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DIFFERENTIABLE AND COMBINATORIAL STRUCTURES ON MANIFOLDS

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1. By extensions of our methods of a previous paper [7], hereafter referred to as GPC, we prove the following theorems. They are all generalizations of results in GPC.

(1.1) Theorem. *Let C^n be a contractible C^∞ compact manifold with simply connected boundary where $n \neq 3, 4, 5, 7$. Then C^n is diffeomorphic to the n-disk D^n.*

(1.2) Corollary. *If $n \neq 4, 5, 7$, there is a unique C^∞ structure up to diffeomorphism on the n-disk.*

(1.3) Corollary. *If $n \neq 4, 7$, and $f\colon S^{n-1} \to E^n$ is a differentiable imbedding of the sphere in euclidean space, then the closure C of the bounded component of $E^n - f(S^{n-1})$ is diffeomorphic to D^n.*

The second corollary is a strong version of the differentiable Schoenflies problem, $n \neq 4, 7$. Mazur's theorem [3] had already implied that C was homeomorphic to D^n.

Two abelian groups, Γ^n and A^n, studied by Milnor [4], Munkres [6], and Thom [8], have been found to be important in the theory of differentiable structures on manifolds. The group Γ^n is the group of diffeomorphisms of S^{n-1} modulo those which can be extended to D^n. It has been identified with the set of differentiable structures on S^n compatible with the standard triangulation of S^n (see GPC for more details and references for all these things discussed in the Introduction). The group A^n is the group of those differentiable structures on S^n which, minus a point, are diffeomorphic to E^n. In GPC it was proved that A^n, $n \neq 4$ is the group of all differentiable structures on S^n.

On the other hand a group $\mathcal{H}^n$ of homotopy spheres of dimension n under "J-equivalence" has been studied by Kervaire and Milnor [2]. They have shown $\mathcal{H}^n$ is finite, $n \neq 3$, and have computed the order of $\mathcal{H}^n$, $3 < n \leqq 15$.

(1.4) Theorem. *For $n \neq 3, 4, 6, 7$, $\mathcal{H}^n = A^n = \Gamma^n$. Also $\mathcal{H}^7 = A^7$, $\Gamma^6 = A^6$.*

(1.5) Corollary. *There are a finite number of differentiable struc-*

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tures on S^n, $n \neq 4, 6$.

We also obtain here

(1.6) THEOREM. *Combinatorial analogues of* (1.1) *and* (1.2) *are valid with diffeomorphism replaced by combinatorial equivalence.*

(1.7) THEOREM. *Combinatorial structures on spheres and cells are unique if their dimension is not* 4, 5, 7. *That is, the Hauptvermutung is true for combinatorial n-manifolds which are homeomorphic to spheres or cells,* $n \neq 4, 5, 7$.

We refer the reader to GPC, beginning, for the proof that (1.1)–(1.7) follow from the next two theorems.

(1.8) THEOREM. *Let* M^n *be an* $(m-1)$*-connected,* C^∞ *closed manifold with* $n \geqq 2m-1$, *and* $(n, m) \notin \{(4, 2), (3, 2), (5, 3), (7, 4)\}$. *Then there exists a non-degenerate* (*nice*) C^∞ *function on* M *with type numbers* M_i *satisfying*

$$M_0 = M_n = 1 , \qquad M_i = 0 , \qquad\qquad 0 < i < m, n - m < i < n .$$

(1.9) THEOREM. *Suppose* M_1^n *and* M_2^n *are J-equivalent* $(m-1)$*-connected closed* C^∞ *manifolds with* $n = 2m$ *or* $n = 2m + 1$ *and* $(n, m) \notin \{(4, 2), (3, 1), (6, 3)\}$. *Then* M_1^n *and* M_2^n *are diffeomorphic.*

REMARK. From the methods used here, it will be clear that all the previous theorems will be true also for $n = 6, 7$ if the following likely hypothesis is true.

Hypothesis. Let f, g: $S^3 \to M^6$ be differentiable imbeddings which are homotopic with

$$\pi_0(M^6) = \pi_1(M^6) = \pi_2(M^6) = 0 .$$

Then f and g are differentiably isotopic.

2. We emphasize that we are assuming the notation and terminology of GPC. We will prove

(2.1) EXTENDED HANDLEBODY THEOREM. *Let* $n \geqq 2s + 1$, $(n, s) \notin \{(4, 1), (3, 1), (5, 2), (7, 3)\}$ *and* $H \in \mathcal{H}(n, k, s)$. *Suppose*

$$V = \chi(H; f_1, \cdots, f_r; s + 1)$$

and $\pi_s(V) = 0$. *Also if* $s = 1$, *let*

$$\pi_1\big(\chi(H; f_1, \cdots, f_{r-k}; s + 1)\big) = 1 .$$

Then $V \in \mathcal{H}(n, r - k, s + 1)$.

We first note that (1.8) follows from (2.1) and the arguments of GPC. Also, since the proof of (1.9) involves nothing beyond straight-forward application of methods of GPC and this paper, we omit it. Hence it remains to prove (2.1). We use the following special case of a very recent theorem of A. Haefliger [1].

(2.2) THEOREM (Haefliger). *Suppose* $f, g: S^k \to M^{2k}$ *are differentiable imbeddings which are homotopic,* $k > 3$ *and*

$$\pi_0(M) = \pi_1(M) = \cdots = \pi_{k-1}(M) = 1 .$$

Then f *and* g *are differentiably isotopic.*

(2.3) THEOREM. *Let* $\gamma \in \pi_k(M^{2k})$ *where* M^{2k} *is a* C^∞ *simply-connected manifold and* $k > 2$. *Then* γ *can be realized by a differentiable imbedding* $f: S^k \to M^{2k}$.

As has been observed by Milnor [5], one can prove this by using the work of Whitney [9]. See also [1].

The following theorem extends Theorem (2.1) of GPC. The proof is the same but uses Theorems (2.2) and (2.3) above, instead of the weaker theorems of Whitney and Wu used there.

(2.4) THEOREM. *Let* $n \geqq 2s + 1$, $(n, s) \notin \{(4, 1), (3, 1), (5, 2), (7, 3)\}$, *let* $\sigma = (M, Q; f_1, \cdots, f_r; s + 1)$ *be a presentation of a manifold* V, *and assume* $\pi_1(Q) = 1$ *if* $n \leqq 2s + 2$, *and* $\pi_2(Q) = 0$ *if* $n = 2s + 1$. *Then for any automorphism* $\alpha: G_r \to G_r$, V *realizes* $f_\sigma \alpha$.

Now (2.1) has been proved in GPC except for the case $n = 2s + 1$, $s > 3$. Thus in proving (2.1), assume $H \in \mathscr{H}(2m + 1, k, m)$, $m > 3$. From §3 of GPC it follows that

$$H \in S_1^m \times D_1^{m+1} + \cdots + S_k^m \times D_k^{m+1} .$$

Let $g_1, \cdots, g_k$ be the corresponding generators of $\pi_m(H)$ and h_i be the generators of $\pi_m(\partial H)$ corresponding to $q_i \times \partial D_i^{m+1}$, $q_i \in S_i^m$. Thus $\{g_1, \cdots, g_k, h_1, \cdots, h_k\}$ is an independent set of generators of the free abelian group $\pi_m(\partial H)$.

We can represent H in the form

$$H = \chi(D^{2m+1}; \psi_1, \cdots, \psi_k)$$

where

$$\psi_i: \partial D_i^m \times D_i^{m+1} \to \partial D^{2m+1}$$

are imbeddings with the images of ψ_i disjoint. Then the above h_i can be represented by ψ_i restricted to $q_i \times \partial D_i^{m+1}$ for some $q_i \in \partial D_i^m$.

On the other hand

$$V = \chi(H; f_1, \cdots, f_r) = \chi(\sigma)$$

where

$$f_i\colon \partial D_i^{m+1} \times D_i^m \to \partial H$$

are imbeddings. Let

$$\pi_m(\partial H) = G_k + H_k$$

where G_k is generated by $g_1, \cdots, g_k$ and H_k by $h_1, \cdots, h_k$, and let

$$\pi_1\colon \pi_m(\partial H) \to G_k$$

be the projection. Since $\pi_m(V) = 0$, $\pi_1 f_\sigma$ is an epimorphism where

$$f_\sigma\colon G_r \to \pi_m(\partial H)$$

is induced by σ. Define an epimorphism $g\colon G_r \to G_k$ by $gD_i = g_i$, $i \leqq k$ and $gD_i = 0$, $i > k$. Then by (4.1) of GPC there is an automorphism α of G_r such that $\pi_1 f_\sigma \alpha = g$. Thus $\pi_1 f_\sigma \alpha(D_1) = g_1$ or

$$f_\sigma \alpha D_1 = g_1 + \sum_1^k a_i h_i \ .$$

Then by (2.4), we can assume $V = \chi(H; f_1, \cdots, f_r)$ where the homotopy class of f_1 restricted to $\partial D_1^{m+1} \times 0$ in $\pi_m(\partial H)$ is $g_1 + \sum_1^k a_i h_i$ for some set of a_i.

Let Γ be the union of the subsets

$$D_2^m \times D_2^{m+1}, \cdots, D_k^m \times D_k^{m+1}$$

of H. Now it is clear that each $h_i \in \pi_m(\partial H)$ is the image of some class in $\pi_m(\partial H - (\partial H \cap \Gamma))$ under the homomorphism induced by inclusion. The same is true also for g_1. This implies that the class $g_1 + \sum_1^k a_i h_i$ has the same property, or that there exists a map

$$\bar{f}_1'\colon \partial D^{m+1} \times 0 \to \partial H - (\partial H \cap \Gamma)$$

which is homotopic in ∂H of the restriction of

$$f_1\colon \partial D_1^{m+1} \times D_1^m \to \partial H$$

to $\partial D_1^{m+1} \times 0$. By (2.3) we can realize $\bar{f}_1'$ by an imbedding. By (2.4) of GPC extended slightly by (2.2) of this paper we obtain a differentiable imbedding

$$f_1'\colon \partial D_1^{m+1} \times D_1^m \to \partial H - (\partial H \cap \Gamma)$$

such that $\chi(H; f_1')$ and $\chi(H; f_1)$ are diffeomorphic.

If we generalize the definition $\chi(M; \varphi_1, \cdots, \varphi_k)$ of GPC to include the case of handles of more than one dimension, we have $\chi(H; f_1')$ is diffeomorphic to $\chi(H_1; f_1', \psi_2, \cdots, \psi_k)$ where $H_1 = \chi(D^{2m+1}; \psi_1)$. Then we have $\chi(H; f_1)$ is diffeomorphic to $\chi(H_2; \psi_2, \cdots, \psi_k)$ where $H_2 = \chi_1(H_1; f_1')$. By the proof of (3.3) of GPC, H_2 is diffeomorphic to D^{2m+1}. Thus V is of form $\chi(H_3; f_2, \cdots, f_r)$ where $H_3 = \chi(H_2; \psi_2, \cdots, \psi_k)$ is in $\mathscr{H}(2m+1, k-1, m)$. By induction on k we get (2.1).

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