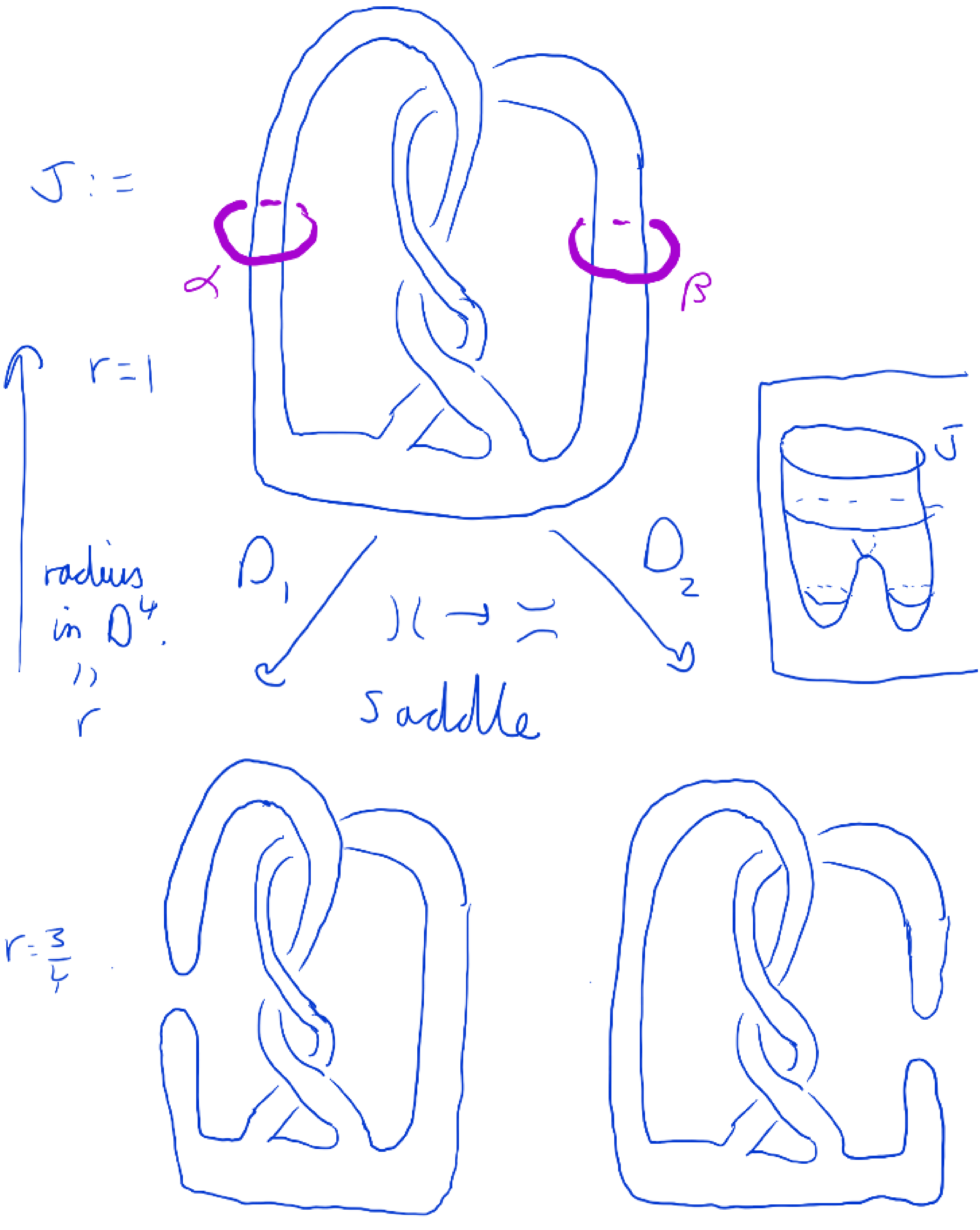
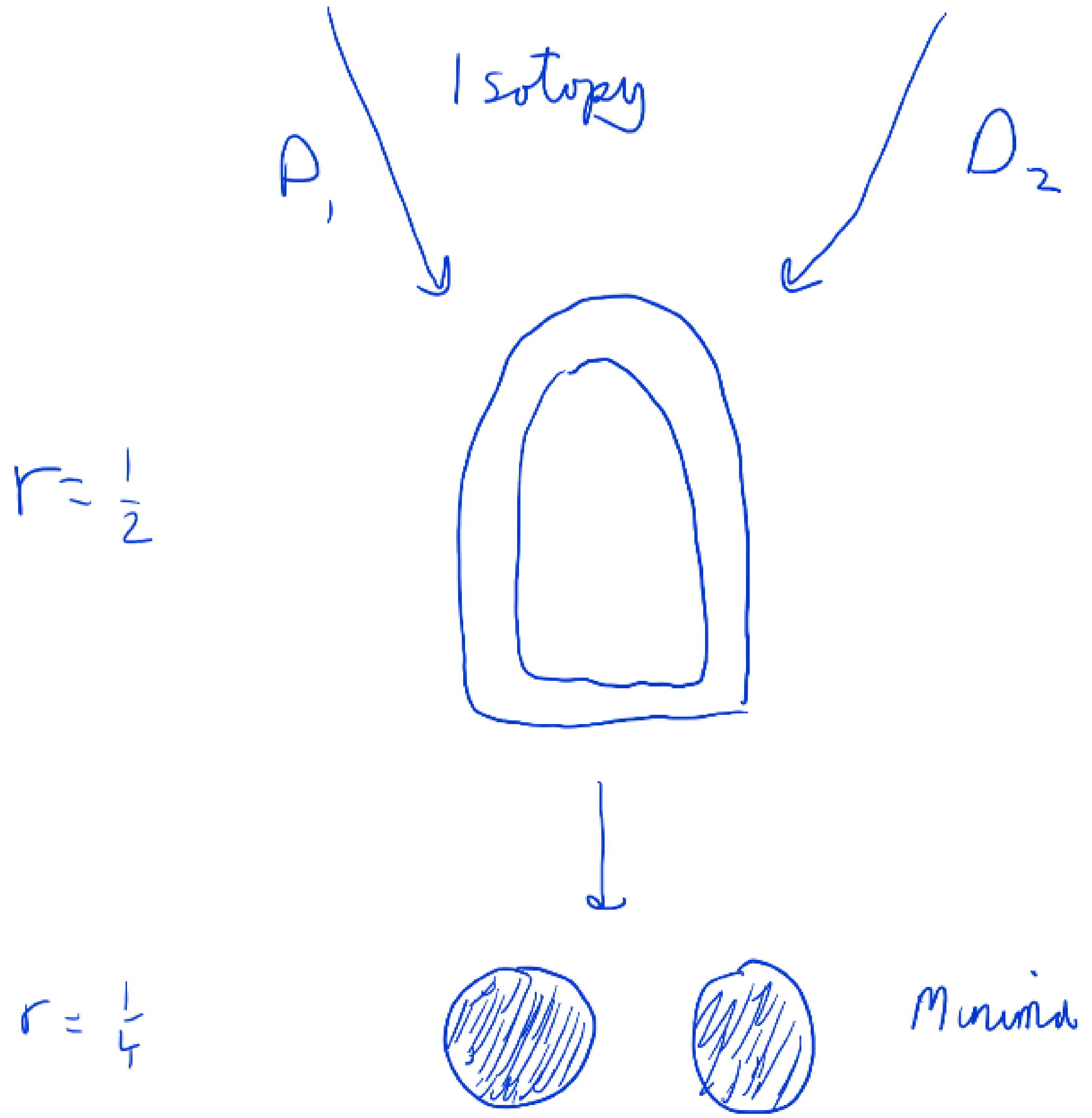
Slice discs 1

A knot J and two slice discs D_1, D_2





Defn

An oriented knot in S^3 is slice if it bounds a locally flat disc $D^2 \hookrightarrow D^4$.

Remark

$\overline{D_1 \text{ and } D_1 \text{ are ambiently isotopic}}$

$$\Big\downarrow \pi$$

But they are not ambiently isotopic rel. ∂.

$$f_i : H_1(S^3 \setminus J ; \mathbb{Z}[t^{\pm 1}]) \to H_1(D^4 \setminus D_i ; \mathbb{Z}[t^{\pm}])$$

$$\begin{array}{ccc} \cong & \cong & \cong \\ \text{Alex. modul} \quad \dfrac{\mathbb{Z}[t^{\pm 1}]}{t-2} \oplus \dfrac{\mathbb{Z}[t^{\pm 1}]}{2t-1} & \longrightarrow & \begin{cases} \dfrac{\mathbb{Z}[t^{\pm}]}{2t-1} & i=1 \\ \dfrac{\mathbb{Z}[t^{\pm 1}]}{t-2} & i=2 \end{cases} \end{array}$$

$$\frac{\mathbb{Z}[t^{\pm 1}]}{t-2} = \langle \alpha \rangle, \qquad \frac{\mathbb{Z}[t^{\pm 1}]}{2t-1} = \langle \beta \rangle$$

$$\ker f_i = \begin{cases} \langle \alpha \rangle & i=1 \\ \langle \beta \rangle & i=2 \end{cases}$$

$$\mathbb{Z}[t^{\pm}]^2 \xrightarrow{tA - A^T} \mathbb{Z}[t^{\pm 1}]^2 \to H_1(S^3 \setminus J ; \mathbb{Z}[t^{\pm}])$$

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \qquad \ast$$

$$tA - A^T = \begin{pmatrix} 0 & 2t-1 \\ t-2 & 0 \end{pmatrix}$$

How many slice discs does J have?

Observe: can $\#$ 2-knot.

so ∞ !

The discs $D_1, \; , D_2$ are

1) homotopy ribbon

$$\pi_1(S^3 \setminus J) \twoheadrightarrow \pi_1(D^4 \setminus D_i)$$

2) $\pi_1(D^4 \setminus D_i) \cong \mathbb{Z} \ltimes \mathbb{Z}[\tfrac{1}{2}]$

$$\left\{ \mathbb{Z} \curvearrowright \mathbb{Z}[\tfrac{1}{2}] \right.$$

$$= \langle a, c \mid a c a^{-1} = c^2 \rangle \quad t \cdot p = (\tfrac{p}{2} \quad i = 1$$

$= B(1,2)$ | $\Big|_{2\cdot p} \quad i=2$

<u>Defn</u>

A slice disc D is G-<u>homotopy ribbon</u> if it is homotopy ribbon and if $\pi_1(D^4 \setminus D) \cong G$.

If $D_1 \sim D_2$ rel ∂

then $\ker f_1 = \ker f_2 \leq H_1(S^3 \setminus S : \mathbb{Z}[t^{\pm 1}])$

so $D_1 \underset{\text{top.}}{\not\sim} D_2$ rel ∂

Slice discs 2 existence and uniqueness theorems

Defn Let G be a group

A slice disc D for K is a

G-homotopy-ribbon disc if

$$\pi_1(S^3 \setminus K) \twoheadrightarrow \pi_1(D^4 \setminus D)$$

and $\pi_1(D^4 \setminus D) \cong G$

Focus on $G = \mathbb{Z}$

and $G = \mathbb{Z} \ltimes \mathbb{Z}[\frac{1}{2}]$

The two solvable ribbon groups

$$\cong \langle a, c \mid aca^{-1} = c^2 \rangle$$

$$= B(1,2) =: \Gamma$$

$\mathbb{Z}$-HR Existence

Theorem (Freedman-Quinn (1984 - 1CM))

A knot K is $\mathbb{Z}$-homotopy ribbon

if and only if $\Delta_K \doteq 1$

Γ-HR Existence

Theorem (Friedl - Teicher 2004)

A knot K is $\mathbb{Z} \ltimes \mathbb{Z}[\frac{1}{2}]$ - homotopy ribbon if and only if there is a surjection $\pi_1(S^3_0(K)) \twoheadrightarrow \Gamma$ such that

$$\mathrm{Ext}^1_{\mathbb{Z}\Gamma}\left(H_1(S^3_0(K);\mathbb{Z}\Gamma), \mathbb{Z}\Gamma\right) = 0.$$

<u>Uniqueness theorems</u>

<u>Theorem</u> (Conway-P.) $(\mathbb{Z})$

Two $\mathbb{Z}$-homotopy ribbon discs to a knot are top. amb. isotopic rel. ∂.

<u>Theorem (Conway-P)</u>

Two Γ-homotopy ribbon D_1, D_2

for a knot K are top. amb isotopic rel ∂ iff

$$\ker\left(f_i : H_1(S^3 \setminus K ; \mathbb{Z}[t^{\pm 1}]) \to H_1(D^4 \setminus D_i ; \mathbb{Z}[t^{\pm 1}])\right)$$

are equal for $i = 1, 2$.

Slice discs 3 proof

Combined statement.

<u>Theorem</u> (Conway - P)

Let D_1, D_2 be two G-homotopy ribbon discs for K with $G = \mathbb{Z}$ or $\mathbb{Z} \ltimes \mathbb{Z}[\frac{1}{2}]$

If $G = \underset{\Gamma}{\underset{\shortparallel}{\mathbb{Z} \ltimes \mathbb{Z}[\tfrac{1}{2}]}}$, assume $\ker f_1 = \ker f_2 \leq H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])$

Then D_1 and D_2 are topologically ambiently isotopic rel ∂.

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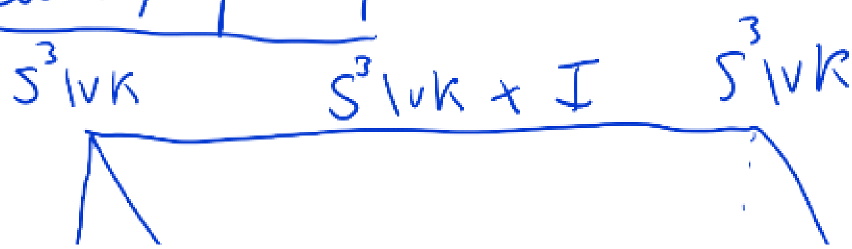
i.e. $\exists\ F : D^4 \times [0,1] \to D^4$

with $F|_{D^4 \times \{t\}}$ a homeomorphism $\forall\, t \in [0,1]$

and $F|_{D^4 \times \{0\}} = \mathrm{Id}$, $F|_{D^4 \times \{1\}}(D_1) = D_2$.

rel ∂ : $F|_{\partial D^4 \times \{t\}} = \mathrm{Id}$ $\forall\, t \in [0,1]$

<u>Idea of proof</u>



1) Construct a cobordism W.

2) Obstruction in $L_5^s(\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Z}$

$\#^k_{S^1} E_8 \times S^1$ to kill surgery obstr.

Surger W to an s-cobordism.

$\Gamma, \mathbb{Z}$ are good group (solvable)

top s-cob thm (FQ)

$\Rightarrow W' \cong (D^4 \setminus \nu D_1) \times I$

$\Rightarrow D^4 \setminus \nu D_1 \xrightarrow[\text{rel } \partial]{\cong} D^4 \setminus \nu D_2$

$$\begin{tikzcd}
D_1 = D^2 \times \{0\} \subset D^2 \times D^2 \arrow[r, "Id"] \arrow[d, phantom, "\shortparallel"] & D^2 \times D^2 \supseteq D^2 \times \{0\} = D_2 \arrow[d, phantom, "\shortparallel"] \\
F : D^4 \arrow[r, "\cong"] & D^4
\end{tikzcd}$$

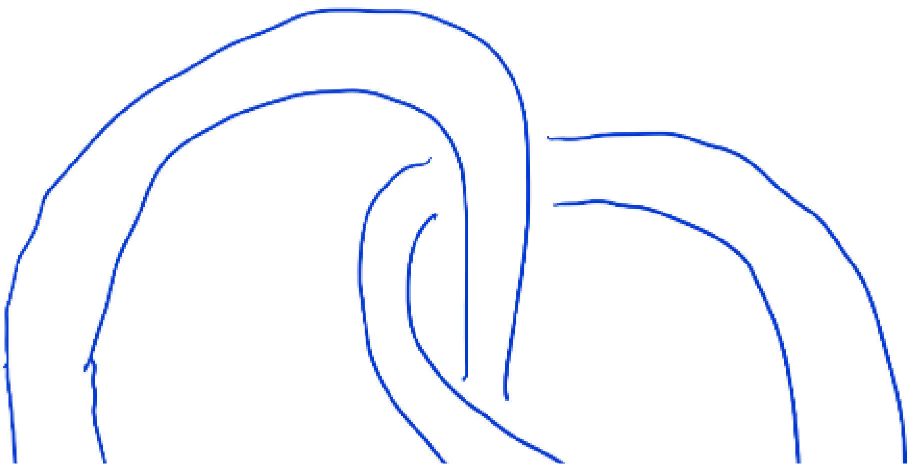
$$D_1 \longmapsto D_2$$

$$F\big|_{\partial D^4} = Id.$$

$$\Rightarrow F \underset{\text{isotopic}}{\simeq} Id$$

Alexander trick

$\Rightarrow D_1, D_2$ amb isotopic.



n	# Γ MR discs
-6	1
-5	1
-4	1
-3	2
-2	2
-1	2
0	2
1	1
2	1
3	1

2 discs iff

$n \equiv 0 \quad (3)$

$n = 0, 5,$

1 digit otherwise

Different question:

How "far away" are two slice discs from each other?

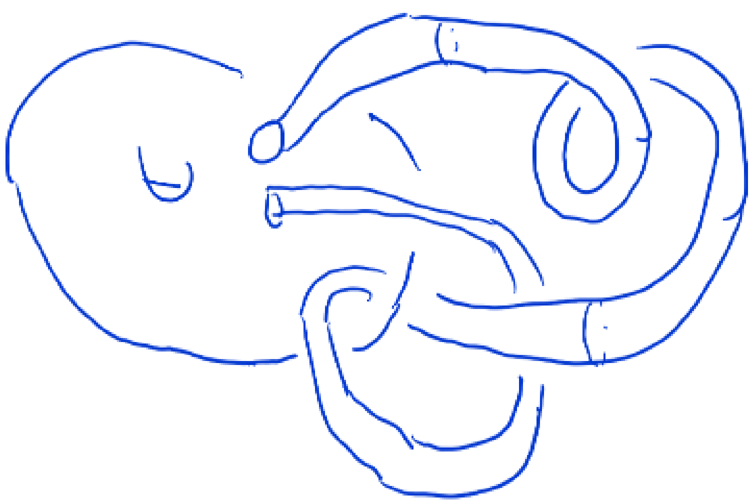
Let

$\Sigma_1, \Sigma_2 \hookrightarrow D^4$ or S^4

Two oriented, compact, connected surfaces, $g(\Sigma_1) = g(\Sigma_2)$

If in D^4, assume $\partial\Sigma_1 = \partial\Sigma_2 \subseteq \partial D^4$.

1-handle stabilisation



$d_1(\Sigma_1, \Sigma_2) = \min \#$ 1h stab^{n}s to Σ_1 and Σ_2 to obtain isotopic surfaces.

Theorem A (AN Miller, P.)

$\forall m \in \mathbb{N}$, there is a 2-knot K $S^2 \subseteq S^4$ with $d_1(K, \text{unknot}) = m$

<u>Example</u>

$$\#^m D_1 \cup_J D_1 =: K$$

$$d_1(K, U) = m.$$

<u>Key prop^n</u>

$$F_1 \subset S^4 \xrightarrow{\text{1h-stab}} F_2$$

$\exists\, p \in \mathbb{Q}[t^{\pm}]$ and a s.e.s

$$0 \to \frac{\mathbb{Q}[t^{\pm}]}{(p)} \longrightarrow H_1(S^4 \setminus F_1 ; \mathbb{Q}[t^{\pm}])$$

$$\to H_1(S^4 \setminus F_2 ; \mathbb{Q}[t^{\pm}]) \to 0$$

Use generating rank of a f.g module over a PID.

K has

$$H_1(S^4 \setminus \nu K ; \mathbb{Q}[t^{\pm}]) = \bigoplus^m \frac{\mathbb{Q}[t^{\pm}]}{(t-2)}.$$

$$gr = m$$

$$\Rightarrow d_1 \geq m.$$

Geometrically check $\leq m$.

Slice discs 5 Generalised stabilisation distance

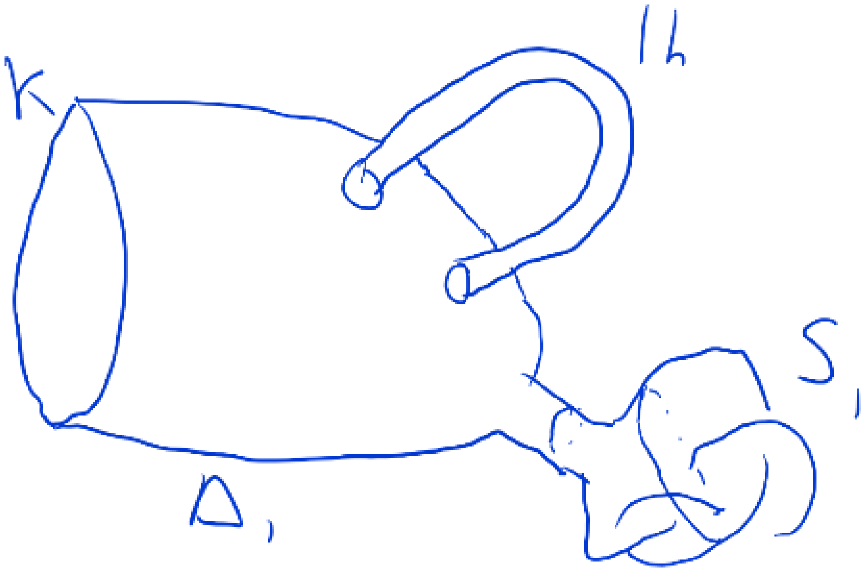
Let $\Delta_1, \Delta_2 \subseteq D^4$ be slice discs for the same knot K.

$d_2(\Delta_1, \Delta_2) :=$ min # 1-h stab^{n}s to obtain Δ_1', Δ_2' such that

$$\Delta_1' \# S_1 \sim \Delta_2' \# S_2$$

for some 2-knots S_1, S_2.

Can add 2-knots; counts as 0.



Theorem B (Miller - P)

$\forall m \in \mathbb{N}$, the knot $\#^m J \subseteq S^3$

has slice discs $\Delta_1 = \natural^m D_1$ $\subseteq D^4$
$\Delta_2 = \natural^m D_2$

with $d_2(\Delta_1, \Delta_2) = m$

Pf uses generating rank arguments to measure difference between $\ker f_1$ and $\ker f_2$.

<u>Theorem C</u> (Miller-P)

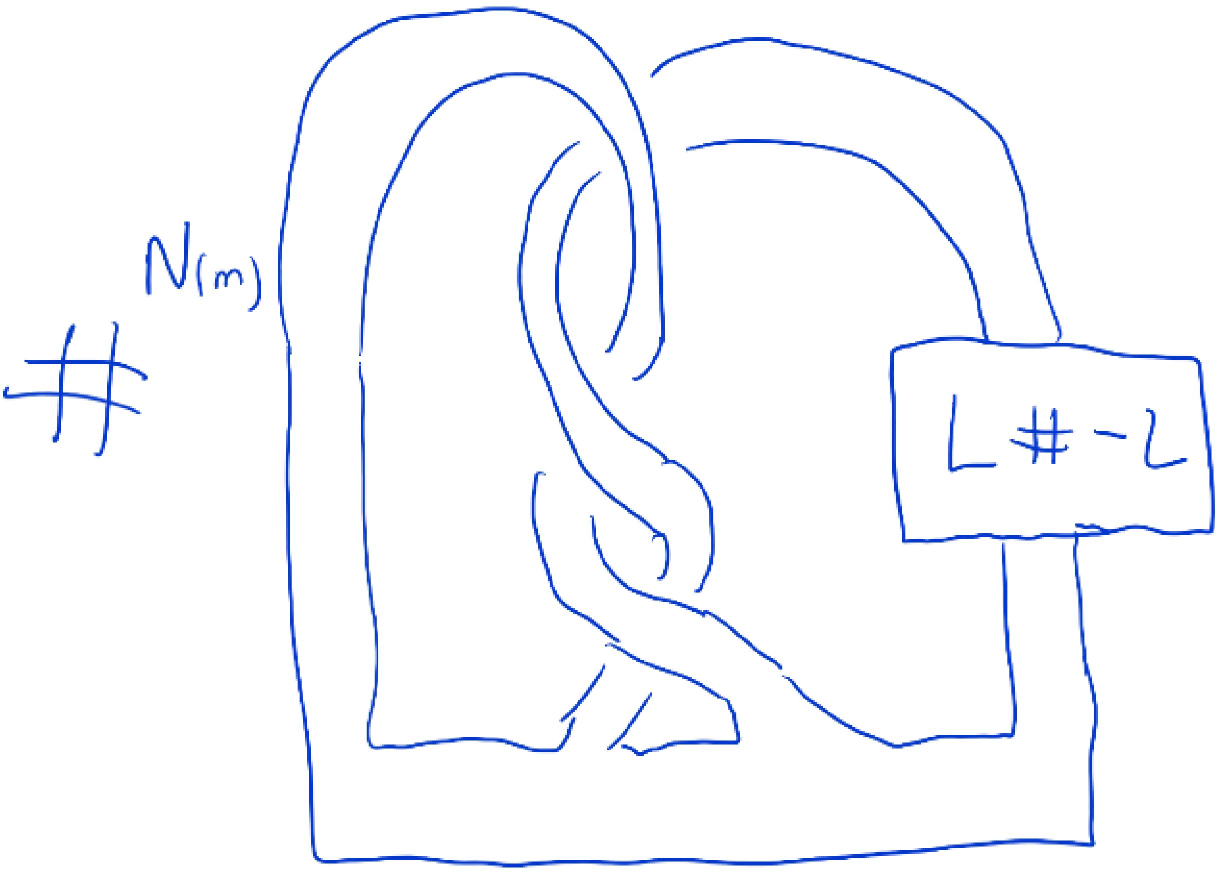
$\forall m \in \mathbb{N}$, $\exists$ $K \subseteq S^3$ and $\Delta_1, \Delta_2 \subseteq D$

with $d_2(\Delta_1, \Delta_2) \geq m$ <u>and</u>

$\ker\left(f_i : H_1(S^3 \setminus\nu K ; \mathbb{Z}[t^{\pm 1}]) \to H_1(D^4 \setminus \nu\Delta_i ; \mathbb{Z}[t^{\pm 1}])\right)$

equal for $i = 1, 2$.

Examples:



L a slice knot.

L#-L has ≥2 slice discs.

$$H_1(S^3 \setminus K) \longrightarrow H_1(D^4 \setminus D)$$

is a metabolise to Bl
form.