ON SUFFICIENTLY LARGE 3-MANIFOLDS

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Introduction

A compact, P^2-irreducible 3-manifold M is said to be *sufficiently large* if it admits a proper 2-sided embedding of a compact surface F not S^2 or P^2, such that $\pi_1(F)$ maps injectively into $\pi_1(M)$. Such an embedding is said to be *incompressible*. Waldhausen has shown in (**12**) that a homotopy equivalence between two closed sufficiently large 3-manifolds is homotopic to a homeomorphism and that two homotopic homeomorphisms are isotopic. He also had similar results for manifolds with boundary. Heil in (**3**) extended these results to the non-orientable case.

If a sufficiently large 3-manifold M contains a separating incompressible surface F then $\pi_1(M)$ is of the form $A *_{\pi_1(F)} B$, and if F is non-separating then $\pi_1(M)$ is of the form $A *_{\pi_1(F)}$. This paper was originally intended to be a short note devoted to proving converses of the above statements, given as Theorem 2.3 and Theorem 2.4 below.

Theorem 2.3 together with results of Waldhausen in (**12**) imply that Neuwirth's Conjecture H in (**7**) is correct. It also provides a partial answer to his Conjecture T.

It was pointed out to the author by G. A. Swarup that it should be possible to extend the methods used in the proofs of these theorems to greatly simplify the proof of Waldhausen's results in (**12**). This the author has done and a complete proof that homotopy equivalent sufficiently large 3-manifolds are homeomorphic is included in this paper.

The plan of the paper is as follows. In § 1 we present new proofs of results of Feustel (**2**) and Jaco (**4**). Theorem 1.3 has also been proved by G. A. Swarup in (**11**). We present these proofs partly for completeness and partly because they seem to be simpler and more direct than the original ones.

In § 2, we prove a theorem which has as immediate corollaries Waldhausen's result in (**12**) and Theorems 2.3 and 2.4 stated above. In § 3, we give some applications of these results showing that most of the interesting theorems on surfaces in 3-manifolds proved in (**12**) are corollaries of Waldhausen's main result.

The results we obtain in §§ 1 and 2 are not as strong as those obtained by other authors, but it follows at once from § 3 that we can in fact prove them in full strength after all.

We work throughout in the piecewise-linear category but all our results hold also in the differential and topological categories. The author would like to express his gratitude to G. A. Swarup for pointing out the possibility of simplifying the proofs of (**12**). He would also like to thank D. B. A. Epstein for a helpful conversation.

1. Some preliminary results

We will need the following definitions.

A 3-manifold M is *P^2-irreducible* if any 2-sphere embedded in M bounds a 3-ball, and M admits no 2-sided embedding of the projective plane.

A map $f: M \to N$, where M and N are manifolds, is *proper* if

$$f^{-1}(\partial N) = \partial M.$$

Note that we shall not use the word 'proper' to mean that the inverse image of a compact set is compact.

A compact surface F, not S^2 or P^2, which is properly embedded in a 3-manifold M is *incompressible* if $\pi_1(F)$ maps injectively into $\pi_1(M)$.

We will also use the standard result that any sufficiently large 3-manifold M is aspherical. For the Projective Plane Theorem in (**1**) implies that $\pi_2(M) = 0$ and either $\pi_1(M)$ is infinite or M is homeomorphic to D^3. We now look at the universal cover of M.

Finally the following classical theorem will be needed. The simplest proof of this result is analogous to our proof of Waldhausen's theorem.

THEOREM 1.1. *If $f: M \to N$ is a proper map of compact surfaces inducing an isomorphism of fundamental groups and if $M \neq S^2$ or P^2 or D^2, then f is properly homotopic to a map g such that either*

(*a*) *g is a homeomorphism, or*

(*b*) *$g(M) \subset \partial N$ and M is $S^1 \times I$ or the Moebius band.*

We start with the following result, which though trivial appears not to be well known.

LEMMA 1.2. *Let F^{n-1} be a closed orientable manifold in the interior of the orientable manifold M^n. Denote by $[F]$ a generator of $H_{n-1}(F, Z)$. Then if $[F]$ is non-zero in $H_{n-1}(M, Z)$, it is also indivisible.*

Proof. If F fails to separate M, we can find a circle in M cutting F transversely in one point. This circle represents an element of $H_1(M)$ dual to an element x of $H^{n-1}(M, \partial M)$ such that $x[F] = 1$. As evaluation is a homomorphism the result follows in this case.

If F separates M into X and Y, then neither X nor Y can be compact with boundary F, as $[F]$ is non-zero in $H_{n-1}(M, Z)$. Therefore we can find a path cutting F transversely at one point and ending either in some boundary component of M or 'at infinity'. The same argument now applies using if necessary duality between homology with infinite chains and cohomology with finite chains.

We can now prove the following two theorems. As we may not quote any of Waldhausen's results in (**12**), our Theorem 1.4 is weaker than the result in (**4**). This will be rectified at the end of § 3.

THEOREM 1.3. *Let M^3 be irreducible and orientable and $F^2 \subset M^3$ a closed incompressible surface. If $\pi_1(F) \subset G \subset \pi_1(M)$, where G is isomorphic to the fundamental group of a closed orientable surface L, then $G = \pi_1(F)$.*

Proof. Let N be the covering space of M determined by $G \subset \pi_1(M)$. Our embedding of F in M lifts to N. As M is aspherical so is N and hence N is homotopy equivalent to L.

So we have $F \xrightarrow{i} N \xrightarrow{f} L$ where f is a homotopy equivalence. Now $f \circ i$ is homotopic to a covering map of some finite degree r, by Theorem 1.1. Hence $i_*: Z \cong H_2(F, Z) \to H_2(N, Z) \cong Z$ is multiplication by r. Lemma 1.2 now tells us that $r = \pm 1$ and hence $G = \pi_1(F)$ as required.

THEOREM 1.4. *Let M^3 be P^2-irreducible and $F^2 \subset M^3$ a closed incompressible surface. If $\pi_1(F) \subset G \subset \pi_1(M)$ where $|G: \pi_1(F)|$ is finite then either*

(a) $G = \pi_1(F)$, or

(b) $|G: \pi_1(F)| = 2$ and F bounds a compact submanifold X of M with $\pi_1(X) = G$.

Note that if G is the fundamental group of a closed surface then $|G: \pi_1(F)|$ is automatically finite.

Proof. Take the covering space N of M determined by $G \subset \pi_1(M)$. Our embedding of F in M lifts to N.

If F fails to separate N then $\pi_1(N) \cong H *_{\pi_1(F)}$ for some group H which contradicts the fact that $|G: \pi_1(F)|$ is finite. Therefore F separates N into X and Y and $\pi_1(N) \cong \pi_1(X) *_{\pi_1(F)} \pi_1(Y)$. As $|G: \pi_1(F)|$ is finite, we deduce that $\pi_1(X) = G$ and $\pi_1(Y) = \pi_1(F)$.

Now if $G \neq \pi_1(F)$ then X has a non-trivial finite covering space $\tilde{X}$ determined by $\pi_1(F) \subset \pi_1(X)$. Our embedding of F in ∂X lifts to $\tilde{X}$ and the inclusion of F in $\tilde{X}$ is a homotopy equivalence. Note that as $\tilde{X}$ is a non-trivial covering space of X, its boundary cannot consist only of F.

Consider the following exact sequence

$$H_3(\tilde{X}, \partial\tilde{X}; Z_2) \to H_2(\partial\tilde{X}; Z_2) \to H_2(\tilde{X}; Z_2).$$

As the dimension of the third group is one, and of the middle group at least two, we deduce that $\tilde{X}$ is compact with two boundary components. Now consider the exact sequence

$$H_2(\tilde{X}, \partial\tilde{X}; Z) \xrightarrow{f} H_1(\partial\tilde{X}; Z) \xrightarrow{g} H_1(\tilde{X}; Z).$$

The map g is onto as the inclusion of F in $\tilde{X}$ induces an isomorphism $H_1(F, Z) \to H_1(\tilde{X}, Z)$. If $\tilde{X}$ is orientable then f and g are dual maps and it follows that the rank of $H_1(\partial\tilde{X}, Z)$ is twice the rank of $H_1(\tilde{X}, Z)$. Hence $\partial\tilde{X}$ consists of two copies of F, say F and F'. Consider the composite $F' \subset \tilde{X} \to F$ where the map $\tilde{X} \to F$ is a homotopy equivalence. As the inclusion of F' in $\tilde{X}$ induces an isomorphism

$$H_2(F', Z) \to H_2(\tilde{X}, Z),$$

the composite map $F' \to F$ is of degree one, so induces an epimorphism of fundamental groups. Now fundamental groups of compact surfaces are Hopfian, so our map must actually induce an isomorphism of fundamental groups. Hence the inclusion of F' in $\tilde{X}$ is also a homotopy equivalence. If $\tilde{X}$ is non-orientable, we get the same result by considering the orientable double cover of $\tilde{X}$. It follows that $|G: \pi_1(F)| = 2$ and that X is compact and bounded by F.

Now consider the projection map $\pi: N \to M$. As X is compact, $\pi^{-1}(F) \cap X$ consists of ∂X and a finite number of compact covering spaces of F. Let L be a component of $\pi^{-1}(F) \cap X$. Then $L \subset X$ and $|\pi_1(X): \pi_1(L)|$ is finite. It follows from what we have already proved that this index must be one or two. Hence as $\pi_1(L) \subset \pi_1(F) \subset \pi_1(X)$ we must have $L \cong F$.

Thus $\pi^{-1}(F) \cap X$ consists of a finite number of copies of F which we denote by $F_1, \ldots, F_n$ and each F_i bounds a submanifold X_i of X. Note that $\pi_1(X_i) \cong G$ for $\pi_1(F_i) \subset \pi_1(X_i) \subset G$ and F_i cannot bound a compact 3-manifold X_i with $\pi_1(X_i) = \pi_1(F_i)$. This follows from the previous argument about $\tilde{X}$. There must be an F_i such that X_i contains no other F_j, and then $\pi \mid X$ will be a homeomorphism. To see this define $S \subset X_i$ by

$S = \{x \in X_i : \exists y \in X_i, y \neq x, \pi(x) = \pi(y)\}$. S is open and closed in X_i and does not meet F_i.

Therefore if $G \neq \pi_1(F)$, F bounds $\pi(X_i)$ in M as required, and this completes the proof of Theorem 1.4.

2. The main result

We are now in a position to prove the following.

THEOREM 2.1. *Let M and N be compact P^2-irreducible 3-manifolds with incompressible boundaries and let $f: M \to N$ be a proper map which is a homotopy equivalence. Let F be an incompressible surface in N which either fails to separate N or separates N into components neither of which has fundamental group equal to $\pi_1(F)$. Then f is properly homotopic to a map g such that either*

(*a*) *$g\,|: g^{-1}(F) \to F$ is a homeomorphism, or*

(*b*) *$g(M) \subset \partial N$. In this case ∂M consists of two copies of F and $\pi_1(M) = \pi_1(F)$, or ∂M consists of one copy of F and $|\pi_1(M): \pi_1(F)| = 2$.*

Remarks. Our conditions on F are simply to ensure that $f^{-1}(F)$ can never be empty.

Using Theorems 3.1 and 3.7, the result in case (*b*) can be improved to say that $M \cong F \times I$ or M is a non-trivial bundle with fibre I over a closed surface.

Proof of 2.1. The proof falls naturally into two cases, the second one being very much easier.

Case 1. *∂F is empty.*

We make f transverse to F by a homotopy fixed on ∂M. Then $f^{-1}(F)$ is a union of closed surfaces in M and by applying the Loop Theorem as in (9), we can suppose that each component is incompressible. Note that no component can be homeomorphic to P^2. Also if a component is homeomorphic to S^2 it must bound a 3-ball in M. Hence, as N is aspherical, we can homotop f so as to remove this component.

Now let L be a component of $f^{-1}(F)$. L is not S^2 or P^2 and

$$f_*: \pi_1(L) \to \pi_1(F)$$

is injective so Theorem 1.1 implies that we can homotop f so that $f\,|: L \to F$ is a covering map. Theorem 1.4 implies that either the map is a homeomorphism or a double covering. In the latter case L bounds a submanifold X of M with $\pi_1(X) = f_*^{-1}(\pi_1(F))$. This implies that we can homotop $f \mid X$ modulo L to a map f' with $f'(X) \subset F$. To construct this homotopy we choose a triangulation of X and define our homotopy on simplices of X working upward in dimension. As $\pi_1(X) = f_*^{-1}(\pi_1(F))$, *we can construct the required homotopy on the* 1-*skeleton of* X. As N and F are aspherical, it is trivial to extend our homotopy over the rest of X. Thus we can homotop f so as to remove the component L from $f^{-1}(F)$. Hence we can suppose that every component of $f^{-1}(F)$ is mapped homeomorphically to F.

Suppose $f^{-1}(F)$ is not connected. We denote the components by $F_1,\ldots, F_n$. We also choose a base-point $e \in F$ and base-points $e_1,\ldots, e_n$ in $F_1,\ldots, F_n$ such that $f(e_i) = e$. We know that $f_*\colon \pi_1(F_i, e_i) \to \pi_1(F, e)$ is an isomorphism for each i.

We now come to the method of arc-chasing. Similar methods have been used by Kneser and Stallings in (**5**) and (**10**). Choose two components F_1 and F_2 of $f^{-1}(F)$ and choose a path λ in M from e_1 to e_2. Then $f(\lambda)$ is a loop in N and so represents an element l of $\pi_1(N, e)$. Let L be a loop in M based at e_1 representing $f_*^{-1}(l^{-1}) \in \pi_1(M, e_1)$. Then the path $\Gamma = \lambda L$ is a path from e_1 to e_2 such that $f(\Gamma)$ is a contractible loop in N.

We may suppose that Γ is immersed in M transversely to $f^{-1}(F)$ except at the end points, and that Γ meets each F_i only in e_i. Then $\Gamma \cap f^{-1}(F)$ divides Γ into a finite number of arcs $\Gamma_1,\ldots, \Gamma_r$ and each $f(\Gamma_i)$ is a loop in N representing some element $g_i \in \pi_1(N, e)$. Now our map of Γ into N defines a map $S^1 \to N$ which is inessential by our construction and so extends to a map $h\colon D^2 \to N$. We can suppose that h is transverse to F, as F is collared in N, so that $h^{-1}(F)$ is a union of circles in the interior of D^2 and of properly embedded arcs. We denote $h^{-1}(F)$ by S. If S contains a circle C then C bounds a 2-disc B in D^2. As $h(C)$ is inessential in N, it must be inessential in F, so we can replace h by a map of h' equal to h outside B and such that $h'(B) \subset F$. Hence we can homotop h further so as to remove C.

By repeating this we can suppose that S contains no circles. Now any arc of S separates D^2 and there must be an arc A such that one of the corresponding components E of D^2 does not meet S in its interior. Then $E \cap \partial D^2$ is an arc and $h(E \cap \partial D^2) = f(\Gamma_i)$ for some i. The disc E defines a homotopy fixed on $S \cap \partial D^2$ of $E \cap \partial D^2$ to A and so g_i lies in $\pi_1(F, e)$.

If the corresponding arc Γ_i has both its end points at the same point e_s of M, we can homotop Γ_i modulo its endpoints to lie in F_s. Hence we can homotop Γ so as to remove these two intersection points of Γ with

$f^{-1}(F)$. Thus we can suppose that there is an arc Γ_i with the property that g_i lies in $\pi_1(F, e)$ and such that Γ_i has distinct endpoints e_s and e_t say.

If we cut M along F_s and F_t there will be a unique component X containing copies of F_s and F_t in its boundary and containing the arc Γ_i. If we take the covering space $\tilde{X}$ of X corresponding to $\pi_1(F_s)$ then our embeddings of F_s and F_t in X will lift to $\tilde{X}$. For $\pi_1(F_s, e_s) = \pi_1(F_t, e_s)$ via the arc Γ_i. The method of proof of the last part of Theorem 1.4 will now show that $\tilde{X}$ is compact with boundary two copies of F_s. It follows that the inclusion of F_s in X is already a homotopy equivalence. This implies that we can homotop $f \mid X$ modulo $F_s \cup F_t$ to a map f' such that $f'(X) \subset F$. As before, we can define this homotopy simplex by simplex of a triangulation of X.

As X does not meet ∂M, we can homotop f modulo ∂M so as to remove F_s and F_t from $f^{-1}(F)$. By repeating this process we can arrange that $f^{-1}(F)$ is connected as required. This completes the proof of Theorem 2.1 in Case 1.

Case 2. ∂F is non-empty.

We first show that either the restriction of f to ∂M is homotopic to a homeomorphism or we can prove case (*b*) of the theorem.

Let L be a component of ∂M and S be the component of ∂N with $f(L) \subset S$. Then $\pi_1(L) \subset f_*^{-1}(\pi_1(S)) \subset \pi_1(M)$ and Theorem 1.4 implies that $\pi_1(L) = f_*^{-1}(\pi_1(S))$ or has index two in it. In this second case L bounds a submanifold X of M with $\pi_1(X) = f_*^{-1}(\pi_1(S))$. As L is a boundary component of M we must have $X = M$. Therefore either $f \mid : L \to S$ is homotopic to a homeomorphism or we have case (*b*) of the theorem as required.

We now suppose that the restriction of f to each boundary component of M is a homeomorphism. Suppose there are two distinct boundary components R_1 and R_2 of M mapped to the same boundary component S of N. Choose a point $e \in S$ and points e_1, e_2 in R_1 and R_2 such that $f(e_i) = e$. As in the proof of Case 1, we can find an arc Γ in M with endpoints at e_1 and e_2 such that $f(\Gamma)$ is a contractible loop in N. Again as in the proof of Case 1 this implies that $\partial M = R_1 \cup R_2$ and that the inclusion of R_1 in M is a homotopy equivalence. As N is aspherical, we can prove case (*b*) of the theorem as required.

Now suppose that $f \mid \partial M$ is a homeomorphism. We make f transverse to F and make $f^{-1}(F)$ incompressible, as usual with no component of $f^{-1}(F)$ being S^2 or P^2. Now choose a component L of $f^{-1}(F)$. Theorem 1.1 tells us that we can homotop f so that L covers F and hence $f \mid : L \to F$

is a homeomorphism, unless $L = D^2$, $S^1 \times I$ or the Moebius band. But we know $f \mid \partial L$ is a homeomorphism, therefore the result is true in these cases also. If $L = D^2$ this is obvious, and if $L = S^1 \times I$ it follows from Theorem 1.1. If L is the Moebius band and the result is false, then Theorem 1.1 implies that we can homotop L into a component circle S of ∂F. But this would imply that the map $f \mid : \partial L \to S$ was of even degree which contradicts the fact that it is a homeomorphism.

As any component L of $f^{-1}(F)$ maps homeomorphically onto F we deduce that $f^{-1}(F)$ must already be connected and our result follows.

The following theorems are easy consequences of this result and its method of proof.

THEOREM 2.2 (Waldhausen and Heil). *Let M and N be compact P^2-irreducible 3-manifolds with incompressible boundaries and suppose that N is sufficiently large. Let $f: M \to N$ be a proper map which is a homotopy equivalence. Then f is properly homotopic to a map g such that either*

(*a*) *g is a homeomorphism, or*

(*b*) *$g(M) \subset \partial N$. In this case either ∂M consists of two copies of a surface F, with $\pi_1(M) = \pi_1(F)$, or ∂M consists of one copy of F with $|\pi_1(M) : \pi_1(F)| = 2$.*

Proof. We first choose a hierarchy for N [see (**12**)]. Our proof is by induction on the length of the hierarchy. Take an incompressible surface F in N, and apply Theorem 2.1. If case (*b*) of Theorem 2.1 holds, then we have proved case (*b*) of Theorem 2.2. If case (*a*) of Theorem 2.1 holds, we cut N along F and cut M along $f^{-1}(F)$ to get a map $f': M' \to N'$ satisfying all the hypotheses of Theorem 2.2 and in addition we can suppose that $f' \mid \partial M'$ is a homeomorphism. We now apply our induction hypothesis for Theorem 2.2 to the map f'. As $f' \mid \partial M'$ is a homeomorphism, f' must satisfy case (*a*) of Theorem 2.2 and therefore we have proved case (*a*) of Theorem 2.2 for the map f. If N has a hierarchy of length one, then in the above we must have $N' \cong D^3$ and $M' \cong D^3$ so f' is still homotopic to a homeomorphism.

Remarks. Waldhausen considers a proper map $f: M \to N$ which induces an injection of fundamental groups. In order to obtain his results we take the covering $\tilde{N}$ of N corresponding to $\pi_1(M) \subset \pi_1(N)$ and apply Theorem 2.2 to the map $\tilde{f}: M \to \tilde{N}$. Note that $\tilde{N}$ need not be compact. But the proof of Case 2 of Theorem 2.1 shows that either we can prove case (*b*) of Theorem 2.2 or the restriction of $\tilde{f}$ to ∂M is a homeomorphism. A simple homology argument now shows that $\tilde{N}$ is compact so $\tilde{N}$ is a finite covering space of N and is also sufficiently large.

Again the result stated here is weaker than Waldhausen's, for in case (*b*) of the theorem Waldhausen says M is a line bundle over a surface. We will be able to prove this also in § 3 of this paper.

THEOREM 2.3. *Let M^3 be closed and P^2-irreducible and suppose $\pi_1(M) \cong A *_C B$, where $C \neq A$ or B and C is isomorphic to the fundamental group of a closed surface L. Then there is an incompressible embedding of L in M separating M into M_1 and M_2 with $\pi_1(M_1) = A$, $\pi_1(M_2) = B$, and $\pi_1(L) = C$.*

THEOREM 2.4. *Let M^3 be closed and P^2-irreducible and suppose $\pi_1(M) \cong A *_C$ where C is isomorphic to the fundamental group of a closed surface L. Then there is an incompressible embedding of L in M such that $\pi_1(L) = C$ and M cut along L is a connected 3-manifold N with $\pi_1(N) \cong A$.*

Proofs of Theorems 2.3 *and* 2.4. The idea is simply to work through the proof of Theorem 2.1 using an appropriate Eilenberg–Maclane space instead of the 3-manifold N. In the first case, we construct an E–M space $K(A *_C B, 1)$ as follows. Choose Eilenberg–Maclane spaces, $K(A, 1)$, $K(B, 1)$, and $K(C, 1)$, which are simplicial complexes. We have simplicial maps $i_0\colon K(C,1) \to K(A,1)$ and $i_1\colon K(C,1) \to K(B,1)$ induced by the inclusion maps of C in A and B. We take $K(A *_C B, 1)$ to be

$$K(A,1) \bigcup_{i_0 \times 0} K(C,1) \times I \bigcup_{i_1 \times 1} K(B,1).$$

In the second case, we have two inclusion maps $i_0, i_1\colon K(C,1) \to K(A,1)$ and we take $K(A *_C, 1)$ to be $K(A,1) \bigcup_{\substack{i_0 \times 0 \\ i_1 \times 1}} K(C,1) \times I$.

Now choose a simplicial map $f\colon M \to K(G,1)$ inducing an isomorphism of fundamental groups where $G = A *_C B$ or $A *_C$ as appropriate. We can choose $t \in (0,1)$ so that $K(C,1) \times \{t\}$ contains no vertices of $f(M)$ and we can suppose that $t = \frac{1}{2}$ without loss of generality. Then

$$f^{-1}(K(C,1) \times \{\tfrac{1}{2}\})$$

is a closed surface possibly not connected in M. Denote this surface by T. As usual we can suppose T is non-empty, incompressible, and has no components homeomorphic to S^2 or P^2. Hence F is not S^2 or P^2. Note that in either case G is infinite, hence M is aspherical. We can now proceed precisely as in the proof of Theorem 2.1.

Remarks. Theorem 2.3 is false if we relax our condition that C be the fundamental group of a closed surface. For let $M = M_1 \cup M_2 \cup M_3$ where $M_1 \cap M_3$ is empty, $M_1 \cap M_2 = \partial M_1$, $M_2 \cap M_3 = \partial M_3$, and M_2 is connected. Then $\pi_1(M) \cong \pi_1(M_1 \cup M_2) *_{\pi_1(M_2)} \pi_1(M_2 \cup M_3)$ and $\pi_1(M_2)$ need not be isomorphic to the fundamental group of a closed surface.

A similar example will give a counter-example to Theorem 2.4.

Theorem 2.3 is also false if we relax the condition that M be P^2-irreducible. In Theorem 2.2 of (**13**) Waldhausen shows that there exists an irreducible closed 3-manifold X which does not admit an incompressibly embedded torus but has $Z \times Z \subset \pi_1(X)$. Let $M = X_1 \# X_2$ where $X_1 \cong X_2 \cong X$. Then $\pi_1(M) \cong \pi_1(X_1) *_{Z \times Z} ((Z \times Z) * \pi_1(X_2))$. Hence if Theorem 2.3 held we could embed a torus incompressibly in M. But by the methods of (**6**), we could isotop this torus so as not to meet the sphere along which X_1 and X_2 are joined and this would contradict our hypothesis on X.

3. Applications

In this section, we apply Theorem 2.2 to prove some of the theorems of (**12**). We can then go back to Theorems 1.4 and 2.2 to prove them in full-strength by applying Theorems 3.1 and 3.7. We need to use the following results to carry out this strengthening. Our first theorem was originally proved by Stallings in (**9**).

THEOREM 3.1. *If X is a compact, P^2-irreducible 3-manifold with a compact surface $F \subset \partial X$ where inclusion in X is a homotopy equivalence, then $X \cong F \times I$.*

Proof. If F is closed then as we showed in the proof of Theorem 1.4, X must have boundary consisting of two copies of F such that the inclusion of each in X is a homotopy equivalence. It follows that there is a proper map $f: F \times I \to X$ which is a homeomorphism on $\partial(F \times I)$ and is a homotopy equivalence. Theorem 2.2 implies that $X \cong F \times I$ as required.

If F is not closed, we show similarly that ∂X is the double of F and the inclusion of each copy of F in X is a homotopy equivalence. We can then define a map $f: F \times I \to X$ which is a homeomorphism on $\partial(F \times I)$ and is a homotopy equivalence. Again this implies our result by Theorem 2.2. Note that ∂X is not incompressible, but as f is a homeomorphism on $\partial(F \times I)$ anyway the proof of Theorem 2.2 goes through.

THEOREM 3.2. *Let F be a closed surface not S^2 or P^2 and let G be an incompressible surface in $F \times I$ with $\partial G \subset F \times 1$ or ∂G empty. Then G is parallel to a surface in $F \times 1$, i.e. there is an embedding of $G \times I$ in $F \times I$ with $G = G \times 0$ and $G \times 1 \cup \partial G \times I \subset F \times 1$.*

Proof. We first note that G must separate $F \times I$, for the map $H_2(G, \partial G; Z_2) \to H_2(F \times I, F \times 1; Z_2)$ is zero, as the second group is zero. Let the complementary components be X and Y with $F \times 0 \subset X$. Then $\pi_1(F \times I) \cong \pi_1(X) *_{\pi_1(G)} \pi_1(Y)$. As the composite

$$\pi_1(F \times 0) \to \pi_1(X) \to \pi_1(F \times I)$$

is an isomorphism we deduce that $\pi_1(X) = \pi_1(F)$ and $\pi_1(Y) = \pi_1(G)$. Theorem 3.1 implies that $Y \cong G \times I$ and this proves the required result.

The final result we need is Theorem 3.7 below. We prove this by quoting the following theorem which was proved in the orientable case by Nielsen in (**8**) and extended to the non-orientable case by Zieschang in (**14**).

THEOREM 3.3. *Let F be a closed surface and $f\colon F \to F$ a map whose n-th composite f^n is homotopic to the identity map. Then f is homotopic to a homeomorphism g where g^n is the identity map of F.*

I am grateful to C. T. C. Wall for suggesting the following application of this result which seems to be of some general interest.

COROLLARY 3.4. *Let F be a closed surface and let G be a torsion-free group which is an extension of $\pi_1(F)$ by Z_n. Then G is the fundamental group of a closed surface.*

Proof. Any extension of a group H by a group K determines a homomorphism $K \to O(H)$, the outer automorphism group of H. If H is centreless, then conversely such a homomorphism determines a unique extension of H by K.

We consider first the case when F is not a torus or Klein bottle and so $\pi_1(F)$ is centreless. Corresponding to G we have a homomorphism $\phi\colon Z_n \to O(\pi_1(F))$. Let α be a generator of Z_n. Then $\phi(\alpha)$ determines a map $f\colon F \to F$ such that f^n is homotopic to the identity. Theorem 3.3 says that we can take f to be a homeomorphism of order n. Hence some extension G' of $\pi_1(F)$ by Z_n acts by homeomorphisms on R^2 extending the standard action of $\pi_1(F)$. But corresponding to G' we have a homomorphism $\phi'\colon Z_n \to O(\pi_1(F))$ and $\phi' = \phi$. It follows that $G' \cong G$ and so G acts on R^2. As G is torsion-free this action must be free. For if the action of $g \in G$ has a fixed point so does the action of any power of g.

But some power of g lies in $\pi_1(F)$ which we know acts freely. This proves the required result.

In the case when F is a torus or Klein bottle the argument is much easier.

Suppose that F is a torus and let $\phi: Z_n \to O(\pi_1(F))$ be the homomorphism corresponding to our extension G. Let α be a generator of Z_n. By choosing a suitable basis of $\pi_1(F)$, we can suppose that $\phi(\alpha)$ has matrix

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the first case we must have $a = 0$ as the matrix must be of finite order. Hence G is abelian and as G is torsion-free we have $G \cong Z \times Z$. In the second case, G has a subgroup of index 2 which is abelian and hence isomorphic to $Z \times Z$, so G itself is isomorphic to the fundamental group of a Klein bottle.

Now suppose that F is a Klein bottle with ϕ and α as before. We know that $\pi_1(F) \cong \{a, b: b^{-1}ab = a^{-1}\}$ and that $\pi_1(F)$ has infinite centre generated by b^2. We also know that $O(\pi_1(F)) \cong Z_2 \times Z_2$ where automorphisms representing the four elements of $O(\pi_1(F))$ are given by $a \to a^{\pm 1}$, $b \to b^{\pm 1}$. Hence $\phi(\alpha)$ can take four possible values and we check that in each case either G is isomorphic to the fundamental group of the Klein bottle or that G cannot be torsion-free. In the case when ϕ is trivial, there exists $g \in G$ projecting down to $\alpha \in Z_n$ such that g commutes with a and b. Now $g^n \in \pi_1(F)$ has the same property so g^n is a power of b^2. Hence $\mathrm{gp}\{b, g\} \cong Z$ with generator h say. We can easily see that $G \cong \{a, h: h^{-1}ah = a^{-1}\}$, proving the required result.

In the general case, G will have a subgroup of index 2 corresponding to $\ker(\phi)$ which will be isomorphic to $\pi_1(F)$. So we will only consider the case $n = 2$.

Case 1. There is $g \in G$ projecting down to $\alpha \in Z_2$ such that $g^{-1}ag = a^{-1}$, $g^{-1}bg = b$. Then g^2 is central in G and hence in $\pi_1(F)$, so $g^2 = b^{2r}$ for some r. Hence $(gb^{-r})^2 = 1$, so $g = b^r$ as G is torsion-free and we have a contradiction.

Case 2. There is $g \in G$ projecting to $\alpha \in Z_2$ such that $g^{-1}ag = a^{\pm 1}$, $g^{-1}bg = b^{-1}$. Again $g^2 = b^{2r}$, so g and b^{2r} commute. But $g^{-1}b^{2r}g = b^{-2r}$, so $r = 0$ which implies $g^2 = 1$ and is again a contradiction.

COROLLARY 3.5. *Let M^n be a manifold homotopy equivalent to a closed surface not S^2 or P^2. If a cyclic group acts freely on M then the quotient space N is also homotopy equivalent to a closed surface.*

Proof. We simply note that M is aspherical and hence so is the quotient space N. Hence $\pi_1(N)$ is torsion-free as N is finite-dimensional. We now apply Corollary 3.4.

THEOREM 3.6. *Let M^3 be compact and P^2-irreducible with incompressible boundary and let F be a closed surface homotopy equivalent to M. Then M is homeomorphic to a bundle with fibre I over F.*

Proof. The boundary of M contains no 2-spheres or projective planes as M is P^2-irreducible. Also note that M is aspherical so F cannot be S^2 or P^2. Finally $H_3(M, Z_2) = 0$ so M has non-empty boundary. Let L be a component of ∂M. Then Theorem 1.4 implies that $\pi_1(L) = \pi_1(F)$ or $|\pi_1(F): \pi_1(L)| = 2$ and $L = \partial M$. In the first case, Theorem 3.1 says that $M \cong F \times I$. In the second case we can show M is a non-trivial bundle with fibre I over F as follows. The composite map

$$L \cong \partial M \subset \xrightarrow{\simeq} F$$

is homotopic to a double covering map and this determines a bundle with fibre I over F and projection π say. Let X be the total space of this bundle. Then the composite map $X \xrightarrow{\pi} F \xrightarrow{\simeq} M$ is a homotopy equivalence. Also the inclusion $L \cong \partial M \subset M$ is homotopic to the map $L \cong \partial X \to M$. Hence we can homotop our map $X \to M$ to be a homeomorphism of ∂X to ∂M. The required result now follows by Theorem 2.2.

THEOREM 3.7. *Let M^3 be a bundle with fibre I over a closed surface not S^2 or P^2. If $\pi: M \to N$ is a covering map then N is also a bundle with fibre I over a closed surface.*

Proof. We first show that N has incompressible boundary. Let L be a component of ∂M and $F = \pi(L)$ and consider the natural map $i_*: \pi_1(F) \to \pi_1(N)$. Now $\pi_1(L)$ is of finite index in $\pi_1(F)$ and injects into $\pi_1(N)$. Hence the kernel of i_* is finite and so trivial.

Now Theorem 1.4 shows that $\pi_1(F) = \pi_1(N)$ or $|\pi_1(N): \pi_1(F)| = 2$, and Theorem 3.1 shows that $N \cong F \times I$ or N is doubly covered by $F \times I$. Our result now follows from Corollary 3.5 and Theorem 3.6.

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