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The topology of four-dimensional manifolds⁽¹⁾

Yu.P. Solov'ev

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Introduction

The main problem of topology of manifolds is the characterization of manifolds by means of algebraic invariants. For example, the orientable closed connected two-dimensional manifolds are completely characterized by their genus g , which determines their diffeomorphism type in a unique way. In a similar fashion non-orientable two-dimensional manifolds are classified. Under transition from two-dimensional to three-dimensional manifolds difficulties grow astronomically, and up to the present there is no classification of three-dimensional manifolds. It is not even known if a closed simply-connected three-dimensional manifold is a standard sphere (the Poincaré conjecture).

It is striking that in dimension four the investigations initiated some forty years ago have been crowned with brilliant achievements in the last decade. In late 1981 Freedman proved the four-dimensional topological Poincaré conjecture asserting that every metrizable topological four-manifold M^4 having the homotopy type of the four-dimensional sphere $S^4 = \{x \in \mathbb{R}^5: |x| = 1\}$

⁽¹⁾These are often called four-manifolds or 4-manifolds. (Ed.)

is homeomorphic to S^4 . As a matter of fact, Freedman managed not only to prove the above conjecture but also to obtain a complete classification of closed simply-connected topological manifolds. The main idea of a solution to this classification problem due to Casson is to construct a certain entirely topological infinite process enabling one to perform the Whitney trick in dimension four. In 1982 Donaldson, using methods of mathematical physics, described smooth four-dimensional manifolds with a positive intersection form. One of the striking corollaries of Donaldson's theorem is the existence of exotic smooth structures on the Euclidean space \mathbb{R}^4 . (A continual family of such structures is now known.)

The results by Freedman and Donaldson have served as a stimulus to the rapid development of the topology of four-dimensional manifolds and its applications to geometry and mathematical physics.

The current survey is devoted to presentation of the main results in the topology of four-dimensional manifolds obtained in the last decade.

CHAPTER I

CLASSIFICATION OF SIMPLY-CONNECTED TOPOLOGICAL FOUR-DIMENSIONAL MANIFOLDS

In this chapter we present Freedman's results on classification of simply-connected topological four-dimensional manifolds. To obtain this classification, it was necessary to encompass the whole of a huge arsenal of techniques of algebraic and geometric topology, as well as to invent a number of new tricks. Taking into account the fact that complicated technical details of the proof of the theorem on classification of simply-connected four-dimensional manifolds are often camouflaging the underlying ideas and methods, in the present survey we have restricted ourselves to a detailed sketch of the proof. At the same time, we have included in the present chapter those fundamental theorems on four-dimensional manifolds that have made it possible to draw near to a solution of the classification problem.

The reader must be warned that by virtue of Cairns' theorem in dimension four there is no difference between smooth (Diff) and piecewise linear (*PL*) manifolds: each *PL*-manifold bears a unique Diff-structure. Therefore, depending on circumstances, we make use of both structures in turn, sometimes without specifying which of them is currently under consideration.

§1. Intersection forms of four-dimensional manifolds

Let M be a closed simply-connected four-dimensional manifold. Using Poincaré duality and the fact that $H_1(M) = \pi_1(M)/[\pi_1(M), \pi_1(M)] = 0$, it is not difficult to show that the integer homology and cohomology of M look

like this:

n	0	1	2	3	4
$H_n(M)$	\mathbb{Z}	0	\mathbb{Z}^m	0	\mathbb{Z}
$H^n(M)$	\mathbb{Z}	0	\mathbb{Z}^m	0	\mathbb{Z}

Here m stands for the second Betti number of M . If x and y are homology classes from $H_2(M)$, then their homology intersection index is defined: $x \cdot y = y \cdot x \in \mathbb{Z}$. By virtue of Poincaré duality, $H_2(M) \cong H^2(M) \cong \text{Hom}(H_2(M), \mathbb{Z})$. Hence, the symmetric bilinear form

$$L_M: H_2(M) \times H_2(M) \rightarrow \mathbb{Z}, \quad L_M(x, y) = x \cdot y$$

is non-degenerate and it can be represented by a symmetric integral unimodular matrix A_M . It is obvious that the class of forms isomorphic to L_M is in one-to-one correspondence with the class of matrices congruent to A_M .

The form L_M can be described in cohomology terms as well. Let $u, v \in H^2(M)$ be any two-dimensional cohomology classes. We set

$$\bar{L}_M(u, v) = \langle u \cup v, [M] \rangle \in \mathbb{Z}.$$

Since $u \cup v = v \cup u$, the form \bar{L}_M is symmetric, and by virtue of Poincaré duality it is non-degenerate. Let $x = Du, y = Dv$, where $D: H^2(M) \xrightarrow{\cong} H_2(M)$ is the Poincaré duality isomorphism. Then $x \cdot y = D(u \cup v)$, hence it follows that $(H_2(M), L_M)$ and $(H^2(M), \bar{L}_M)$ are isomorphic as inner product spaces. This isomorphism enables us to identify the forms L_M and \bar{L}_M ; in what follows we will denote them by the same symbol L_M and consider L_M in case of need either as an intersection form or as a cohomology cup product form.

Thus, the homology structure of a closed simply-connected four-dimensional manifold M is completely determined by the integral non-degenerate form L_M . In this connection it is useful to recall briefly a classification of such forms (see [36], [41]).

Let H be a free Abelian group of a finite rank. We say that a bilinear symmetric form $L: H \times H \rightarrow \mathbb{Z}$ is a form of *type I* (or an *odd form*) if there is an $x \in H$ such that $L(x, x)$ is an odd number. Otherwise L is said to have *type II* (or to be *even*). Clearly, L has type II if and only if the associated quadratic function $q_L(x) = \frac{1}{2} L(x, x)$ takes values in \mathbb{Z} . The rank of the

Abelian group H is called the *rank* $\text{rk } L$ of the form L . Since the form L is non-degenerate, $\text{rk } L$ equals the rank of any matrix A_L representing it. The matrix A_L can be diagonalized over the rational number field, and the *signature* $\sigma(L)$ of the form L is defined as the signature of the matrix A_L over \mathbb{Q} , that is, the number of positive terms minus the number of negative terms in the diagonalization of A_L . If $\text{rk } L = |\sigma(L)|$, then L is called *definite*;

if $\text{rk } L > |\sigma(L)|$, then L is called *indefinite*. An element $x \in H$ such that $L(x, y) \equiv q_L(y) \pmod 2$ for all $y \in H$ is called *characteristic*.

We now define several canonical forms. We denote by $\pm I$ the form represented by the matrix (± 1) , by U the form represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and by E_8 the form represented by the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

The same symbols will stand for the corresponding inner product spaces.

The following classification theorems hold.

Theorem 1.1 (van der Blij). *Every symmetric non-degenerate form $L: H \times H \rightarrow \mathbb{Z}$ possesses characteristic elements, and if $x \in H$ is a characteristic element of it, then $L(x, x) \equiv \sigma(L) \pmod 8$. In particular, the signature of a symmetric non-degenerate form L of type II is divisible by 8.*

Theorem 1.2 (Serre). *Let L be an indefinite non-degenerate symmetric form of rank $r = p + n$ and signature $\sigma = p - n$ ($p, n > 0$).*

(i) *If L is a form of type I, then $L \cong pI \oplus n(-I)$.*

(ii) *If L is a form of type II, then $L \cong aE_8 \oplus bU$, where $a = (1/8)\sigma$ and $b = (1/2)(r - |\sigma|)$. ■*

A complete classification of non-degenerate definite forms is not known. There are only the following partial results. For $n \geq 1$ we denote by I_n the set of isomorphism classes of positive definite non-degenerate forms of rank n of type I, and for $n = 8k$ we denote by II_n the set of isomorphism classes of positive definite non-degenerate forms of rank n of type II. Let $c(I_n) = \text{card}(I_n)$ and $c(II_n) = \text{card}(II_n)$. We have the following information on the integers $c(I_n)$ and $c(II_n)$:

n	≤ 8	9	11	12	13	14	15	16	17	...	28	29	30
$c(I_n)$	1	2	2	3	3	4	5	5	7	...	≥ 209	$\geq 3 \cdot 10^6$	$\geq 9 \cdot 10^8$

$n = 8k$	8	16	24	32	40
$c(II_n)$	1	2	24	$\geq 8 \cdot 10^7$	$\geq 8 \cdot 10^{61}$

Let us consider a few examples. Let P be $\mathbb{C}P^2$ with the usual orientation, and let Q be $\mathbb{C}P^2$ with the opposite orientation. Then $L_P \cong I$, $L_Q \cong -I$. For $M = S^2 \times S^2$ we have $L_M \cong U$. If $M = M_1 \# M_2$ is a connected sum of two manifolds, then $L_M \cong L_{M_1} \oplus L_{M_2}$.

Let M be a closed simply-connected manifold, $H = H^2(M)$, and let L_M be the cup product form. A two-dimensional cohomology class $x \in H$ is a characteristic element if and only if its mod 2 reduction is the second Stiefel–Whitney class $w_2(M)$. Hence it follows that L_M has type II if and only if $w_2(M) = 0$, that is, in the case where M is a spinor manifold. In dimension 4, the property of being a spinor manifold is equivalent to almost parallelizability. To verify this, we observe that if $\tau : M \rightarrow BSO$ is a map corresponding to the tangent bundle, then obstructions to τ being homotopic to a constant map are in groups $H^k(M; \pi_k(BSO))$. Moreover, $w_2(M) = \tau^*(w_2)$, where w_2 is the generator of the group $H^2(BSO) \cong \pi_2(BSO) \cong \mathbb{Z}/2$. Therefore, if $w_2(M) = 0$, then the restriction of the tangent bundle τ_M to the three-dimensional skeleton of M is trivial. Thus, $\tau_M|_{M-3\text{-pt}} \simeq 0$, and hence M is almost parallelizable.

Thus, the type of the form of a smooth closed simply-connected manifold M is determined by the second Stiefel–Whitney class. If $w_2(M) \neq 0$, then L_M is of type I, and if $w_2(M) = 0$, then L_M is of type II. In particular, by Theorem 1.1 the signature of such a manifold is divisible by 8. (We recall that the signature of a manifold is defined as the signature of its form L_M .)

In 1952 Rokhlin [4] discovered the following astonishing property of four-dimensional spinor manifolds.

Theorem 1.3 (Rokhlin). *If M is an oriented closed smooth four-dimensional manifold with $w_2(M) = 0$, then $\sigma(M) \equiv 0 \pmod{16}$.*

Proof. Consider the Whitehead J -homomorphism

$$J: \pi_{k-1}(SO(m)) \rightarrow \pi_{m+k-1}(S^m),$$

defined by

$$[f] \in \pi_{k-1}(SO(m)) \mapsto [J(f)] \in \pi_{m+k-1}(S^m),$$

where $J(f): S^{m+k-1} \rightarrow S^m$ is the map obtained from

$$(f, \text{id}): S^{k-1} \times S^{m-1} \rightarrow S^{m-1}, \quad (f, \text{id})(x, y) = f(x)(y)$$

by passing over to the join

$$S^{k-1} * S^{m-1} = S^{m+k-1} \xrightarrow{J(f)} \Sigma S^{m-1} = S^m.$$

With the help of Steenrod algebra calculations one can show that

$$J: \pi_3(SO(m)) \rightarrow \pi_{m+3}(S^m)$$

is an epimorphism for sufficiently large m . But $\pi_3(SO(m)) \cong \mathbb{Z}$ and $\pi_{m+3}(S^m) \cong \mathbb{Z}/24$. Hence if $J(x) = 0$ for $x \in \pi_3(SO(m))$, then $x \equiv 0 \pmod{24}$.

Now let M be a closed smooth four-dimensional manifold with $w_2(M) = 0$. As was shown above, M is almost parallelizable. Consequently, the stable principal normal bundle ν is trivial on $M - \text{pt}$ and the obstruction extending a trivialization f of the bundle $\nu|_{M - \text{pt}}$ to a trivialization of the bundle ν is a cohomology class $e(\nu, f) \in H^4(M; \pi_3(SO(m)) \cong \pi_3(SO(m)) = \mathbb{Z}$. It is verified straightforwardly that $J(e(\nu, f)) = 0$, whence it follows that $e(\nu, f)$ is divisible by 24. On the other hand, $e(\nu, f) = \pm p_1(M)$, where $p_1(M)$ is the Pontryagin number of M . Hence $p_1(M) \equiv 0 \pmod{48}$. But according to the Hirzebruch formula, $p_1(M) = 3\sigma(M)$, so that $\sigma(M) \equiv 0 \pmod{16}$. ■

§2. The Pontryagin–Whitehead and Novikov–Wall theorems

In the preceding section we have linked to every closed simply-connected four-dimensional manifold M its intersection form L_M . Now we ask the question: to what extent the manifold M itself is determined by the form L_M . The first result in this direction was the following theorem obtained in 1949 by Pontryagin [3] and independently by Whitehead [55].

Theorem 2.1 (Pontryagin–Whitehead). *Let M_1 and M_2 be closed simply-connected four-dimensional manifolds and let L_{M_1} and L_{M_2} be their intersection forms. There exists a homotopy equivalence $M_1 \rightarrow M_2$ compatible with the orientations if and only if the pairs $(H_2(M_1), L_{M_1})$ and $(H_2(M_2), L_{M_2})$ are isomorphic as inner product spaces.*

Proof. We denote by E the interior of a four-dimensional ball embedded in M_1 . Making use of the exact homology sequence of the pair $(M_1, M_1 - E)$ with coefficients in \mathbb{Z} and the Poincaré duality, it is not difficult to show that $H_i(M_1 - E) \cong H_i(\text{pt})$, $i \neq 2$. In addition, $H_2(M_1 - E) \cong H_2(M_1)$ is a free-Abelian group. Let $m = \text{rk } H_2(M_1 - E)$. Since $\pi_1(M_1 - E) = 0$, by the Hurewicz theorem

$$\pi_2(M_1 - E) \cong H_2(M_1 - E) \cong H_2(M_1).$$

Therefore, there exists a map

$$f: \underbrace{S^2 \vee \dots \vee S^2}_m \rightarrow M_1 - E,$$

inducing isomorphism in homology

$$H_*(S^2 \vee \dots \vee S^2) \cong H_*(M_1 - E).$$

Since the manifold $M_1 - E$ is an absolute neighbourhood retract, it has the homotopy type of a CW -complex. Hence, by Whitehead's theorem, f is a homotopy equivalence.

We now observe that M_1 is obtained from $M_1 - E$ by attaching to it a four-dimensional cell. Thus, M_1 has the homotopy type of a space obtained

from the wedge (one-point union) $\underbrace{S^2 \vee \dots \vee S^2}_m$ by attaching to it a four-dimensional cell with the help of a map

$$g: S^3 \rightarrow S^2 \vee \dots \vee S^2.$$

Let us denote this space by $(S^2 \vee \dots \vee S^2) \cup_g E^4$. The homotopy type of M is entirely determined by the homotopy class of the map g in $\pi_3(S^2 \vee \dots \vee S^2)$.

In order to calculate the group $\pi_3(S^2 \vee \dots \vee S^2)$, let us embed $S^2 = \mathbb{C}P^1$ in the infinite-dimensional complex projective space $\mathbb{C}P^\infty$ and use the embedding

$$\underbrace{S^2 \vee \dots \vee S^2}_m \subset \underbrace{S^2 \times \dots \times S^2}_m \subset \underbrace{\mathbb{C}P^\infty \times \dots \times \mathbb{C}P^\infty}_m.$$

For the sake of brevity we put

$$B = S^2 \vee \dots \vee S^2, \quad K = \mathbb{C}P^\infty \times \dots \times \mathbb{C}P^\infty.$$

Then the embedding $B \subset K$ provides isomorphisms

$$H_4(K) \cong H_4(K, B) \cong \pi_4(K, B) \cong \pi_3(B).$$

Indeed the first of these isomorphisms is obtained from the exact homology sequence for the pair (K, B) , the second one from the relative Hurewicz theorem, and the third one from the exact homotopy sequence for the pair.

Obviously, $H_4(K)$ is a free \mathbb{Z} -module. The dual module

$$H^4(K) \cong \text{Hom}(H_4(K), \mathbb{Z})$$

has a basis formed by products $u_i \cup u_j$, $i \leq j$, where u_1, \dots, u_m is a basis for the cohomology group

$$H^2(K) \cong H^2(B) \cong H^2(M_1 - E) \cong H^2(M_1).$$

We extend the embedding $B \rightarrow K$ to a map $B \cup_g E^4 \rightarrow K$. Let $v_i \cup v_j \in H^4(B \cup_g E^4) \cong H^4(M_1)$ stand for the images of elements $u_i \cup u_j \in H^4(K)$ under the homomorphism

$$H^4(K) \rightarrow H^4(B \cup_g E^4),$$

induced by the map $B \cup_g E^4 \rightarrow K$. By calculating the value of $v_i \cup v_j$ at the fundamental class $[M_1]$ of M_1 , we obtain an integral symmetric matrix

$$\langle v_i \cup v_j, [M_1] \rangle.$$

It is easily seen that the corresponding bilinear form on $H^2(M_1) \cong H_2(M_1)$ is nothing but the intersection form. At the same time, the matrix $\langle v_i \cup v_j, [M_1] \rangle$ completely determines the homotopy class of the attaching map in $\pi_3(B) \cong \pi_4(K, B) \cong H_4(K)$. Thus, classes of attaching maps $[g] \in \pi_3(S^2 \vee \dots \vee S^2)$ are in one-to-one correspondence with non-degenerate integral forms, as required. ■

In 1964 Novikov [1] and independently Wall [54] significantly strengthened the Pontryagin–Whitehead theorem. To formulate their result, we recall that closed manifolds M_1^n, M_2^n are said to be *cobordant* if there exists a compact manifold W^{n+1} with boundary $\partial W^{n+1} = M_1^n \cup M_2^n$; the triple $(W^{n+1}; M_1^n, M_2^n)$ is called a *cobordism*, and the manifold W^{n+1} is called a *membrane* spanned over M_1^n, M_2^n . A cobordism $(W^{n+1}; M_1^n, M_2^n)$ is called an *h-cobordism* if both inclusions $i_1 : M_1^n \hookrightarrow W^{n+1}, i_2 : M_2^n \hookrightarrow W^{n+1}$ are homotopy equivalences, and an *s-cobordism* if both i_1, i_2 are simple homotopy equivalences in the Whitehead sense. (As to the latter definition, see [35].)

Theorem 2.2 (Novikov–Wall). *Two homotopically equivalent smooth closed simply-connected four-dimensional manifolds are h-cobordant.*

Combining this theorem with the Pontryagin–Whitehead theorem, we deduce that closed simply-connected four-dimensional manifolds M_1 and M_2 are *h-cobordant* if and only if their intersection form L_{M_1} and L_{M_2} are isomorphic.

Proof. We use the following classical construction. Let M be a closed smooth four-dimensional manifold embedded with the normal bundle ν in the $(N+4)$ -dimensional sphere S^{N+4} for sufficiently large positive N . Assuming that ν is endowed with a Riemannian metric, let us denote by $D(\nu)$ (respectively, $S(\nu)$) the bundle associated with ν , with unit disk (respectively, unit sphere) as a standard fibre. Then the space $T(\nu) = D(\nu)/S(\nu)$, called the *Thom space* of the bundle ν , is defined. Since $D(\nu)$ and $S(\nu)$ lie in the sphere S^{N+4} , contracting the complement $S^{N+4} - D^0(\nu)$ of the space $D(\nu)$ to a point we get a degree 1 map

$$c: S^{N+4} \rightarrow T(\nu),$$

called the *Pontryagin–Thom map*. The homotopy class $[c] \in \pi_{N+4}(T(\nu))$ of this map is called the *normal invariant* of the bundle ν .

Now let M_1, M_2 be given four-dimensional manifolds, and let ν_1, ν_2 be their normal bundles in a sphere S^{N+4} . We observe that in order to prove the theorem it suffices to construct a homotopy equivalence $\psi : M_1 \rightarrow M_2$ meeting the following conditions:

(i) the map ψ is covered by a normal bundle isomorphism $\Psi : \nu_1 \rightarrow \nu_2$, that is, the following diagram commutes:

$$\begin{array}{ccc} \nu_1 & \xrightarrow{\Psi} & \nu_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{\psi} & M_2; \end{array}$$

(ii) the induced map between Thom spaces $T\Psi : T(\nu_1) \rightarrow T(\nu_2)$ sends the normal invariant $\alpha_1 \in \pi_{N+4}(T(\nu_1))$ to the normal invariant $\alpha_2 \in \pi_{N+4}(T(\nu_2))$.

Indeed, if such a homotopy equivalence ψ has been constructed, then there is a homotopy $\Gamma : S^{N+4} \times I \rightarrow T(v_2)$ connecting the maps

$$S^{N+4} \xrightarrow{c_1} T(v_1) \xrightarrow{\psi} T(v_2)$$

and

$$S^{N+4} \xrightarrow{c_2} T(v_2) \xrightarrow{id} T(v_2).$$

By a small perturbation Γ can be made transversal to a submanifold $M_2 \subset T(v_2)$. Then $\Gamma^{-1}(M_2)$ becomes a membrane spanned over M_1 and M_2 . After performing a finite series of spherical surgeries of $\Gamma^{-1}(M_2)$ we obtain the required h -cobordism.

Thus, suppose we have a homotopy equivalence $\phi : M_1 \rightarrow M_2$. First of all, we show that it is covered by a normal bundle isomorphism. Indeed, we identify M_1 and M_2 by means of a homotopy equivalence ϕ and consider two bundles v_1 and v_2 over M_1 (that is, v_1 and ϕ^*v_2). Let γ be the principal $O(N)$ -bundle corresponding to $v_1 - v_2$. We wish to show that it is trivial.

Since $\pi_1(M_1) = 0$, γ is a principal $SO(N)$ -bundle. The first obstruction $o_1(\gamma)$ to its trivialization, that is, to the existence of a non-vanishing section, is in the group $H^2(M_1; \pi_2(BSO(N))) = H^2(M_1; \mathbb{Z}/2)$. The obstruction $o_1(\gamma)$ is functorial and it is an invariant of $SO(N)$ -bundles. Therefore, it is covered by a universal class belonging to the group $H^2(BSO(N); \mathbb{Z}/2) \cong \mathbb{Z}/2$.

It is easily verified that this universal class is non-trivial, and hence it coincides with the universal Stiefel–Whitney class w_2 . Further, $w_2(v_1 - v_2) = w_2(v_1) - w_2(v_2)$. By virtue of Wu’s formulae, the Stiefel–Whitney classes of a normal bundle are homotopy invariant. Therefore, $w_2(v_1 - v_2) = 0$, that is, $o_1(\gamma) = 0$. Since $H^3(M_1; BSO(N)) = 0$, the next obstruction $o_2(\gamma)$ is in the group $H^4(M_1; \pi_4(BSO(N))) \cong \mathbb{Z}$. We know already that the bundle γ is trivial over the two-dimensional skeleton $M_1^{[2]}$ of M_1 , therefore γ is induced from a bundle $\tilde{\gamma}$ over $M_1/M_1^{[2]} \simeq S^4$. Let us calculate its Pontryagin class $p_1(\tilde{\gamma})$. Inducing $\tilde{\gamma}$ on M_1 , we obtain the bundle $v_1 - v_2$. But according to Hirzebruch’s formula $p_1(v_1 - v_2) = 0$, and the homomorphism $H^4(S^4) \rightarrow H^4(M_1)$ is an isomorphism. Hence, $p_1(\tilde{\gamma}) = 0$, so it follows that $\tilde{\gamma} = 0$. Thus, $\gamma = 0$, that is, $v_1 = v_2$.

Thus we have proved that any homotopy equivalence ϕ of simply-connected four-dimensional manifolds M_1 and M_2 is covered by an isomorphism of their normal bundles $\Phi : v_1 \xrightarrow{\cong} v_2$. We will now be concerned with the normal invariants of normal bundles. Generally speaking, the homotopy equivalence $\phi : M_1 \rightarrow M_2$ does not send $\alpha_1 \in \pi_{N+4}(T(v_1))$ to $\alpha_2 \in \pi_{N+4}(T(v_2))$, and we have to modify it. We denote by $T^{[k]}(v_1)$, $T^{[k]}(v_2)$ respectively the Thom spaces of the bundles $v_1|_{M_1^{[k]}}$, $v_2|_{M_2^{[k]}}$. (Here $v_i|_{M_i^{[k]}}$ stands for the restriction of the bundle v_i to the k -dimensional skeleton of the manifold M_i , $i = 1, 2$.) Let $T\Phi : T(v_1) \rightarrow T(v_2)$ be the map of Thom spaces induced by the bundle isomorphism $\Phi : v_1 \rightarrow v_2$. Since normal invariants are homotopy classes of degree 1 maps, the image of α_1

in $\pi_{N+4}(T(v_1)/T^{[2]}(v_1)) = \pi_{N+4}(S^{N+4})$ goes to the image of α_2 in $\pi_{N+4}(T(v_2)/T^{[2]}(v_2)) = \pi_{N+4}(S^{N+4})$. Taking into account that N is chosen to be sufficiently large and we are dealing with stable homotopy groups, we can assume that the element $\theta(\varphi) = (T\Phi)^*(\alpha_2) - \alpha_1$ is in the group $\pi_{N+4}(T^{[2]}(v_1))$.

If $\theta(\varphi) = 0$, then α_1 goes to α_2 and everything is proved. Therefore we suppose that $\theta(\varphi) \neq 0$. We must change φ so that the equality $\theta(\psi) = 0$ holds in the group $\pi_{N+4}(T^{[2]}(v_1)/T^{[0]}(v_1))$ for a new homotopy equivalence ψ . This would complete the proof, for $\pi_{N+4}(T^{[0]}(v_1)) = \pi_{N+4}(S^N) = 0$ and $\pi_{N+4}(T^{[2]}(v_1)/T^{[0]}(v_1)) = \pi_{N+4}(\bigvee_{i=1}^m S_i^{N+2})$, where $m = \dim H_2(M_1)$.

The homotopy equivalence $\varphi : M_1 \rightarrow M_2$ can be changed by an element $\beta \in \pi_4(M_2^{[2]}) \cong \pi_4(\bigvee_{i=1}^m S_i^2)$. The new homotopy equivalence ψ will still be covered by a normal bundle isomorphism. Indeed, since $\beta^*p_1(v_2 | M_2^{[2]}) = p_1(\beta^*(v_2)) = 0$, we have $\beta^*(v_2) = 0$. Hence in order to prove the theorem it remains to discover an element $\beta \in \pi_4(M_2^{[2]}) = \pi_4(\bigvee_{i=1}^m S_i^2)$ such that $\theta(\psi) = 0$. We consider as β the composition $\beta = \varepsilon \circ \eta \in \pi_4(\bigvee_{i=1}^m S_i^2)$, where η is a generator of the group $\pi_4(S^3)$ and ε is an element of $\pi_3(\bigvee_{i=1}^m S_i^2)$.

Clearly, by a suitable choice of ε we can ensure that $\Sigma^N\beta = \theta(\varphi)$. But having changed the homotopy equivalence φ to an element β satisfying $\Sigma^N\beta = \theta(\varphi)$, by the same token we are changing $(T\Phi)^*(\alpha_2) - \alpha_1$, viewed as an element of $\pi_{N+4}(T^{[2]}(v_1)/T^{[0]}(v_1))$, to $\theta(\varphi)$. Hence,

$$\theta(\psi) = (T\Phi)^*(\alpha_2) - \alpha_1 - \theta(\varphi) = \theta(\varphi) - \theta(\varphi) = 0,$$

as required. ■

Thus, we have proved that any two closed simply-connected four-dimensional manifolds M_1 and M_2 with isomorphic intersection forms L_{M_1} and L_{M_2} are h -cobordant. If the dimension of the manifolds M_1 and M_2 were higher than four, then one could proceed further and prove that M_1 and M_2 are PL -homeomorphic. This follows from the so-called h -cobordism theorem of Smale [45]. But Smale's proof is inappropriate for four-dimensional manifolds. To unravel the situation, in the next section we will analyse Smale's proof and elucidate those difficulties of principle that prevent us from proving the h -cobordism theorem for four-dimensional manifolds.

§3. Handle body theory. The h -cobordism theorem

In this section we will somewhat digress from our main topic and be concerned with handle body theory and the h -cobordism theorem. We recall

that by virtue of Cairns' theorem [12] every four-dimensional *PL*-manifold bears a unique smooth structure. Therefore, in order to avoid unnecessary technical details, we will consider here *PL*-manifolds only. Nevertheless, all the following theorems and proofs can be transferred to other categories of manifolds as well.

Let W^w be an arbitrary piecewise linear manifold, and let H be a w -dimensional ball such that $W \cap H = \partial W$ and there exists an isomorphism (that is, a piecewise linear homeomorphism) $h : D^p \times D^q \rightarrow H$ such that $h(\partial D^p \times D^q) = H \cap W$. Then we say that H is a p -handle or a *handle of index p on W* (notation: $H^{(p)}$). The disk $h(D^p \times 0)$ is called the *core* of the p -handle H , and the disk $h(0 \times D^q)$ the *co-core* of it. The sphere $h(\partial D^p \times 0)$ is called the *attaching sphere*, and the sphere $h(0 \times \partial D^q)$ the *belt sphere* of H . The isomorphism h is called the *characteristic map* of H , and the embedding $f = h\partial D^p \times D^q$ the *attaching map*.

Let $(W; M_0, M_1)$ be a cobordism and let H be a handle on the manifold W . If $H \cap W \subset M_1$, then H is said to be a *handle on the cobordism W* . By putting $M_2 = \partial W' - M_0$, where $W' = W \cup H$, we obtain a new cobordism $(W'; M_0, M_2)$, which will be said to be obtained from the initial cobordism by *attaching a handle H* . For a cobordism W' obtained by a consecutive attaching of handles in such a way that a handle $H^{(r_j)}$ is attached to a cobordism $W \cup H^{(r_1)} \cup H^{(r_2)} \cup \dots \cup H^{(r_{j-1})}$, the following notation is often used:

$$W' = W \cup H^{(r_1)} \cup H^{(r_2)} \cup \dots \cup H^{(r_m)}.$$

In most of the applications of handle body theory, handles on cobordisms are used. At the same time it is clear that for $M_0 = \emptyset$ the notion of a handle on cobordism reduces to the notion of a handle on a manifold.

If $W^{n+1} = W \cup H^{(r)} \cup H^{(r+1)}$, then both the belt sphere $S_1^{n-r} = h^{(r)}(0 \times \partial D^{n-r+1})$ of the handle $H^{(r)}$ and the attaching sphere $S_2^r = h^{(r+1)}(\partial D^{r+1} \times 0)$ of the handle $H^{(r+1)}$ lie in $M_2 = \partial(W \cup H^{(r)}) - M_0$ and have complementary dimensions therein. Therefore by a small perturbation of the attaching map of the handle $H^{(r+1)}$ to a general position we can make the spheres S_1 and S_2 meet transversally at finitely many points. Since characteristic maps endow the sphere S_1 and S_2 with standard orientations, the intersection index $S_1 \cdot S_2$ is defined. We will call this index the *incidence index* of the handles $H^{(r)}$ and $H^{(r+1)}$ and denote it by $(H^{(r)}, H^{(r+1)})$. If S_1 and S_2 meet transversally at a unique point, then $H^{(r)}$ and $H^{(r+1)}$ are said to be *complementary handles*. If the incidence index $\varepsilon(H^{(r)}, H^{(r+1)})$ equals ± 1 , then the handles $H^{(r)}$ and $H^{(r+1)}$ are said to be *algebraically complementary*.

A *handle decomposition* of a closed manifold W^{n+1} is a decomposition $W = H_0 \cup H_1 \cup \dots \cup H_k$, where H_0 is a disk of dimension $n+1$ and H_{i+1} is a handle on $W_i = \bigcup_{j \leq i} H_j$. If $(W; M_0, M_1)$ is a cobordism, then

a handle decomposition of the cobordism on M_0 is a decomposition $W = (M_0 \times I) \cup H_1 \cup \dots \cup H_k$, where $M_0 \times I$ is an obvious cobordism and H_{i+1} is a handle on the cobordism $(M_0 \times I) \cup \bigcup_{j \leq i} H_j$. Any smooth or piecewise linear manifold (and, therefore, cobordism) can be shown to admit a handle decomposition. For topological manifolds this is no longer true (see [11]).

There are several basic rules for treating handles.

I (*regrouping*). If $W' = W \cup H^{(r)} \cup H^{(s)}$ and $s \leq r$, then $W' \cong W \cup H^{(s)} \cup H^{(r)}$ with $H^{(s)}$ and $H^{(r)}$ non-intersecting.

II (*cancellation*). If for $W' = W \cup H^{(r)} \cup H^{(r+1)}$ the handles $H^{(r)}$ and $H^{(r+1)}$ are complementary, then there is an isomorphism $h : W' \rightarrow W$ that is the identity outside any neighbourhood of the union $H^{(r)} \cup H^{(r+1)}$.

III (*decomposition*). Let $W' = W \cup B^{n+1}$ and let $B^{n+1} \cap W = B \cap M_1 = B_1$ be a face of the disk B . Then $W' = W \cup H^{(r)} \cup H^{(r+1)}$, where $H^{(r)}$ and $H^{(r+1)}$ are complementary handles.

IV (*attaching handles*). Let $W^{n+1} = W' \cup H^{(r-1)} \cup H_1^{(r)} \cup H_2^{(r)}$, where $r \geq 2$ and $n \geq r+1$. If the manifold $\partial(W' \cup H^{(r-1)}) - M_0$ is simply-connected, then there is another handle decomposition $W^{n+1} = W' \cup H^{(r-1)} \cup H_1^{(r)} \cup H_2^{(r)}$ of W^{n+1} with $\varepsilon(H^{(r-1)}, H_2^{(r)}) = \varepsilon(H^{(r-1)}, H_1^{(r)}) \pm \varepsilon(H^{(r-1)}, H_1^{(r)})$.

V (*algebraic cancellation*). Let $W' = W^{n+1} \cup H^{(r)} \cup H^{(r+1)}$. Suppose that M is simply-connected, $n-r \geq 3$, $r \geq 2$, $n \geq 5$. If the handles $H^{(r)}$ and $H^{(r+1)}$ are algebraically complementary, then $W' \cong W$.

There is a basic difference between the rule I–IV and the rule V. The first four rules are proved by means of general position arguments. Rule V is based upon the so-called *Whitney trick*, which is as follows.

Theorem 3.1 (Whitney [56]). *Let P^p and Q^q be oriented submanifolds of an oriented manifold M^m , where $p+q = m$. Suppose also that the submanifolds P^p and Q^q meet transversally, $x, y \in P^p \cap Q^q$, and $(P \cdot Q)_x = -(P \cdot Q)_y$. If either $p \geq 3$, $q \geq 3$, and $\pi_1(M) = 0$, or $p = 2$, $q \geq 3$, and $\pi_1(M-Q) = 0$, then there is an isotopy of M sending P to a submanifold P' which meets Q transversally in such a way that $P' \cap Q = P \cap Q - \{x, y\}$. Moreover, the support of the above isotopy is contained in a compact set containing no other intersection points.*

Sketch of the proof. We join the points x and y in P and Q , respectively, by curves α and β not passing through other intersection points. We show that there is a two-dimensional disk $D^2 \subset M$ such that $\partial D^2 = \alpha \cup \beta$ and $D^2 \cap (P \cup Q) = \partial D^2$ (this disk D^2 is called the *Whitney disk*). Indeed, if $p \geq 3$, then by virtue of the condition $\pi_1(M) = 0$ there is a map $f : D^2 \rightarrow M$ such that $f(\partial D^2) = \alpha \cup \beta$. By putting this map into general position, we obtain the desired disk. In a similar though somewhat more cumbersome way the case $p = 2$ is analysed.

We now consider a regular neighbourhood (N, B_1, B_2) of the Whitney disk D^2 in (M, P, Q) . We can show that $(S, S_1, S_2) = \partial(N, B_1, B_2)$ is a trivial linkage. Hence there is an unknotted sub-ball B'_1 such that $\partial B'_1 = S_1$ and $B'_1 \cap B_2 = \emptyset$. Therefore there is an isotopy of the ball N that is stationary on $\partial N = S$ and sends B_1 to B'_1 . Extending it to the whole manifold M by means of the identity map, we get the required isotopy. ■

Corollary 3.2. *Under the hypotheses from Whitney's theorem, the submanifold P can be transformed by means of an isotopy to a submanifold P' transversal to Q in M such that if $n = P \cdot Q$, then $P' \cap Q = \{x_1, \dots, x_{|n|}\}$ with $(P' \cdot Q)_{x_i} = \text{sign } n$ for $n \neq 0$ and $P' \cap Q = \emptyset$ for $n = 0$. ■*

Using the handle body theory, Smale [45] proved the following remarkable theorem in 1961.

Theorem 3.3 (*h-cobordism theorem*). *Let $(W^{n+1}; M_0, M_1)$ be a simply-connected h-cobordism and let $n \geq 5$. Then $W \cong M_0 \times I$. In particular, M_0 and M_1 are isomorphic.*

Proof. By virtue of the rule I and the transversality of the product cobordism, W^{n+1} can be represented as

$$W^{n+1} = (M_0 \times I) \cup \left(\bigcup_{i=1}^{a_0} H_i^{(0)} \right) \cup \left(\bigcup_{j=1}^{a_1} H_j^{(1)} \right) \cup \dots \cup \left(\bigcup_{k=1}^{a_n} H_k^{(n)} \right) \cup \left(\bigcup_{l=1}^{a_{n+1}} H_l^{(n+1)} \right),$$

where all the handles of the same index do not meet pairwise. Now, using the connectedness of W^{n+1} , we can show that each 0-handle cancels with a 1-handle. Cancelling a_0 0-handles with a_1 1-handles, we obtain a new handle decomposition containing no 0-handles and $a_1 - a_0$ 1-handles ($a_1 - a_0 \geq 0$ from homology considerations).

Since the attaching to the cobordism $W = (M_0 \times I) \cup H_1 \cup \dots \cup H_t$ of a trivial cobordism $M_1 \times I$ does not change W ,

$$W = (M_0 \times I) \cup H_1 \cup \dots \cup H_t \cup (M_1 \times I).$$

We put $W_{i+1}^* = (M_1 \times I) \cup_{j \geq i+1} H_j$. The factor permutation automorphism $D^p \times D^q \rightarrow D^q \times D^p$ enables us to consider a p -handle H_i as a q -handle H_i^* on W_{i+1}^* . Hence W_i^* provides a dual handle decomposition of our cobordism, where any p -handle turns into an $(n+1-p)$ -handle. In particular, to the cancellation of 0-handles with 1-handles in the dual handle decomposition there corresponds the cancellation of $(n+1)$ -handles with n -handles in the original decomposition. Thus, by virtue of the connectedness of W and the fact that $M_0 \neq \emptyset, M_1 \neq \emptyset$, we can assume that our cobordism has neither 0-handles nor $(n+1)$ -handles. Further, by using the simple-connectedness of W , the rule IV, and the dimensional restriction $n \geq 4$, we can kill such 1-handles with the help of 2-handles, and n -handles with the help of $(n-1)$ -handles. After these operations, the following equalities hold in a decomposition of W : $a_0 = a_1 = a_n = a_{n+1} = 0$.

To proceed further, we need to apply the rule V. Making use of the simple-connectedness of W and the fact that the embedding $M_0 \subset W$ is a homotopy equivalence, for any s -handle $H^{(s)}$, $2 \leq s \leq n-3$, we can construct, by means of attaching and subtraction of $(s+1)$ -handles, a handle $H^{(s+1)}$ that is algebraically complementary to $H^{(s)}$. After this, in the case $n \geq 5$ we can use the rule V, which makes it possible to cancel the handles $H^{(s)}$ and $H^{(s+1)}$. Thus, we can assume that there are only handles of indices $n-2$ and $n-1$ in a decomposition. But if we now apply the rule V to the dual decomposition, then by the same token all the $(n-1)$ -handles are excluded.

Thus, after all the above operations only handles of index $n-2$ may appear in a decomposition. But since $H_{n-2}(W, M_0) = 0$, it is impossible that only handles of index $n-2$ are present in a decomposition. Hence under exclusion of $(n-1)$ -handles all the $(n-2)$ -handles have disappeared as well. Thus, we remain with $W \cong M_0 \times I$, as required. ■

The h -cobordism theorem generalizes to the non-compact case (under the same dimensional restrictions). We will give a few definitions before formulating it. Let W be a non-compact manifold. A set $U \subset W$ is called a *neighbourhood of infinity* in W if $W - \dot{U}$ is compact. A manifold W is said to be *connected at infinity* if it has arbitrarily small connected neighbourhoods of infinity. A manifold W is called *simply-connected at infinity* if for every neighbourhood of infinity U there is a simply-connected neighbourhood of infinity V contained in U . A triple $(W; M_0, M_1)$ is called a *proper h -cobordism* if W is a non-compact manifold with a compact boundary $\partial W \supset M_0 \cup M_1$ and both inclusions $i_0: M_0 \hookrightarrow W$, $i_1: M_1 \hookrightarrow W$ are homotopy equivalences.

Theorem 3.4 (Siebenmann [42]). *Let $(W^{n+1}; M_0, M_1)$ be either a simply-connected smooth or a piecewise linear proper h -cobordism, and let $n \geq 5$. Suppose that W is simply-connected at infinity. In addition, if $C = \partial W - (M_0 \cup M_1)$ is non-empty, suppose that $C \cong (C \cap M_0) \times I$. Then $W \cong M_0 \times I$. ■*

The proof of the h -cobordism theorem presented above does not go over to four-dimensional manifolds. The reason for that is that Whitney's theorem is no longer true in dimension four. To convince ourselves of that, let us return once again to the proof. If $m = 4$, then there is no immersed Whitney disk. In this case we can only find an immersed disk $f: D^2 \rightarrow M$ such that $f(\partial D^2) = \alpha \cup \beta$ and $D^2 \cap (P \cup Q) = \partial D^2 \cup \{\text{transversal intersection points } x_1, \dots, x_k \text{ for some } k\}$. Without having eliminated self-intersection points of D^2 and points of its intersection with $P \cup Q$, we cannot speak about any isotopy of submanifolds.

An extensive list of publications is devoted to attempts to find either a proper substitute for the Whitney trick or a way to proceed without it. A survey of results in this direction is contained in [38].

§4. Casson handles

In 1973 Casson [13] proposed a clever approach to surmounting difficulties related to the Whitney trick in dimension 4. An infinite construction devised by him enables us to turn all the intersection points out “to infinity”. As a result, we get the necessary results but only up to proper homotopy type.

Theorem 4.1 (Casson). *Let $(W^4, \partial W^4)$ be a simply-connected smooth four-dimensional manifold with boundary and let $f_i : D^2 \rightarrow W^4, i = 1, \dots, n$, be maps of a disk D^2 such that $f_i|_{\partial D^2} : \partial D^2 \rightarrow \partial W^4$ are immersions with non-intersecting images. Suppose that all the intersection indices $f_i \cdot f_j$ vanish (they are defined for $i \neq j$) and there are homology classes $\beta_i \in H_2(W)$ with even squares $\beta_i^2 = \beta_i \cdot \beta_i$ such that $\beta_i \cdot f_j = \delta_{ij}$. Then there are disjoint open sets $V_i \subset W$ such that:*

- (1) *the pair $(V_i, V_i \cap \partial W)$ has proper homotopy type of the pair $(D^2 \times \mathbb{R}^2, S^1 \times \mathbb{R}^2)$;*
- (2) *$V_i \cap \partial W$ is an open tubular neighbourhood of the image $f_i(\partial D^2)$ in ∂W ;*
- (3) *the map f_i is rel ∂ homotopic to a map into V_i .*

The manifolds V_i are called *Casson handles*.

Proof of Theorem 4.1. It is based on several lemmas, which are of interest on their own because they contain a description of Casson handles. We call an immersion $f : D^2 \rightarrow W$ *normal* if $f(S^1)$ is immersed in ∂W , f is transversal to ∂W in $f(S^1)$, and all the self-intersections are transversal double points in $\text{int } W$.

Lemma 4.2. *Let W be a smooth simply-connected four-dimensional manifold and $f : D^2 \rightarrow W$ a normal immersion. Suppose there is a homology class $\beta \in H_2(W)$ with $f \cdot \beta = 1$. Then f is regularly rel ∂ homotopic to a normal immersion $g : D^2 \rightarrow W$ such that $W - g(D^2)$ is simply connected.*

Proof. Since $f \cdot \beta = 1$, a straightforward homology group calculation shows that $H_1(W - f(D^2)) = 0$, so the group $\pi_1(W - f(D^2))$ is perfect. Let z be a meridian of $f(D^2)$ and let $w \in \pi_1(W - f(D^2))$ (Fig. 1, p. 182). We will show first how to cancel the commutator $[z, z^w] = z(w^{-1}zw)z^{-1}(w^{-1}zw)^{-1}$. With this purpose in mind, let us construct an immersion $g : D^2 \rightarrow W$ by pushing $f(D^2)$ through along the loop w so as to make the “finger” return back near the original position. Consider a four-dimensional disk in $\text{int } W$ partly containing $f(D^2)$ and a “finger” approaching it (Fig. 2). By connecting $f(S^2)$ and the “finger” by an arc and applying the inverse Whitney trick, we obtain a Whitney disk A and two more intersection points, p and q .

It is obvious from the construction that the immersions f and g are regularly homotopic. An effect of this regular homotopy is that the group $\pi_1(W - g(D^2))$ is isomorphic to a quotient group of $\pi_1(W - f(D^2))$ by a normal subgroup generated by $[z, z^w]$. Indeed, $\pi_1(W - (g(D^2) \cup A)) \cong \pi_1(W - f(D^2))$, since $W - (g(D^2) \cup A)$ is homeomorphic to $W - (f(D^2) \cup \text{arc})$ where *arc* stands for an arc connecting $f(D^2)$ with the “finger” (Fig. 2, p. 182).

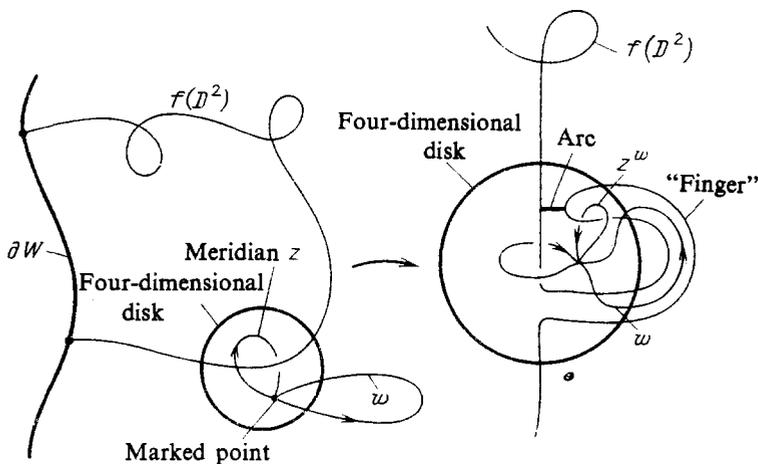


Fig. 1

Moreover, $W - g(D^2)$ results from attaching int A to $W - (g(D^2) \cup A)$, which is homotopically equivalent to attaching a 2-handle cancelling a meridian of A .

We now consider a neighbourhood of p (Fig. 3), where z and z^w can be homotopically mapped on a 2-torus, complementary to intersecting sheets of an immersed disk $g(D^2)$. This torus is punctured by the disk A at exactly one point, and also z and z^w form a π_1 -basis of the punctured torus. Hence, $[z, z^w] \in \pi_1(W - (g(D^2) \cup A))$ is a meridian of A , that is, $W - g(D^2)$ is obtained from $W - (g(D^2) \cup A)$ by cancelling $[z, z^w]$, as was claimed.

Four-dimensional disk

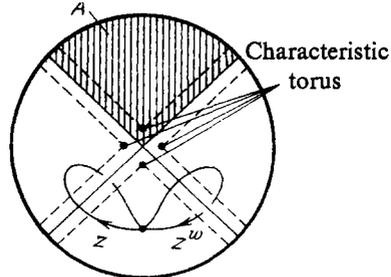
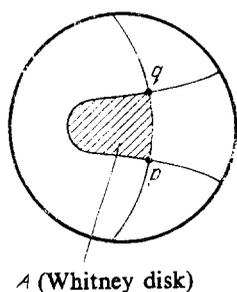
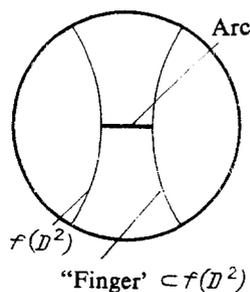


Fig. 2

Fig. 3

To complete the proof of Lemma 4.2, let us make use of the fact that the group $\pi_1(W - f(D^2))$ is perfect, and because of the simple-connectedness of W it is generated by conjugates of z . Thus, in order to cancel $\pi_1(W - f(D^2))$ it suffices to cancel z . Since $\pi_1(W - f(D^2))$ is perfect, we can represent z in the form

$$z = [z^{u_1}, z^{v_1}] \dots [z^{u_k}, z^{v_k}].$$

But since $[z, z^w]^u = [z^u, z^{wu}]$, each of the commutators $[z^{u_1}, z^{v_1}], \dots, [z^{u_k}, z^{v_k}]$ can be cancelled by the construction described above. ■

Remark. The two-dimensional torus appearing in the proof of Lemma 4.2 will be called the *characteristic torus* of the corresponding double point of the immersed disk $f(D^2)$.

Lemma 4.3. *Let W^4 be a smooth simply-connected four-dimensional manifold and let $f_i : D^2 \rightarrow W, i = 1, \dots, n$, be normal immersions such that $f_i \cdot f_j = 0$ for all $i \neq j$ and there are classes $\beta_i \in H_2(W)$ with $f_i \cdot \beta_j = \delta_{ij}$. Then f_i are rel ∂ regularly homotopic to disjoint normal immersions $g_i : D^2 \rightarrow W$ such that*

$$\pi_1(W - \bigcup_{i=1}^n g_i(D^2)) = 0.$$

Proof. Consider a normal immersion f_1 . Using the preceding lemma, let us rel ∂ regularly deform f_1 to g_1 in such a way that $\pi_1(W - g_1(D^2)) = 0$ and g_1 still remains a normal immersion. Then by the Hurewicz theorem $\pi_2(W, W - g_1(D^2)) \cong H_2(W, W - g_1(D^2))$. Since $g_1 \cdot f_2 = f_1 \cdot f_2 = 0$, the map f_2 represents 0 in $H_2(W, W - g_1(D^2))$. Because of this, we can deform f_2 rel ∂ so that it does not meet $g_1(D^2)$. Once again using Lemma 4.2, we deform f_2 rel ∂ in $W - g_1(D^2)$ to a normal immersion g_2 with $\pi_1(W - g_1(D^2) \cup g_2(D^2)) = 0$. Continuing this process, we deform f_1, \dots, f_n rel ∂ to normal immersions g_1, \dots, g_n such that for all $i \neq j$ we have $g_i(D^2) \cap g_j(D^2) = \emptyset$ and $\pi_1(W - (g_1(D^2) \cup \dots \cup g_n(D^2))) = 0$. ■

Lemma 4.4. *Let W^4 be a smooth simply-connected four-dimensional manifold, $f : D^2 \rightarrow W$ a normal immersion, and N a regular neighbourhood of $f(D^2)$ in W . We put $\partial^+ N = \{\text{boundary of } N \text{ in } W\}$ and $\partial^- N = N \cap W$ (Fig. 4).*

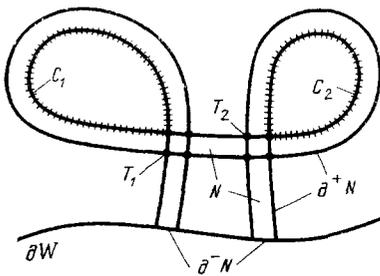


Fig. 4

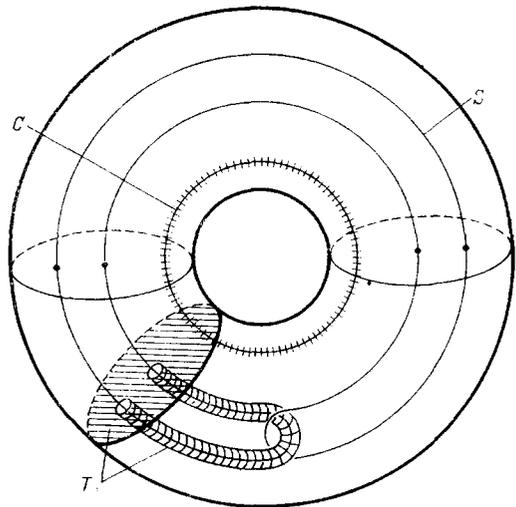


Fig. 5

Then in $\partial^+ N$ there are disjoint framed circles C_1, \dots, C_n such that if \hat{N} is a result of attaching 2-handles to N by C_1, \dots, C_n , then the pair $(\hat{N}; f(S^1))$ is diffeomorphic to B^4 , a circle unknotted in ∂B^4 . Here each circle C_i can be chosen so as to meet exactly one characteristic torus transversally in $\partial^+ N$. Framings of circles C_1, \dots, C_n can be chosen so as to be compatible with any spinor structure of the manifold W .

Proof. We remark first that the differential topological type of the pair $(N, f(D^2))$ is completely determined by the number p of positive double points and the number q of negative double points of the immersion f . Therefore it suffices to prove the lemma for the case $p = 1, q = 0$ only, because the general case can be reduced to that by means of obvious arguments using the connected sum and orientation inversion operations. Thus, let f have a single positive double point. Consider the model represented in Fig. 5. On this model, inside $S^3 = \partial(B^2 \times B^2)$ there are represented the solid torus $B^2 \times S^1$, a circle S embedded in $\text{int}(B^2 \times S^1)$, a punctured two-dimensional torus $T_0 \subset B^2 \times S^1$ with boundary $\partial B^2 \times t_0, t_0 \in S^1$, and a circle C having no points in common with S and intersecting T_0 transversally at a single point.

Let $A = B^2 - \frac{1}{2}B^2 \cong S^1 \times \left[\frac{1}{2}, 1 \right]$ be an annulus lying in the second factor B^2 of the handle $B^2 \times B^2$. Consider the space $T = T_0 \cup \partial T_0 \times \left[\frac{1}{2}, 1 \right] \cup B^2 \times t_0 \times \frac{1}{2}$. It is not hard to verify that T is a two-dimensional torus situated in $\partial(B^2 \times A) \cong S^2 \times S^1$. We will show that there is a diffeomorphism of pairs $(N, f(\partial D^2)) \cong (B^2 \times A, S)$ sending a characteristic torus from inside $\partial^+ N$ onto T . By virtue of the uniqueness of regular neighbourhoods, in order to prove this it suffices to find in $B^2 \times A$ a normally immersed two-dimensional disk D with exactly one double point such that $\partial D = S$ and $B^2 \times A$ is an abstract regular neighbourhood of D . Let us cut our model by a horizontal plane into two disks $B^2 \times B^1_+$ and $B^2 \times B^1_-$ with $B^1_+ \cap B^1_- = S^0 \subset S^1$. Extending this partition to $B^2 \times A$, by the same token we cut $B^2 \times A$ into two disks $B^2 \times A_\pm = B^2 \times \left(B^1_\pm \times \left[\frac{1}{2}, 1 \right] \right)$, each of which contains one of the semicircles S_\pm ($S_+ \cup S_- = S$). Joining the four cutting points on the semicircles S_+, S_- by two arcs I_1, I_2 that are immersed in the disk $B^2 \times \left[\frac{1}{2}, 1 \right] \times S^0$ in an unknotted way, we get two links \hat{S}_+ and \hat{S}_- . Obviously, \hat{S}_+ is a trivial knot in $\partial(B^2 \times A_+)$, bounding an unknotted two-dimensional disk D_+ in $B^2 \times A_+$. On the other hand, \hat{S}_- is a Hopf link (Fig. 6), and boundaries of regular neighbourhoods of each component are parallel to T . Hence there is a diffeomorphism

$$\theta: B^2 \times B^2 \rightarrow B^2 \times A_-,$$

such that $\hat{S}_- = \theta(S^1 \times 0 \cup 0 \times S^1)$ and $T = \theta(S^1 \times S^1)$. Then $\theta(B^2 \times 0 \cup 0 \times B^2)$ is the union of two immersed disks D_1 and D_2 with a unique double point $\theta(0, 0)$, the characteristic torus of which is T . Therefore, starting with the two-dimensional disk $D = D_+ \cup D_1 \cup D_2$, we obtain a diffeomorphism $(N, f(D^2)) \cong (B^2 \times A, S)$ which maps a characteristic torus from $\partial^+ N$ onto T .

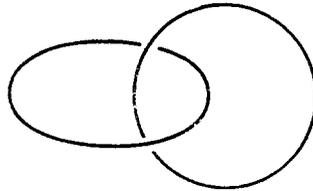


Fig. 6. Hopf links

Now we are able to prove the first two statements of the lemma. Indeed, $B^4 = B^2 \times B^2$ is nothing but $B^2 \times A$ with a 2-handle $B^2 \times \frac{1}{2} B^2$ attached along the circle $C' = 0 \times \partial(\frac{1}{2} B^2)$, which is isotopic in $\partial(B^2 \times A)$ to C . Like C , the circle C' intersects T transversally at a single point. Finally, S is obviously unknotted in S^3 .

In order to see that a framing of C can be made compatible with any spinor structure of the manifold W , let us choose a ball neighbourhood B^4 of the double point of the immersion f in such a way that $X = \overline{N - B^4}$ is a regular neighbourhood of part of the immersed disk $P = \overline{f(D^2) - B^4}$. Then $X \cong P \times D^2$, and any homeomorphism of X that is an automorphism of the above D^2 -bundle structure and is stationary on $\partial P^2 \times D^2$ extends to a homeomorphism of N that is stationary on a neighbourhood of $f(S^1)$ in ∂N . It is not difficult to choose an automorphism h such that the framing of the circle $h(C)$ is compatible with the given spinor structure. ■

Proof of the Casson theorem 4.1. Let $f_1, \dots, f_n : D^2 \rightarrow W$ satisfy the conditions of the theorem. Let us approximate $F_i, i = 1, \dots, n$, by normal immersions, which we will still denote by f_i , and then with the help of Lemma 4.3 deform f_i regularly to normal immersions g_i such that $g_i(D^2) \cap g_j(D^2) = \emptyset$ for all $i \neq j$ and $\pi_1(W - \bigcup_{i=1}^n g_i(D^2)) = 0$.

Let N be a regular neighbourhood of the immersion of $\bigcup_{i=1}^n g_i(D^2)$ in W .

We put $W' = W - \overset{\circ}{N}$, $\partial^- N = N \cap \partial W$, $\partial^+ N = W' \cap N$. It is easy to see that N consists of n connected components N_i , where N_i is a regular neighbourhood of $g_i(D^2)$. Let C_1, \dots, C_m be a family of circles in $\partial^+ N$ obtained by applying Lemma 4.4 successively to N_1, \dots, N_n . Since the manifold W' is simply-connected, there is a map $h_1 : (D^2, S^1) \rightarrow (W', \partial^+ N)$

such that $h_1|S^1$ is a diffeomorphism of S^1 onto C_1 . Let us construct a manifold \widehat{W}' attaching a 2-handle to W' in an abstract way along the framed circle C_1 in $\partial W'$, and define the relative homology intersection index $h_1 \cdot h_1 \in \mathbb{Z}$ as the absolute intersection index $\widehat{h}_1 \cdot \widehat{h}_1$ in \widehat{W}' , where $\widehat{h}_1 = h_1 \cup$ (core of 2-handle) in $H_2(\widehat{W}')$.

We show now that h_1 can be modified in order to ensure that $h_1 \cdot h_1 = 0$. First of all, we modify h_1 in order to make the intersection index $h_1 \cdot h_1$ even.

If the manifold W is endowed with a spinor structure, then $h_1 \cdot h_1$ is automatically even because the framing of the circle C_1 agrees with the spinor structure on W , and therefore on W' . If, on the contrary, W has no spinor structure, then there is $x \in H_2(W)$ such that $x \cdot x$ is odd. By assumption, there are classes $\beta_i \in H_2(W)$, $i = 1, \dots, n$, such that all the integers $\beta_i \cdot \beta_i$ are even and $\beta_i \cdot g_j = \delta_{ij}$. If $\beta = b_1\beta_1 + \dots + b_n\beta_n$, then $\beta \cdot \beta$ is even and $(x + \beta) \cdot (x + \beta) = x \cdot x + 2x \cdot \beta + \beta \cdot \beta$ is odd. Properly choosing the class β , namely, putting $b_i = -x \cdot g_i$, we get $(x + \beta) \cdot g_i = 0$ for all $i = 1, \dots, n$. This means that the class $x + \beta$ comes from $x' \in H_2(W')$ and $x' \cdot x' = (x + \beta) \cdot (x + \beta)$ is odd. We use this class in order to modify $h_1 \in H_2(W', \partial W')$ in the case of odd intersection index $h_1 \cdot h_1$, namely, we put $h'_1 = h_1 + x'$. Then $h'_1 \cdot h'_1 = h_1 \cdot h_1 + 2h_1 \cdot x' + x' \cdot x'$ is an even integer.

Now we will achieve the equality $h_1 \cdot h_1 = 0$. Denoting by $\tau_i \in H_2(\partial^+ N)$ the homology class of the characteristic torus from Lemma 4.4, $1 \leq i \leq m$, we find that $\tau_i \cdot C_j = \delta_{ij}$. For the sake of brevity we also denote by τ_i the image of class τ_i in $H_2(W')$; then $\tau_i \cdot \tau_j = 0$ and $h_1 \cdot \tau_j = \delta_{1j}$. Thus, if $h_1 \cdot h_1 = 2k$, then

$$(h_1 - k\tau_1) \cdot (h_1 - k\tau_1) = h_1 \cdot h_1 - 2kh_1 \cdot \tau_1 + k^2\tau_1 \cdot \tau_1 = 0$$

and hence $h_1 - k\tau_1$ is a suitable modification of h_1 .

Finally, using the Hurewicz theorem and the equality $h_2 \cdot h_2 = 0$, let us represent h_1 by a normal immersion $h_1 : D^2 \rightarrow W'$ with $h_1(\partial D^2) = C_1$. Since $g_1 \cdot \tau_j = \delta_{1j}$, Lemma 4.2 (with τ_1 instead of β) enables us to deform h_1 so that $\pi_1(W' - h_1(D^2)) = 0$.

Now consider C_2 . As before, we can find a map $h_2 : D^2 \rightarrow W'$ such that $h_2|\partial D^2 : \partial D^2 \rightarrow C_2$ is a diffeomorphism, $h_2 \cdot h_1 = 0$ and $h_2 \cdot \tau_j = \delta_{2j}$. Without loss of generality we may assume that $h_2 \cdot h_1 = 0$. Indeed, if $h_2 \cdot h_1 = -k$, then $(h_2 + k\tau_1) \cdot (h_2 + k\tau_1) = h_2 \cdot h_2 + 2kh_2 \cdot \tau_1 + k^2\tau_1 \cdot \tau_1 = 0$ and $(h_2 + k\tau_1) \cdot h_1 = 0$. Therefore, once again using the Hurewicz theorem and Lemma 4.2, we find a normal immersion $h_2 : D^2 \rightarrow (W' - h_1(D^2))$ with $h_2(\partial D^2) = C_2$, $\pi_1(W' - (h_1(D^2) \cup h_2(D^2))) = 0$ and $h_2 \cdot \tau_j = \delta_{2j}$.

Proceeding in this way, we obtain disjoint normal immersions

$$h_1, \dots, h_m : D^2 \rightarrow W' \text{ with } h_j(\partial D^2) = C_j \text{ and } \pi_1(W' - \bigcup_{i=1}^m h_i(D^2)) = 0.$$

In addition, we have homology classes τ_j in $H_2(W')$ such that $h_i \cdot \tau_j = \delta_{ij}$ and $\tau_i \cdot \tau_j = 0$ for all i, j . This means that for the manifold W' and the immersions h_i all the conditions of Theorem 4.1 are fulfilled. Hence the

construction described above can be repeated as many times as desired. As a result, we obtain in W a sequence $N = N(1), N(2), N(3), \dots$ of compact submanifolds of codimension zero (Fig. 7) with the following property. If we put $P(n) = N(1) \cup \dots \cup N(n)$ and $W(n) = W - \mathring{P}(n)$, then $N(n+1)$ is a regular neighbourhood in the manifold $W(n)$ of a finite union $D(n)$ of disjoint normally immersed disks whose boundaries form in $\partial^+ W(n) = \partial W(n) - \mathring{P}(n)$ a family $C(n)$ of disjoint circles.

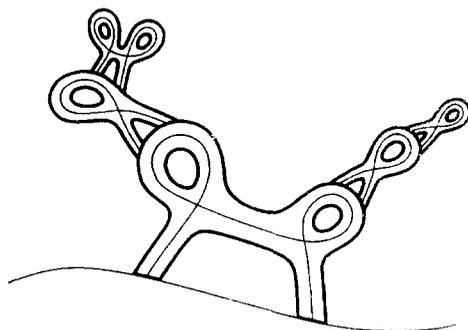


Fig. 7

We put $V = \bigcup_n \mathring{P}(n)$. Then, by definition, a Casson handle V_i is a connected component of V containing $f_i(S^1)$.

To complete the proof of Theorem 4.1, it remains only to verify the homotopy properties of V_i . We omit this verification here, leaving it to the reader (see [13]). ■

The following remarkable corollary follows at once from Theorem 4.1.

Theorem 4.5. *Let W^4 be a simply-connected four-dimensional manifold and let $\alpha, \beta \in H_2(W)$ be such that $\alpha \cdot \alpha = \beta \cdot \beta = 0$ and $\alpha \cdot \beta = 1$. Then there is an open set $V \subset W$ having proper homotopy type of $S^2 \times S^2$ -pt and such that $\alpha, \beta \in \text{im}(H_2(V) \rightarrow H_2(W))$.*

Proof. We represent the homology classes $\alpha, \beta \in H_2(W)$ by maps $f, g : S^2 \rightarrow \text{int } W^4$ in general position and pick a four-dimensional disk $B^4 \subset \text{int } W^4$ that forms a regular neighbourhood of a point of positive transversal intersection of $f(S^2)$ and $g(S^2)$. Cutting a disk \mathring{B}^4 from W^4 , we obtain a manifold $W_0^4 = W^4 - \mathring{B}^4$ and maps $f_0, g_0 : (D^2, \partial D^2) \rightarrow (W_0, \partial W_0)$ corresponding to homology classes $\alpha, \beta \in H_2(W_0)$. It is obvious that $f_0 \cdot g_0 = \alpha \cdot \beta - 1 = 0$, $f_0 \cdot \beta = 1$, $g_0 \cdot \alpha = 1$, $\alpha \cdot \alpha = \beta \cdot \beta = 0$. Because of this, applying Theorem 4.1 to maps f_0 and g_0 , we obtain open sets V_1 and V_2 and put $V = \mathring{B}^4 \cup V_1 \cup V_2$. ■

If the Casson handles V_1 and V_2 were *diffeomorphic* to open standard 2-handles, then such a V would be diffeomorphic to a submanifold of $S^2 \times S^2 - \text{pt}$ and we could perform smooth four-dimensional surgery without any restrictions. It is not known at present whether Casson handles are diffeomorphic to standard 2-handles. However, at the end of 1981 Freedman [24] managed to prove that Casson handles are *homeomorphic* to standard 2-handles. The rest of our survey is devoted to a presentation of this result of Freedman and its applications.

§5. Geometric control theorem. Casson handle design

In this section we will give an intrinsic (not depending on an embedding in a specific manifold) definition of Casson handles and formulate the so-called geometric control theorem, which will play a central role in the presentation to follow.

Definition 5.1. A *kinky handle* is a pair obtained from a standard 2-handle by finitely many self-plumbings away from the attaching region. A *self-plumbing* is an identification of $D_0^2 \times D^2$ with $D_1^2 \times D^2$, where $D_0^2, D_1^2 \subset D^2$ are disjoint subdisks of the first factor disk D^2 of the handle $D^2 \times D^2$.

Figure 8 shows a kinky handle with one self-plumbing. Figure 9 illustrates a kinky handle with one positive and one negative self-plumbing. It is easy to show that (1) a kinky handle k , as an absolute space, is a finite connected sum $\# S^1 \times D^3$; (2) as a pair, a kinky handle $(k, \partial^- k)$ is determined up to a diffeomorphism by the numbers p of positive self-plumbings and n of negative self-plumbings.

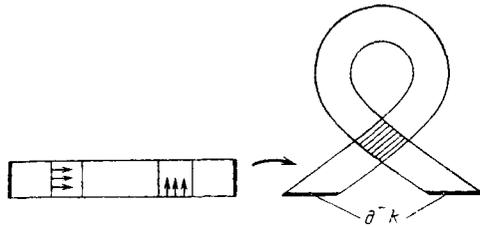


Fig. 8

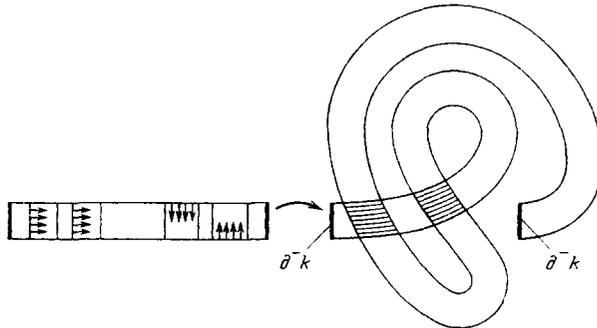


Fig. 9

We denote by C the core of a kinky handle k , that is, the image under the self-plumbing $\pi : D^2 \times D^2 \rightarrow k$ of the 2-handle core $D^2 \times 0$. In the case of one self-plumbing a pair (k, C) can be obtained from $(B^4; (B^4 \cap (w, x)\text{-plane}) \cup (B^4 \cap (y, z)\text{-plane}))$ by the attaching of a 1-handle pair $(D^1 \times D^3, D^1 \times D^1; S^0 \times D^3, S^0 \times D^1)$ (Fig. 10). We observe that the two planes (w, x) and (y, z) meet ∂B^4 in a Hopf link (see Fig. 6) and that there is a torus in ∂B^4 , $\left\{ (w, x, y, z) : w^2 + y^2 = \frac{1}{2}, x^2 + z^2 = \frac{1}{2} \right\}$, that separates the components. In Fig. 10 the torus appears as $S^0 \times S^0 = 4$ points. Its image in the kinky handle will be called the *characteristic torus*.

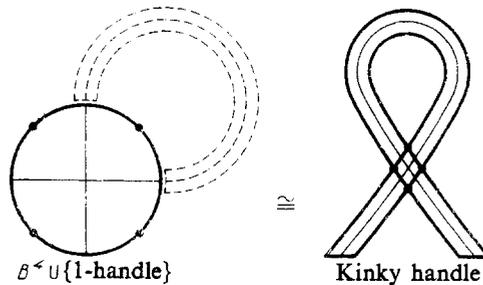


Fig. 10

Definition 5.2. Let us call a kinky handle (k, ∂^-k) a *one-stage tower*. Then an $(n+1)$ -stage tower $(T_{n+1}, \partial^-T_{n+1})$ is a union of n -stage towers (T_n, ∂^-T_n) of kinky handles. These kinky handles are attached to the zero-framed link whose circles represent 1-handles from $T_n - T_{n-1}$.

Each kinky handle (k, ∂^-k) is a cylinder of a piecewise smooth map

$$\rho : \overline{(\partial k - \partial^-k)}, \partial(\partial^-k) \rightarrow (C, \partial C).$$

If one kinky handle is attached to another, say (k^1, ∂^-k^1) is attached to (k, ∂^-k) , and (L, ∂^-L) is the resulting space, we can consider the connected piecewise smooth subcomplex $J = C \cup C^1 \cup \{\text{map cylinder of } \partial C^1\}$. It is easy to see that (L, ∂^-L) itself is a map cylinder of a piecewise smooth map

$$\rho' : \overline{(\partial L - \partial^-k)}, \partial(\partial^-k) \rightarrow (J, \partial C).$$

Definition 5.3. Let T_n be an n -stage tower. We denote by T_n^- the interior of T_n united with ∂^-T_n . Let

$$T_1 \subset T_2 \subset T_3 \subset \dots$$

be inclusions of towers as specified by their inductive definition. Then there is a corresponding sequence of embeddings

$$T_1^- \subset T_2^- \subset T_3^- \subset \dots$$

We define the Casson handle CH to be the direct limit $\text{ind lim } T_i^-$ with respect to the above inclusions, endowed with the direct limit topology.

In what follows the symbol CH will stand for an arbitrary Casson handle. It is not known at present if all Casson handles are diffeomorphic. At the same time it is not difficult to prove the following statement.

Theorem 5.4. *The interior of any Casson handle CH is diffeomorphic to \mathbb{R}^4 . ■*

Starting with Definition 5.3, it is not difficult to show that any Casson handle is an everywhere dense open subset of a standard 2-handle $(D^2 \times D^2, \partial D^2 \times D^2)$ (for details, see [15]). In the simplest (unramified) case $CH \cong (D^2 \times D^2 - (D^2 \times \partial D^2 \cup C(W)), \partial D^2 \times D^2)$, where W is the Whitehead continuum and $C(W)$ is the cone over W . Thus among possible compactifications of a CH there is a standard handle H . There is another useful compactification—the so-called Shapiro–Bing compactification. We will call a handle CH compactified according to Shapiro and Bing a *Shapiro–Bing handle* and reserve the letter K for it. To describe a Shapiro–Bing handle, we observe that a Casson handle is specified up to a diffeomorphism by a so-called \pm -labelled tree, that is, an oriented finitely-branching tree with base-point $*$. In addition, it is assumed that each edge of such a tree is given a label $+$ or $-$. Indeed, let Q be such a tree. To construct a Casson handle CH_Q from it, we assign to each vertex of Q a kinky handle such that the self-plumbing number of it equals the number of branches leaving the vertex. The sign on each branch corresponds to the sign of the associated self-plumbing.

For a given \pm -labelled tree Q we define a certain set \overline{Wh}_Q . For this purpose, we embed in $D^2 \times \partial D^2$ a ramified Whitehead link with one component for every edge labelled by plus leaving $*$, and one mirror image component for every edge labelled by minus leaving $*$. By thickening all the circles of this link we obtain smoothly embedded solid tori with normal framing.

We now consider a \pm -labelled tree Q . Corresponding to each first level vertex of Q we have already found normally framed solid tori embedded in $D^2 \times \partial D^2$. In each of these solid tori there is embedded a Whitehead link, ramified according to the number of plus and minus labelled branches leaving that vertex. Thickening the above Whitehead links, we obtain second level normally framed solid tori. Then Q determines third level normally framed solid tori, embedded in the second level, and so on. We denote by X_n the disjoint closed solid tori forming the n -th level. With the help of Q we can construct an infinite chain of inclusions

$$\dots X_{n+1} \subset X_n \subset \dots \subset X_1 \subset D^2 \times \partial D^2.$$

We observe that X_n is nothing but the attaching region of a disjoint union of 2-handles W_n , relatively embedded in $(D^2 \times D^2, D^2 \times \partial D^2)$.

Let $Wh_Q^* = \bigcap_{n=1}^{\infty} X_n$, $\overline{Wh}_Q^* = \bigcap_{n=1}^{\infty} W_n$, let Wh_Q be the set of connected components of the space Wh_Q^* , and \overline{Wh}_Q the set of connected components of \overline{Wh}_Q^* . We define a Shapiro–Bing handle K (or, in more detail, K_Q) as the quotient space $K_Q = (D^2 \times D^2 / \overline{Wh}_Q, \partial D^2 \times D^2)$. This means that every element from \overline{Wh}_Q is declared a point and the resulting set is endowed with a quotient topology. We put

$$\partial^- K_Q = \partial D^2 \times D^2 \text{ and } Fr^+(K_Q) = Fr(K_Q) - \partial D^2 \times \text{int } D^2.$$

Proposition 5.5. *The space $Fr^+(K_Q)$ is homeomorphic to $D^2 \times \partial D^2 / Wh_Q$.*

Proof.

$$Fr^+(K_Q) = \partial^+(D^2 \times D^2) / \overline{Wh}_Q \cap \partial(D^2 \times D^2) = D^2 \times \partial D^2 / Wh_Q. \blacksquare$$

Theorem 5.6 (Geometric control, [24]). *Given any 6-stage tower $(T_6, \partial^- T_6)$ there is a Shapiro–Bing handle K and an embedding of it $(K, \partial^- K) \subset (T_6, \partial^- T_6)$ satisfying the following conditions:*

- (i) $\partial^- K = \partial^- T_6$,
- (ii) $Fr(K) \cap \partial T_6 = \partial^- T_6$.

This embedding is smooth on the Casson handle $(K - Fr^+(K), \partial^- K)$. \blacksquare

The geometric control theorem is of key importance for the proof of Freedman’s theorem claiming that any Casson handle $(CH, \partial^- CH)$ is homeomorphic to the standard open 2-handle $(D^2 \times \mathbb{R}^2, \partial D^2 \times \mathbb{R}^2)$. Indeed, although the Casson handle CH still remains “terra incognita”, its compactification K has a well arranged frontier $Fr^+ K = D^2 \times \partial D^2 / Wh$. Theorem 5.6 enables us to begin investigation of CH by putting $Fr^+ K$ into $T_6 \subset CH_Q \subset K_Q$ for an arbitrary Q . To be sure, we will describe very little by means of this embedding, only a minor part of the handle CH_Q in codimension 1, but this is the very beginning. It turns out that we can embed uncountably many such frontiers in CH_Q (indexed by the Cantor continuum). Together those frontiers are placed in a Casson handle so as to specify its topology completely.

Let us proceed to arrange these frontiers coherently in a certain space \mathcal{D}_Q depending on a tree Q . The space \mathcal{D}_Q will be called a *Casson handle design*.

In order to construct \mathcal{D}_Q we need a new type of labelled tree. Such a tree $S = S(Q)$ is constructed from a \pm -labelled tree Q by means of the following procedure. There is a base point in S from which a single edge called “decimal point” (\cdot) emerges. From the second vertex of this tree two more edges emerge, their ends give rise to two edges each, and so on. Thus, three edges adjoin each vertex of the tree S , one edge entering and two edges leaving the vertex. The edges are named by initial segments of infinite base 3 fractions corresponding to the standard representation of points of the Cantor set $CS \subset [0, 1]$ (Fig. 11). In what follows the symbol e will be reserved for any finite base 3 fraction consisting of 0’s and 2’s. We observe that $\cdot 0$ and $\cdot 00$

are different edges and therefore different fractions. The symbol c will denote an infinite fraction of 0's and 2's, that is, $c \in CS$.

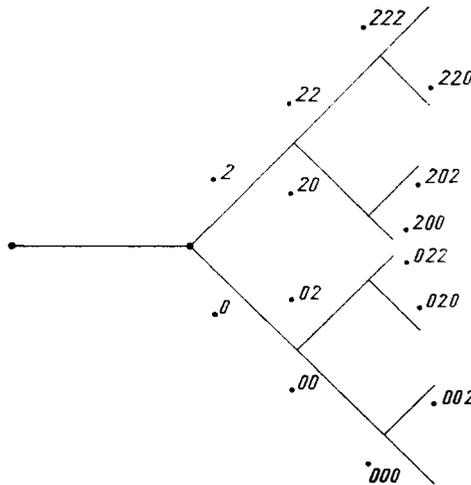


Fig. 11

Each edge e of the tree S carries a label τ_e , where τ_e is an ordered finite disjoint union of 6-stage towers with an ordered collection of standard loops generating the fundamental groups. The first requirement imposed upon the labels τ_e is that if the base 3 fraction e' is formed by adding either a zero or a two to the end of the base 3 fraction e , then the number of connected components of $\tau_{e'}$ is equal to the total number of standard generators of fundamental groups in all the components of τ_e . This condition means that any branch of S , that is, any path beginning at the base point and heading out toward infinity, is represented by some \pm -labelled tree Q' . Each edge in the branch determines six stages of Q' . Equivalently we may assume that each branch of S determines the Shapiro–Bing handle $K_{Q'}$. The second requirement on the labels τ_e is that the branch $B_{.222\dots}$ represents the \pm -labelled tree Q .

Let e be a finite base 3 fraction ending in 2. We denote by e^0 the fraction that results from e when the last 2 is changed to 0, and by e^k the fraction $\underbrace{2 \dots 2}_k e^0$. We associate with the branch $e^0 \cup e^1 \cup e^2 \cup \dots$ a disjoint union of Shapiro–Bing handles $(\Pi K)_e$. The third and final condition on the labels τ_e is that for every finite fraction e ending in 2 there is an embedding

$$((\Pi K)_e, \partial^-(\Pi K)_e) \subset (\tau_e, \partial^-\tau_e).$$

Let Q be a labelled \pm -tree. If a τ -labelled tree S satisfies the three above requirements, then it is called the τ -labelled tree associated with Q . The following important statements hold (for their proofs, see [24]).

Proposition 5.7. For any \pm -labelled tree Q there exists an associated τ -labelled tree S . ■

Proposition 5.8. Let $\{Q_c\}$ be a family of \pm -labelled trees associated with the branches B_c of the tree $S(Q)$, $c \in CS$. Then there is a family of embeddings $\{i_c\}$ such that if $c, c' \in CS$, $c' < c$, then

$$i_{c'}(K_{Q_{c'}}) \subset i_c(K_{Q_c}) \subset K_{Q.222\dots} = K_{Q_1} = K_Q.$$

Also, if c' and c are base 3 fractions agreeing in the first n places, then the first $6(n+1)$ stages of $i_{c'}(K_{Q_{c'}})$ and $i_c(K_{Q_c})$ are identical. All the embeddings i_c are normalized at ∂^- by requiring that

$$K_{Q_c} \cap \partial K_Q = \partial^- K_{Q_c} \cap \partial^- K_Q = \partial^- K_Q$$

for every $c \in CS$. ■

We are now able to construct the design $\mathcal{D}(S(Q))$ or, for the sake of brevity, \mathcal{D}_Q . It will be defined as the quotient space of a closed subset A of the standard 2-handle $H = (D^2 \times D^2, \partial D^2 \times D^2)$. We give a collar of the boundary of each factor disk of H a radial coordinate, running through values from zero to one. We denote the closure of this collar by D' , and let S_r^1 be the circle with radial coordinate r .

For every $c \in CS \subset [0, 1]$ we have a branch B_c of $S(Q)$ that determines a \pm -labelled tree Q_c . As explained earlier, Q_c specifies a defining sequence X_n^c for a decomposition of a solid torus. In our coordinates for H we will denote this torus by $D' \times S_c^1$. Each of the X_n^c is a disjoint union of embedded solid tori. Suppose the expressions for c and c' coincide at the first n places. Then for $k \leq 6(n+1)$ the collections $X_k^{c'}$ and X_k^c will be isotopic after identifying $D' \times S_c^1$ and $D' \times S_{c'}^1$. Here it is convenient to re-index the sets in X_n^c by putting ${}^{\text{new}}X_n^c = {}^{\text{old}}X_{6(n+1)}^c$. We do this without further indications.

Because of the stated connection between $X_n^{c'}$ and X_n^c we can construct a countable four-dimensional defining sequence \bar{X}_n consisting of the disjoint union of products of the form {solid torus \times interval}, whose intersection $\bigcap_{n=1}^{\infty} \bar{X}_n$ is equal to $\bigcup_{c \in CS} Wh_{Q_c}$, where $Wh_{Q_c} \subset D' \times S_c^1$. We call the set

$[0, .1] \cup [.2, 1]$ the first complement, and $(.1, .2)$ the first third. Further, we call $[0, .01] \cup [.02, 1] \cup [.2, .2] \cup [.22, 1]$ the second complement, and $(.01, .02) \cup (.21, .22)$ the second third. We continue this terminology according to the standard construction of the Cantor set. We represent $D' \times D^2$ as $D' \times D' \cup D' \times S^1 \times [0, 1]$. Then the set $\bar{X}_1 = X_1^0 \times [0, 1] \subset D' \times S^1 \times [0, 1]$ is defined as the product of the submanifold $X_1^0 \subset D^2 \times S^1 \times 0$ by the segment $[0, 1]$. We put

$$\bar{X}_2 = X_2^0 \times [0, .1] \cup X_2^2 \times [.2, 1].$$

Similarly, \bar{X}_n is defined as the radial thickening of the n -th stage of the defining sequence X_n^e , where e is a fraction of n places of 0's and 2's (e may terminate in any number $k \leq n$ of 0's).

We put $\mathcal{B} = \bigcup_{k=1}^{\infty} (\text{int } \bar{X}_k) \cap D' \times S^1 \times (k\text{-th third}) = \bigcup_{k=1}^{\infty} \mathcal{B}_k$. We define a set A as the difference $A = H - (\mathcal{B} \cup \text{int } D' \times \text{int } D')$. Then $\mathcal{D}_Q = A / \overline{Wh}_Q$, where \overline{Wh}_Q is the collection of closed subsets of A consisting of the components of $\bigcup_{n=1}^{\infty} \bar{X}_n$. We endow \mathcal{D}_Q with the quotient topology. It is more convenient to consider A and \mathcal{D}_Q as pairs $(A, \partial^- A)$, $(\mathcal{D}_Q, \partial^- \mathcal{D}_Q)$, where $\partial^- A = \partial^- \mathcal{D}_Q = \partial D^2 \times D^2 \subset H$. We define $\text{Fr}^+(\mathcal{D}_Q)$ to be the image of $D^2 \times \partial D^2$ under the natural projection. The family of embeddings $\{i_c\}$ from Proposition 5.8 determines an embedding

$$f: (\mathcal{D}_Q; \partial^- \mathcal{D}_Q, \text{Fr}^+(\mathcal{D}_Q)) \rightarrow (K_Q; \partial^- K_Q, \text{Fr}^+(K_Q)).$$

Now let us see how close we have approached our final goal—the construction of a homeomorphism $h: \dot{H} \rightarrow CH$, where $\dot{H} = (D^2 \times \dot{D}^2, \partial D^2 \times \dot{D}^2)$ is the standard open 2-handle. Let $\dot{A} = A - D^2 \times \partial D^2$, $\dot{\mathcal{D}} = \mathcal{D} - \text{Fr}^+ \mathcal{D} = \dot{A} / \overline{Wh}$ and let $g: \dot{\mathcal{D}} \rightarrow CH$ be the restriction to $\dot{\mathcal{D}}$ of the embedding $f: D \rightarrow K$. In order to construct the desired homeomorphism $h: \dot{H} \rightarrow CH$, we have to extend the map $j: \dot{A} \xrightarrow{\pi} \dot{\mathcal{D}} \xrightarrow{g} CH$ somehow to a map H :

$$\begin{array}{ccc} j: \dot{A} & \xrightarrow{\pi} & \dot{\mathcal{D}} \xrightarrow{g} CH \\ \cap & & \uparrow \\ H & \xrightarrow{h'} & \end{array},$$

and then to try to approximate it by a homeomorphism. (Let us recall that a map $\gamma: X \rightarrow Y$ of compact metric spaces is said to be *approximable by homeomorphisms* if for every $\varepsilon > 0$ there is a homeomorphism $h: X \rightarrow Y$ with $\text{dist}(\gamma, h) = \sup_{x \in X} \text{dist}_Y(\gamma(x), h(x)) < \varepsilon$).

To realize this plan, we will act as follows. We denote by $\{\text{gaps}\}$ the collection of closures of the components of $CH - g(\dot{\mathcal{D}})$ and form the quotient space $CH/\{\text{gaps}\}$ by declaring each gap to be a point. Then j induces a map of quotient spaces

$$\bar{j}: H / \{\text{holes}\} \rightarrow CH / \{\text{gaps}\},$$

where $\{\text{holes}\} = (D' \times D') \cup \{\text{closure of components of } \mathcal{B}\}$. The map \bar{j} is a homeomorphism over a small neighbourhood of the subspace $\partial^- CH / \{\text{gaps}\}$.

Consider two maps a and b :

$$\begin{array}{ccccccc} \dot{H} & \xrightarrow{\text{proj}} & \dot{H} / \{\text{holes}\} & \xrightarrow{\bar{j}} & CH / \{\text{gaps}\} & \xleftarrow{b=\text{proj}} & CH. \\ \downarrow & & \downarrow & & \uparrow & & \\ & & a & & & & \end{array}$$

Unfortunately there is no hope of approximating these maps by homeomorphisms, since the inverse images $a^{-1}(x)$ and $b^{-1}(x)$ have

non-vanishing fundamental groups. However, one can get rid of this inconvenience. Indeed, suppose there is a set {disks} of disjoint topologically flat disks embedded in A that stick together the fundamental groups of subspaces from {holes}. Also suppose that the set {disks} can be chosen so that its image under $g \circ \pi$ will still consist of disjoint topologically flat disks embedded in CH that stick together the fundamental groups of subspaces from {gaps}. We denote by {holes⁺} the collection of components of {holes} \cup {disks}, and by {gaps⁺} the collection of components of {gaps} \cup {disks}. If the set {disks} with such properties exists, then there are defined maps

$$\dot{H} \xrightarrow{\alpha} CH/\{\text{gaps}^+\} \xleftarrow{\beta} CH,$$

which one can hope to approximate by homeomorphisms.

§6. Freedman's theorem and its corollaries

Thus, in order to reduce the problem of constructing a homeomorphism $h : \dot{H} \rightarrow CH$ to approximation problems, we have to construct the set {disks} of disjoint disks d_k^j embedded in A meeting the conditions listed above. They will be constructed so that the boundaries ∂d_k^j lie in $\text{Fr}(\mathcal{B})$ and establish a one-to-one correspondence with the components of $\overline{\mathcal{B}}$. The subscript k in the disk notation d_k^j corresponds to the k in the decomposition $\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k$. The index j numbers components of \mathcal{B}_k ; thus, \mathcal{B}_k^j are the connected components of the space \mathcal{B}_k .

To construct the disks d_k^j , it is useful to introduce the sets $\overline{\mathcal{B}}_{k-1, k}^j$, which are defined as the connected components of $\overline{X}_{k-1} \cap D' \times S^1 \times$ (the closure of the k -th middle third). Let $\overline{\mathcal{B}}_{k-1, k} = \bigcup_j \overline{\mathcal{B}}_{k-1, k}^j$. The spaces $\overline{\mathcal{B}}_k^j$ and $\overline{\mathcal{B}}_{k-1, k}^{j'}$ are homeomorphic to $S^1 \times D^3$ and any two $\overline{\mathcal{B}}_k^j$ and $\overline{\mathcal{B}}_{k-1, k}^{j'}$ intersect in $D' \times \partial D^2 \times (2l+1)/2^k \subset D' \times$ (collar ∂D^2). These intersections are solid tori, which we denote respectively by b_k^j and $b_{k-1, k}^{j'}$. The solid tori b_k^j and $b_{k-1, k}^{j'}$ are nothing but the intersections of X_k and X_{k-1} with $D' \times \partial D^2 \times (2l+1)/2^k$. Thus $\bigcup_{j < j'} b_k^j \subset S_{k-1, k}^{j'}$ is a regular neighbourhood in the solid torus of some iterated Whitehead link. Since each inclusion $b_k^j \subset b_{k-1, k}^{j'}$ is null-homotopic, there is a collection of immersed disks Δ_k^j with $\partial \Delta_k^j = c_k^j$, where c_k^j is the core of b_k^j .

We can assume that each $\Delta_k^j \cap (\bigcup_{j < j'} b_k^{j'})$ consists of a collar c_k^j on $\partial \Delta_k^j$ and several closed subdisks $\omega_k^{ij} \subset \Delta_k^j$. Let $\delta_k^j = \overline{\Delta_k^j - gc_k^j}$. The set $\{\delta_k^j\}$ will be endowed with a function θ taking values in the radial coordinate $[0, 1]$ of the product $D' \times \partial D^2 \times [0, 1]$; $\theta = \prod_{j, k} \theta_k^j$, where $\theta_k^j : (\delta_k^j, \partial \delta_k^j) \rightarrow ([0, 1], 0)$. After this we define disks d_k^j by letting $d_k^j = \delta_k^j + \theta_k^j(\delta_k^j)$, where $+$ is interpreted as parallel translation in the radial $[0, 1]$ coordinate.

Before constructing θ_k^j we list the properties that the disks d_k^j must satisfy.

(1) Each disk d_k^j is embedded in $(\dot{A} - \bigcup_{l=k}^{\infty} \mathcal{B}_{l, l+1}) \stackrel{\text{def}}{=} \dot{A} - J_k \subset \dot{H}$.

(2) Distinct disks do not intersect, that is, $d_k^j \cap d_{k'}^{j'} = \emptyset$ for $k \neq k'$ and $j \neq j'$.

(3) No disk d_k^j intersects any component ω of \overline{Wh} in more than one point.

(4) No component $\omega \in \overline{Wh}$ intersects more than one disk of the collection $\{d_{k'}^{j'}, k' \leq k, j \text{ arbitrary}\}$.

We will construct θ by induction. The base for the induction is $k = -1$ with $\{d_{-1}^j\} = \emptyset$. Let us assume now that all $\theta_k^{j'}$ have been defined so that $\{d_k^{j'}\}$ satisfy conditions (1)–(4). The function θ_{k+1}^j will be defined as the limit of a certain uniformly convergent sequence $\{\theta_{k+1}^j, j = 0, 1, 2, \dots\}$, all the terms of which are zero on the boundary $\partial\delta_{k+1}^j$.

We fix a particular value of j ; the value of $k+1$ is already fixed. By (1) there is a real number $r > 0$ such that the previously constructed disks $\{d_k^{j'}\}$ do not come within distance r of $\overline{\mathcal{B}}_{k, k+1}$. Let $\overline{\mathcal{B}}_{k, k+1} = \overline{X}_n \cap (D' \times S^1 \times \times \{\text{points distant } \leq r \text{ from the closed } (k+1)\text{st middle third}\})$, that is, $\overline{\mathcal{B}}_{k, k+1}$ is a thickening of $\overline{\mathcal{B}}_{k, k+1}$. We consider the inverse image pattern on

$\prod \delta = \prod_j \delta_{k+1}^j$ (Fig. 12).

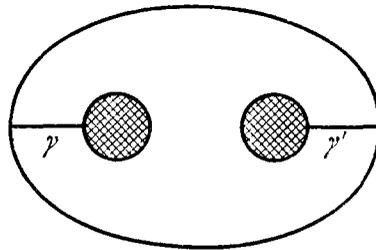


Fig. 12. Arcs γ and γ' are inverse images of double curves of $\prod \delta$; the hatched region is the inverse image of $\prod_j b_k^j$.

Although the situation shown in Fig. 12 is the simplest, it is locally the general case; one can get rid of closed arc-preimages and triple points by a small perturbation. The function $\prod_j \theta_{k+1}^j = \theta_{k+1}$ must be chosen so that the following requirements are met.

(a) If γ, γ' are oriented arcs, paired under the immersion, then θ_{k+1} is increasing on one of them and decreasing on the other.

(b) On hatched disks θ_{k+1} assumes one of two values, $\pm \frac{1}{2} \left(\left(\frac{1}{3} \right)^{k+1} + r' \right)$,

where $0 < r' < \min\left(r, \left(\frac{1}{3}\right)^{k+1}\right)$ is some fixed element of CS^- (the Cantor set with end points deleted).

(c) The values of the function ${}_0\theta_{k+1}$ are in $[-r, r]$.

Let ${}_0d_{k+1} = \delta_{k+1} + {}_0\theta_{k+1}(\delta)$. According to (a), the disks ${}_0d_{k+1}$ are embedded, and by (c) they are disjoint from the previously constructed disks. Condition (b) implies that ${}_0d_{k+1} \subset \dot{A}$ and that ${}_0d_{k+1} \cap J_{k+1} = \emptyset$. For the second assertion we note that the projection

$$\pi: D' \times \partial D^2 \times [0, 1] \rightarrow D' \times \partial D^2 \times \frac{2m+1}{2^{k+1}}$$

carries the subspace

$$L = \left(\bar{X}_{k+1} \cap \left(D' \times \partial D^2 \times \left[\frac{2m}{2^{k+1}}, \frac{2m+2}{2^{k+1}} \right] \right) \right)$$

onto $\prod_j b_{k+1}^j$. To omit $\bar{\mathcal{B}}_{l, l+1}$, $l \geq k+1$, it is only necessary to make sure that ${}_0\theta_{k+1}$ lifts to CS^- level, that is, the intersection $\delta \cap \pi(\bar{\mathcal{B}}_{l, l+1}) \cap \delta \cap \pi L$ —the inverse image of $\prod_j b_{k+1}^j$ (Fig. 12). Unfortunately $\prod_0 d_{k+1}^j$ does not satisfy the conditions (3) and (4).

Let us introduce a sub-induction, assuming that the maps ${}_q\theta$ have been defined so that (I) $\left\{ \prod_j {}_q\theta_{k+1}^j \right\}$ are disjoint embeddings onto $\bar{\mathcal{B}}_{k, k+1} \cap (- J_{k+1})$ and (II) all the intersections of $\prod_j d_{k+1}^j$ with \bar{X}_{q+k+1} are horizontal, that is, have constant radial coordinates.

We will construct the map ${}_{q+1}\theta_{k+1}$ with the corresponding properties. (Below we omit the subscript $k+1$ in the notation ${}_{q+1}\theta_{k+1}$.) Let

$$0 < \varepsilon_{k+1} \leq \min \left\{ \frac{1}{3} \min \{ |v_{k+1}^i - v_{k+1}^j|, i, j \in \{1, \dots, n_{k+1}\} \text{ with } i \neq j \}, \varepsilon_k/2 \right\}.$$

We now change ${}_q\theta$ to ${}_q\theta^{-1}(X_{q+k+1})$ to make ${}_{q+1}\theta_{k+1}$ assume distinct constant values from CS^- on the components of the set ${}_q\theta^{-1}(X_{q+k+2})$ and $\text{sup dist}({}_q\theta, {}_{q+1}\theta) \leq \varepsilon_{k+1}$. Finding the nearby new values is possible, since the Cantor set is a perfect set. Also there is no difficulty in preserving property (I). The set ${}_{q+1}\theta^{-1}(J_{k+1})$ is contained in ${}_q\theta^{-1}(X_{q+k+1}) - {}_q\theta^{-1}(X_{q+k+2})$. Since $\text{dist}({}_q\theta(S), J_{k+1}) > 0$, the number ε_{k+1} can be chosen so that ${}_{q+1}\theta^{-1}(J_{k+1}) = \emptyset$.

Thus by our sub-induction for any $q \geq 0$ we obtain a sequence $\{ {}_q\theta_{k+1}^j \}$ satisfying (I) and (II). Since these sequences are uniformly convergent, there are continuous limiting functions $\{ \theta_{k+1}^j \}$. These functions satisfy (I). The limiting disks of $\{ d_{k+1}^j \}$ are graphs over flat disks and thus flat, although not smooth.

We now return to the main induction. The disks $\{d_{k+1}^j\}$ can be separated into $\overline{\mathcal{D}}_{k+1, k+2}$ and hence satisfy the conditions (1) and (2). Because of this separation, the condition (4) could only fail if the two disks d_{k+1}^j and $d_{k+1}^{j'}$ meet the same component $\omega \in \overline{Wh}$. This effect and any possible violation of (3) are ruled out simultaneously by showing that $\prod_j \theta_{k+1}^j = \theta_{k+1}$ becomes an injection $\theta_{k+1} | \Gamma: \Gamma \rightarrow [0, 1]$ when restricted to the set Γ of points p such that $\{q \theta_{k+1}^j(p), q = 0, 1, 2, \dots\}$ is infinite. The latter remark will suffice, since any point carried from $\delta + \theta(\delta)$ to \overline{Wh}^* must be, by construction, a limit point.

The elements of Γ are indexed by the infinite branches γ of a certain tree S . We have $\theta_{k+1}(\gamma) = \sum v$, where the values of v depend on the choice of a branch γ . If $\gamma \neq \gamma'$, then these sums are different, so that $\theta_{k+1}(\gamma) \neq \theta_{k+1}(\gamma')$.

This completes the main induction step. We now set $\theta = \prod_k \theta_k$ and $d = \prod_{k,j} \delta_k^j + \theta_k(\delta_k^j)$. As a result, we obtain the desired family of disjointly

embedded disks killing the fundamental groups of the subspaces from the set {holes}.

As explained at the end of preceding section, after having constructed the set of embedded disks $\{d_k^j\}$, we get at once the maps

$$\dot{H} \xrightarrow{\alpha} CH/\{\text{gaps}^+\} \xleftarrow{\beta} CH,$$

which are homeomorphisms near the boundary ∂^- . The following approximation theorems hold.

Theorem 6.1 (Edwards and Kirby, see [13]). *Let $\pi : M^n \rightarrow N^n$ be a continuous map between closed topological manifolds of an arbitrary dimension n . Suppose that π is approximable by homeomorphisms and that for some closed set $C \subset N$ the restriction $\pi|_{\pi^{-1}(C)} : \pi^{-1}(C) \rightarrow C$ is already a homeomorphism. Then there is a homeomorphism $h : M^n \rightarrow N^n$ that coincides with π on $\pi^{-1}(C)$ and approximates the map π . ■*

Theorem 6.2 (Edwards, see [24]). *The map $\alpha : \dot{H} \rightarrow CH/\{\text{gaps}^+\}$ is approximable by homeomorphisms. Hence for any Casson handle CH there is a homeomorphism of pairs $CH/\{\text{gaps}^+\} \cong_{\text{Top}} \dot{H}$. ■*

Theorem 6.3 (Freedman [24]). *Let $f : S^n \rightarrow S^n$ be a continuous map of an n -dimensional sphere onto itself. Suppose that for all $\epsilon > 0$ there are only finitely many sets $f^{-1}(\text{pt})$ having diameter greater than ϵ and that the singular set $S(f) = \{x \in S^n : \text{card}(f^{-1}(x)) > 1\}$ is nowhere dense in S^n . Then f is approximable by homeomorphisms. ■*

Now we are in a position to prove the central result of the present survey.

Theorem 6.4 (Freedman [24]). *For any Casson handle CH there is a homeomorphism of pairs $H \cong_{\text{Top}} CH$. ■*

Proof. Let $\dot{\mathcal{D}}$ be the design of the Casson handle CH . The attaching region $\partial^- \dot{\mathcal{D}} = \partial D^2 \times \dot{D}^2$ of the design $\dot{\mathcal{D}}$ has a collar $\partial D^2 \times \dot{D}^2 \times [0, \varepsilon]$. The embedding $g : \dot{\mathcal{D}} \rightarrow CH$ shows that $\partial^- CH$ also has a closed collar $W = \partial D^2 \times \dot{D}^2 \times [0, \varepsilon]$ with $\partial D^2 \times \dot{D}^2 \times 0 = \partial^- CH$ that is disjoint with subspaces from $\{\text{gaps}\}$. By Theorem 5.4, the interior $CH - \partial^- H$ is homeomorphic (actually, diffeomorphic) to \mathbb{R}^4 . By Theorem 6.2, the difference $CH/\{\text{gaps}^+\} - \partial^-(CH/\{\text{gaps}^+\})$ is also homeomorphic to \mathbb{R}^4 . Consider the proper map of pairs $\beta : CH \rightarrow CH/\{\text{gaps}^+\}$. Composing β with the homeomorphism given by Theorem 6.2, we obtain a map of pairs $\bar{\beta} : CH \rightarrow \dot{H}$. Since $W \cap \{\text{gaps}^+\} = \emptyset$, $\bar{\beta}|_W$ is a homeomorphism and $\bar{\beta}^{-1}(\beta(W)) = W$.

We now delete the attaching regions and form one-point compactifications of both image and inverse image. As a result, $\bar{\beta}$ turns into a map $f : S^4 \rightarrow S^4$ between the one-point compactifications of spaces homeomorphic to \mathbb{R}^4 .

We now verify that f satisfies the hypothesis of Theorem 6.3. According to the geometric control theorem, only finitely many subsets from $\{\text{gaps}\}$ have diameter larger than any fixed $\varepsilon > 0$. Further, we have constructed the disks

d_k^j for an arbitrary subset of $\{\text{gaps}\}$ within the blocks $\bar{\mathfrak{B}}_{k-1, k}^j$ whose diameters also tend to zero. Consequently, the diameter of the disks $g \circ \pi(d_k^j) \subset CH$ also tends to zero. Thus, only finitely many subsets from $\{\text{gaps}\}$ have diameter greater than $\varepsilon > 0$.

We now observe that the singular set $S(\beta)$ of β is contained in the singular set $S(\alpha)$ of α , $S(\beta) \subset S(\alpha)$. (Their difference is equal to $\beta(S(\dot{A} - \dot{\mathcal{D}}))$.) We wish to prove that $S(\alpha)$ is nowhere dense. For this purpose we denote by $D^*(f)$ the set $D^*(f) = \bigcup_{x \in S(f)} f^{-1}(x)$ and show that if $p \in \overline{D^*(\alpha)} - D^*(\alpha)$ is a limit point, then there are points q in $\dot{H} - \overline{D^*(\alpha)}$ that are arbitrarily close to p . By the construction of the disks d_k^j , each element $\bar{\mathfrak{B}}_k^j \cup d_k^j \in \{\text{holes}^+\}$ is contained in the block $\bar{\mathfrak{B}}_{k-1, k}^j$. Therefore, any limit point p is approached by a sequence $\bar{\mathfrak{B}}_{k-1, k}^j$ as k tends to infinity. Thus $p \in \overline{Wh^*}$. Since $p \notin D^*(\alpha)$, it will have radial coordinates lying in CS^- , the Cantor set minus end points. But on the level r only a one-dimensional set lies in $\overline{D^*(\alpha)}$. Hence p may be approximated by points $q \in (r = \text{level} - \overline{D^*(\alpha)})$.

Applying Theorem 6.3, we conclude that the map f is approximable by homeomorphisms. Let \hat{C} denote the image under one-point compactification of the half-open collar $C^- = D^2 \times \mathbb{R} \times (0, \varepsilon] \subset CH - \partial^- CH$. Since $f : W \rightarrow f(W)$ is a homeomorphism, $f|_{\hat{C}}$ maps \hat{C} homeomorphically onto its image, that is, $f^{-1}(f(\hat{C})) = \hat{C}$. Hence, setting $f(\hat{C}) = C$, we can apply Theorem 6.1. As a result, we obtain a homeomorphism $h : S^4 \rightarrow S^4$ meeting

the condition $h|\hat{C} = f|\hat{C}$. In particular, $h(\infty) = \infty$, where ∞ is the compactifying point.

We now remove the compactifying points from the spheres S^4 . This results in a homeomorphism $h| : CH - \partial^- CH \rightarrow \dot{H} - \partial^- \dot{H}$, which coincides with f on C^- . Splicing the homeomorphisms $f|C$ and $h|CH - \partial^- CH$ together on C^- , we obtain the required homeomorphism of pairs

$$\bar{h}: f|W \cup h|S^4 - \{\infty\}: (CH, \partial^- H) \rightarrow (\dot{H}, \partial^- \dot{H}). \blacksquare$$

As the first corollary of Freedman's theorem we obtain the following statement.

Theorem 6.5. *Let W^4 be a simply-connected four-dimensional manifold and let $\alpha, \beta \in H_2(W)$ be such that $\alpha \cdot \alpha = \beta \cdot \beta = 0$ and $\alpha \cdot \beta = 1$. Then W^4 is homeomorphic to $W^4 \cong_{\text{Top}} \tilde{W}^4 \# S^2 \times S^2$, where \tilde{W}^4 is simply-connected. \blacksquare*

The meaning of Theorem 6.5 is that in dimension four the topological Whitney trick has begun working. Hence, we obtain by standard means the theorem on five-dimensional topological h -cobordism.

Theorem 6.6. *Let $(W^5; M, M')$ be a simply-connected smooth five-dimensional h -cobordism. Then W is homeomorphic to $M \times [0, 1]$. Moreover, this homeomorphism can be smoothed over the complement to a flat four-dimensional cell in M . \blacksquare*

Theorem 6.6 has a non-compact version, but its proof is quite non-trivial and cumbersome. Here we give only the formulation of it.

Theorem 6.7 (Freedman [24]). *Let $(W; V, V')$ be a simply-connected smooth proper h -cobordism of dimension 5, where W is simply-connected at infinity. If the set $C = \partial W - (V \cup V')$ is non-empty, then we assume also that C already has product structure $C \cong_{\text{Diff}}(C \cap V) \times I$. Then W is homeomorphic to $V \times I$ (extending the product structure on C). \blacksquare*

Theorem 6.8. *Every topological four-dimensional manifold V that is properly homotopically equivalent to \mathbb{R}^4 is homeomorphic to \mathbb{R}^4 .*

Proof. It follows from the smoothing theorem [31], [32] that for any $0 < n < \infty$ a non-compact n -dimensional topological manifold V can be smoothed if and only if the classifying map $f : V \rightarrow B \text{Top}(n)$ of its topological tangent microbundle admits a lifting

$$\begin{array}{ccc} & & B \text{O}(n) \\ & \nearrow f & \downarrow \\ V & & B \text{Top}(n) \\ & \searrow f & \\ & & \end{array}$$

In spite of the fact that for $n = 4$ the structure of the space $B \text{Top}(4)$ is not known, the contractible manifold V can have no obstruction to the lifting.

Thus, every manifold V that is properly homotopically equivalent to \mathbb{R}^4 admits a smoothing V_Σ .

Now we will construct a proper h -cobordism W between V_Σ and \mathbb{R}^4 . To do this, we set $(W; V_\Sigma, \mathbb{R}^4) \cong (V_\Sigma \times [0, 1]) \cup \mathring{B}^4 \times 1$; $V_\Sigma \times 0, B^4 \times 1$, where \mathring{B}^4 is the interior of smooth ball in \mathbb{R}^4 . Applying Theorem 6.7 on proper h -cobordism, we find that $\mathbb{R}^4 \cong_{\text{Top}} V_\Sigma \cong_{\text{Top}} V$, as desired. ■

Theorem 6.9. *Every three-dimensional manifold Σ^3 with integer homology of a three-dimensional sphere is the boundary of a contractible topological four-dimensional manifold Δ^4 .*

Proof. In the paper [23] Freedman constructed a proper smooth embedding of the punctured Poincaré homology sphere $P^- = P - \text{pt}$ to a smooth manifold V that is properly homotopically equivalent to \mathbb{R}^4 . In an exactly similar way it is proved that any punctured three-dimensional homology sphere $\Sigma^- = \Sigma - \text{pt}$ can be smoothly embedded in the same manifold. By Theorem 6.8 any such homology sphere Σ^- can be topologically flatly embedded in \mathbb{R}^4 . A well-known theorem of Kirby and Chernavskii [29], [5] states that, apart from surfaces in three-dimensional manifolds, no topological embedding of codimension 1 can have isolated points at which it is not locally flat. Hence by taking the one-point compactification of the pair (\mathbb{R}^4, Σ^-) we obtain a topologically flat embedding $\Sigma \hookrightarrow S^4$, since it is evidently flat apart from the compactification point.

Further, the embedding $P^- \hookrightarrow V$ from [23] separates V into two contractible pieces. The same is true also for $\Sigma^- \hookrightarrow V \cong_{\text{Top}} \mathbb{R}^4$. Hence, $\Sigma \hookrightarrow S^4$ is the boundary of a topological contractible four-dimensional manifold, as required. ■

§7. Classification of simply-connected topological four-dimensional manifolds

Definition. A manifold M is said to be *almost smooth* if it can be endowed with a smooth structure on a complement to a point.

In 1982 Quinn [40] showed that every simply-connected closed topological four-dimensional manifold is almost smooth. This observation enables us to use arguments from smooth topology while investigating four-dimensional manifolds.

The central result of this section is the following classification theorem.

Theorem 7.1 (Freedman [24]).

(Existence.) For a given integral non-degenerate quadratic form L there is a closed simply-connected topological four-dimensional manifold M that realizes L as the intersection form

$$L: H_2(M) \times H_2(M) \rightarrow \mathbb{Z}.$$

(Uniqueness.) If the form L is even, then any two closed simply-connected topological four-dimensional manifolds M and M' realizing L are homeomorphic.

If L is odd, then there are exactly two non-homeomorphic closed simply-connected topological four-dimensional manifolds M and M' realizing L . One of these manifolds M has vanishing Kirby–Siebenmann obstruction (and hence $M \times S^1$ is smoothable), and the other manifold M' has non-vanishing Kirby–Siebenmann obstruction (in this case $M' \times S^1$ is non-smoothable).

The proof of the classification theorem is based on the following classical result of Milnor.

Theorem 7.2. For every symmetric integral unimodular matrix $A = (m_{ij})$ there is a smooth four-dimensional manifold N^4 with boundary such that

- (i) N^4 is simply-connected;
- (ii) the matrix of the intersection form

$$L_N: H_2(N^4) \times H_2(N^4) \rightarrow \mathbb{Z}$$

is congruent to the matrix A ;

- (iii) the boundary ∂N^4 is a three-dimensional homology-sphere. ■

Proof of the classification Theorem 7.1:

Existence. Once a basis of two-dimensional homology is chosen, the form L determines a symmetric integral unimodular matrix A_L . Applying Theorem 7.2, we can associate with this matrix a smooth simply-connected four-dimensional manifold N with boundary ∂N equal to the three-dimensional homology sphere $\Sigma = \partial N$. By Theorem 6.9, $\Sigma = \partial \Delta^4$, where Δ is a compact contractible topological manifold. We set $M = N \cup_{\Sigma} \Delta^4$. Easy calculations based on the theorems of Van Kampen and Mayer-Vietoris establish that M is simply-connected and has L as the intersection form. Finally, the inclusion $N \hookrightarrow M - x$, where x is a point of Δ^4 , is a homotopy equivalence. Since M is smooth, there is a lifting of the classifying map for the topological tangent bundle of N :

$$\begin{array}{ccc} & \xrightarrow{\tau'_N} & BO(4) \\ N & \xrightarrow{\tau_N} & B\text{Top}(4) \end{array}$$

Since $N \hookrightarrow M - x$ is a homotopy equivalence, the above lifting extends to

$$\begin{array}{ccc} & \xrightarrow{\tau'_{M-x}} & BO(4) \\ M - x & \xrightarrow{\tau_{M-x}} & B\text{Top}(4), \end{array}$$

that is, $M - x$ can be given a smooth structure. Thus, beginning with a symmetric integral unimodular form L , we have constructed an almost-smooth manifold M with $L_M = L$.

Uniqueness. Let M and M' be closed simply-connected topological four-dimensional manifolds, both having the same intersection form L . According to Quinn they are almost smooth. Then by Theorem 2.1 there is a homotopy equivalence $f: M \rightarrow M'$. (The fact that M and M' are almost smooth rather than smooth manifolds does not affect the proof of Theorem 2.1.) Compare the stable normal bundles of M and M' . Their difference is an element of $[M, G/Top]$. Consider the fibration

$$G/PL \rightarrow G/Top \rightarrow B(Top/PL).$$

The obstruction to lifting an element from $[M, G/Top]$ to $[M, G/PL]$ is nothing but the Kirby–Siebenmann obstruction (M) , which lies in $[M, B(Top/PL)] \cong [M, K(\mathbb{Z}/2, 4)] \cong H^4(M; \mathbb{Z}/2) \cong \mathbb{Z}/2$. If this obstruction vanishes, then arguments similar to those used in the proof of the Novikov–Wall Theorem 2.2 show that f is topologically h -cobordant to the identity map id_M . Let $W = \overline{W}$ -arc, where arc is a flat arc joining non-smoothable points of M and M' . It is easily seen that $(W; M' - pt, M - pt)$ is a proper simply-connected topological h -cobordism, simply-connected at infinity. It has a smooth structure at each end, and the only obstruction to extension of the structure lies in the zero group

$$H^4(W, \partial W; \mathbb{Z}/2) \cong H^3(M - pt) \cong H^3(M) \cong H_1(M) \cong 0.$$

Therefore the smooth structure extends to the whole of W . Applying Theorem 6.7 to the proper h -cobordism $(W; M' - pt, M - pt)$, we obtain a homeomorphism $(M' - pt) \cong_{Top} (M - pt)$, which extends to a homeomorphism between the one-point compactifications $M' \cong_{Top} M$.

In the case where L is even, M and M' are spinor manifolds. According to [43] the obstruction $\kappa(M)$ for a closed spinor four-dimensional manifold M equals $\kappa(M) = \frac{1}{8} \sigma(M) \pmod{2}$. Thus, for a class $[f] \in [M, B(Top/PL)]$ we have $[f] = \frac{1}{8} [\sigma(M) - \sigma(M')] \pmod{2} = 0$. This explains why in the even case there is only one manifold.

In the case where L is of type I and indefinite, by Serre's Theorem 1.2 it is isomorphic to $nI \oplus m(-I)$. One of the manifolds realizing this form is a smooth manifold $N = (\#_n \mathbb{C}P^2) \#_m (\#_m - \mathbb{C}P^2)$. Hence it is sufficient to construct a non-smooth manifold realizing L . For if such a manifold is constructed, then there are at least two classes of homeomorphic manifolds realizing L . There cannot be more than two classes, because for any homotopy equivalence $f: M' \rightarrow M''$ the element $\langle f \rangle \in [M'', B(G/PL)]$ is non-zero only if one manifold has zero and the other non-zero Kirby–Siebenmann obstruction.

In order to construct a non-smooth manifold realizing $nI \oplus m(-I)$ one has to replace in N one $\mathbb{C}P^2$ smooth connected summand by a non-smooth manifold of the same homotopy type. For this purpose we consider in the

three-dimensional sphere S^3 that bounds the four-ball B^4 a torus knot of type $(2, 3)$ (trifolium) with framing $+1$ and attach along it a standard 2-handle H^2 to B^4 . According to [33], $\partial(B^4 \cup H^2)$ is a three-dimensional Poincaré homology sphere P^3 . Pasting it with the help of Theorem 6.9 to the contractible topological manifold Δ^4 , we obtain a non-smoothable closed four-dimensional manifold Ch^4 having the homotopy type of $\mathbb{C}P^2$.

It remains to consider odd definite intersection forms L only. The preceding considerations reduce the realization problem for L to the construction of two framed links ω_0 and ω_1 with $R(\partial K(\omega_0)) = 0$, $R(\partial K(\omega_1)) = 1$, both with matrix representing L . Here $K(\omega_i)$ is the manifold with boundary obtained by attaching 2-handles to the four-ball relative to ω_i , and $R \in \mathbb{Z}/2$ is the Rokhlin invariant [33]. Pasting the boundaries of $K(\omega_i)$ to contractible topological manifolds, we obtain two almost smooth simply-connected manifolds with zero and non-zero Kirby–Siebenmann obstruction, as required. ■

We single out a special case of Theorem 7.1 as an assertion.

Theorem 7.3 (the topological four-dimensional Poincaré conjecture). *If Σ^4 is a topological four-dimensional manifold homotopy equivalent to the standard sphere S^4 , then Σ^4 is homeomorphic to S^4 .* ■

Another important special case of Theorem 7.1 is the following.

Theorem 7.4. *There is a closed simply-connected almost parallelizable, almost smooth four-dimensional manifold Rh^4 with intersection matrix E_8 .* ■

Corollary 7.5. *Either the manifold Rh^4 is not homeomorphic to a polyhedron or the three-dimensional Poincaré conjecture is false.*

Proof. Suppose Rh^4 is a simplicial polyhedron. By Rokhlin's Theorem 1.3 Rh^4 fails to be a PL -manifold, hence there is at least one vertex whose link lk is not a combinatorial triangulation of the standard sphere S^3 . At the same time the link lk is a three-dimensional homology manifold with the homotopy type of S^3 . But all three-dimensional homology manifolds are manifolds; hence, lk is a three-dimensional homotopy sphere. If $\text{lk} \cong_{\text{Top}} S^3$, then Moise's theorem [37] tells us that $\text{lk} \cong_{PL} S^3$. So the assumption that Rh^4 is polyhedral leads to the conclusion that some link lk is a three-dimensional homotopy sphere not homeomorphic to S^3 . ■

CHAPTER II

GEOMETRIC METHODS

In this chapter we consider smooth four-dimensional manifolds. The approach used by Freedman does not enable us to understand what kinds of integral unimodular forms can be realized as intersection forms of smooth

four-dimensional manifolds. Very strong and unexpected results in that direction were obtained in 1982 by Donaldson. He proved that if the intersection form L_X of a smooth simply-connected four-dimensional manifold is positive definite, then it reduces over \mathbb{Z} to a sum of squares. Perhaps the most astonishing thing about Donaldson's work is that a theorem of topological nature is proved by involving ideas and methods from differential geometry and mathematical physics—the gauge field theory.

§1. Gauge fields and Donaldson's theorem

Let G be a Lie group with Lie algebra $\text{Alg}(G)$, let X be a smooth manifold, and (P, π, X, G) a principal G -bundle over X . Suppose that the group G acts on the manifold P on the left and let $L_g : P \rightarrow P$ ($g \in G$) denote this action.

Definition 1.1. A connection in the principal bundle (P, π, X, G) is a 1-form $A : TP \rightarrow \text{Alg}(G)$ on P with values in the Lie algebra $\text{Alg}(G)$ satisfying the following conditions:

(i) $L_g^* A = \text{Ad}(g) \circ A$;

(ii) if $Y \in \text{Alg}(G)$ is an element of the Lie algebra $\text{Alg}(G)$ and \tilde{Y} is a vector field on P given by the formula

$$\tilde{Y}(p) = \left. \frac{d}{dt} L_{\exp(tY)}(p) \right|_{t=0}, \quad p \in P,$$

then

$$A(\tilde{Y}) = Y.$$

The connection A determines a decomposition $TP = T_vP \oplus T_hP$ of the tangent space to P into vertical and horizontal vectors if we put $T_hP = \ker A$.

Let pr_h denote the projection of TP onto the horizontal subbundle T_hP . The form

$$F(A)(t_1, t_2) = dA(\text{pr}_h(t_1), \text{pr}_h(t_2)), \quad t_1, t_2 \in TP,$$

is called the *curvature of the connection* A .

We now consider the adjoint representation $\text{Ad} : G \rightarrow GL(\text{Alg}(G))$ of the group G in the vector space $\text{Alg}(G)$ and denote by

$$\mathfrak{g} = P \times_{\text{Ad}} \text{Alg}(G)$$

the vector bundle over X associated with this representation. Then $F(A)$ can be viewed as a form in $\Omega^2(\mathfrak{g}) = \Gamma(\Lambda^2(X)) \otimes \Gamma(\mathfrak{g})$. Indeed, let $x \in X$. We choose a basis $t^i \in T_x(X)$ of the tangent space at the point x and denote by $e_i \in T_x^*(X)$ the dual basis of the cotangent space. We now take a point $p \in P$ meeting the condition $\pi(p) = x$ and choose horizontal vectors $\tilde{t}^i \in T_p(P)$ for which $\pi_*(\tilde{t}^i) = t^i$. We put

$$F(A)(x) = \sum_{i,j} F(A)(\tilde{t}^i, \tilde{t}^j) \otimes (e_i \wedge e_j).$$

It is not difficult to see that $F(A)(x)$ is well defined and is a 2-form on X with values in \mathfrak{g} .

Remark. The curvature form $F(A) \in \Omega^2(\mathfrak{g})$ is often called a *gauge field*, and the corresponding connection A a *gauge potential*.

A *morphism of a principal bundle* P is a diffeomorphism $f: P \rightarrow P$ compatible with the action of G , that is, commuting with the map L_g :

$$f \circ L_g = L_g \circ f.$$

This diffeomorphism sends fibres of the principal map P to fibres and in this way induces a diffeomorphism $\bar{f}: X \rightarrow X$ of the base. If $\bar{f} = \text{id}_X$, then f is called a *gauge transformation*.

The following statements are easy to verify.

Proposition 1.1. *Let $f: P \rightarrow P$ be a gauge transformation and A a connection. Then f^*A is a connection as well.*

Proposition 1.2. *Let A_0 be a fixed connection in a principal bundle (P, π, X, G) . Then an arbitrary connection A in this bundle can be represented as $A = A_0 + h$, where $h \in \Omega^1(\mathfrak{g}) = \Gamma(\Delta^1(X)) \otimes \Gamma(\mathfrak{g})$. In other words, the set \mathcal{A} of all connections in the bundle P bears an affine space structure.*

According to Proposition 1.2, the set of gauge transformations forms a group \mathcal{G} acting on the affine space of connections \mathcal{A} .

Let E be the vector bundle associated with the principal bundle P . Then the connection A determines a first order differential operator

$$d_A: \Omega^0(E) \rightarrow \Omega^1(E),$$

called the *covariant derivative*. If U_α is a trivializing neighbourhood in X (that is, $E|_{U_\alpha} \simeq U_\alpha \times V$) and A_α is the restriction of the form A to U_α (that is, $A_\alpha: U_\alpha \rightarrow T^*(U_\alpha) \otimes \text{Alg}(G)$), then the covariant derivative is given by the operator $d + A_\alpha$. The operator d_A extends to an operator

$$d_A: \Omega^p(E) \rightarrow \Omega^{p+1}(E), \quad \alpha \otimes s \mapsto d\alpha \oplus s + \alpha \wedge d_A s,$$

where $\alpha \in \Gamma(\Lambda^p(X))$, $s \in \Gamma(E)$.

Now let X be an oriented Riemannian four-dimensional manifold and $d_A: \Omega^p(E) \rightarrow \Omega^{p+1}(E)$ the covariant derivative operator. The metric on X determines a formally adjoint operator $d_A^* = *d_A*$: $\Omega^{p+1}(E) \rightarrow \Omega^p(E)$, where $*$ is the Hodge operator. For all connections $A \in \mathcal{A}$ the so-called *Bianchi identity* is valid: $d_A F(A) = 0$. The equation $d_A^* F(A) = 0$ (which is far from being valid for all connections) is called the *Yang-Mills equation*.

A connection A on P is called *self-dual* if $F(A) = *F(A)$, and *anti-self-dual* if $F(A) = -*F(A)$. In both these cases

$$d_A^* F(A) = *d_A * F(A) = \pm *d_A F(A) = 0$$

by virtue of the Bianchi identity. Hence (anti-)self-dual connections automatically satisfy the Yang-Mills equations.

The Yang-Mills equations describe critical points of the Yang-Mills (action) functional

$$YM(A) = \|F(A)\|_{L^2}^2 = \int_X |F(A)|^2 d\mu.$$

Self-dual connections provide absolute minima of this functional. If $G = SU(2)$, then $YM(A) = -8\pi^2 c_2(P)$, where $c_2(P)$ is the second Chern class of the associated vector bundle of rank 2.

The existence of self-dual connections was proved under rather general conditions by Taubes [48]. Let X be a compact oriented Riemannian four-dimensional manifold with a positive definite intersection form L_X , and let P be a principal $SU(2)$ -bundle over X with $c_2(P) \leq 0$. Then P admits an irreducible self-dual connection.

As an example of self-dual connection, consider $X = \mathbb{R}^4$ and $G = SU(2)$. Then in terms of quaternionic coordinates $x \in \mathbb{H} \cong \mathbb{R}^4$, making use of the identification $SU(2) \cong \text{Im } \mathbb{H}$, one can show that a 1-instanton solution of self-dual equations is given by the formula [11]

$$A_\lambda = \text{Im} \left(\frac{x d\bar{x}}{\lambda^2 + |x|^2} \right) \text{ with } F(A_\lambda) = \frac{\lambda^2 dx \wedge d\bar{x}}{(\lambda^2 + |x|^2)^2}$$

with action $8\pi^2$.

Theorem 1.3. *Let A be a self-dual $SU(2)$ -connection on \mathbb{R}^4 with action $8\pi^2$. Then up to a gauge transformation and a parallel transport on \mathbb{R}^4 , the connection A equals A_λ for some $\lambda \in \mathbb{R}$.*

The Yang-Mills functional and equations are invariant under a conformal change of metric on X and under gauge transformations on P . The set of classes of solutions of the Yang-Mills equations with respect to gauge invariance is called the *moduli space* and denoted by \mathcal{M} . Investigation of the topological structure of the moduli space led Donaldson to establishing remarkable links between the geometry of gauge theories and the topology of smooth four-dimensional manifolds.

Theorem 1.4 (Donaldson [14]). *Let X be a compact smooth simply-connected oriented four-dimensional manifold with a positive definite intersection form Q on $H^2(X; \mathbb{Z})$. Then there is an integral basis in $H^2(X; \mathbb{Z})$ such that*

$$Q(u, u) = u_1^2 + u_2^2 + \dots + u_r^2, \quad r = \text{rk } H^2(X; \mathbb{Z}).$$

Proof of Theorem 1.4. Let $r = \text{rk } H^2(X; \mathbb{Z})$ and

$$2n = \# \{u \in H^2(X; \mathbb{Z}) : Q(u, u) = 1\}.$$

The proof consists in constructing an oriented cobordism between X and n copies of the complex projective plane $\mathbb{C}P^2$. (Section 2 is devoted to this construction.) Let p of these $\mathbb{C}P^2$ carry canonical orientation given by the complex structure, and $q = n - p$ the opposite orientation. Then by virtue of the invariance of signature relative to the cobordism we have

$$r = \sigma(X) = (p - q) \cdot \sigma(\mathbb{C}P^2) = p - q \leq n.$$

Let $\{\pm x_1, \pm x_2, \dots, \pm x_n\} = \{u \in H^2(X; \mathbb{Z}) : Q(u, u) = 1\}$. Then $Q(x_i, x_j) \in \mathbb{Z}$ and by virtue of the Cauchy–Buniakowski inequality $|Q(x_i, x_j)| < 1$ for $i \neq j$. Hence, $Q(x_i, x_j) = 0$ for $i \neq j$, that is, $\{x_1, \dots, x_n\}$ are orthogonal. Hence it follows that $n \leq r$. It follows from the inequalities $r \leq n$ and $n \leq r$ that $n = r$ and $\{x_1, \dots, x_n\}$ is an orthonormal basis for $H^2(X; \mathbb{Z})$. Thus, for $u \in H^2(X; \mathbb{Z})$

$$u = \sum_{i=1}^n Q(u, x_i) x_i = \sum_{i=1}^n u_i x_i,$$

where $u_i \in \mathbb{Z}$ and $\{x_1, \dots, x_n\}$ is a basis for $H^2(X; \mathbb{Z})$. Hence,

$$Q(u, u) = Q\left(\sum_{i=1}^n u_i x_i, \sum_{j=1}^n u_j x_j\right) = \sum_{i,j=1}^n u_i u_j Q(x_i, x_j) = \sum_{i,j=1}^n u_i u_j \delta_{ij} = \sum_{i=1}^n u_i^2,$$

as required. ■

§2. Construction of cobordism

The cobordism needed for the proof of Donaldson's theorem results from the moduli space of self-dual connections.

Let X be a compact smooth simply-connected four-dimensional manifold with positive definite intersection form. Let us fix a Riemannian metric on X and denote by P the principal $SU(2)$ -bundle on X with $c_2(P) = -1$. Using the covariant derivative of a fixed smooth connection A_0 in the bundle P , we can define Sobolev spaces $L_n^p(V)$ of sections of an arbitrary associated vector bundle V . We denote by \mathcal{A} the affine space of connections on P . If a connection A_0 is chosen, then any element $A \in \mathcal{A}$ is of the form $A = A_0 + h$, where $h \in \Omega^1(\mathfrak{g})$. Therefore, when considering sections $h \in L_3^2(\Omega^1(\mathfrak{g}))$ we endow the space of connections with a topology. Further, the group of gauge transformations \mathcal{G} is isomorphic to the groups of sections $\Gamma(P \times_{\text{Ad}} SU(2))$. By taking L_4^2 -sections, we obtain a Banach–Lie group of gauge transformations \mathcal{G} , which acts smoothly on the affine space \mathcal{A} according to the formula

$$g(A) = A - (d_A g)g^{-1}.$$

We denote by \mathcal{B} the quotient space \mathcal{A}/\mathcal{G} ; let $p : \mathcal{A} \rightarrow \mathcal{B}$ be the natural projection and $p(A) = [A]$.

We recall that a connection on a bundle P is called *reducible* if its holonomy group is a proper subgroup of $SU(2)$. Since X is simply-connected

and P is non-trivial, the only possible reduction is the reduction on the subgroup $U(1) \subset SU(2)$. We denote by $\Gamma_A \subset \mathcal{G}$ the subgroup of covariantly constant sections corresponding to the connection A . Then A is reducible if and only if $\Gamma_A \cong U(1)$. Equivalence classes of irreducible connections form an open subset $\mathcal{B}^* \subset \mathcal{B}$. Using the implicit function theorem for Banach spaces, it is not difficult to obtain the following statement.

Theorem 2.1.

- (1) The set \mathcal{B} is a Hausdorff space.
- (2) \mathcal{B}^* is a Banach manifold whose charts are the slices

$$T_{A, \varepsilon} = \{A + a \mid d_A^* a = 0, \|a\|_{L^2_3} < \varepsilon\}$$

of the action of the gauge group \mathcal{G} .

- (3) The map $p: p^{-1}(\mathcal{B}^*) \rightarrow \mathcal{B}^*$ is a principal $\mathcal{G}/\{\pm 1\}$ -bundle.
- (4) If the connection A is reducible, then Γ_A acts on $T_{A, \varepsilon}$ and the map $T_{A, \varepsilon}/\Gamma_A \rightarrow \mathcal{B}$ is a homeomorphism onto a neighbourhood of an element $[A] \in \mathcal{B}$ that is smooth outside the set of fixed points.

We now denote by \mathcal{M} the subspace of equivalence classes of self-dual connections on P and call it the moduli space. If $A \in \mathcal{B}$ is reducible to a connection in the principal $U(1)$ -bundle $Q \subset P$, then, because $\pi_1(X) = 0$, the equivalence class $[A]$ is completely determined by the curvature $F(A) \in \Omega^2$. If A is self-dual, then $F(A)$ is a self-dual closed 2-form. In other words,

$$dF(A) = 0 \text{ and } d^*F(A) = *d*F(A) = 0,$$

because $*F(A) = F(A)$. Hence, the form $F(A)$ is harmonic and by virtue of Hodge's theorem it is determined by its cohomology class $2\pi ic_1(Q)$. The reduction to $U(1)$ is well defined modulo the Weyl group; consequently, the element $[A] \in \mathcal{M}$ is described by the cohomology classes $\pm c_1(Q)$. Since $c_2(P) = -c_1(Q)^2 = -1$, there are exactly n distinct points in the space \mathcal{M} that represent self-dual connections, where

$$n = \frac{1}{2} \# \{u \in H^2(X; \mathbb{Z}) \mid L_X(u, u) = 1\}.$$

According to Taubes' theorem, there are also irreducible connections on X .

We now consider an elliptic complex

$$(1) \quad \Omega^0(\mathfrak{g}) \xrightarrow{d_A} \Omega^1(\mathfrak{g}) \xrightarrow{d_A^-} \Omega^2_-(\mathfrak{g}),$$

and let H^p_A , $0 \leq p \leq 2$, be the cohomology groups of this complex. The Atiyah–Singer index theorem implies that the following equality holds [8]:

$$-\sum_{p=0}^2 (-1)^p \dim H^p_A = 8 |c_2(P)| - \frac{\dim R}{2} (\chi(X) - \sigma(X)).$$

In particular, for $G = SU(2)$ we have $\dim G = 3$. In addition, if X is a simply-connected manifold with positive intersection form, then $\chi(X) = 2 + \sigma(X)$, that is, $\chi(X) - \sigma(X) = 2$. Hence it follows that in the case under consideration

$$-\sum_{p=0}^2 (-1)^p \dim H_A^p = 5.$$

Theorem 2.2. *Let A be a self-dual connection in P . Then there exist a neighbourhood U of $0 \in H_A^1$ and a smooth map $\varphi : U \rightarrow H_A^2$ such that:*

(1) *if A is irreducible, then a neighbourhood of the point $[A] \in \mathcal{M}$ is diffeomorphic to the set $\varphi^{-1}(0) \subset H_A^1$;*

(2) *if A is reducible, then a neighbourhood of the point $[A] \in \mathcal{M}$ is diffeomorphic to $\varphi^{-1}(0)/\Gamma_A$.*

The proof of this theorem is based on ideas applied by Kuranishi to the investigation of moduli of complex structures. Let

$$\Phi: \mathcal{A} \rightarrow L_2^2(\Omega_-^2(\mathfrak{g}))$$

be a map given by

$$\Phi(A + a) = F_-(A + a) = d_A^- a + \frac{1}{2} [a, a].$$

It is easy to see that the connection $A + a$ is self-dual if and only if

$$\Phi(A + a) = 0 \in L_2^2(\Omega_-^2(\mathfrak{g})).$$

Consider the differential $D\Phi$ of the above map at the point A . Being restricted to the slice $T_{A, \mathfrak{E}}$, this differential is a Fredholm operator

$$d_A^-: \ker d_A^* (\subset L_3^2(\Omega^1(\mathfrak{g}))) \rightarrow L_2^2(\Omega_-^2(\mathfrak{g})),$$

and therefore Φ is a Fredholm map [46]. After being composed with a local diffeomorphism, Φ can be represented in the form

$$\Phi(x) = (D\Phi)_{Ax} + \varphi(x).$$

Now Theorem 2.2 is obtained by applying to Φ the standard technique of non-linear analysis (for details, see [22]).

From Theorems 2.1 and 2.2 an important corollary follows.

Corollary. *Let A be an irreducible connection (which means that $H_A^0 = 0$), and in addition let $H_A^2 = 0$. Then the moduli space \mathcal{M} is a smooth five-dimensional manifold in a neighbourhood of $[A]$. If A is a reducible connection, the group Γ_A acts on H_A^1 by multiplication by a unitary complex number; therefore, if $H_A^2 = 0$, then by the index theorem $H_A^1/\Gamma_A \simeq \mathbb{C}^3 + S^1$ and $\dim H_A^0 = \dim \Gamma_A = 1$.*

Thus, we have shown that at "good" points the moduli space \mathcal{M} is a smooth five-dimensional manifold. A description of the global structure of \mathcal{M} is based on the following key assertion.

Theorem 2.3. Let $\tilde{A}_i \in \mathcal{A}$ be a sequence of self-dual connections on a bundle P . Then we can choose a subsequence of \tilde{A}_i such that one of the following two conditions holds:

- (1) the connections \tilde{A}_i are gauge equivalent to connections $A_i \in \mathcal{A}$ converging in the C^∞ -topology to a self-dual connection A_∞ ; hence, $[\tilde{A}_i] \rightarrow [A_\infty] \in \mathcal{M}$;
- (2) there are points $x \in X$ and trivializations ρ_i of the bundle $P|_K$ on the complement K to an arbitrary geodesic ball centered at x such that $\rho_i^* \tilde{A}_i \rightarrow \theta$ in $C^\infty(K)$, where θ is a trivial flat connection.

The proof of this theorem involves the following two lemmas.

Lemma 2.4. Let $L, C > 0$ and let $\{f_i\}$ be a sequence of integrable functions on X with $f_i \geq 0$ and $\int_x f_i d\mu \leq L$. Then there exist a subsequence of $\{f_i\}$, a finite point set $\{x_1, \dots, x_l\} \subset X$, and a countable family $\{B_\alpha\}$ of geodesic balls of X such the balls of half-radii cover $X \setminus \{x_1, \dots, x_l\}$ and for each α one has $\int_{B_\alpha} f_i d\mu < C$.

Lemma 2.5. Let h_i be a sequence of metrics in the ball B^4 close enough to the Euclidean metric that converges to a metric h_∞ in $C^\infty(\bar{B}^4)$. Also, let \tilde{A}_i be a sequence of connections on a trivial bundle on B^4 that are self-dual with respect to the metrics h_i . Then there is a constant C depending neither on h_i nor on \tilde{A}_i such that if

$$\int_{B^4} |F(\tilde{A}_i)|^2 d\mu \leq C,$$

then there is a subsequence in $\{\tilde{A}_i\}$ such that connections A_i gauge equivalent to \tilde{A}_i converge in $C^\infty(\frac{1}{2}\bar{B}^4)$ to a connection A_∞ that is self-dual with respect to h_∞ .

The reader will find proofs of both above assertions in the book [22] (see also [52]).

To obtain Theorem 2.3, consider on a geodesic ball $B \subset X$ of radius r a geodesic coordinate system χ . Thus, χ determines a diffeomorphism $\chi : B_r^4 \rightarrow B$ from a Euclidean ball of radius r to B . By taking the inverse image of the metric h and continuing it over the Euclidean unit ball by means of homothety, we obtain a metric

$$h_r = \chi^* h(rx) = r^2 (\delta_{ij} + r^2 O(|y|^2)) dy_i dy_j.$$

We choose r so small as to make the metric $r^{-2} h_r$ on B^4 satisfy the conditions of Lemma 2.5. By virtue of conformal invariance, every connection \tilde{A}_i is self-dual with respect to the metric h_r .

We now take the constant C in Lemma 2.4 as in Lemma 2.5 and put

$$f_i = |F(\tilde{A}_i)|^2, L = 8\pi^2. \text{ Then, by virtue of Lemma 2.5, on every ball } \frac{1}{2} \bar{B}_\alpha$$

some sequence converges (after gauge transformations) to $A_\infty(\alpha)$. Using the Cantor diagonal process, one can achieve convergence simultaneously for all numbers α .

The gauge transformations participating in this process determine connections $A_i(\alpha) \rightarrow A_\infty(\alpha)$ in $C^\infty\left(\frac{1}{2}B_\alpha\right)$ and transition functions $g_i(\alpha, \beta): \frac{1}{2}B_\alpha \cap \frac{1}{2}B_\beta \rightarrow SU(2)$ satisfying the relation

$$(2) \quad A_i(\alpha) = -dg_i(\alpha, \beta) g_i(\alpha, \beta)^{-1} + g_i(\alpha, \beta) A_i(\beta) g_i(\alpha, \beta)^{-1}.$$

The compactness of $SU(2)$ provides a uniform bound for dg_i in the formula (2), and hence one can select from the sequence a uniformly convergent subsequence. Repeated application of the formula (2) provides convergence in the C^∞ -topology. Using the Cantor diagonal process, we obtain a subsequence $(A_i(\alpha), g_i(\alpha, \beta))$, which converges to $(A_\infty(\alpha), g_\infty(\alpha, \beta))$ simultaneously for all (α, β) . This determines a self-dual connection on the bundle Q over $X \setminus \{x_1, \dots, x_l\}$. Moreover, if a set $K \subset X \setminus \{x_1, \dots, x_l\}$ is compact, then by induction on the number of balls covering K one can construct isomorphisms $\rho_i: Q|_K \rightarrow P|_K$ such that $\rho_i^*: A_i \rightarrow A_\infty$ in $C^\infty(K)$ (see [52]).

We now consider punctured balls B'_j centered at points x_j ($1 \leq j \leq l$) of sufficiently small radii. Since

$$\int_{B'_j} |F(\bar{A}_i)|^2 d\mu \leq 8\pi^2,$$

by Fatou's lemma

$$\int_{B'_j} |F(A_\infty)|^2 d\mu \leq 8\pi^2.$$

Hence, by Uhlenbeck's theorem on removable singularity [52] both the connection A_∞ and the bundle Q can be extended to the whole manifold X . By definition of the points x_j

$$\lim \int_{B'_j} |F(\bar{A}_i)|^2 d\mu > \frac{c}{2}$$

for all balls B'_j . Hence, for a sufficiently small ball,

$$\int_{B'_j} |F(A_\infty)|^2 d\mu < \lim \int_{B'_j} |F(\bar{A}_i)|^2 d\mu.$$

On the other hand, since all the connections considered are self-dual, the integrands are nothing but Chern forms. Hence they can be calculated modulo $8\pi^2\mathbb{Z}$ with the help of integrals over the boundary (the so-called

Chern–Simons invariants). Therefore, uniform convergence on the boundary ∂B_j implies that

$$\int_{B_j} |F(A_\infty)|^2 d\mu = \lim \int_{B_j} |F(\tilde{A}_i)|^2 d\mu \pmod{8\pi^2\mathbb{Z}}.$$

But since

$$\int_{B_j} |F(A_\infty)|^2 d\mu > 0 \quad \text{and} \quad \int_{B_j} |F(\tilde{A}_i)|^2 d\mu \leq 8\pi^2,$$

we get only two possibilities:

(a) $l = 0,$

(b) $\lim \int_{B_j} |F(\tilde{A}_i)|^2 d\mu = 8\pi^2 \quad \text{and} \quad \int_X |F(A_\infty)|^2 d\mu < 8\pi^2,$

hence it follows that Q is a trivial bundle and A_∞ is a flat connection. Thus, Theorem 2.3 has been proved completely.

Let us make an important remark. Theorem 2.3 demonstrates that degenerate self-dual connections on a bundle P are those connections whose curvatures are concentrated in a neighbourhood of some point.

We now show that one can attach a boundary to the manifold \mathcal{M} . Let $\beta: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function approximating the characteristic function $\chi_{[-1,1]}$, and let

$$R_A(x, s) = \int_X \beta\left(\frac{d(x, y)}{s}\right) \cdot |F(A)|^2 d\mu_y,$$

where $d(x, y)$ is the geodesic distance on X . We put

$$(3) \quad \lambda(A) = \frac{1}{K} \min \{s \mid \exists x \in X: R_A(x, s) = 4\pi^2\},$$

where the number K is chosen so that $\lambda(A_1) = 1$ for an instanton A_1 . This function is introduced as a measure of the curvature concentration: if we substitute $\chi_{[-1,1]}$ for β , then $\lambda(A)$ becomes the radius of the smallest ball containing half of the action. In any case, the ball of radius $\lambda(A)$ contains more than half of the action, and hence any sequence $[A_i] \in \mathcal{M}$ without a convergent subsequence must satisfy the condition $\lambda(A_i) \rightarrow 0$ by Theorem 2.3. Thus, the function λ measures distance from the boundary.

Theorem 2.6. *There is a constant $\lambda_0 > 0$ such that if a self-dual connection A on a bundle P meets the condition $\lambda(A) < \lambda_0$, then the minimum of the function (3) is reached at a single point $x(A) \in X$.*

Proof. We take a small geodesic ball of radius r centred at a point x , providing a minimum for A , and transfer from it both metric and connection to a Euclidean ball of radius $r/\lambda(A)$. For every sequence A_i of connections with $\lambda(A_i) \rightarrow 0$ the sequence \tilde{A}_i of inverse images satisfies by construction the

condition $\lambda(\tilde{A}_i) = 1$. Hence by Theorem 2.3 and Lemma 2.5 one can choose from \tilde{A}_i a subsequence that converges to a self-dual connection on \mathbb{R}^4 . It follows from the classification presented above (Theorem 1.3) that the limit connection is an instanton A_1 . Since $\lambda(\tilde{A}_i) = 1$, it follows from Theorem 2.3 that every subsequence of \tilde{A}_i is convergent, and because the limit is unique, $\tilde{A}_i \rightarrow A_1$. The function R_{A_1} has a unique non-degenerate minimum. Then the same is true for $R_{\tilde{A}}$ if $\lambda(A)$ is small enough. But every two minima for A should be located at a distance not exceeding $2\lambda(A)$, because balls of radius $\lambda(A)$ centred at any of the minima contain more than half of the action. Hence, the uniqueness of the minimum for $R_{\tilde{A}}$ yields the uniqueness of the minimum for R_A as well. The theorem is proved. ■

We now put

$$\mathcal{M}_{\lambda_0} = \{[A] \in \mathcal{M} \mid \lambda(A) < \lambda_0\}$$

and define a map

$$p: \mathcal{M}_{\lambda_0} \rightarrow X \times (0, \lambda_0),$$

specifying it by the formula

$$p(A) = (x(A), \lambda(A)).$$

Theorem 2.7.

- (1) *The space $\mathcal{M} \setminus \mathcal{M}_{\lambda_0}$ is compact.*
- (2) *\mathcal{M}_{λ_0} is a smooth manifold.*
- (3) *p is a smooth covering map.*

Proof. The first statement follows at once from Theorem 2.3. From the same theorem it follows that if $\lambda(A) \rightarrow 0$, then $[A] \rightarrow 0$ in $C^\infty(X \setminus B(x(A), r))$. Using this fact, one can show that $H_A^2 = 0$. But now the second statement follows from Theorem 2.2. Since the minimum of R_A is non-degenerate, p is a smooth map. By Theorem 2.3, p is a proper map. Therefore, in order to prove the third statement it suffices to verify that the differential of p is an isomorphism. This is done by means of Taubes' implicit function theorem [48]. ■

Theorem 2.8. *The map p is a diffeomorphism.*

This is the most lengthy technical part of Donaldson's proof, which relies on subtle estimates of the curvature. The idea of the proof is to show that any two self-dual connections A and B for which $x(A) = x(B)$ and the number $\lambda(A) = \lambda(B)$ is sufficiently small can be joined in \mathcal{M} by a short path (for details, see [14]). ■

It follows from Theorems 2.7 and 2.8 that the moduli space \mathcal{M} can be compactified by points of the manifold X :

$$\overline{\mathcal{M}} = \mathcal{M} \setminus \mathcal{M}_{\lambda_0} \cup \overline{\mathcal{M}}_{\lambda_0} = \mathcal{M} \setminus \mathcal{M}_{\lambda_0} \cup X \times [0, \lambda_0].$$

We will show in the sequel that \mathcal{M} can be made into a manifold by means of a small perturbation; then $\overline{\mathcal{M}}$ will become a manifold with boundary x .

Now let us proceed to modifying \mathcal{M} into a smooth manifold. We have noted earlier that if $H_A^2 = 0$ for all self-dual connections A , then the moduli space \mathcal{M} is a smooth manifold except for some $n(Q)$ points corresponding to reduced connections. Generally speaking, the two-dimensional cohomology does not vanish for every connection A , therefore there may exist a subset $K \subset \mathcal{M}$ such that for elements of it $H_A^2 \neq 0$. By Theorem 2.7 the set K is compact.

It turns out that the space \mathcal{M} can be perturbed so as to make it a manifold in the general case as well. A perturbation of neighbourhoods of reducible connections is made "point-blank": the finite-dimensional map $\varphi(x)$ in the expansion $\Phi(x) = (D\Phi_A)(x) + \varphi(x)$ can be replaced by a similar map with a surjective differential. Then, as we pointed out in Theorem 2.2, a neighbourhood of $[A]$ is diffeomorphic to the space \mathbb{C}^3/S^1 —a cone over $\mathbb{C}P^2$. Hence it can be assumed that $K \subset \mathcal{M} \cap \mathcal{B}^*$.

To construct a perturbation of the set K , we observe that the map $p^{-1}(\mathcal{B}^*) \rightarrow \mathcal{B}^*$ is a principal $\mathcal{G}/\{\pm 1\}$ -bundle. Since the group $\mathcal{G}/\{\pm 1\}$ acts on Banach spaces $L_3^2(\Omega_+^2(\mathfrak{g}))$ and $L_2^2(\Omega_-^2(\mathfrak{g}))$, we get vector bundles $\mathcal{E}^s \subset \mathcal{E}^2$ with norms and connections associated with the principal bundle $p^{-1}(\mathcal{B}^*) \rightarrow \mathcal{B}^*$. There exists a canonical section $\Phi = F_-(A)$ of the bundle \mathcal{E}^2 , and we will search for perturbations $\sigma \in C^\infty(\mathcal{B}^*, \mathcal{E}^3)$ such that zero is a non-degenerate value of $\Phi + \sigma$.

Theorem 2.9. *There is a section $\sigma \in C^\infty(\mathcal{B}^*, \mathcal{E}^3)$ with support in a neighbourhood of K such that $(\Phi + \sigma)^{-1}(0)$ is a smooth five-dimensional manifold.*

Proof. We cover K by finitely many slices $T_{A,\varepsilon}$ and take open sets U_1, U_2 such that $K \subset U_1$ and $\overline{U_1} \subset U_2$. Let σ be a bounded section of the bundle \mathcal{E}^3 with support in U_2 . Then the set

$$\hat{K} = \{[A] \in \overline{U_1} \mid \|(\Phi + \sigma)(A)\|_{L^3} \leq R\}$$

is compact. This follows from the fact that U_2 is covered by finitely many slices and on each of them $\Phi(A) = d_A^- a + \frac{1}{2}[a, a] + \sigma(A)$, where $d_A^- a = 0$ and $\|a\|_{L_3^2} < \varepsilon$. Hence, L_3^2 -bounds on $\sigma(A), a, (\Phi + \sigma)(A)$ by virtue of ellipticity arguments result in an L_4^2 -bound for a . The required property now follows from the compactness of the embedding $L_4^2 \subset L_3^2$. Thus, if $\Phi + \sigma$ vanishes on $\overline{U_1}$ in a non-degenerate way, then this remains true also for sections $\Phi + \sigma'$ close to $\Phi + \sigma$ in the topology of uniform convergence together with derivatives on compact sets. The space of all such non-degenerate perturbations is dense. Indeed, for each point we take a slice on which there is a decomposition $\Phi + \sigma = L + \psi$, where L is linear and ψ is finite-dimensional. By virtue of compactness, we can choose a finite subcover

formed by those slices. We modify the function $\Phi + \sigma$, dividing it by a regular value of ψ and extending with the help of a cut-off function. By Sard's theorem such perturbations can be made in the L^2_3 norm as close to $\Phi + \sigma$ as desired.

The section itself vanishes outside \overline{U}_1 in a non-degenerate way. Using the density of the perturbation space, we choose a sufficiently small σ so as to make $\Phi + \sigma$ vanish in a non-degenerate way on $\overline{U_2 \setminus U_1}$. Then $\Phi + \sigma$ is everywhere non-degenerate. Consequently, $\mathcal{M}^\sigma = (\Phi + \sigma)^{-1}(0)$ is a five-dimensional manifold with boundary X and n singularities whose neighbourhoods are homeomorphic to \mathbb{C}^3/S^1 .

We now show that the resulting manifold $\mathcal{M}^\sigma \cap \mathcal{B}^*$ is orientable. For the manifold $\mathcal{M}^\sigma \cap \mathcal{B}^*$ consider the Stiefel–Whitney class $w_1(\ker \nabla(\Phi + \sigma))$. We can avoid singularities by using gauge transformations $\mathcal{G}_0 \subset \mathcal{G}$ that are stationary at a fixed point $x_0 \in X$. The group \mathcal{G}_0 acts freely on \mathcal{A} and results in a quotient space $\pi: \tilde{\mathcal{B}} \rightarrow \mathcal{B}$. Over \mathcal{B}^* the projection π is a principal $SO(3)$ -bundle; hence, the manifold $\mathcal{M}^\sigma \cap \mathcal{B}^*$ is orientable if and only if $\pi^{-1}(\mathcal{M}^\sigma \cap \mathcal{B}^*)$ is orientable.

The vector bundle $\ker \nabla(\Phi + \sigma)$, being restricted to an arbitrary compact subset $C \subset \pi^{-1}(\mathcal{M}^\sigma \cap \mathcal{B}^*)$, determines an element of the K -functor $K(C)$, the so-called index class of the family of Fredholm operators $d_{A_1}^* + d_{A_2}^* + (\nabla\sigma)A$. Considering the deformation $d_{A_1}^* + d_{A_2}^* + t(\nabla\sigma)A$, $0 \leq t \leq 1$, we see that the index class does not depend on the choice of the section σ . Since the Stiefel–Whitney class w_1 can be omitted through KO , in order to prove orientability it suffices to consider an element $\text{ind}(d_{A_1}^* + d_{A_2}^*) \in KO(C)$, where C is a closed loop in $\tilde{\mathcal{B}}$.

We embed the group $SU(2)$ in $SU(3)$ in the standard way. Let \tilde{A} be an $SU(3)$ -connection and $\tilde{\mathfrak{g}}$ the Lie algebra bundle associated with this embedding. Then $\tilde{\mathfrak{g}}$ splits into the Whitney sum $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R} \oplus V$, where V is a complex vector bundle of rank 2 and \mathbb{R} is a trivial bundle. This splitting is obviously compatible with the connection. Hence, $w_1(\text{ind}(d_{\tilde{A}}^* + d_{\tilde{A}}^-)) = w_1(\text{ind}(d_A^* + d_A^-))$, and so the loop C is subject to homotopy in the space of classes of gauge-equivalent $SU(3)$ -connections.

We will show that \mathcal{B}_3 is simply-connected. Since the group \mathcal{G}_0 of gauge transformations of the principal $SU(3)$ -bundle preserving the point x_0 acts freely on the space of connections, $\pi_1(\mathcal{B}_3) \cong \pi_0(\mathcal{G}_0)$. Therefore in order to establish that \mathcal{B}_3 is simply-connected it suffices to verify that $\pi_0(\mathcal{G}_0) = 0$, that is, the group \mathcal{G}_0 is connected. The principal bundle P is trivial over a complement to a point and, in particular, over the two-dimensional skeleton of the manifold X . Since $\pi_2(SU(3)) = 0$, every transformation from \mathcal{G}_0 can be deformed to a transformation that is the identity on the two-dimensional skeleton. Contracting the two-dimensional skeleton of X to a point, we obtain a four-dimensional sphere S^4 . The homotopy type of \mathcal{G}_0 on S^4 does not depend on the Chern class $c_2(P)$ (see [9]), and hence it suffices to consider a trivial bundle. But $\pi_4(SU(3)) = 0$, and therefore the group \mathcal{G}_0 is connected.

Thus $\mathcal{M}^\sigma \cap \mathcal{B}^*$ is an orientable five-dimensional manifold. Adding a boundary to it with the help of the procedure described in Theorem 2.8, we obtain a cobordism between X and n copies of $\mathbb{C}P^2$, needed for the proof of Donaldson's Theorem 1.4. ■

§3. Further results

(1) *Non-simply-connected manifolds and orbifolds.*

A detailed study of moduli space orientation and virtuoso use of transversality arguments enabled Donaldson to generalize his theorem to the case of non-simply-connected manifolds.

Theorem 3.1 [18]. *Let X be a smooth compact oriented four-dimensional manifold with definite intersection form L_X . Then L_X can be diagonalized over the ring of integers \mathbb{Z} .*

A somewhat weaker result was obtained independently by Furuta [25].

There are also results for certain indefinite forms. The strongest among them is of the following form.

Theorem 3.2 [16]. *Let X be a smooth compact oriented four-dimensional manifold such that the first integral homology group $H_1(X; \mathbb{Z})$ is without 2-torsion. Suppose that the positive part of the intersection form L_X of X has rank 2. Then*

$$L_X \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Fintushel and Stern [19] transferred the technique of Yang-Mills fields to four-dimensional orbifolds, that is, spaces with finitely many singularities modelled over spaces \mathbb{R}^4/Γ , where Γ is a finite group. Orbifolds appear quite naturally as quotient spaces either of smooth four-dimensional manifolds by action of a finite group or of five-dimensional manifolds by action of the circle. Orbifolds are rational homology manifolds, and analysis over them greatly resembles analysis over smooth manifolds—the principal modification is that there appear additional terms in the index formula, taking account of singularities. Generalizing the technique of $SO(3)$ -connections to orbifolds, Fintushel and Stern obtained strong restrictions on the existence of orbifolds with a prescribed intersection form. This enabled them to prove that the group θ_H^3 of homology 3-spheres modulo homology cobordism has infinitely many elements.

Theorem 3.3 [20]. *The Poincaré homology sphere P has infinite order in θ_H^3 , that is, no connected sum $P \# \dots \# P$ bounds an acyclic smooth four-dimensional manifold. In addition, $\theta_H^3(P) \neq 0$.*

This result is very unexpected, because according to Freedman [24] the Poincaré homology sphere bounds an acyclic topological four-dimensional manifold.

(2) *Exotic structures on \mathbb{R}^4 .*

It follows from the Freedman theorem that a direct sum decomposition of the intersection form of a four-dimensional manifold can be realized as a topological decomposition of the corresponding manifold into a connected sum. Donaldson's Theorem 1.4 provides very strong restrictions on those intersection forms admitting a smooth decomposition. As a consequence of these restrictions, "exotic \mathbb{R}^4 " come into being—smooth manifolds homeomorphic but not diffeomorphic to Euclidean space \mathbb{R}^4 (see [26]). The first examples of such exotic \mathbb{R}^4 were obtained as open subsets in $S^2 \times S^2$ or $\mathbb{C}P^2$. Then Gompf [27] found countably many exotic \mathbb{R}^4 in this way.

Let us give here an example of an exotic \mathbb{R}^4 . It follows from Donaldson's Theorem 1.4 that if the intersection form L_X of a four-dimensional manifold is even and positive definite, then X is not smooth. In particular, the manifold F_8^4 realizing the intersection form $L_{F_8} = E_8 \oplus E_8$ is not smooth. At the same time, the Kummer surface

$$K = \{(z_0 : z_1 : z_2 : z_3) \in \mathbb{C}P^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

with the intersection form $L_K = -2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a smooth manifold. Since

$S^2 \times S^2$ with $L_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is also a smooth manifold, one would like, by

means of surgery on K , to remove

$$3(S^2 \times S^2) = S^2 \times S^2 \# S^2 \times S^2 \# S^2 \times S^2$$

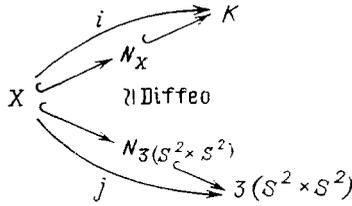
with $L_{3(S^2 \times S^2)} = 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and thus to obtain, after reversing the orientation, the smooth manifold F_8^4 . We will show that such a surgery is impossible. (And from this there emerges—in a quite unforeseen way—the existence of an exotic \mathbb{R}^4 , denoted in the sequel by \mathbb{R}_{FD}^4 .)

Let $a_i, b_i \in H_2(K; \mathbb{Z})$, $i = 1, 2, 3$, be generators of the two-dimensional homology of the Kummer surface. In order to bring about surgery on this surface, we make use of two theorems, which may be considered as corollaries of Theorems 6.5 and 6.8 from Ch. I.

Theorem 3.4. *Homologies generated by elements $a_i, b_i \in H_2(K; \mathbb{Z})$ are represented by an edged topological embedding of $X = 3(S^2 \times S^2) - D^4$ in K . Moreover, this embedding is smoothly equivalent to an embedding of X in $3(S^2 \times S^2)$, representing the homology of the manifold $3(S^2 \times S^2)$.*

Theorem 3.5. *Let V be a non-compact simply-connected four-dimensional manifold without boundary satisfying the condition $H_2(V; \mathbb{Z}) = 0$ and having a single end homeomorphic to $S^2 \times [0, \infty)$. Then V is homeomorphic to \mathbb{R}^4 .*

Theorem 3.4 claims that the following commutative diagram occurs:



In this diagram N_X and $N_{3(S^2 \times S^2)}$ are neighbourhoods of the images of X in K and $3(S^2 \times S^2)$, respectively. We recall that an *edged embedding* of a manifold X is an embedding such that the image of the boundary ∂X has a neighbourhood of the form $\partial X \times [0, 1)$. The existence of an edging for $j(X) \subset 3(S^2 \times S^2)$ implies that the manifold $V = 3(S^2 \times S^2)$ has a unique end homeomorphic to $S^3 \times [0, \infty)$. Hence, by Theorem 3.5, V is homeomorphic to \mathbb{R}^4 .

We will now use Donaldson's theorem and show that the smooth structure inherited by V from $3(S^2 \times S^2)$ is not standard, that is, V is not diffeomorphic to \mathbb{R}^4 with standard smooth structure. Suppose there exists a smoothly immersed three-dimensional sphere S^3 encircling the manifold X in its collar. Cutting off from K a manifold $i(X)$ along this sphere and gluing the cut by a four-dimensional disk D^4 , we can construct a smooth manifold with intersection form $E_8 \oplus E_8$, that is, the smooth manifold F_8^4 . But this is prohibited by Donaldson's theorem. Hence, there is no smoothly embedded sphere S^3 in the collar of $i(X)$. By virtue of the diffeomorphism $N_X \cong N_{3(S^2 \times S^2)}$ there is no smoothly embedded sphere S^3 in the collar of $j(X)$ either. Denoting by C the compact set $3(S^2 \times S^2) - (j(X) \cup \text{collar})$, we see that there is no smoothly embedded sphere S^3 encircling C in the manifold

$V \cong_{\text{top}} \mathbb{R}^4$. Hence, the manifold V is not diffeomorphic to four-dimensional Euclidean space with standard smooth structure \mathbb{R}_{st}^4 , because there are arbitrarily large smooth three-dimensional spheres in \mathbb{R}_{st}^4 . Putting $\mathbb{R}_{FD}^4 = V$, we thus deduce that \mathbb{R}_{FD}^4 is not diffeomorphic to \mathbb{R}_{st}^4 , as required.

An indicator of exotic smooth structure in \mathbb{R}^4 is the following striking property of \mathbb{R}_{FD}^4 , which certifies that the latter space is not diffeomorphic to \mathbb{R}_{st}^4 : there is a compact set C in the space \mathbb{R}_{FD}^4 that cannot be encircled by any smoothly embedded three-dimensional sphere. Of course, there are arbitrarily large continuously embedded three-dimensional spheres in \mathbb{R}_{FD}^4 . (To construct them, we pick any norm in \mathbb{R}_{FD}^4 and consider the sets $\{x \in \mathbb{R}_{FD}^4: \|x\| = R\}$ for large R .) Therefore, in the smooth manifold \mathbb{R}_{FD}^4 spheres are extremely "jagged" near infinity.

Further advance in constructing exotic \mathbb{R}^4 is due to Taubes [50], who generalized the basics of Yang-Mills theory to the so-called asymptotically periodic four-dimensional manifolds. An asymptotically periodic four-dimensional manifold is a non-compact manifold whose end has periodic

configuration $W_1 \cup W_2 \cup \dots \cup W_n \cup \dots$, where all the W_i are diffeomorphic to an open manifold W . The simplest examples of periodic manifolds are those manifolds whose end is of the form $Y^3 \times (0, \infty)$ and $W = Y \times (0, 1)$. Taubes showed that if certain conditions imposed on the homology of a manifold W and representations $\pi_1(W) \rightarrow SU(2)$ are met, then asymptotically periodic manifolds behave from the viewpoint of anti-self-dual equations as compact manifolds. This observation enabled him to generalize Donaldson's theorem to asymptotically periodic manifolds and to prove the following assertion with the help of the resulting generalization.

Theorem 3.6 [27], [50]. *There is a family $\mathbb{R}_{s,t}$ of smooth four-dimensional manifolds parametrized by points $(s, t) \in \mathbb{R}^2$ every one of which is homeomorphic but not diffeomorphic to the Euclidan space \mathbb{R}^4 .*

Thus, there exist moduli for smooth structures on the topological manifold \mathbb{R}^4 . Moreover, Taubes' family does not contain all exotic \mathbb{R}^4 : none of the $\mathbb{R}_{s,t}$ can be embedded in the standard \mathbb{R}^4 , while the violation of the h -cobordism theorem in dimension 4 (see Theorem 3.7 below) implies that such exotic \mathbb{R}^4 should exist.

In the spirit of Taubes' work, Donaldson and Sullivan generalized the Yang-Mills theory to quasiconformal four-dimensional manifolds for which the coordinate transition functions are quasiconformal maps of domains in \mathbb{R}^4 . This enables us to transfer to such manifolds all the results obtained for smooth four-dimensional manifolds. In particular, there are exotic quasiconformal structures on \mathbb{R}^4 . All we have said above about quasiconformal manifolds extends a fortiori to Lipschitz four-dimensional manifolds in sharp contrast to higher dimensions: according to Sullivan, every topological manifold has a unique Lipschitz structure in dimensions exceeding 4 [47].

(3) *The Donaldson invariant.*

We restrict ourselves to the case of smooth simply-connected four-dimensional manifolds X . For an arbitrary bundle E over X the rational cohomology ring of the space \mathcal{B}^* is generated by classes c/α , where c is the rational characteristic class of the universal bundle on the product $\mathcal{B}^* \times X$ and α is a homology class of X . In particular, all rational cohomologies of the manifold \mathcal{B}^* are in even dimensions. The invariants constructed by Donaldson are in homologies of \mathcal{B}^* , hence it seems reasonable, in order to get interesting results, to assume that the moduli spaces under consideration are of even dimension. We put $\sigma(X) = b_2^+(X) - b_2^-(X)$, where $\sigma(X)$ is the signature of X , that is, the signature of its intersection form L_X , and $b_2^+(X)$, $b_2^-(X)$ are the dimensions of the positive and the negative parts of this quadratic form. As we noted earlier,

$$\dim \mathcal{M} = 8 |c_2(E)| - \frac{1}{2} \dim G \cdot (\chi(X) - \sigma(X)).$$

Since $\sigma = b_2^+ - b_2^-$, we have $\chi = 2 + b_2^+ + b_2^-$, and so

$$\dim \mathcal{M} = 8 | c_2(E) | - \dim G \cdot (1 + b_2^-).$$

In what follows we shall be interested in the cases $G = SU(2)$ and $G = SO(3)$, where $\dim G = 3$. Hence, for $G = SU(2)$ and $SO(3)$

$$\dim \mathcal{M} = 8 | c_2(E) | - 3 (1 + b_2^-),$$

and the moduli space \mathcal{M} is of even dimension exactly when b_2^- is odd.

Manifolds with $b_2^- = 1$ form a very special class, because in this case there appear reducible connections in the moduli space for a family of metrics of codimension 1. Consider the two-dimensional moduli space for $SU(2)$ -connections on such a manifold with the Chern class $c_2 = 1$. We can assign to a typical metric on X a homology class in \mathfrak{B}^* that changes in the presence of reducible solutions only. As a result, we get a differential-topological invariant Γ_X of X lying in $H^2(X; \mathbb{Z})$. For algebraic surfaces X this invariant can be calculated in a number of cases by using a holomorphic description of anti-self-dual connections with the help of stable bundles. Every rational surface has $b_2^- = 1$; this property is shared also by some irrational surfaces, in particular the family $D_{p,q}$ (p and q are coprime integers) constructed by Dolgachev. The difference between the complex geometry of rational and irrational surfaces is reflected in stable bundles, and hence in the structure of moduli spaces and in the Γ invariant. As a result we obtain the following statement.

Theorem 3.7 [17]. *The surfaces $D_{p,q}$ are homotopically equivalent (and therefore homeomorphic and h -cobordant) but not diffeomorphic to the connected sum $\mathbb{C}P^2 \# 9 \overline{\mathbb{C}P^2}$.*

Thus, the h -cobordism theorem cannot be extended to smooth four-dimensional manifolds. Moreover, using the Γ -invariant, Okonek and Van de Ven [39] have shown that for homotopically equivalent manifolds $D_{p,q}$ there are infinitely many diffeomorphism types.

§4. Floer homology

Let M be a closed connected oriented three-dimensional manifold. It is well known that every topological three-dimensional manifold can be endowed with a unique smooth structure, so we can view M in any of these two categories of manifolds. An important algebraic invariant of M is the fundamental group $\pi_1(M)$. Unfortunately, application of this invariant to the classification of three-dimensional manifolds meets serious difficulties: firstly, it is not known whether $\pi_1(M) = 0$ implies that $M = S^3$ (the Poincaré conjecture), and secondly, one lacks a natural characterization of those groups capable of serving as fundamental groups of three-dimensional manifolds. We can arrive at alternative invariants of three-dimensional manifolds by studying

representations of $\pi_1(M)$ in non-Abelian Lie groups G . Let

$$\mathcal{R} = \text{Hom} (\pi_1 (M), G)/\text{ad} (G)$$

be the set of equivalence classes of representations of $\pi_1(M)$ in G . While \mathcal{R} itself depends on the homotopy type of M only, the corresponding smooth bundles together with a smooth structure on M may lead to new topological invariants arising, such as the Reidemeister torsion [35]. Recently Casson has defined a new topological invariant for three-dimensional homology spheres M , that is, closed three-dimensional manifolds such that $H_1 (M; \mathbb{Z}) = 0$. To define this invariant, we need a certain amount of information on representations of discrete groups Γ in $SU(2)$ and the Hager decomposition of three-dimensional manifolds.

The space $\mathcal{R} (\Gamma)$ of representations of a discrete group Γ in $SU(2)$ is made into a topological space by means of the simple convergence topology. We denote by $\tilde{\mathcal{R}} (\Gamma)$ the open subset of $\mathcal{R} (\Gamma)$ formed by irreducible representations. The group $SU(2)$ acts on the space $\mathcal{R} (\Gamma)$ by conjugations. This action factors to an effective action of the group $SU(2)/\{\pm 1\} = SO(3)$, which is free on the space $\tilde{\mathcal{R}}(\Gamma)$; thus we obtain a principal bundle

$$\pi: \tilde{\mathcal{R}} (\Gamma) \rightarrow \mathcal{R}^* (\Gamma) = \tilde{\mathcal{R}} (\Gamma)/ \text{ad} (SU (2)).$$

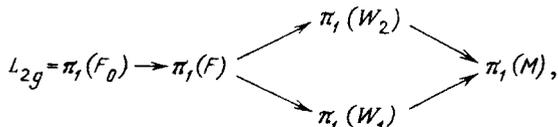
It is not difficult to show that if L is a free group of rank k , then $\mathcal{R}(L)$ is homeomorphic to the product $SU(2)^k$ and it is endowed with a smooth manifold structure in such a way that the tangent space to $\mathcal{R}(L)$ at a trivial representation identifies naturally with $H^1(L; su(2))$.

The fundamental group Γ_g of an oriented surface of genus g is the quotient group of the free group L_{2g} with generators $a_1, b_1, \dots, a_g, b_g$ by the normal subgroup generated by $\delta = [a_1, b_1] \dots [a_g, b_g]$. To the canonical projection $L_{2g} \rightarrow \Gamma_g$ there corresponds an inclusion $\mathcal{R} (\Gamma_g) \hookrightarrow \mathcal{R} (L_{2g})$, whose image in $\mathcal{R} (L_{2g})$ has the form $\partial^{-1}(1)$, where $\partial: \mathcal{R} (L_{2g}) \rightarrow S^3$ is given by $\partial (\rho) = \rho (\delta)$, $\rho \in \mathcal{R} (L_{2g})$. Calculating the differential for the commutator map $(x, y) \mapsto [x, y]$, we find that for $g > 1$ the sets $\tilde{\mathcal{R}} (\Gamma_g)$ and $\mathcal{R}^* (\Gamma_g)$ are submanifolds of the manifolds $\mathcal{R} (L_{2g})$ and $\mathcal{R}^* (L_{2g})$ of dimension $6g - 3$ and $6g - 6$ respectively.

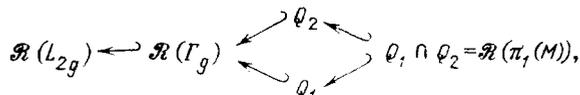
We now recall certain details about the Hager decomposition of a three-dimensional manifold (for the details, see [53]). Let f be an ordered Morse function on a closed oriented three-dimensional manifold M , and let t be a regular value of f separating critical points of indices 1 and 2. The sets $W_1 = \{f \leq t\}$ and $W_2 = \{f \geq t\}$ have their intersection $F = \{f = t\}$ as a common boundary, and their union is the manifold M . Such a subdivision $W_1 \cup_F W_2$ is called a *Hager decomposition* of genus g of the manifold M (here g is the genus of F). The order of W_1 and W_2 is of significance: it enables us to orient F like the boundary of W_1 with outer normal. If $M' = W'_1 \cup_{F'} W'_2$, then by taking connected sums along the boundaries we can form the Hager decomposition $W_1 \# W'_1 \cup_{F \# F'} W_2 \# W'_2$ of the

manifold $M \# M'$. If $M' = S^3$ is the unit sphere in \mathbb{C}^2 with standard Hager decomposition $\{|z_1| \leq |z_2|\} \cup \{|z_1| \geq |z_2|\}$, then we obtain on $M = M \# S^3$ the elementary stabilization of the Hager decomposition of the initial manifold M . The stabilization is obtained by repeating this process. The following statement is true (the Reidemeister–Singer theorem): any two Hager decompositions of the same manifold are stably isomorphic.

Let $W_1 \cup_F W_2$ be a Hager decomposition of genus g of a manifold M , and let F_0 be the surface F with a disk cut out. Consider the Van Kampen diagram



where all homomorphisms are surjections. At the level of representation spaces, the above diagram induces the following diagram of embeddings:



It is not difficult to show that $\pi_1(W_i)$ is a free subgroup of rank g ; hence, Q_i is a half-dimensional submanifold of $\mathcal{R}(L_{2g})$.

By fixing in $\pi_1(F_*)$ a basis $(a_1, b_1, \dots, a_g, b_g)$ that is symplectic for the orientation of F_* , we can orient the manifolds $\mathcal{R}(L_{2g})$, $\tilde{\mathcal{R}}(\Gamma_g)$, $\mathcal{R}^*(\Gamma_g)$ and $\tilde{Q}_i \rightarrow Q_i^*$ in a coordinated way. Let $(Q_1, Q_2)_{\mathcal{R}(L_{2g})}$ denote the intersection index of Q_1 and Q_2 in $\mathcal{R}(L_{2g})$. The properties of representation spaces yield at once the following assertion.

Proposition 4.1. *A manifold M is a homology sphere if and only if $(Q_1, Q_2)_{\mathcal{R}(L_{2g})} = \pm 1$. In this case Q_1 and Q_2 are transversal at a point corresponding to the trivial representation.*

In particular, if M is a homology sphere, then the trivial representation determines an isolated point and the intersection $Q_1 \cap Q_2$ is compact. Hence, in this case one can define the intersection index of Q_1^* and Q_2^* in $\mathcal{R}^*(\Gamma_g) = \mathcal{R}^*(\pi_1(F))$, denoted by $(Q_1^*, Q_2^*)_{\mathcal{R}^*(\pi_1(F))}$.

Definition 4.2. Let M be a three-dimensional homology sphere. The number

$$\lambda(M) = \frac{1}{2} (-1)^g (Q_1^*, Q_2^*)_{\mathcal{R}^*(\pi_1(F))} / (Q_1, Q_2)_{\mathcal{R}(L_{2g})}$$

is called the Casson invariant of M .

Casson has shown that $\lambda(M)$ does not depend on the Hager decomposition $W_1 \cup_F W_2$ of M and that it is an integer. The presence of the multiplier $1/2$ in the definition is motivated by the fact that $(Q_1^*, Q_2^*)_{\mathcal{R}^*(\pi_1(F))}$ is an even number (mainly because the map $SU(2) \rightarrow SO(3)$ has degree 2).

Let us put the properties of the Casson invariant together into the following theorem.

Theorem 4.3. *Let \mathcal{S} be the set of oriented diffeomorphism classes of three-dimensional homology spheres. Then there is a map $\lambda: \mathcal{S} \rightarrow \mathbb{Z}$ with the following properties:*

- 1) $\lambda(M) = 0$ if all representations of $\pi_1(M)$ in $SU(2)$ are trivial;
- 2) the Rokhlin invariant $\rho(M)$ of M is the modulo 2 reduction of $\lambda(M)$;
- 3) $\lambda(-M) = -\lambda(M)$;
- 4) $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$.

In 1987 Floer [21] showed that the Casson invariant can be obtained by means of the infinite-dimensional Morse theory as the Euler characteristic

$\sum_{k=0}^7 (-1)^k \text{rk } HF_k(M)$ of 8-periodic homology theory $HF_*(M)$. The construction of the homology groups $HF_*(M)$ is based on a study of gauge fields for the three-dimensional and four-dimensional manifolds.

Let M be a three-dimensional manifold and $\pi: P \rightarrow M$ be a principal bundle with structure group $SU(2)$. Using the obstruction theory, it is not difficult to show that every principal $SU(2)$ -bundle over a three-dimensional manifold is topologically trivial, that is, it admits a representation in the form of a product $P = M \times SU(2)$. If such a trivialization has been chosen, then we can identify the space of connections $\mathcal{A}(P)$ with the space $\Omega^1(M) \otimes su(2)$ of 1-forms on M taking values in $su(2)$. Under this identification the zero element of $\Omega^1(M) \otimes su(2)$ corresponds to the trivial connection θ on P . The gauge transformation group is identified with the map space $C^\infty(M, SU(2))$ acting on \mathcal{A} by the rule

$$g(A) = gAg^{-1} + (dg)g^{-1}, \quad g \in \mathcal{G}, \quad A \in \mathcal{A}.$$

We will assume that \mathcal{A} and \mathcal{G} are endowed with topologies induced by the Sobolev norms L_k^2 . If $k+1 > 3/p$, then \mathcal{G} acts continuously on \mathcal{A} . As before, in this case the orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ is an infinite-dimensional manifold, apart from those points corresponding to irreducible connections. Irreducible connections form an open dense set \mathcal{B}^* in \mathcal{B} .

In what follows we will need a description of the tangent and cotangent spaces to $\mathcal{B}(P)$ in terms of classes $[A]$ of equivalent connections. Since $\mathcal{A}(P)$ is an affine space, $T_A \mathcal{A}(P) = \Omega^1(M) \otimes su(2)$ and an easy calculation shows that

$$T_A(\mathcal{G}(P) \cdot A) = \text{im } \{d_A: \Omega^0(M) \otimes su(2) \rightarrow \Omega^1(M) \otimes su(2)\},$$

where d_A is the covariant derivative on sections of the bundle \mathfrak{g}_P . If A is an element of an open dense set $\mathcal{A}^*(P) \subset \mathcal{A}(P)$ of irreducible connections, then $\mathcal{G}(P)$ acts on A with a stabilizer $\{\pm 1\}$, so

$$T_{[A]} \mathcal{B}(P) = \Omega^1(M) \otimes su(2) / d_A(\Omega^0(M) \otimes su(2)).$$

In order to obtain the cotangent space to $\mathcal{B}(P)$, we first observe that there is a non-degenerate bilinear pairing

$$\Omega^1(M) \otimes su(2) \times \Omega^2(M) \otimes su(2) \rightarrow \mathbb{R}, \quad (a, b) \mapsto - \int_M \text{tr}(a \wedge b);$$

hence $T_A^* \mathcal{A}(P) \cong \Omega^2(M) \otimes su(2)$. Cotangent vectors on $\mathcal{B}(P)$ are cotangent vectors in $\Omega^2(M) \otimes su(2)$ on $\mathcal{A}(P)$ that vanish along $\mathcal{G}(P)$ -orbits, that is,

$$- \int_M \text{tr}(d_A a \wedge b) = 0$$

for all $a \in \Omega^0(M) \otimes su(2)$. Using Stokes's theorem, we deduce from this that there is a natural isomorphism

$$T_{[A]}^* \mathcal{B}(P) = \ker \{d_A: \Omega^2(M) \otimes su(2) \rightarrow \Omega^3(M) \otimes su(2)\}.$$

Now consider the case where M is a oriented three-dimensional homology sphere. The Chern–Simons invariant of classes of gauge equivalent connections is determined by the function

$$CS: \mathcal{B}(P) \rightarrow S^1 = \mathbb{R}/\mathbb{Z}.$$

This is a function playing the role of the Morse function in the definition of Floer homology. To define the function CS , let us agree that $CS(\theta) = 0$, where θ is a trivial connection, and that the derivative $(4\pi^2)^{-1}DCS_{[A]} \in T_{[A]}^* \mathcal{B}(P)$ is the curvature 2-form $F^A (= dA + A \wedge A$ in terms of 1-forms on $P) \in \Omega^2(M) \otimes su(2)$. It follows from the Bianchi identity that $d_A F^A = 0$, and therefore the non-degenerate pairing

$$\Omega^1(M) \otimes su(2) \times \Omega^2(M) \otimes su(2) \rightarrow \mathbb{R}, \quad A \mapsto F^A,$$

determines in a natural way a 1-form on $\mathcal{B}(P)$. This 1-form on $\mathcal{B}(P)$ is closed but not exact. The Chern–Simons function is obtained from it by integration. We can obtain a less abstract definition of the function CS as follows. Let A_4 be a connection on the principal $SU(2)$ -bundle over $M \times [0, 1]$ such that $A_4|_{M \times \{0\}}$ is a trivial connection and $A_4|_{M \times \{1\}} = A$. Then

$$CS(A) = - \frac{1}{8\pi^2} \int_{M \times [0, 1]} \text{tr}(F^{A_4} \wedge F^{A_4}).$$

The construction of the Floer chain complex is based on those critical points of CS lying in $\mathcal{B}^*(P)$. By definition, the derivative of the function CS is a curvature, hence $[A]$ is a critical point of CS if and only if $F^A = 0$, that is, when A is flat. To classify such connections, we recall the notion of the holonomy of A . Let $y \in M$ and $p \in \pi^{-1}(y)$ be distinguished points in M and P , and let $\gamma: t \mapsto y(t)$ be a closed loop in M such that $y(0) = y(1) = y$. We define a horizontal lifting $\eta: t \mapsto p(t) \in P$ of the loop γ by requiring that $\eta(0) = p$, $\pi(\eta(t)) = y(t)$, and $A\left(\frac{d}{dt} \eta(t)\right) = 0$. This determines the horizontal lift in a unique way. Then $\eta(1) = R_{g_\gamma}(\eta(0))$ for some element

$g_\gamma \in SU(2)$, and this element g_γ is called the *holonomy of A along γ* . The correspondence $\gamma \mapsto g_\gamma$ is a representation of the group of closed loops in M with a distinguished point y in the group $SU(2)$. Any change of the distinguished point p or a change of the connection A to a gauge equivalent one only results in a conjugation in the group $SU(2)$. If the connection is flat, then the holonomy along contractible loops is trivial. Hence, the holonomy is determined by a representation of the fundamental group $\pi_1(M) \rightarrow SU(2)$. It is called the *monodromy representation*. A well-known theorem states that flat connections are, up to gauge transformations, in one-to-one correspondence with elements of the space $\mathcal{R}(M) = \text{Hom}(\pi_1(M), SU(2))/\text{ad } SU(2)$.

The space $\mathcal{R}(M)$ is always compact, because the group $\pi_1(M)$ is finitely generated. We will assume that the Hessian of CS is an isomorphism at critical points. In the general case it is necessary first to perturb the Chern–Simons function so that the new function has finitely many non-degenerate critical points. Since a perturbed function is no longer canonical, one has to show that the homology of the resulting complex does not depend on the perturbation.

In order to construct the homology theory from the function CS , we must clear up the indices of the critical points and gradient curves. As we see now, gradient curves lead us to a four-dimensional gauge theory. Indeed, the metric on the oriented three-dimensional manifold M determines the Hodge operator $*$: $\Omega^k(M) \otimes su(2) \rightarrow \Omega^{3-k}(M) \otimes su(2)$. By combining it with the natural pairing

$$\Omega^1(M) \otimes su(2) \times \Omega^2(M) \otimes su(2) \rightarrow \mathbb{R},$$

we obtain a $\mathcal{G}(P)$ -invariant Riemannian metric on $\mathcal{A}(P)$, which induces the Riemannian metric on $\mathcal{B}(P)$. It is easy to verify that the gradient vector field of the function CS is given by the correspondence $A \mapsto 4\pi^2 * F^A$. The equation for gradient curves is of the form

$$\frac{dA_t}{dt} = * F^{A_t},$$

where $t \mapsto A_t$ is some family of connections on M . The above equation does not determine a flow for the same reasons that the heat transfer equation cannot be solved. Nevertheless, there exist both stable and unstable manifolds at critical points of the vector field $A \mapsto * F^A$, and they are exactly what one needs to construct the Floer homology.

It is not difficult to verify that there is a natural isomorphism between families of connections on M , that is, maps $\mathbb{R} \rightarrow \mathcal{B}(P)$: $t \mapsto A_t$ and classes of gauge equivalence of connections A_4 on a trivial bundle over the four-dimensional manifold $M \times \mathbb{R}$. In terms of connections on $M \times \mathbb{R}$ the equation

$$\frac{dA_t}{dt} = * F^{A_t}$$

takes the form

$$F^{A_4} = -*F^{A_4},$$

that is, it becomes the celebrated anti-self-duality equation for four-dimensional manifolds. Solutions of this equation, subordinate to certain boundary conditions, are exactly the minima of the Yang-Mills functional

$$YM: \mathcal{B}(P) \rightarrow \mathbb{R}: A \mapsto -\frac{1}{8\pi^2} \int_{M \times \mathbb{R}} \text{tr}(F^A \wedge *F^A).$$

Taubes [50] developed a theory for this equation over non-compact periodic manifolds similar to $M \times \mathbb{R}$, and this theory plays a key role in the definition of the boundary operator in the Floer complex.

A module basis in the Floer complex is given by irreducible flat connections. The Hessian of the function CS at an irreducible flat connection A can be written as a self-adjoint operator on

$$\begin{aligned} T_{[A]} \mathcal{B}(P) &\cong (T_A \mathcal{G}(P) \cdot A)^\perp = \\ &= (\text{im } \{d_A: \Omega^0(M) \otimes su(2) \rightarrow \Omega^1(M) \otimes su(2)\})^\perp = \\ &= \ker \{d_A^*: \Omega^1(M) \otimes su(2) \rightarrow \Omega^0(M) \otimes su(2)\}. \end{aligned}$$

It is easy to see that the above operator is nothing but

$$-*d_A: \ker d_A^* \rightarrow \ker d_A^*.$$

This is a self-adjoint Fredholm operator with discrete spectrum unbounded both from below and from above. Hence, it is impossible to define an index in a naive way, but there is a procedure to resolve this problem. The index difference $i(a) - i(b)$ for two critical points $a = [A]$, $b = [B]$ equals the number of eigenvalues, crossing over from + to -, minus the number of transitions from - to + in a one-parameter family of spectra of Hessians $t \mapsto \text{Hess}(CS_{A_t})$, where A_t is a path from $[A]$ to $[B]$. This number $i(a, b, A_t)$ of eigenvalues passing through zero, taken with the opposite sign, is called the *spectral flow* of a family of operators and it depends only on the homotopy class of the path $t \mapsto A_t$. The spectral flow and its connection with the index theory have been studied in detail by Atiyah, Patodi and Singer [10]. For every t , the connection A_t should be irreducible, because at a reducible connection the correspondence $A \mapsto \ker d_A^* \subset \Omega^1(M) \otimes su(2)$ is no longer continuous, and hence the spectrum in this case does not transform continuously.

It is easy to see that the spectral flow of a family of Hessians equals the spectral flow of the following family of self-adjoint elliptic operators on M taken with the opposite sign:

$$D_{A_t} = *d_{A_t} + d_{A_t}*: \Omega^{\text{odd}}(M) \otimes su(2) \rightarrow \Omega^{\text{odd}}(M) \otimes su(2),$$

under the assumption that all the A_t are irreducible. An advantage of this description is that both the operator D_A and its spectrum depend continuously on A even if A is reducible.

Now we are in a position to define the index of a critical point $a \in \mathcal{B}(P)$. In principle, it could be done this way: we join the point θ corresponding to the trivial connection to a by a path A_t and take the spectral flow $i(\theta, a, A_t)$. Unfortunately, the operator D_θ has zero eigenvalue of multiplicity three, and hence the spectral flow in this case is not well defined. To eliminate this shortcoming, we consider a typical connection θ' in a small neighbourhood of θ . It can be chosen so that the operator $D_{\theta'}$ has three simple eigenvalues whose absolute values are less than that of any other eigenvalue. We denote by $p(\theta')$ the number of positive eigenvalues among those three that have just appeared, and put

$$i(a) = i(\theta', a, A_t) + p(\theta') \pmod{8},$$

where A_t is a path joining θ' and a . It is not difficult to see that the number $i(a)$ does not depend on the choice of θ' . Since $\pi_1(\mathcal{B}(P)) = \mathbb{Z}$, it follows from the results of [10] that the spectral flow along a closed loop is divisible by 8. Therefore the index $i(a)$ does not depend on the choice of A_t .

Now we construct the Floer chain complex C_j , $j \in \mathbb{Z}/8$, by defining C_j as the free module over \mathbb{Z} generated by irreducible flat connections with index equal to j . The boundary operator $d: C_j \rightarrow C_{j-1}$, as is usual in Morse theory, is determined by the number of gradient curves taken with sign. We noticed above that parametrized gradient curves A_t are elements of the moduli space of instantons (anti-self-dual connections) on the four-dimensional manifold $M \times \mathbb{R}$. We denote by $\mathcal{M}_k(a, b)$ the moduli space of instantons A_t on $M \times \mathbb{R}$ satisfying the following conditions:

$$\lim_{t \rightarrow -\infty} A_t = a, \quad \lim_{t \rightarrow \infty} A_t = b \quad \text{and} \quad i(a, b, A_t) = k \in \mathbb{Z}.$$

One can arrange that $\mathcal{M}_k(a, b)$ is a smooth manifold by a small perturbation of the self-duality equations.

Theorem 4.4.

- (i) $\mathcal{M}_k(a, b)$ is a smooth k -dimensional manifold (possibly, an empty one);
- (ii) there is defined a proper free action of the group \mathbb{R} on the manifold $\mathcal{M}_k(a, b)$, induced by translations on $M \times \mathbb{R}$;
- (iii) $\mathcal{M}_k(a, b)$ is canonically oriented;
- (iv) $\mathcal{M}_k(a, b)$ has finitely many connection components.

The solutions A_t are determined by their initial values A_0 , therefore the map $\{A_t\} \rightarrow A_0$ determines an immersion of the manifold $\mathcal{M}_k(a, b)$ in $\mathcal{B}(P)$. It follows from Theorem 4.4 that the set of non-parametrized gradient curves $\mathcal{M}_1(a, b)/\mathbb{R}$ is a compact oriented zero-dimensional manifold, that is, a finite set of signed points. Let $i(a) - i(b) = 1$ and let $\langle \partial a, b \rangle \in \mathbb{Z}$ be the sum of

the signs of the finite set $\widehat{\mathcal{M}}_1(a, b) = \mathcal{M}_1(a, b)/\mathbb{R}$. We define an operator $d : C_k \rightarrow C_{k-1}$ by putting

$$da = \sum_{i(b)=k-1} \langle \partial a, b \rangle b.$$

It turns out that d is a boundary operator in the Floer complex. The linearity of d is obvious. The relation $d^2 = 0$ follows from the fact that the matrix elements

$$\langle \partial \partial a, c \rangle = \sum_{b \in \mathcal{B}^*} \langle \partial a, b \rangle \cdot \langle \partial b, c \rangle$$

correspond to the sum of products of signs in the decomposition

$$\widehat{\mathcal{M}}_1(a, c) = \bigcup_{b \in \mathcal{B}^*} \widehat{\mathcal{M}}_1(a, b) \times \widehat{\mathcal{M}}_1(b, c).$$

By definition, $HF_*(M) = H_*(C, d)$, where (C, d) is the Floer complex.

In conclusion, we note an important property of the Floer homology $HF_*(M)$. We denote by \overline{M} the manifold M with opposite orientation. Then there is a non-degenerate pairing

$$HF_j(M) \times HF_{3-j}(\overline{M}) \rightarrow \mathbb{Z}.$$

It is obtained as follows. First of all, the Chern–Simons functional CS satisfies the relation

$$CS_M(A) = -CS_{\overline{M}}(A),$$

and hence by changing the orientation to the opposite one the Hessian CS changes sign. Hence, $i_M(\theta', a, A_t) = -i_{\overline{M}}(\theta', a, A_t)$ for the spectral flow of the Fredholm operator D_{A_t} . Further, it is obvious that

$$\rho_M(\theta') = 3 - \rho_{\overline{M}}(\theta'),$$

therefore

$$i_M(a) = 3 - i_{\overline{M}}(a),$$

that is,

$$C_j(M) = C_{3-j}(\overline{M}).$$

Now let $a \in C_j(M)$ and $b \in C_{j-1}(M)$ be basis vectors, that is, irreducible flat connections. Then both incidence coefficients $\langle \partial_M a, b \rangle$ and $\langle \partial_{\overline{M}} b, a \rangle$ are equal to the number of gradient curves passing from a to b and counted with signs. We can verify that these signs coincide, that is, the coefficients $\langle \partial_M a, b \rangle$ and $\langle \partial_{\overline{M}} b, a \rangle$ are equal. Thus, we get a natural bilinear pairing

$$HF_j(M) \times HF_{3-j}(\overline{M}) \rightarrow \mathbb{Z}.$$

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Translated by V. Pestov

Moscow State University

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