of differentiable maps $\Delta^k \rightarrow O(q)$.

LEMMA (Lashof and Rothenberg [11; proof of 1.1]). *Let M be a differential manifold, and $f\colon \Delta^k \rightarrow M$ be a map which is differentiable on $\dot{\Delta}^k$. Then f is homotopic* mod $\dot{\Delta}^k$ *to a differentiable map.*

Now let ξ^q/K be an open block bundle over the ordered simplicial complex K of dimension k. An r-simplex of the associated $\widetilde{PD}^k_q$-bundle is a PD isomorphism

$$g\colon E(\xi \mid \sigma^r) \longrightarrow E(\varepsilon^q(R)/\Delta^r)\ , \qquad \sigma^r \in K.$$

The associated $\widetilde{PD}_q$-bundle over $\boldsymbol{K}$ is constructed similarly (cf. § 3).

An *orthogonal structure* on ξ^q is a subcomplex of the associated $\widetilde{PD}^k_q$-bundle which is principal under left O^k_q action. An open block bundle with an orthogonal structure is called a *t-vector bundle* (triangulated vector bundle). An *isomorphism* of t-vector bundles ξ, η is a homeomorphism $h\colon E(\xi) \rightarrow E(\eta)$, which is an isomorphism of open block bundles, and which preserves orthogonal structures. A t-vector bundle over K determines a principal O_q-bundle over $\boldsymbol{K}$ unique up to isomorphism, and we now prove the converse of this. We first show how to costruct a block bundle from an O_q-bundle:

THEOREM 6.3. *There is a* c.s.s. *complex BO_q classifying for principal O_q-bundles and a fibration*

$$\widetilde{PD}_q/O_q \longrightarrow BO_q \longrightarrow \mathcal{G}_q .$$

PROOF. Let $\widetilde{V}_q^{PD}$ be the c.s.s. complex with k-simplex a PD isomorphism from $E(i_*(\sigma^k))$, where $\sigma^k \in \mathcal{G}_q$, to $E(\varepsilon^q(R)/\Delta^k)$. Now $\widetilde{V}_q^{PD}$ is contractible by a similar argument to that used for $\widetilde{V}_q$, and O_q acts freely on the left of $\widetilde{V}_q^{PD}$ by composition. Thus $BO_q = \widetilde{V}_q^{PD}/O_q$ is classifying for left (and hence for right) principal O_q-bundles. Now $\widetilde{PD}_q \to \widetilde{V}_q^{PD} \to \mathcal{G}_q$ is a fibration, and factoring by O_q yields the theorem.

Given an O_q-bundle ξ over K, by Theorem 6.3 we get a block bundle ξ_1/K. $i_*(\xi_1)$ is an open block bundle which, by the proof of 6.3, has an orthogonal structure isomorphic as a vector bundle with the geometrical realization of ξ (constructed as in § 3). Similar considerations show uniqueness, and we have a bijection between isomorphism classes of O_q-bundles over K and isomorphism classes of t-vector bundles over K.

Remark. (1) By the above and Proposition 6.2, there is a bijection between isomorphism classes of t-vector bundles and ordinary vector bundles over K.

(2) Several of the above constructions echo parts of Lashof and Rothenberg, cf. [11; §§ 1–3].

Now let ξ be an open block bundle. The associated $\widetilde{PD}_q^k/O_q^k$-bundle is defined as follows. A simplex of the total space is an equivalence class of simplexes of the associated $\widetilde{PD}_q^k$-bundle under the equivalence

$$g \sim g' \Longleftrightarrow g'g^{-1} \in O_q .$$

One verifies at once (cf. § 5) that orthogonal structures on ξ correspond bijectively with cross-sections of the associated $\widetilde{PD}_q^k/O_q^k$-bundle. Combining this with the tubular neighbourhood theorem for open block bundles (analogy of Theorem 4.4 (a)) which follows from § 5 and 4.4 (a), we have (cf. 5.6)

THEOREM 6.4. *Let $M^n \subset Q^{n+q}$ be a compact submanifold. Concordance classes of t-vector bundles on M in Q correspond bijectively with homotopy classes of cross-sections of the $\widetilde{PD}_q/O_q$-bundle associated with any normal open block bundle on M in Q.*

Now let α be a smoothing of Q (i.e., a compatible differential structure), and denote by $\Gamma(M, Q_\alpha)$ the set of equivalence classes of smoothings of Q concordant to α under the equivalence of concordance preserving M as smooth submanifold.

The following theorem is close to Lashof and Rothenberg [11; Th. 7.1], and we therefore omit the proof. The essential ingredients are the Cairns-Hirsch product theorem [7] and Theorem 6.3 above.

THEOREM 6.5. *Let ξ^q/K, $|K| = M$, be an open block bundle, and suppose α is a smoothing of $E(\xi)$. Then there is a bijection*

$$\varphi\colon \{\text{concordance classes of orthogonal structures of } \xi\} \longrightarrow \Gamma(M, E(\xi)_\alpha)\ ,$$

and the normal bundle of a representative of $\varphi(x)$ is isomorphic as a vector bundle with a representative of x.

Combining Theorems 6.4 and 6.5 with the (smooth) concordance extension theorem [7; Th. 1.2], we deduce

THEOREM 6.6. *Let $M^n \subset Q^{n+q}$ be a compact submanifold, and let α be a smoothing of Q. Then there is a bijection between $\Gamma(M, Q_\alpha)$ and the set of homotopy classes of cross-sections of the $\widetilde{PD}_q/O_q$-bundle associated with any normal open block bundle on M in Q.*

Now write $\Gamma^q_n = (\Sigma^n, \Sigma^{n+q}_s)$, where s denotes the standard smoothing of Σ^{n+q}.

COROLLARY 6.7. $\Gamma^q_n \cong \pi_n\,(\widetilde{PD}_q, O_q)$

PROOF. By Theorem 6.5, and the fact that a normal block bundle on Σ^n in Σ^{n+q} is unique and trivial, we have

$$\Gamma^q_n \cong [\Sigma^n, \widetilde{PD}_q/O_q]\ .$$

This, and the (easily proved) fact that $\pi_1(\widetilde{PD}_q/O_q) = 0$, gives the result.

Remark. The stable version of Corollary 6.5 was first proved by Hirsch [6], and a theorem similar to 6.6 appeared in the preprint to [4], cf. also Haefliger [3].

Theorem 6.5 gives an obstruction theory to smoothing M in Q, with coefficients in the Γ^q_i. Wall has proved that $\Gamma^2_n = 0$ for all n (see [15]) and so we have

COROLLARY 6.8. $M^n \subset Q^{n+2}_\alpha \Rightarrow$ *M smoothable in Q_α.*

QUEEN MARY COLLEGE, LONDON
UNIVERSITY OF WARWICK, COVENTRY

REFERENCES

1. W. BROWDER, *Open and closed disc bundles*, Ann. of Math., 84 (1966), 218–230.
2. E. FADELL, *Generalised normal bundles for locally-flat imbeddings*, Trans. Amer. Math. Soc., 114 (1965), 488–513.
3. A. HAEFLIGER, *Lissage des Immersions* I (Topology, 1967) and II (to appear).
4. ——— and C. T. C. WALL, *Piecewise-linear bundles in the stable range*, Topology, 4 (1965), 209–214.

5. M. W. HIRSCH, *On tubular neighbourhoods of manifolds*, Proc. Cam. Phil. Soc., 62 (1966), 177 and 183.

6. ———, *Obstruction theories for smoothing manifolds and maps*, Bull. Amer. Math. Soc., 69 (1963), 352-356.

7. ——— and B. MAZUR, Smoothing of piecewise-linear manifolds, mimeographed notes, Cambridge University.

8. J. F. P. HUDSON and E. C. ZEEMAN, *On regular neighbourhoods*, Proc. Lon. Math. Soc., 14 (1964), 719-745.

9. ———, *Correction* to "On regular neighbourhoods", (to appear).

9A. D. M. KAN, *A combinatorial definition of homotopy groups*, Ann. of Math., 67 (1958), 282-312.

10. N. H. KUIPER and R. K. LASHOF, *Microbundles and bundles*, Invent. Math., 1 (1966), 1-17.

11. R. K. LASHOF and M. ROTHENBERG, *Microbundles and smoothing*, Topology, 3 (1965), 357-388.

12. J. LEVINE, *A classification of differentiable knots*, Ann. of Math., 82 (1965), 15-50.

13. J. W. MILNOR, Microbundles and differentiable structures, mimeographed notes, Princeton University, 1961.

14. C. P. ROURKE and B. J. SANDERSON, *Block bundles*, Bull. Amer. Math. Soc., 72 (1966), 1036-1039.

14A. ———, *Block bundles*: II (transversality), Ann. of Math. (to appear).

14B. ———, *An embedding without a normal microbundle*, Invent. Math., 3 (1967), 293-299.

14C. ———, *A note on the homotopy theory of Δ-sets*, (to appear).

14D. J. STASHEFF, *A classification theorem for fibre spaces*, Topology, 2 (1963), 239-246.

15. C. T. C. WALL, *On PL submanifolds of codimension* 2, Proc. Cam. Phil. Soc., 63 (1967), 5-8.

16. J. H. C. WHITEHEAD, *On C^1-complexes*, Ann. of Math., 41 (1940), 809-824.

17. ———, *Combinatorial homotopy*: I, Bull. Amer. Math., Soc., 55 (1949), 213-245.

18. E. C. ZEEMAN, *Unknotting combinatorial balls*, Ann. of Math., 78 (1963), 501-526.

19. ———, Seminar on combinatorial topology, Inst. Hautes Etudes Sci., Paris 1963-5.

20. ———, *Unknotting Spheres*, Ann. of Math., 72 (1960), 350-361.

(Received May 18, 1966)
(Revised June 7, 1967)