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Simply-connected 4-manifolds with a given boundary

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Abstract

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Let M be a closed, oriented, connected 3-manifold and let $(\mathbb{Z}^n, L)$ be a symmetric bilinear form which presents $H_*(M;\mathbb{Z})$. Let $\mathscr{V}_L(M)$ be the set of all oriented homeomorphism types of compact, 1-connected, oriented 4-manifolds with boundary M and intersection pairing isomorphic to $(\mathbb{Z}^n, L)$. We will give a complete description of the sets $\mathscr{V}_L(M)$.

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Introduction

In [2] Freedman classified closed, 1-connected, oriented 4-manifolds up to orientation preserving homeomorphism (modulo a few technical details that are removed in either Quinn [4] or Freedman and Quinn [3]). This result has been generalized in a number of ways. In particular, Vogel [6] and Boyer [1] have both given partial results on classifying compact, 1-connected, oriented 4-manifolds with specified boundary. The goal of this paper is to complete the analysis given by Boyer.

Let M be a closed, oriented, connected 3-manifold which will be fixed throughout this discussion. Let $(\mathbb{Z}^n, L)$ be any symmetric bilinear form presenting $H_*(M;\mathbb{Z})$. Define $\mathscr{V}_L(M)$ to be the set of all oriented homeomorphism types of compact, 1-connected, oriented 4-manifolds with boundary M and intersection pairing isomorphic to $(\mathbb{Z}^n, L)$. With these definitions Boyer gives essentially the uniqueness half of the classification. If L is an odd form, then Boyer constructs an injective map $c_L^t \times \Delta : \mathscr{V}_L(M) \to B_L^t(M) \times \mathbb{Z}/2\mathbb{Z}$, where $\Delta : \mathscr{V}_L(M) \to \mathbb{Z}/2\mathbb{Z}$ is the Kirby–Siebenmann invariant and $B_L^t(M)$ is a double coset space described in more

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detail below. If L is even, then the analysis is more subtle. Boyer constructs an injective map $\hat{c}_L : \mathscr{V}_L(M) \to \hat{B}_L(M)$, where $\hat{B}_L(M)$ is again a double coset space.

Let $T_1(M)$ denote the torsion subgroup of $H_1(M;\mathbb{Z})$ and let $l_M : T_1(M) \times T_1(M) \to \mathbb{Q}/\mathbb{Z}$ be the link pairing. For an Abelian group A let A^* denote the dual group.

Definition. A bilinear form $(\mathbb{Z}^n, L)$ presents $H_*(M;\mathbb{Z})$ if there is an exact sequence

$$0 \longrightarrow H_2(M;\mathbb{Z}) \xrightarrow{h} \mathbb{Z}^n \xrightarrow{\mathrm{ad}(L)} [\mathbb{Z}^n]^* \xrightarrow{\partial} H_1(M;\mathbb{Z}) \longrightarrow 0$$

such that

(i) if $\mathrm{ad}(L)(\xi_i) = m_i\eta_i$ $(i = 1, 2)$ where $m_1m_2 \neq 0$, then

$$l_M(\partial\eta_1, \partial\eta_2) \equiv -L(\xi_1, \xi_2)/m_1m_2 \pmod 1,$$

(ii) if $\beta \in H_2(M;\mathbb{Z})$ and $\eta \in [\mathbb{Z}^n]^*$, then $\partial(\eta)\cdot\beta = \eta(h(\beta))$.

Such an exact sequence is called a *presentation of* $H_*(M;\mathbb{Z})$ *by* $(\mathbb{Z}^n, L)$.

Note that if V is a compact, oriented, 1-connected 4-manifold with boundary M and we fix an isomorphism $H_2(V;\mathbb{Z}) \cong \mathbb{Z}^n$, then we obtain a presentation of $H_*(M;\mathbb{Z})$ by $(H_2(V;\mathbb{Z}),\cdot)$ from the long exact sequence of the pair (V, M) which we will call the *geometric presentation* of V. In particular $\mathscr{V}_L(M)$ is empty unless L presents $H_*(M;\mathbb{Z})$.

The heart of the constructions will be the following result of Freedman and Quinn [3, Theorem 10.5] (which for simplicity has been specialized to the case at hand). Call a subgroup B of a free Abelian group A a *summand* if the quotient A/B is free.

Theorem (Freedman and Quinn). *Let W be a closed, 1-connected 4-manifold and $(V, \partial V)$ a compact, 1-connected 4-manifold with connected boundary. Let $h : V \to W$ be a map that preserves intersection numbers with $h_*H_2(V) \subset H_2(W)$ a summand. If ω_2 vanishes on $H_2(W)$ or does not vanish on the subspace perpendicular to $h_*H_2(V)$, then h is homotopic to a π_1-negligible embedding.*

Roughly the construction is as follows. Fix a particularly nice 1-connected 4-manifold V_0 with $\partial V_0 = M$. Then the 1-connected 4-manifolds with boundary M are just the complements of the π_1-negligible embeddings of V_0 in closed, 1-connected 4-manifolds. These embeddings by the theorem above are constructible by homotopy data.

Odd forms

Suppose L is an odd form (which will be fixed throughout this discussion). The classifying space $B_L^!(M)$ may be described as follows. Let $A(M)$ denote the set of all presentations of $H_*(M;\mathbb{Z})$ by $(\mathbb{Z}^n, L)$. Any two presentations differ by a pair of automorphisms (α_1, α_2) from $(H_1(M;\mathbb{Z}), H_2(M;\mathbb{Z}))$ to itself satisfying two compatibility conditions derived from (i) and (ii) above. Boyer defines $A(M)$ to be the group of all such pairs so this definition is equivalent to Boyer's. Define $B_L^t(M)$ to be the double coset space obtained by modding out by the pairs of maps induced by homeomorphisms of M and automorphisms of the form L. This coset space is studied in detail in Boyer [1]. However since we need only show existence and not uniqueness it is enough to find two 4-manifolds (differing in their Kirby–Siebenmann invariants) whose geometric presentation is the given one.

Fix a presentation

$$0 \longrightarrow H_2(M;\mathbb{Z}) \xrightarrow{h} \mathbb{Z}^n \xrightarrow{\mathrm{ad}(L)} [\mathbb{Z}^n]^* \xrightarrow{\partial} H_1(M;\mathbb{Z}) \longrightarrow 0$$

of $H_*(M)$ by $(\mathbb{Z}^n, L)$. The construction problem we are faced with is not changed if we switch to a different integral basis for $\mathbb{Z}^n$ and adjust the maps accordingly, therefore we may choose a particularly nice one. If $b_1(M) = k$, then $\mathrm{ad}(L)$ has a k-dimensional kernel and we may assume the basis $\{e_1, e_2, \ldots, e_n\}$ is chosen so that $\{e_1, e_2, \ldots, e_k\}$ is a basis for $\ker(\mathrm{ad}(L))$. In this basis for $\mathbb{Z}^n$ and the dual basis $\{e_i^*\}$ for $[\mathbb{Z}^n]^*$ $\mathrm{ad}(L)$ has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$$

where A is a symmetric integral matrix with $|\det(A)| = |T_1(M)|$.

Fix also a smooth, compact, 1-connected 4-manifold V_0 with boundary M. Assume further that the intersection form of V_0 is odd. For simplicity we may further assume that V_0 has a handlebody structure with only one 0-handle and some 2-handles. As above choose a basis $\{f_1, f_2, \ldots, f_m\}$ for $H_2(V_0;\mathbb{Z})$ in which the first k basis elements are in the kernel of $i_*: H_2(V_0;\mathbb{Z}) \to H_2(V_0, M;\mathbb{Z})$. After taking connected sums with $S^2 \times S^2$ sufficiently many times we may assume $m \geqslant n$. We may further assume this basis is chosen so that $\partial(f_i^*) = \partial(e_i^*)$ for $1 \leqslant i \leqslant n$ and $\partial(f_i^*) = 0$ for $i > n$. With respect to the basis $\{f_i\}$ and the Poincaré dual basis $\{f_i^*\}$ for $H_2(V_0, M;\mathbb{Z})$, i_* has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

where B is also a symmetric integral matrix with $|\det(B)| = |T_1(M)|$. After making handle passes we may assume this basis corresponds to the handlebody structure assumed on V_0.

We wish to find a closed, 1-connected 4-manifold W and a π_1-negligible embedding $h': V_0 \to W$ for which the complement $W - \mathrm{int}(h'(V_0))$ gives the presentation above. To see what W should be let (Y, M) be a hypothetical Poincaré pair with intersection form $(\mathbb{Z}^n, L)$. View V_0 as built from M by attaching 2-handles dual to the basis above, then a 4-handle. Adding only the 2-handles to (Y, M) will produce a space W_0 homotopy equivalent to a wedge of 2-spheres. We wish to describe the intersection form on this space. (Alternately one can simply define W by the formula we will derive below. Then one reverses this calculation to check that the complement is correct.)

A basis for $H_2(W_0)$ can be built as follows. Start with the basis $\{e_1, e_2, \dots, e_n\}$ for $H_2(Y)$. For $1 \leqslant i \leqslant n$ build classes by taking the dual class $e_i^* \in H_2(Y, M)$ and the dual $f_i^* \in H_2(V_0, M)$ and joining their boundaries by a surface Σ_i in M. For $i > n$ build classes by taking the dual to f_i and adjoining a null-homology of the boundary in M. Denote this basis by $\{e_1, e_2, \dots, e_n, g_1, \dots, g_m\}$.

Lemma 1. *With respect to the basis $\{e_1, e_2, \dots, e_n, g_1, \dots, g_m\}$ the intersection form on $H_2(W_0)$ has the following form*

$$Q = \begin{array}{c} \begin{array}{ccccc} k & n-k & k & n-k & m-n \end{array} \\ \begin{pmatrix} 0 & 0 & I & 0 & 0 \\ 0 & A & 0 & I & 0 \\ I & 0 & * & * & * \\ 0 & I & * & & \\ 0 & 0 & * & \multicolumn{2}{c}{A^{-1}-B^{-1}} \end{pmatrix} \end{array}$$

where B^{-1} is the inverse to B in $\mathrm{GL}(m-k, \mathbb{Q})$ *and A^{-1} is the inverse to A in* $\mathrm{GL}(n-k, \mathbb{Q})$ *extended by zeroes to be $(m-k) \times (m-k)$.*

Proof. Most of the entries specified are obvious from the construction. To see that the lower rightmost block is $A^{-1} - B^{-1}$ note that it is enough to compute the intersection of Ng_i with Ng_j where $N = |T_1(M)|$. For $1 \leqslant i \leqslant n$ a representative for Ng_i may be described as follows. Take N copies of e_i^* and cap off the boundary in M by a surface Σ' and take N copies of f_i^* in V_0 and cap off the boundary by the surface obtained from $-\Sigma'$ by adding a copy of Σ_i to each boundary component. Thus Ng_i is expressed as a union of a class α_i in Y and a class β_i in M. For $i > n$ we have a similar representative with $\alpha_i = 0$. The class α_i has $e_j \cdot \alpha_i = N\delta_{ij}$ therefore $\alpha_i = NA^{-1}e_i + \gamma$ where γ is some element of $\ker(\mathrm{ad}(L))$. Since γ does not affect intersection numbers this shows that $\alpha_i \cdot \alpha_j = N^2(A^{-1})_{ij}$. A similar result holds for the β_i thus $g_i \cdot g_j = (A^{-1})_{ij} - (B^{-1})_{ij}$, where by convention $(A^{-1})_{ij} = 0$ if either $i > n$ or $j > n$.

Note that the matrices A^{-1} and B^{-1} have rational coefficients however the difference is always integral since the inverse is up to sign the lift of the link pairing. Specifically $l_M(\partial(e_i^*), \partial(e_j^*)) \equiv -(A^{-1})_{ij} \pmod 1$. The unspecified entries depend on the details of the construction above. By choosing the Σ_i, $1 \leqslant i \leqslant k$, correctly we may arrange that the off-diagonal unspecified entries are zero and the diagonal ones are 0 or 1. Since the intersection form of V_0 is odd we may assume the diagonal entries are zero by altering the caps. □

Lemma 2. *The matrix above is unimodular.*

Proof. By elementary cancelations it is enough to show that the matrix

$$Q=\begin{pmatrix} A & I & 0 \\ I & \multicolumn{2}{c}{A^{-1}-B^{-1}} \\ 0 & & \end{pmatrix}$$

is invertible. Break B^{-1} into blocks as

$$B^{-1}=\begin{pmatrix} D & E^{\mathrm{T}} \\ E & F \end{pmatrix}.$$

Then Q becomes

$$Q=\begin{pmatrix} A & I & 0 \\ I & A^{-1}-D & -E^{\mathrm{T}} \\ 0 & -E & -F \end{pmatrix}.$$

Subtracting A times the second column from the first column gives

$$\begin{pmatrix} 0 & I & 0 \\ DA & A^{-1}-D & -E^{\mathrm{T}} \\ EA & -E & -F \end{pmatrix}$$

thus it is enough to show that

$$\begin{pmatrix} DA & -E^{\mathrm{T}} \\ EA & -F \end{pmatrix}$$

is invertible. Over $\mathbb{Q}$ this matrix can be written as

$$B^{-1}\begin{pmatrix} A & 0 \\ 0 & -I \end{pmatrix}.$$

The first term has determinant $\pm|T_1(M)|^{-1}$ and the second has determinant $\pm|T_1(M)|$, hence this matrix is invertible. □

From these two lemmas and the theorem of Freedman and Quinn above the existence is now clear. Let W be either of the two closed, oriented, 1-connected 4-manifolds whose intersection pairing is given by the matrix Q above (with the unspecified entries taken to be zero). Let $-V_0$ denote V_0 with the opposite orientation. The construction of W_0 includes an implicit map $h_*: H_2(-V_0)\to H_2(W)$ onto a summand and preserving intersection numbers. Since V_0 is homotopy equivalent to a wedge of 2-spheres this map is realized by a map $h: -V_0\to W$. Further ω_2 does not vanish on the perpendicular subspace to $h_* H_2(-V_0)$ since L and hence A is odd. Therefore h is homotopic to a π_1-negligible embedding $h': -V_0\to W$. Let $V=W-\operatorname{int}(h'(-V_0))$. Then V is a compact, oriented, 1-connected 4-manifold whose geometric presentation is the given one. Starting with the other closed, oriented, 1-connected 4-manifold whose intersection pairing is given by the matrix Q produces a second example with opposite Kirby–Siebenmann invariant.

Theorem 3. *If L is an odd symmetric bilinear form presenting $H_*(M;\mathbb{Z})$, then the map $c_L^t \times \Delta : \mathscr{V}_L(M) \to B_L^t(M) \times \mathbb{Z}/2\mathbb{Z}$ is bijective.*

Even forms

The case of even forms is almost the same as that of odd forms. Suppose we are given a presentation of $H_*(M;\mathbb{Z})$ by $(\mathbb{Z}^n, L)$. If V is any compact, oriented, 1-connected 4-manifold with intersection form L and boundary M, then V has a unique spin structure. This spin structure induces a spin structure on M. As shown in Boyer and sketched below only a special subset of the spin structures on M can occur in this way. The classifying space $\hat{B}_L(M)$ is again a double coset space. Start with all the pairs of a presentation of $H_*(M;\mathbb{Z})$ by $(\mathbb{Z}^n, L)$ and a spin structure on M of the type required and again quotient by homeomorphisms of M and automorphisms of L. As before since we are only interested in showing existence not uniqueness it will be enough to show that any such pair is realized by some 4-manifold V.

Before mimicking the argument above we must first discuss quadratic enhancements of the link pairing following [1,5]. Define a quadratic enhancement q of the link pairing to be a function $q : T_1(M) \to \mathbb{Q}/\mathbb{Z}$ satisfying

(i) $q(a+b) = q(a) + q(b) + l_M(a, b)$, for all $a, b \in T_1(M)$,

(ii) $q(ma) = m^2 q(a)$, for all $a \in T_1(M)$ and $m \in \mathbb{Z}$.

Let $I^1(M) = H^1(M;\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \subset H^1(M;\mathbb{Z}/2\mathbb{Z})$. With these definitions Taylor [5] shows the following

Proposition (Taylor). (1) *For any spin structure σ on M there is an associated quadratic enhancement q_σ of l_M and any quadratic enhancement q of l_M arises in this way.*

(2) *Two spin structures on M induce the same quadratic enhancement if and only if they differ by an element of $I^1(M)$.*

(3) *Let V be a compact*, 1-*connected*, *spin* 4-*manifold with boundary M and σ the spin structure induced by V. Let $\xi \in H_2(V;\mathbb{Z})$ have image $m\eta \in H_2(V, M;\mathbb{Z})$ with $m \neq 0$. Then $q_\sigma(\partial\eta) \equiv -(\xi \cdot \xi)/2m^2$* (mod 1).

As an immediate corollary we see that a presentation of $H_*(M)$ by $(\mathbb{Z}^n, L)$ determines a quadratic enhancement. If V is any closed, 1-connected, oriented 4-manifold with boundary M and intersection form L, it must induce a spin structure on M with this enhancement, i.e., in the appropriate orbit of the action of $I^1(M)$ on the spin structures. Boyer denotes this subset by $\mathrm{Spin}_L(M)$. To show existence we need only show that for any presentation and any spin structure

$\sigma \in \mathrm{Spin}_L(M)$ there is a closed, 1-connected, oriented 4-manifold with boundary M realizing this data.

We proceed as for odd forms except that we choose the manifold V_0 with the additional property that V_0 is spin and the induced spin structure on M is σ. The calculation above goes through exactly as before producing a unimodular matrix Q (again with the unspecified entries taken to be zero). Further since L and V_0 induce the same quadratic enhancement of the link pairing we have by part (3) of the proposition above that the diagonal entries of $A^{-1} - B^{-1}$ are even and Q determines an even form.

Let W be the unique closed, oriented, 1-connected 4-manifolds whose intersection pairing is given by the matrix Q above. The construction of W_0 includes an implicit map $h_* : H_2(-V_0) \to H_2(W)$ onto a summand and preserving intersection numbers. Since V_0 is homotopy equivalent to a wedge of 2-spheres this map is realized by a map $h : -V_0 \to W$. Further $\omega_2(W) = 0$ since Q is even. Therefore h is homotopic to a π_1-negligible embedding $h' : -V_0 \to W$. Let $V = W - \mathrm{int}(h'(-V_0))$. Then V is a compact, oriented, 1-connected 4-manifold whose geometric presentation is the given one. Further W has a unique spin structure which must induce the unique spin structures on $-V_0$ and V. The spin structure on $-V_0$ induces the structure σ on M hence so must V.

Theorem 4. *If L is an even symmetric bilinear form presenting $H_*(M;\mathbb{Z})$, then the map $\hat{c}_L : \mathscr{V}_L(M) \to \hat{B}_L(M)$ is bijective.*

References

[1] S. Boyer, Simply-connected 4-manifolds with a given boundary, Trans. Amer. Math. Soc. 298 (1986) 331–357.

[2] M. Freedman, The topology of 4-manifolds, J. Differential Geom. 17 (1982) 357–453.

[3] M. Freedman and F. Quinn, Topology of 4-Manifolds (Princeton University Press, Princeton, NJ, 1990).

[4] F. Quinn, Ends of maps III, Dimensions 4 and 5, J. Differential Geom. 17 (1982) 503–521.

[5] L. Taylor, Relative Rochlin invariants, Topology Appl. 18 (1984) 259–280.

[6] P. Vogel, Simply-connected 4-manifolds, Seminar Notes 1 (Aarhus University, Aarhus, 1982) 116–119.