Inventiones math. 3, 293—299 (1967)

An Embedding Without a Normal Microbundle

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In this paper we construct a piecewise linear embedding of $S^{19}\times I$ in S^{29} with no topological normal microbundle (and hence in particular no piecewise linear one). By restricting the normal block bundle of this embedding to $S^{19}\times\{0\}$ we have a p.l. embedding of a 19-sphere in a 28-manifold with no (topological) normal microbundle. Since the 28-manifold is parallelizable we also have a p.l. immersion of S^{19} in S^{28} without a normal microbundle.

These results improve on results of HIRSCH [5], which showed existence of embeddings without normal disc bundles, and together with results of [9] justify our claim (see also [12]) that block bundles are the correct bundle theory for the p.l. category.

The result is obtained as follows. In Theorem A we prove a result about the linking classes (defined below) of the boundary link of a p.l. "ribbon" (i.e. an embedding of $S^n\times I$ in S^{n+q}) which has a normal microbundle. In Theorem B we prove that the suspension of a p.l. link is the boundary of a p.l. ribbon. Finally we use HAEFLIGER's results [3] and known homotopy groups to find a link which is a suspension but fails to satisfy the conclusions of Theorem A and hence bounds a ribbon without a normal microbundle.

In [11] we will use this example to prove the existence of a differentiably embedded framed sphere (with ordinary differential structure) $S^{18}\subset S^{27}$ whose normal bundle is non-standard as a topological microbundle.

Notation. We work in the categories of topological and piecewise linear manifolds. Objects and maps in the latter category will be prefixed "p.l.". All embeddings and isotopies will be locally flat (in the p.l. case, since we always have codimension $\geqq 3$, this is no restriction at all by ZEEMAN [15]).

The following are standard objects in both categories: R^n is the subspace of Hilbert space with coordinates $x_i=0$ for $i>n$, $S^{n-1}=\{x:x\in R^n, |x|=1\}$, $R^n_+=\{x:x\in R^n, x_n\geqq 0\}$, $u_n^+\in R^n$ is the point with coordinates $x_n=+1$ and $x_i=0$, $i\neq n$, u_n^- is defined similarly. The n-sphere Σ^n is defined inductively by $\Sigma_0=\{u_1^+\}\cup\{u_1^-\}$, $\Sigma^n=\Sigma^{n-1}*(\{u_{n+1}^+\}\cup\{u_{n+1}^-\})$, where $*$ denotes geometric join, $\Sigma^n_+=\Sigma^n\cap R^{n+1}_+$ and $\Sigma^n_-=\Sigma^n\cap R^{n+1}_-$. I is the unit interval $[0,1]\subset R$ and $I^1=[-1,+1]$. An orientation for Σ^n and S^n is determined by the outward normal and the orientation of R^{n+1}. $\Sigma^n\times\{0\}$ and $\Sigma^n\times\{1\}$ are oriented by identification with Σ^n.

A *link* in Σ^{n+q} (or R^{n+q}) is an embedding $f\colon \Sigma^n \times \delta I \to \Sigma^{n+q}$ (or R^{n+q}). We also refer to a pair of oriented subspheres $(\Sigma_0^n, \Sigma_1^n) \subset \Sigma^{n+q}$ as a link; the embedding f determines the pair $(\Sigma_0^n, \Sigma_1^n) = (f(\Sigma^n \times \{0\}), f(\Sigma^n \times \{1\}))$. Suppose $q \geqq 3$, then each of Σ_0^n, Σ_1^n is unknotted by STALLINGS [13] and we define the linking classes of the link as follows. $\Sigma^{n+q} - \Sigma_1^n$ has the homotopy type of S^{q-1} and the orientations of Σ^{n+q} and Σ_1^n determine a homotopy equivalence $h\colon \Sigma^{n+q} - \Sigma_1^n \to S^{q-1}$. Define the *first linking class* $\alpha_1(\Sigma_0^n, \Sigma_1^n) \in \pi_n(S^{q-1})$ to be the element determined by $h|\Sigma_0^n$. Define the *second linking class* $\alpha_2(\Sigma_0^n, \Sigma_1^n) = \alpha_1(\Sigma_1^n, \Sigma_0^n)$. We refer to the pair

$$(\alpha_1(\Sigma_0^n, \Sigma_1^n), \alpha_2(\Sigma_0^n, \Sigma_1^n))$$

simply as the *linking class* of (Σ_0^n, Σ_1^n). It is readily proved by applying Stalling's theorem in the next dimension that the linking class is invariant under isotopy of the link. Given a link $f\colon \Sigma^n \times \delta I \to R^{n+q}$, we define the *suspension* of f, $\Sigma f\colon \Sigma^{n+1} \times \delta I \to R^{n+q+2}$ as follows. $\Sigma f|\Sigma^n \times \delta I = f$, $\Sigma f(u_{n+2}^{\pm} \times \{0\}) = u_{n+q+1}^{\pm}$ and $\Sigma f(u_{n+2}^{\pm} \times \{1\}) = u_{n+q+2}^{\pm}$, then define $\Sigma f|\Sigma^{n+1} \times \{0\}$ to be the join of $\Sigma f|\Sigma^n \times \{0\}$ with $\Sigma f|(u_{n+1}^{+} \cup u_{n+1}^{-}) \times \{0\}$ and similarly for $\Sigma f|\Sigma^{n+1} \times \{1\}$.

A *normal bundle* of an embedding $f\colon M^n \to Q^{n+q}$ is a topological $(R^q, \{0\})$-bundle ξ^q with total space $E(\xi) \subset Q^{n+q}$ and zero section $f(M)$. We write $E_0(\xi)$ for $E(\xi) - f(M)$. The "Kister-Mazur theorem" [8] states that any (topological) microbundle contains a bundle which is unique up to isomorphism and so we prove that our embedding has no normal bundle. We do this for convenience — the Kister-Mazur theorem is not essential to the paper.

Theorem A. *Let* $g\colon \Sigma^n \times I^1 \to R^{n+q+1}$, *q odd* $\geqq 3$, *be a p.l. embedding such that*

a) *g has a normal bundle*

b) $g|\colon \Sigma^n \times \{0\} \to R^{n+q+1}$ *is the standard inclusion*

c) *the second linking class* $\alpha_2(\Sigma^n, g(\Sigma^n \times \{1\}))$ *is zero.*

Then the first linking class $\alpha_1(\Sigma^n, g(\Sigma^n \times \{1\}))$ *is of order* 2.

Proof. By collaring a regular neighbourhood of $g(\Sigma^n \times I^1)$ mod $g(\Sigma^n \times \delta I^1)$ (see [6], [15]) one has a p.l. embedding $g_1\colon \Sigma^n \times [-2, +2] \to R^{n+q+1}$ which extends g. It follows easily from the p.l. isotopy extension theorem [6] that g_1 has a normal bundle, say η. Hence Σ^n has a normal bundle ξ such that $\xi \cong \eta \oplus \varepsilon$ (where ε is the trivial line bundle) and

$$E(\xi|\{x\}) = E(\eta|g_1(\{x\} \times (-2, +2))) \qquad \text{for each } x \in \Sigma^n.$$

Lemma. *There exists a map* $t\colon E(\xi) \to \Sigma^n \times R^{q+1}$ *satisfying*

i) $t|E(\xi|\Sigma_+^n)$ *is a homeomorphism,*

ii) $tg(x) = x$ *for* $x \in \Sigma_+^n \times I^1 \cup \Sigma_-^n \times \{0\}$,

iii) $t_0 = t \mid E_0(\xi) \to \Sigma^n \times (R^{q+1} - \{0\})$ *is a fibre homotopy equivalence,*

iv) *Let* $p\colon \Sigma^n \times (R^{q+1} - \{0\}) \to R^{q+1} - \{0\}$ *be the projection, and suppose given a never-zero section* s *of* ξ. *Then* $p \circ t \circ s$ *determines an element of* $\pi_n(S^q)$ *which coincides with the first linking class of* $(s(\Sigma^n), \Sigma^n)$.

Proof of Lemma. It follows from local triviality that the image of a section of a bundle over a manifold is locally flat and hence, as is implicit in the lemma, $s(\Sigma^n) \subset R^{n+q+1}$ is locally flat. By similar considerations, it is easily checked that the links and isotopies defined in what follows are all locally flat.

An inclusion of $\Sigma^n \times R^{q+1}$ in R^{n+q+1} which is standard on $\Sigma^n \times \{0\}$ provides Σ^n with a trivial normal bundle. Now using results of [*I*] (in particular Proposition 4.8) one can find a homotopy H of the inclusion $E(\xi) \subset R^{n+q+1}$ such that $H_t^{-1}(\Sigma^n) = \Sigma^n$ and $H_1 \mid E_0(\xi) \to \Sigma^n \times (R^{q+1} - \{0\})$ is a fibre homotopy equivalence. $t = H_1 \mid E(\xi)$ now satisfies conditions (iii) and (iv). It is a simple matter to modify t so that in addition conditions (i) and (ii) are satisfied.

Proof of Theorem A (*continued*). Now let s^+, s^- be the sections of ξ given by $g \mid \Sigma^n \times \{\pm 1\}$ and let $S^\pm = s^\pm(\Sigma^n)$. Let $\beta \in \pi_n(S^q)$ be the first linking class of (S^-, Σ^n) and let $b\colon \Sigma^n_+ \to R^{q+1} - \{0\}$ with $b(\delta \Sigma^n_+) = \{u_1^+\}$ represent β. Define new sections s_*^+ and s_*^- by:

$$s_*^+(x) = \begin{cases} s^+(x) & \text{if } x \in \Sigma^n_- \\ t^{-1}(x, b(x)) & \text{if } x \in \Sigma^n_+ \end{cases}$$

$$s_*^-(x) = \begin{cases} s^-(x) & \text{if } x \in \Sigma^n_- \\ t^{-1}(x, -b(x)) & \text{if } x \in \Sigma^n_+ . \end{cases}$$

The links (S^+, Σ^n) and (Σ^n, S^-) are isotopic (slide along $g(\Sigma^n \times I^1)$) and thus have linking class $(0, \beta)$. Similarly (S_*^+, Σ^n) and (Σ^n, S_*^-) are isotopic and have linking class $(\beta, 2\beta)$ (to see that (Σ^n, S_*^-) has second linking class 2β, use the fact that the antipodal map on S^q is of degree $+1$ since q is odd).

We assert that s_*^+ and s^- are homotopic sections (where the homotopy is via never-zero sections). For let t' be a fibre homotopy inverse of t. Then $s_*^+ \simeq t' t s_*^+ \simeq t' t s^- \simeq s^-$ and the middle homotopy exists since $(t s_*^+(\Sigma^n), \Sigma^n)$ $(t s^-(\Sigma^n), \Sigma^n)$ both have first linking class β by (iv) and construction.

So the links (S_*^+, Σ^n) (S^-, Σ^n) are isotopic, and it follows that $2\beta = 0$, as required.

For Theorem B we will need the following

Proposition. Given any p.l. link $f\colon \Sigma^n \times \delta I \to R^{n+q}$, $q \geqq 3$, there exists a p.l. embedding $g\colon \Sigma^n \times I \to R_+^{n+q+1}$ extending f.

Proof. Choose a small p.l. disc $D^n \subset \Sigma^n$ and pipe $\Sigma_0^n = f(\Sigma^n \times \{0\})$ to $\Sigma_1^n = f(\Sigma^n \times \{1\})$ by a p.l. $(n+1)$-pipe $P^{n+1} \subset R^{n+q}$ which meets Σ_0^n in $D_0^n = f(D^n \times \{0\})$ and Σ_1^n in D_1^n. Choose P^{n+1} to respect the orientations of D_0^n and D_1^n. This implies that f extends to a p.l. embedding $g_1: \Sigma^n \times \delta I \cup D^n \times I \to R^{n+q}$ such that $g_1(D^n \times I) = P^{n+1}$. Now $\mathrm{cl}(\Sigma^n - D^n)$ is a p.l. n-disc B^n say [*16*; Theorem 3]. Choose a cone structure on $B^n \times I$ with vertex $p \in \mathrm{int}(B^n \times I)$ and choose a point $q \in R_+^{n+q+1} - R^{n+q}$. Define $g_2(p) = q$ and extend $g_1 | \delta(B^n \times I)$ conewise to give an embedding g_2: $B^n \times I \to R_+^{n+q+1}$. Finally define $g = g_1 \cup g_2$.

Theorem B. *If f: $\Sigma^n \times \delta I \to R^{n+q}$ is any p.l. link, then there is a p.l. ribbon g: $\Sigma^{n+1} \times I \to R^{n+q+2}$ s.t. $g | \Sigma^{n+1} \times \delta I = \Sigma f$.*

Proof. Define the half space $T_+^{n+q+1} \subset R^{n+q+2}$ by $x_{n+q+1} = -x_{n+q+2}$, $x_{n+q+1} \geqq 0$. Extend f to an embedding g_1: $\Sigma^n \times I \to T_+^{n+q+1}$ by the Proposition. Extend g_1 conewise in four steps to form g;

1) Note that $\Sigma_-^{n+1} = \Sigma^n * u_{n+2}^-$. Define $g(u_{n+2}^- \times \{0\}) = u_{n+q+1}^-$ and extend $g_1 | \Sigma_n \times \{0\}$ conewise to $\Sigma_-^{n+1} \times \{0\}$.

2) Similarly define $g(u_{n+2}^+ \times \{1\}) = u_{n+q+2}^+$ and extend conewise to $\Sigma_+^{n+1} \times \{1\}$.

3) Note that $\Sigma_-^{n+1} \times I = (u_{n+2}^- \times \{1\}) * (\Sigma^n \times I \cup \Sigma_-^{n+1} \times \{0\})$. Define $g(u_{n+2}^- \times \{1\}) = u_{n+q+2}^-$ and extend conewise to $\Sigma_-^{n+1} \times I$.

4) Finally define $g(u_{n+2}^+ \times \{0\}) = u_{n+q+1}^+$ and extend conewise as in 3) to $\Sigma_+^{n+1} \times I$.

(See Fig. 1.)

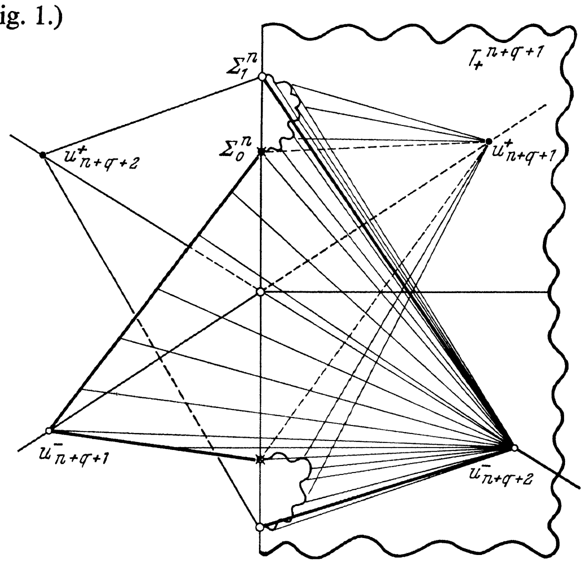

Fig. 1

Remark. The restriction of g to $\Sigma_-^{n+1}\times\{0\}\cup\Sigma^n\times I\cup\Sigma_+^{n+1}\times\{1\}$ followed by projection onto the hyperplane $T^{n+q+1}\subset R^{n+q+2}$ containing T_+^{n+q+1} determines an immersion of Σ^{n+1} in T^{n+q+1}.

Now let L_n^q denote the set of p.l. isotopy classes of p.l. links $f\colon \Sigma^n\times\delta I\to R^{n+q}$ and denote by N_n^q the subset of links whose second linking class is zero. We need the following result (see [3; Corollary 10.3]).

Theorem C. (Haefliger). *There is an exact sequence*

$$N_n^q \xrightarrow{\lambda} \pi_n(S^{q-1}) \xrightarrow{\sigma} \pi^{n-q+1} \longrightarrow N_{n-1}^q$$

where λ gives the first linking class (up to sign) and σ is repeated suspension. The sequence is valid for $3q\geqq n+6$ and π^r denotes the stable r-stem of the homotopy groups of spheres.

Now let $\Sigma\colon L_n^q\to L_{n+1}^{q+1}$ be the function given by suspension and E denote Freudenthal suspension. It is readily verified that $\Sigma(N_n^q)\subset N_{n+1}^{q+1}$ and that the following diagram commutes up to sign.

$$\begin{array}{ccccc} N_n^q & \xrightarrow{\lambda} & \pi_n(S^{q-1}) & & \\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle E} & \searrow{\scriptstyle\sigma} & \\ & & & & \pi^{n-q+1}. \\ & & & \nearrow{\scriptstyle\sigma'} & \\ N_{n+1}^{q+1} & \xrightarrow{\lambda'} & \pi_{n+1}(S^q) & & \end{array}$$

We now results of Toda [14]. A cyclic group of order n is abbreviated "n".

$$\begin{array}{ccccc} N_{18}^9 & \xrightarrow{\lambda} & \pi_{18}S^8=24+24+2 & & \\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle E} & \searrow{\scriptstyle\sigma} & \\ & & & & \pi^{10}=6. \\ & & & \nearrow{\scriptstyle\sigma'} & \\ N_{19}^{10} & \xrightarrow{\lambda'} & \pi_{19}S^9=24+2 & & \end{array}$$

Consider the element $\alpha\in\pi_{19}S^9$ which is six times the generator of 24. $\sigma'\alpha=0$. On the other hand E lies in an exact sequence (see James [7])

$$\pi_{18}S^8 \xrightarrow{E} \pi_{19}S^9 \longrightarrow \pi_{10}S^8=2$$

and so $\alpha=E\beta$ say, $\beta\in\pi_{18}S^8$. It follows that there is an element α_1 of N_{19}^{10} which is in the image of Σ and has first linking class α, not of order 2.

Choose a representative $f_1=\Sigma f\colon \Sigma^{19}\times\delta I\to R^{29}$ for α_1. By Theorem B f_1 spans a ribbon $g_1\colon \Sigma^{19}\times I\to R^{29}$. Extend g_1 to an embedding $g\colon \Sigma^{19}\times I^1\to R^{29}$ and p.l. isotope g by Zeeman [15] so that $g|\Sigma^{19}\times\{0\}$ is standard. g now satisfies all the conditions of Theorem A except condition (a), but fails to satisfy the conclusion, thus we have;

Example 1. There is a p.l. embedding $\Sigma^{19} \times I^1 \to \Sigma^{29}$ with no normal bundle.

Let η be a normal block bundle on $g(\Sigma^{19} \times I^1)$ in R^{29} and let $\eta_1 = \eta | \Sigma^{19}$, see [*9*, §§ 1,4], then η and $\eta_1 \times I^1$ are equivalent by [*9*, § 1]. Now suppose Σ^{19} has a normal bundle in $E(\eta_1) = M^{28}$, then the last equivalence furnishes $\Sigma^{19} \times I^1$ with a normal bundle in R^{29}, and we have;

Example 2. There is a p.l. 28-manifold M^{28} and a p.l. embedding $\Sigma^{19} \subset M^{28}$ with no normal bundle.

Now by remarks above $M^{28} \times I^1$ is a p.l. submanifold of R^{29}, consequently, by [*4*], M^{28} p.l. immerses in Σ^{28}. Let f: $\Sigma^{19} \to \Sigma^{28}$ be the restriction of this immersion. In [*4*] "induced neighbourhoods" of a p.l. immersion are defined in a topologically invariant way, so it makes sense to talk about a "normal bundle" to this immersion. But since an induced neighbourhood has the same germ as $\Sigma^{19} \subset M^{28}$, we have;

Example 3. There is a p.l. immersion f: $\Sigma^{19} \to \Sigma^{28}$ with no normal bundle.

Finally we remark that an explicit geometrical construction for Example 1 is obtained from Haefliger's construction in [*2*; 8.12] and our construction in Theorem B. The remark following Theorem B gives an explicit construction for Example 3.

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(*Received January 20, 1967*)